TAMENESS IN GEOMETRY AND ARITHMETIC: BEYOND O-MINIMALITY

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ABSTRACT

The theory of o-minimal structures provides a powerful framework for the study of geometrically tame structures. In the past couple of decades a deep link connecting o-minimality to algebraic and arithmetic geometry has been developing. It has been clear, however, that the axioms of o-minimality do not fully capture some algebro-arithmetic aspects of tameness that one may expect in structures arising from geometry. We propose a notion of *sharply o-minimal structures* refining the standard axioms of o-minimality, and outline through conjectures and various partial results the potential development of this theory in parallel to the standard one.

We illustrate some applications of this emerging theory in two main directions. First, we show how it can be used to deduce Galois orbit lower bounds—notably including in non-abelian contexts where the standard *transcendence methods* do not apply. Second, we show how it can be used to derive effectivity and (polynomial-time) computability results for various problems of unlikely intersection around the Manin–Mumford, André–Oort, and Zilber–Pink conjectures.

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1. TAME GEOMETRY AND ARITHMETIC

1.1. O-minimal structures

The theory of o-minimal structures was introduced by van den Dries as an attempt to provide a framework of tame topology in the spirit of Grothendieck's "Esquisse d'un Programme" [42]. We refer the reader to this book for a general introduction to the subject and its history. For us, an o-minimal structure will always be an expansion of the ordered real field $\mathbb{R}_{\mathrm{alg}} := \{\mathbb{R}, +, \cdot, <\}$. Briefly, such an expansion is *o-minimal* if all definable subsets of $\mathbb{R}$ consist of finite unions of points and intervals.

Despite their apparent simplicity, it turns out that the axioms of o-minimality provide a broad framework of tame topology. In particular, one has good notions of dimension, smooth stratification, triangulation, and cell-decomposition for every definable set in an o-minimal structure. On the other hand, several natural and important structures turn out to be o-minimal. A few examples of particular importance for us in the present paper are $\mathbb{R}_{\mathrm{alg}}$, $\mathbb{R}_{\mathrm{an}}$, $\mathbb{R}_{\mathrm{an,exp}}$, and $\mathbb{R}_{\mathrm{Pfaff}}$. We will say a bit more on these in later sections.

1.2. Pila–Wilkie counting theorem

In [37], Pila and Wilkie discovered a "counting theorem" that would later find deep applications in arithmetic geometry. The theorem concerns the asymptotic density of rational (or algebraic) points in a definable set—as a function of height. We introduce this first, to motivate a broader discussion of the connection between tame geometry and arithmetic.

For $x \in \mathbb{Q}$, we denote by $H(x)$ the standard height of x. For a vector $x \in \mathbb{Q}^n$, we denote by $H(x)$ the maximum among the heights of the coordinates of x. For a set $A \subset \mathbb{R}^n$, we denote the set of $\mathbb{Q}$-points of A by $A(\mathbb{Q}) = A \cap \mathbb{Q}^n$ and denote

$$A(\mathbb{Q}, H) := \big\{x \in A(\mathbb{Q}) : H(x) \leqslant H\big\}. \tag{1.1}$$

For a set $A \subset \mathbb{R}^n$, we define the algebraic part A^{alg} of A to be the union of all connected semialgebraic subsets of A of positive dimension. We define the transcendental part A^{trans} of A to be $A \setminus A^{\mathrm{alg}}$.

Theorem 1 (Pila and Wilkie [37]). *Let $A \subset \mathbb{R}^m$ be a set definable in an o-minimal structure. Then for every $\epsilon > 0$ there exists a constant $C(A, \epsilon)$ such that for every $H \geqslant 1$,*

$$\#A^{\mathrm{trans}}(\mathbb{Q}, H) = C(A, \epsilon) H^{\epsilon}. \tag{1.2}$$

1.3. Transcendence methods, auxiliary polynomials

The use of transcendental (as opposed to algebraic) methods in the study of arithmetic questions has a long history. A common theme in these methods, running through the work of Schneider, Lang, Baker, Masser, and Wüstholz to name a few, is the use of *auxiliary polynomials*. We refer to [28] for a broad treatment of this subject.

The usefulness of polynomials in this context stems from their dual algebraic/analytic role. Suppose one is interested in the set $A(\mathbb{Q}, H)$ for some analytic set A. On the one hand, if a polynomial P, say, with integer coefficients, is evaluated at $x \in A(\mathbb{Q}, H)$ then $P(x)$ is again rational, and one can estimate its height in terms of H and the height of P. On the other hand, polynomials are extremely well-behaved analytic functions, and a variety of analytic methods may be used to prove upper bounds on the restriction of P to an analytic set A assuming it is appropriately constructed (say to vanish to high order at some points of A). One concludes from such an argument that P must vanish at every point in $A(\mathbb{Q}, H)$, for otherwise the height bound would contradict the upper bound.

The proof of the Pila–Wilkie counting theorem follows this classical line. However, it is fairly unique in the realm of transcendence methods in that the degrees of the auxiliary polynomials P are *independent* of the height, depending in fact only on ε. It is this unusual feature that makes it possible to prove the Pila–Wilkie theorem in the vast generality of o-minimal structures: polynomials of a given degree form a definable family, and the general machinery of o-minimality gives various finiteness statements uniformly for all such polynomials.

1.4. Beyond Pila–Wilkie theorem: the Wilkie conjecture

By contrast with the Pila–Wilkie theorem, most transcendence methods require the degrees of the auxiliary polynomials to depend on the height H of the points being considered—sometimes logarithmically and in some cases, such as the Schneider–Lang theorem, even linearly. A famous conjecture that seems to fall within this category is due to Wilkie.

Conjecture 2 (Wilkie [37]). *Let $A \subset \mathbb{R}^m$ be a set definable in $\mathbb{R}_{\exp}$. Then there exist constants $C(A), \kappa(A)$ such that for all $H \geqslant 3$,*

$$\#A^{\text{trans}}(\mathbb{Q}, H) = C(A)(\log H)^{\kappa(A)}. \tag{1.3}$$

The conclusion of the Wilkie conjecture is known to fail for general o-minimal structures, for instance, in $\mathbb{R}_{\text{an}}$ [40]. To achieve such asymptotics, it seems one would have to use auxiliary polynomials of degrees $d = (\log H)^q$, and o-minimality places no restrictions on the geometric complexity as a function of d.

In formulating his conjecture, Wilkie was probably influenced by Khovanskii's theory of fewnomials [25]. The latter implies fairly sharp bounds for the number of connected components of sets defined using algebraic and exponential functions (and more generally Pfaffian functions) as a function of the degrees of the equations involved. Below we attempt to axiomatize what it would mean for an arbitrary o-minimal structure to satisfy such sharp complexity bounds.

2. SHARPLY O-MINIMAL STRUCTURES

In this section we introduce *sharply o-minimal structures*, which are meant to endow a standard o-minimal structure with an appropriate notions comparable to dimension and degree in the algebraic case, and provide suitable control over these parameters under the basic logical operations. We first introduce the notion of a *format-degree* filtration (abbreviated *FD*-filtration) on a structure $\mathcal{S}$. This is a collection $\Omega = \{\Omega_{\mathcal{F},D}\}_{\mathcal{F},D\in\mathbb{N}}$ such that each

$\Omega_{\mathcal{F},D}$ is a collection of definable sets (possibly of different ambient dimensions), with

$$\Omega_{\mathcal{F},D} \subset \Omega_{\mathcal{F}+1,D} \cap \Omega_{\mathcal{F},D+1} \quad \forall \mathcal{F}, D \in \mathbb{N} \tag{2.1}$$

and $\bigcup_{\mathcal{F},D} \Omega$ is the collection of all definable sets in $\mathcal{S}$. We call the sets in $\Omega_{\mathcal{F},D}$ sets of *format* $\mathcal{F}$ and *degree* D. However, note that the format and degree of a set are not uniquely defined since Ω is a filtration rather than a partition.

We now come to the notion of a sharply o-minimal structure.

Definition 3 (Sharply o-minimal structure). A sharply o-minimal structure is a pair $\Sigma := (\mathcal{S}, \Omega)$ consisting of an o-minimal expansion of the real field $\mathcal{S}$ and an FD-filtration Ω; and for each $\mathcal{F} \in \mathbb{N}$, a polynomial $P_{\mathcal{F}}(\cdot)$ such that the following holds:

If $A \in \Omega_{\mathcal{F},D}$ then

(1) if $A \subset \mathbb{R}$, it has at most $P_{\mathcal{F}}(D)$ connected components,

(2) if $A \subset \mathbb{R}^{\ell}$ then $\mathcal{F} \geqslant \ell$,

(3) $A^c, \pi_{\ell-1}(A), A \times \mathbb{R}$, and $\mathbb{R} \times A$ lie in $\Omega_{\mathcal{F}+1,D}$.

Similarly if $A_1, \dots, A_k \subset \mathbb{R}^{\ell}$ with $A_j \in \Omega_{\mathcal{F}_j,D_j}$ then

$$(4) \quad \bigcup_i A_i \in \Omega_{\mathcal{F},D}, \quad (5) \quad \bigcap_i A_i \in \Omega_{\mathcal{F}+1,D},$$

where $\mathcal{F} := \max_j \mathcal{F}_j$ and $D = \sum_j D_j$. Finally,

(6) if $P \in \mathbb{R}[x_1, \dots, x_\ell]$ then $\{P = 0\} \in \Omega_{\ell, \deg P}$.

Given a collection $\{A_\alpha\}$ of sets generating a structure $\mathcal{S}$, and associated formats and degrees $\mathcal{F}_\alpha, D_\alpha$ one can consider the minimal FD-filtration Ω satisfying the axioms (2)–(6) above. We call this the FD-filtration generated by $\{(A_\alpha, \mathcal{F}_\alpha, D_\alpha)\}$. This will be sharply o-minimal if and only if axiom (1) is satisfied.

Definition 4 (Reduction of FD-filtrations). Let Ω, Ω' be two FD-filtrations on a structure $\mathcal{S}$. We say that Ω is *reducible* to Ω' and write $\Omega \leqslant \Omega'$ if there exist functions $a : \mathbb{N} \to \mathbb{N}$ and $b : \mathbb{N} \to \mathbb{N}[D]$ such that

$$\Omega_{\mathcal{F},D} \subset \Omega'_{a(\mathcal{F}),[b(\mathcal{F})](D)} \quad \forall \mathcal{F}, D \in \mathbb{N}. \tag{2.2}$$

We say that Ω, Ω' are equivalent if $\Omega \leqslant \Omega' \leqslant \Omega$.

We will usually try to prove that certain measures of complexity of definable sets depend polynomially on the degree, thinking of the format as constant. If one can prove such a statement for Ω'-degrees, and $\Omega \leqslant \Omega'$, then the same statement holds for Ω-degrees and in this sense Ω is reducible to Ω'.

Remark 5 (Effectivity). One can require further that a sharply o-minimal structure is *effective*, in the sense that the polynomial $P_{\mathcal{F}}(D)$ in Definition 3 is given by some explicit primitive recursive function of $\mathcal{F}$. Similarly, one may require a reduction $\Omega \leqslant \Omega'$ to be effective. Unless otherwise stated, all constructions in this paper are effective in this sense.

2.1. Examples and nonexamples

2.1.1. The semialgebraic structure

Consider the structure $\mathbb{R}_{\text{alg}}$ with the FD-filtration Ω generated by all algebraic hypersurfaces $\{P = 0\}$ with the format given by the ambient dimension and the degree given by $\deg P$. Then $(\mathbb{R}_{\text{alg}}, \Omega)$ is a sharply o-minimal structure. This is not an immediate statement: it follows from the results on effective cell decomposition, or elimination of quantifiers, in semialgebraic geometry [3].

Perhaps a more natural notion of format and degree in the semialgebraic category is as follows. Define $\Omega'_{\mathcal{F},D}$ to be the subsets of $\mathbb{R}^\ell$ with $\ell \leqslant \mathcal{F}$, that can be written as a union of basic sets

$$\{P_1 = \cdots = P_k = 0,\ Q_1 > 0, \ldots, Q_l > 0\} \tag{2.3}$$

with the sum of the degrees of the P_i and Q_j, over all basic sets, bounded by D. This is not sharply o-minimal according to our definition because it does not satisfy axiom (3), for instance. However, it is equivalent to Ω defined above.

2.1.2. The analytic structure $\mathbb{R}_{\text{an}}$

Not surprisingly, $\mathbb{R}_{\text{an}}$ is not sharply o-minimal with respect to any FD-filtration. Assume the contrary. Let $\omega_1 = 1$ and $\omega_{n+1} = 2^{\omega_n}$, and let $\Gamma = \{y = f(z)\} \subset \mathbb{C}^2$ denote the graph of the holomorphic function $f(z) = \sum_{j=1}^{\infty} z^{\omega_j}$ restricted to the disc of radius $1/2$ (which is definable in $\mathbb{R}_{\text{an}}$). Then by axioms (1), (5) and (6), the number of points in

$$\Gamma \cap \left\{ y = \varepsilon + \sum_{j=1}^{n} z^{\omega_j} \right\} \tag{2.4}$$

should be polynomial in ω_n, with the exact polynomial depending on the format and degree of Γ. But it is, in fact, $\omega_{n+1} = 2^{\omega_n}$ for $0 < \varepsilon \ll 1$, and we have a contradiction for $n \gg 1$.

2.1.3. Pfaffian structures

Let $B \subset \mathbb{R}^\ell$ be a domain, which for simplicity we take to be a product of (possibly infinite) intervals. A tuple $f_1, \ldots, f_m : B \to \mathbb{R}$ of analytic functions is called a *Pfaffian chain* if they satisfy a triangular system of algebraic differential equations of the form

$$\frac{\partial f_i}{\partial x_j} = P(x_1, \ldots, x_\ell, f_1, \ldots, f_i), \quad \forall i, j. \tag{2.5}$$

They are called *restricted* if B is bounded and $f_1, \ldots, f_m$ extend as real analytic functions to $\bar{B}$. A Pfaffian function is a polynomial $Q(x_1, \ldots, x_\ell, f_1, \ldots, f_m)$. We denote the structure generated by the Pfaffian functions by $\mathbb{R}_{\text{Pfaff}}$, and its restricted analog by $\mathbb{R}_{\text{rPfaff}}$.

Khovanskii [25] proved upper bounds for the number of connected components of systems of Pfaffian equations. This was later extended by Gabrielov and Vorobjov to sets defined using inequalities and quantifiers [21]. However, their results fall short of establishing the sharp o-minimality of $\mathbb{R}_{\text{rPfaff}}$. The problem is that for Gabrielov–Vorobjov's notion of format and degree, if $A \in \Omega_{\mathcal{F},D}$ then they are only able to show that $A^c \in \Omega_{P_{\mathcal{F}}(D), P_{\mathcal{F}}(D)}$

rather than $A^c \in \Omega_{\mathcal{F}+1,P_{\mathcal{F}}(D)}$ as required by our axioms. This is a fundamental difficulty, as it is essential in our setup that the format never becomes dependent on the degree.

In [13] the first author and Vorobjov introduce a modified notion of format and degree and prove the following.

Theorem 6. *There is an FD-filtration Ω on $\mathbb{R}_{\mathrm{rPfaff}}$ that makes it into a sharply o-minimal structure. Moreover, Gabrielov–Vorobjov's standard filtration is reducible to Ω.*

We conjecture that this theorem extends to the structure $\mathbb{R}_{\mathrm{Pfaff}}$, and this is the subject of work in progress by the first author and Vorobjov utilizing some additional ideas of Gabrielov [19].

2.2. Cell decomposition in sharply o-minimal structures

We recall the notion of a *cell* in an o-minimal structure. A cell $C \subset \mathbb{R}$ is either a point or an open interval (possibly infinite). A cell $C \subset \mathbb{R}^{\ell+1}$ is either the graph of a definable continuous function $f : C' \to \mathbb{R}$ where $C' \subset \mathbb{R}^\ell$ is a cell, or the area strictly between two graphs of such definable continuous functions $f, g : C' \to \mathbb{R}$ satisfying $f < g$ identically on C'. One can also take $f = -\infty$ and $g = \infty$ in this definition.

We say that a cell $C \subset \mathbb{R}^\ell$ is *compatible* with $X \subset \mathbb{R}^\ell$ if it is either strictly contained, or strictly disjoint from X. The following cell decomposition theorem can be viewed as the raison d'être of the axioms of o-minimality.

Theorem 7 (Cell decomposition). *Let $X_1, \dots, X_k \subset \mathbb{R}^\ell$ be definable sets. Then there is a decomposition of $\mathbb{R}^\ell$ into pairwise disjoint cells that are pairwise compatible with $X_1, \dots, X_k$.*

Given the importance of cell decomposition in the theory of o-minimality, it is natural to pose the following question.

Question 8. If $\mathcal{S}$ is sharply o-minimal and $X_1, \dots, X_k$ have format $\mathcal{F}$ and degree D, can one find a cell-decomposition where each cell has format $\mathrm{const}(\mathcal{F})$, and the number of cells and their degrees are bounded by $\mathrm{poly}_{\mathcal{S},\mathcal{F}}(k, D)$?

We suspect the answer to this question may be negative. Since cell decomposition is perhaps the most crucial construction in o-minimality, this is a fundamental problem. The following result rectifies the situation.

Theorem 9. *Let $(\mathcal{S}, \Omega)$ be sharply o-minimal. Then there exists another FD-filtration Ω' with $(\mathcal{S}, \Omega')$ sharply o-minimal such that $\Omega \leqslant \Omega'$, and in Ω' the following holds.*

Let $X_1, \dots, X_k \in \Omega'_{\mathcal{F},D}$, all subsets of $\mathbb{R}^\ell$. Then there exists a cell decomposition of $\mathbb{R}^\ell$ compatible with each X_j such that each cell has format $\mathrm{const}(\mathcal{F})$*, the number of cells is* $\mathrm{poly}_{\mathcal{F}}(k, D)$*, and the degree of each cell is* $\mathrm{poly}_{\mathcal{F}}(D)$.

In the structure $\mathbb{R}_{\mathrm{rPfaff}}$, Theorem 9 is one of the main results of [13]. The general case is obtained by generalizing the proof to the general sharply o-minimal case, and is part of the PhD thesis of Binyamin Zack-Kutuzov.

2.3. Yomdin–Gromov algebraic lemma in sharply o-minimal structures

Let $I := (0,1)$. For $f : I^n \to \mathbb{R}^m$ a C^r-smooth map, we denote

$$\|f\|_r := \sup_{x\in I^n} \max_{|\alpha|\leqslant r} \|f^{(\alpha)}(x)\|. \tag{2.6}$$

The Yomdin–Gromov algebraic lemma is a result about C^r-smooth parametrizations of bounded norm for definable subsets of I^n. A sharply o-minimal version of this lemma is as follows.

Lemma 10. *Let $(\mathcal{S}, \Omega)$ be sharply o-minimal. Then there is a polynomial $P_{\mathcal{F},r}(\cdot)$ depending on the pair $(\mathcal{F}, r)$, such that for every $A \in \Omega_{\mathcal{F},D}$ the following holds. There exist a collection of maps $\{f_\alpha : I^{n_\alpha} \to A\}$ of size at most $P_{\mathcal{F},r}(D)$ such that $\bigcup_\alpha f_\alpha(I^{n_\alpha}) = A$; and $\|f_\alpha\|_r \leqslant 1$ and $n_\alpha \leqslant \dim A$ for every α.*

In the algebraic case, this result is due to Gromov [23], based on a similar but slightly more technically involved statement by Yomdin [44]. In the general o-minimal case, but without complexity bounds, the result is due to Pila and Wilkie [37]. In the restricted Pfaffian case, this result is due to the first author with Jones, Schmidt, and Thomas [6] using Theorem 9 in the $\mathbb{R}_{\mathrm{rPfaff}}$ case. The general case follows in the same way.

The following conjecture seems plausible, though we presently do not have an approach to proving it in this generality.

Conjecture 11. *In Lemma 10, one can replace $P_{\mathcal{F},r}(D)$ by a $P_{\mathcal{F}}(D,r)$, i.e., by a polynomial in both D and r, depending only on $\mathcal{F}$.*

In the structure $\mathbb{R}_{\mathrm{alg}}$ this was conjectured by Yomdin (unpublished) and by Burguet [14], in relation to a conjecture of Yomdin [45, CONJECTURE 6.1] concerning the rate of decay of the tail entropy for real-analytic mappings. The conjecture was proved in [9] by complex-analytic methods. We will say more about the possible generalization of these methods to more general sharply o-minimal structures in Section 3.

2.4. Pila–Wilkie theorem in sharply o-minimal structures

We now state a form of the Pila–Wilkie counting theorem, Theorem 1, with explicit control over the asymptotic constant.

Theorem 12. *Let $(\mathcal{S}, \Omega)$ be sharply o-minimal. Then for every $\epsilon > 0$ and $\mathcal{F}$ there is a polynomial $P_{\mathcal{F},\epsilon}(\cdot)$ depending on $(\mathcal{S}, \Omega)$, such that for every $A \in \Omega_{\mathcal{F},D}$ and $H \geqslant 2$,*

$$\#A^{\mathrm{trans}}(\mathbb{Q}, H) = P_{\mathcal{F},\epsilon}(D) \cdot H^\epsilon. \tag{2.7}$$

This result is based on Lemma 10, in the same way as the classical Pila–Wilkie theorem is based on the o-minimal reparametrization lemma. This reduction is carried out in [6] using Theorem 9 in the $\mathbb{R}_{\mathrm{rPfaff}}$ case. The general case follows in the same way.

2.5. Polylog counting in sharply o-minimal structures

We state a conjectural sharpening of the Pila–Wilkie theorem, in line with the Wilkie conjecture, in the context of sharply o-minimal structures. For $A \subset \mathbb{R}^\ell$, let

$$A(g,h) := \{x \in \mathbb{A} \cap \bar{\mathbb{Q}}^\ell : [\mathbb{Q}(x) : \mathbb{Q}] \leqslant g, h(x) \leqslant h\}, \tag{2.8}$$

where $h(\cdot)$ denotes the logarithmic Weil height.

Conjecture 13. *Let $(\mathcal{S}, \Omega)$ be sharply o-minimal. Then there is a polynomial $P_{\mathcal{F}}(\cdot,\cdot,\cdot)$ depending only on $(\mathcal{S}, \Omega)$ and $\mathcal{F}$, such that for every $A \in \Omega_{\mathcal{F},D}$ and $g, h \geqslant 2$,*

$$\# A^{\mathrm{trans}}(g,h) \leqslant P_{\mathcal{F}}(D,g,h). \tag{2.9}$$

The conjecture sharpens Pila–Wilkie in two ways. First, we replace the subpolynomial term H^ε by a polynomial in $h \sim \log H$. Second, we count algebraic points of arbitrary degree, and stipulate polynomial growth with respect to the degree as well.

Conjecture 13 is currently known only for the structure of *restricted elementary functions* $\mathbb{R}^{\mathrm{RE}} := (\mathbb{R}, +, \cdot, <, \exp|_{[0,1]}, \sin|_{[0,\pi]})$ where it is due to [8] (with a minor technical improvement in [5]).

Combining the various known techniques in the literature, it is not hard to see that Conjecture 11 implies Conjecture 13 in a general sharply o-minimal structure. In Section 5 we will see that Conjecture 13 has numerous applications in arithmetic geometry, going beyond the standard applications of the Pila–Wilkie theorem. We also discuss some partial results in the direction of Conjecture 13 in Section 4.4.

3. COMPLEX ANALYTIC THEORY

In this section we consider holomorphic analogs of the standard cell decomposition of o-minimality. We fix a sharply o-minimal structure $(\mathcal{S}, \Omega)$ throughout. We also assume that $\mathcal{S}$ admits cell-decomposition in the sense of Theorem 9, as we may always reduce to this case.

3.1. Complex cells

We start be defining the notion of a *complex cell*. This is a complex analog of the cells used in o-minimal geometry.

3.1.1. Basic fibers and their extensions

For $r \in \mathbb{C}$ (resp. $r_1, r_2 \in \mathbb{C}$) with $|r| > 0$ (resp. $|r_2| > |r_1| > 0$), we denote

$$\begin{gathered} D(r) := \{|z| < |r|\}, \quad D_\circ(r) := \{0 < |z| < |r|\}, \quad D_\infty(r) = \{|r| < |z| < \infty\}, \\ A(r_1, r_2) := \{|r_1| < |z| < |r_2|\}, \quad * := \{0\}. \end{gathered} \tag{3.1}$$

For any $0 < \delta < 1$, we define the δ-extensions by

$$\begin{gathered} D^\delta(r) := D(\delta^{-1} r), \quad D^\delta_\circ(r) := D_\circ(\delta^{-1} r), \quad D^\delta_\infty(r) := D_\infty(\delta r), \\ A^\delta(r_1, r_2) := A(\delta r_1, \delta^{-1} r_2), \quad *^\delta := *. \end{gathered} \tag{3.2}$$

For any $0 < \rho < \infty$, we define the $\{\rho\}$-extension $\mathcal{F}^{\{\rho\}}$ of $\mathcal{F}$ to be $\mathcal{F}^{\delta}$ where δ satisfies the equations

$$\begin{aligned} \rho &= \frac{2\pi\delta}{1-\delta^2} \quad \text{for } \mathcal{F} \text{ of type } D, \\ \rho &= \frac{\pi^2}{2|\log\delta|} \quad \text{for } \mathcal{F} \text{ of type } D_\circ, D_\infty, A. \end{aligned} \tag{3.3}$$

The motivation for this notation comes from the following fact, describing the hyperbolic-metric properties of a domain $\mathcal{F}$ within its $\{\rho\}$-extension.

Fact 14. *Let $\mathcal{F}$ be a domain of type $A, D, D_\circ, D_\infty$ and let S be a component of the boundary of $\mathcal{F}$ in $\mathcal{F}^{\{\rho\}}$. Then the length of S in $\mathcal{F}^{\{\rho\}}$ is at most ρ.*

3.1.2. The definition of a complex cell

Let $\mathcal{X}, \mathcal{Y}$ be sets and $\mathcal{F} : \mathcal{X} \to 2^{\mathcal{Y}}$ be a map taking points of $\mathcal{X}$ to subsets of $\mathcal{Y}$. Then we denote

$$\mathcal{X} \odot \mathcal{F} := \{(x, y) : x \in \mathcal{X}, y \in \mathcal{F}(x)\}. \tag{3.4}$$

If $r : \mathcal{X} \to \mathbb{C} \setminus \{0\}$ then for the purpose of this notation we understand $D(r)$ as the map assigning to each $x \in \mathcal{X}$ the disc $D(r(x))$, and similarly for $D_\circ, D_\infty, A$.

We now introduce the notion of a complex cell of *length* $\ell \in \mathbb{Z}_{\geqslant 0}$. If $U \subset \mathbb{C}^n$ is a definable domain, we denote by $\mathcal{O}_d(U)$ the space of definable holomorphic functions on U. As a shorthand we denote $\mathbf{z}_{1..\ell} = (\mathbf{z}_1, \dots, \mathbf{z}_\ell)$.

Definition 15 (Complex cells). A complex cell $\mathcal{C}$ of *length* zero is the point $\mathbb{C}^0$. A complex cell of length $\ell + 1$ has the form $\mathcal{C}_{1..\ell} \odot \mathcal{F}$ where the *base* $\mathcal{C}_{1..\ell}$ is a cell of length ℓ, and the *fiber* $\mathcal{F}$ is one of $*$, $D(r)$, $D_\circ(r)$, $D_\infty(r)$, $A(r_1, r_2)$ where $r \in \mathcal{O}_d(\mathcal{C}_{1..\ell})$ satisfies $|r(\mathbf{z}_{1..\ell})| > 0$ for $\mathbf{z}_{1..\ell} \in \mathcal{C}_{1..\ell}$; and $r_1, r_2 \in \mathcal{O}_d(\mathcal{C}_{1..\ell})$ satisfy $0 < |r_1(\mathbf{z}_{1..\ell})| < |r_2(\mathbf{z}_{1..\ell})|$ for $\mathbf{z}_{1..\ell} \in \mathcal{C}_{1..\ell}$.

Next, we define the notion of a δ-extension (resp. $\{\rho\}$-extension).

Definition 16. The cell of length zero is defined to be its own δ-extension. A cell $\mathcal{C}$ of length $\ell + 1$ admits a δ-extension $\mathcal{C}^\delta := \mathcal{C}^\delta_{1..\ell} \odot \mathcal{F}^\delta$ if $\mathcal{C}_{1..\ell}$ admits a δ-extension, and if the function r (resp. r_1, r_2) involved in $\mathcal{F}$ admits holomorphic continuation to $\mathcal{C}^\delta_{1..\ell}$ and satisfies $|r(\mathbf{z}_{1..\ell})| > 0$ (resp. $0 < |r_1(\mathbf{z}_{1..\ell})| < |r_2(\mathbf{z}_{1..\ell})|$) in this larger domain. The $\{\rho\}$-extension $\mathcal{C}^{\{\rho\}}$ is defined in an analogous manner.

As a shorthand, when say that $\mathcal{C}^\delta$ is a complex cell (resp. $\mathcal{C}^{\{\rho\}}$) we mean that $\mathcal{C}$ is a complex cell admitting a δ (resp $\{\rho\}$) extension.

3.1.3. The real setting

We introduce the notion of a *real* complex cell $\mathcal{C}$, which we refer to simply as *real cells* (but note that these are subsets of $\mathbb{C}^\ell$). We also define the notion of *real part* of a real cell $\mathcal{C}$ (which lies in $\mathbb{R}^\ell$), and of a *real* holomorphic function on a real cell. Below we let $\mathbb{R}_+$ denote the set of positive real numbers.

Definition 17 (Real complex cells). The cell of length zero is real and equals its real part. A cell $\mathcal{C} := \mathcal{C}_{1..\ell} \odot \mathcal{F}$ is real if $\mathcal{C}_{1..\ell}$ is real and the radii involved in $\mathcal{F}$ can be chosen to be real holomorphic functions on $\mathcal{C}_{1..\ell}$; The real part $\mathbb{R}\mathcal{C}$ (resp. positive real part $\mathbb{R}_+\mathcal{C}$) of $\mathcal{C}$ is defined to be $\mathbb{R}\mathcal{C}_{1..\ell} \odot \mathbb{R}\mathcal{F}$ (resp. $\mathbb{R}_+\mathcal{C}_{1..\ell} \odot \mathbb{R}_+\mathcal{F}$) where $\mathbb{R}\mathcal{F} := \mathcal{F} \cap \mathbb{R}$ (resp. $\mathbb{R}_+\mathcal{F} := \mathcal{F} \cap \mathbb{R}_+$) except the case $\mathcal{F} = *$, where we set $\mathbb{R}* = \mathbb{R}_+ * = *$; A holomorphic function on $\mathcal{C}$ is said to be real if it is real on $\mathbb{R}\mathcal{C}$.

3.2. Cellular parametrization

We now state a result that can be viewed as a complex analog of the cell decomposition theorem. We start by introducing the notion of prepared maps.

Definition 18 (Prepared maps). Let $\mathcal{C}, \hat{\mathcal{C}}$ be two cells of length ℓ. We say that a holomorphic map $f : \mathcal{C} \to \hat{\mathcal{C}}$ is *prepared* if it takes the form $\mathbf{w}_j = \mathbf{z}_j^{q_j} + \phi_j(\mathbf{z}_{1..j-1})$ where $\phi_j \in \mathcal{O}_d(\mathcal{C}_{1..j})$ for $j = 1, \dots, \ell$.

Since our cells are always centered at the origin, it is the images of cellular maps that should be viewed as analogous to the cells of o-minimality. The additional exponent q_j in Definition 18 is needed to handle ramification issues that are not visible in the real context.

Definition 19. For a complex cell $\mathcal{C}$ and $F \in \mathcal{O}_d(\mathcal{C})$ we say that F is *compatible* with $\mathcal{C}$ if F vanishes either identically or nowhere on $\mathcal{C}$. For a cellular map $f : \hat{\mathcal{C}} \to \mathcal{C}$, we say that f is compatible with F if f^*F is compatible with $\hat{\mathcal{C}}$.

We will be interested in covering (real) cells by prepared images of (real) cells.

Definition 20. Let $\mathcal{C}^{\{\rho\}}$ be a cell and $\{f_j : \mathcal{C}_j^{\{\sigma\}} \to \mathcal{C}^{\{\rho\}}\}$ be a finite collection of cellular maps. We say that this collection is a *cellular cover* of $\mathcal{C}$ if $\mathcal{C} \subset \bigcup_j (f_j(\mathcal{C}_j))$. Similarly, we say it is a *real* cellular cover if $\mathbb{R}_+\mathcal{C} \subset \bigcup_j (f_j(\mathbb{R}_+\mathcal{C}_j))$.

Finally, we can state our main conjecture on complex cellular parametrizations.

Conjecture 21 (Cellular Parametrization Theorem, CPT). *Let $\rho, \sigma \in (0, \infty)$. Let $\mathcal{C}^{\{\rho\}}$ be a (real) cell and $F_1, \dots, F_M \in \mathcal{O}_d(\mathcal{C}^{\{\rho\}})$ (real) holomorphic functions, with $\mathcal{C}^{\{\rho\}}$ and each F_j having format $\mathcal{F}$ and degree D. Then there exists a (real) cellular cover $\{f_j : \mathcal{C}_j^{\{\sigma\}} \to \mathcal{C}^{\{\rho\}}\}$ such that each f_j is prepared and compatible with each F_k. The number of cells is* $\operatorname{poly}_{\mathcal{F}}(D, M, \rho, 1/\sigma)$*, and each of them has format* $\operatorname{const}(\mathcal{F})$ *and degree* $\operatorname{poly}_{\mathcal{F}}(D)$.

The main result of [9] is that Conjecture 21 holds in the structure $\mathbb{R}_{\text{alg}}$ (we assume there for technical convenience that the functions are bounded rather than just definable, but this does not seem to be a serious obstacle). We remark that there are significant difficulties with extending this proof to the general sharply o-minimal case.

3.3. Analytically generated structures

We say that a sharply o-minimal structure $(\mathcal{S}, \Omega)$ is *analytically generated* if there is a collection of complex cells $\{\mathcal{C}_\alpha\}$ admitting a $1/2$-extension, and associated formats and degrees $(\mathcal{F}_\alpha, D_\alpha)$ such that $\mathcal{S}$ is generated by $\{\mathcal{C}_\alpha\}$ and Ω is generated by $\{(\mathcal{C}_\alpha, \mathcal{F}_\alpha, D_\alpha)\}$. We fix such a structure $\mathcal{S}$ below. Assuming the CPT, one can prove the following analog of Theorem 9 giving a cell decomposition by real parts of complex analytic cells.

Theorem 22. *Let $(\mathcal{S}, \Omega)$ be sharply o-minimal and assume that it satisfies the CPT. Then there exists another FD-filtration Ω' with $(\mathcal{S}, \Omega')$ sharply o-minimal such that $\Omega \leqslant \Omega'$, and in Ω' the following holds.*

Let $X_1, \dots, X_k \in \Omega'_{\mathcal{F},D}$, all subsets of $\mathbb{R}^\ell$. Then there exists a real cellular cover $\{f_j : \mathcal{C}_j^{\{\sigma\}} \to \mathbb{C}^\ell\}$ such that each f_j is prepared, and each $f_j(\mathbb{R}_+\mathcal{C}_j^{\{\sigma\}})$ is compatible with each X_i. The number of cells is $\operatorname{poly}_{\mathcal{F}}(D, k, 1/\sigma)$, and each of them has format $\operatorname{const}(\mathcal{F})$ and degree $\operatorname{poly}_{\mathcal{F}}(D)$.

In particular, the cells $f_j(\mathbb{R}_+\mathcal{C}_j) \subset \mathbb{R}^\ell$ form a cell-decomposition of $\mathbb{R}^\ell$ compatible with $X_1, \dots, X_k$. In addition, each cell admits "analytic continuation" to a complex cell $\mathcal{C}_j$ with a $\{\sigma\}$-extension.

In [9] it is shown that from a parametrization of the type provided by Theorem 22 one can produce C^r-smooth parametrizations, with the number of maps depending polynomially on both D and r. In particular, the conclusion of Theorem 22 implies Conjectures 11 and 13. It therefore seems that proving the CPT in a general analytically-generated sharply o-minimal structure provides a plausible approach to these two conjectures.

We remark that a different complex analytic approach, based on the notion of *Weierstrass polydiscs*, was employed in [8] to prove the Wilkie conjecture in the structure $\mathbb{R}^{\mathrm{RE}}$. This may also give an approach to proving Conjecture 13 in general, but it does not seem to be applicable to Conjecture 11.

3.4. Complex cells, hyperbolic geometry, and preparation theorems

The main motivation for introduction the notion of $\{\rho\}$-extensions of complex cells is that one can use the hyperbolic geometry of $\mathcal{C}$ inside $\mathcal{C}^{\{\rho\}}$ to control the geometry of holomorphic functions defined on complex cells. This is used extensively in the proof of the algebraic CPT in [9], but also gives statements of independent interest. We illustrate two of the main statements.

For any hyperbolic Riemann surface X, we denote by $\operatorname{dist}(\cdot,\cdot; X)$ the hyperbolic distance on X. We use the same notation when $X = \mathbb{C}$ and $X = \mathbb{R}$ to denote the usual Euclidean distance, and when $X = \mathbb{C}P^1$ to denote the Fubini–Study metric normalized to have diameter 1. For $x \in X$ and $r > 0$, we denote by $B(x, r; X)$ the open r-ball centered at x in X. For $A \subset X$, we denote by $B(A, r; X)$ the union of r-balls centered at all points of A.

Lemma 23 (Fundamental lemma for $\mathbb{C} \setminus \{0, 1\}$). *Let $\mathcal{C}^{\{\rho\}}$ be a complex cell and let $f : \mathcal{C}^{\{\rho\}} \to \mathbb{C} \setminus \{0, 1\}$ be holomorphic. Then one of the following holds:*

$$f(\mathcal{C}) \subset B\big(\{0, 1, \infty\}, e^{-\Omega_\ell(1/\rho)}; \mathbb{C}P^1\big) \quad \textit{or} \quad \operatorname{diam}\big(f(\mathcal{C}); \mathbb{C} \setminus \{0, 1\}\big) = O_\ell(\rho). \tag{3.5}$$

The fundamental lemma for $\mathbb{C} \setminus \{0, 1\}$ implies the Great Picard Theorem: indeed, taking $\mathcal{C}$ to be a small punctured disc $D_\circ$ around the origin, it implies that any function

$f : D_\circ \to \mathbb{C} \setminus \{0,1\}$ has an image of small diameter in $\mathbb{C}P^1$, hence is bounded away from some $w \in \mathbb{C}P^1$, and it follows elementarily that f is meromorphic at the origin.

If $f : \mathcal{C}^{\{\rho\}} \to \mathbb{C} \setminus \{0\}$ is a bounded holomorphic map then we may decompose it as $f = \mathbf{z}^{\boldsymbol{\alpha}(f)} \cdot U(\mathbf{z})$, where $U : \mathcal{C}^{\{\rho\}} \to \mathbb{C} \setminus \{0\}$ is a holomorphic map and the branches of $\log U : \mathcal{C}^{\{\rho\}} \to \mathbb{C}$ are univalued. The following lemma shows that U enjoys strong boundedness properties when restricted to $\mathcal{C}$.

Lemma 24 (Monomialization lemma). *Let $0 < \rho < \infty$ and let $f : \mathcal{C}^{\{\rho\}} \to \mathbb{C} \setminus \{0\}$ be a holomorphic map. If $\mathcal{C}^{\{\rho\}}, f \in \Omega_{\mathcal{F},D}$ then there exists a polynomial $P_{\mathcal{F}}(\cdot)$ such that $|\boldsymbol{\alpha}(f)| \leqslant P_{\mathcal{F}}(D)$ and*

$$\operatorname{diam}\bigl(\log U(\mathcal{C}); \mathbb{C}\bigr) < P_{\mathcal{F}}(D) \cdot \rho, \quad \operatorname{diam}\bigl(\operatorname{Im} \log U(\mathcal{C}); \mathbb{R}\bigr) < P_{\mathcal{F}}(D). \tag{3.6}$$

The monomialization lemma is proved in this form for the structure $\mathbb{R}_{\text{alg}}$ in **[9]**, but the proof extends to the general sharply o-minimal case. It is also shown in **[9]** that the monomialization lemma in combination with the CPT gives an effective version of the subanalytic preparation theorem of Parusinski **[33]** and Lion–Rolin **[26]**, which is a key technical tool in the theory of the structure $\mathbb{R}_{\text{an}}$.

3.5. Unrestricted exponentials

One of the milestones in the development of o-minimality is Wilkie's theorem on the model-completeness of $\mathbb{R}_{\exp}$ **[43]**, which, together with Khovanskii's theory of fewnomials, established the o-minimality of $\mathbb{R}_{\exp}$. Wilkie's methods were later used by van den Dries and Miller to establish the o-minimality of $\mathbb{R}_{\text{an,exp}}$. This latter structure plays a key role in many of the applications of o-minimality to arithmetic geometry, since it contains the uniformizing maps of (mixed) Shimura varieties restricted to an appropriate fundamental domain. We conjecture a sharply o-minimal version of the theorems of Wilkie and van den Dries, Miller as follows.

Conjecture 25. *Let $(\mathcal{S}, \Omega)$ be an analytically generated sharply o-minimal structure. Let $\mathcal{S}_{\exp}$ denote the structure generated by $\mathcal{S}$ and the unrestricted exponential, and let $\Omega_{\exp}$ be the FD-filtration of $\mathcal{S}_{\exp}$ generated by Ω and by the graph of the unrestricted exponential (say with format and degree 1). Then $(\mathcal{S}_{\exp}, \Omega_{\exp})$ is sharply o-minimal.*

It is perhaps plausible to make the same conjecture even without the assumption of analytic generation. However, the analytic case appears to be sufficient for all (currently known) applications, and the availability of the tools discussed in this section make the conjecture seem somewhat more amenable in this case. In particular, Lion–Rolin **[26]** have a geometric approach to the o-minimality of $\mathbb{R}_{\text{an,exp}}$ using the subanalytic preparation theorem as a basic tool. The CPT provides a sharp version of the subanalytic preparation theorem, thus suggesting a possible path to the proof of Conjecture 25.

4. SHARPLY O-MINIMAL STRUCTURES ARISING FROM GEOMETRY

The fundamental motivation for introducing the notion of sharply o-minimal structures is the expectation that structures arising naturally from geometry should indeed be tame in this stronger sense. We start by motivating the discussion with the example of Abel–Jacobi maps, and then state some general conjectures.

4.1. Abel–Jacobi maps

Recall that for C a compact Riemann surface of genus g and $\omega_1, \ldots, \omega_g$ a basis of holomorphic one-forms on C, there is an associated lattice of periods $\Lambda \subset \mathbb{C}^g$, a principally polarized abelian variety $\mathrm{Jac}(C) \simeq \mathbb{C}^g/\Lambda$ and, for any choice of base point $p_0 \in C$, an *Abel–Jacobi* map

$$u_C : C \to \mathrm{Jac}(C), \quad u_C(p) = \int_{p_0}^{p} (\omega_1, \ldots, \omega_g) \bmod \Lambda. \tag{4.1}$$

To discuss definability properties of u_C, we choose a semi-algebraic (or even semi-linear) fundamental domain $\Delta \subset \mathbb{C}^g$ for the Λ-action and consider u_C as a map $u_C : C \to \Delta$.

Proposition 26. *There is an analytically generated sharply o-minimal structure where every u_C is definable.*

Indeed, after covering C by finitely many charts $\phi_j : D \to C$, where ϕ_j are algebraic maps extending to some neighborhood of $\bar{D}$, it is enough to show that the structure generated by these $\phi_j^* u_C$ is sharply o-minimal. Moreover, it is enough to show instead that the lifts

$$\tilde{u}_{C,j} : D \to \mathbb{C}^g, \quad \tilde{u}_{C,j}(z) = \int_0^z \phi^*(\omega_1, \ldots, \omega_g) \tag{4.2}$$

are definable. Indeed, $\tilde{u}_{C,j}(\bar{D})$ being compact meets finitely many translates of Δ, and the further projection $\mathbb{C}^g \to \Delta$ restricted to some ball containing $\tilde{u}_{C,j}(D)$ is thus definable in any sharply o-minimal structures (even in $\mathbb{R}_{\mathrm{alg}}$). The sharp o-minimality of the structure generated by all these $\tilde{u}_{C,j}$ follows from Theorem 6, since these functions, as indefinite integrals of algebraic one-forms, are restricted-Pfaffian (see, e.g., [27] for the elliptic case).

The construction above, however, is not uniform over C of a given genus. More precisely, while we do have $\tilde{u}_{j,C} \in \Omega_{\mathcal{F},D}$ for some uniform $\mathcal{F}$, D, the number of algebraic charts $\phi_j : D \to C$ may tend to infinity as C approaches the boundary of the moduli space $\mathcal{M}_g$ of compact genus g curves. However, we do have the following.

Proposition 27. *There is a sharply o-minimal structure where every $u_C \in \Omega_{\mathcal{F},D}$ for some uniform $\mathcal{F} = \mathcal{F}(g)$ and $D = D(g)$.*

To prove this, we replace the covering $\phi_j : D \to C$ by a covering $\phi_j : \mathcal{C}_j^{1/2} \to C$, where each $\mathcal{C}_j$ is a one-dimensional complex cell and $\phi_j(\mathcal{C}_j)$ covers C. By the removable singularity theorem, we may assume each $\mathcal{C}_j$ is either a disc or an annulus. Moreover, $\#\{\phi_j\}$ and their degrees are $\mathrm{poly}_{\mathcal{F}}(g)$ by the algebraic CPT. Here we use the fact that a genus g curve can always be realized as an algebraic curve of degree $d = \mathrm{poly}(g)$. The same construction as above now shows that each $\tilde{u}_{C,j} : \mathcal{C} \to \mathbb{C}^g$, if univalued, is restricted Pfaffian of format $\mathcal{F} = \mathcal{F}(g)$ and degree $D = \mathrm{poly}_g(D)$. In general, we have

$$\tilde{u}_{C,j}(z) = u'_{C,j}(z) + (a_{C,j,1}, \ldots, a_{C,j,g}) \log z \tag{4.3}$$

where $u'_{C,j}$ is univalued and $a_{C,j,k}$ is the residue of $\phi_j^* \omega_k$ around the annulus. Since $\log z$, understood for instance as having a branch cut in the negative real line, is restricted Pfaffian with uniform format and degree over every annulus, this proves the general case.

Finally, one should check that the projection $\mathbb{C}^g \to \Delta$, restricted to $\tilde{\phi}_j(\mathcal{C}_j)$ is definable (say in $\mathbb{R}_{\mathrm{alg}}$) with format and degree depending only on g. Equivalently, one should check that $\tilde{\phi}_j(\mathcal{C}_j)$ meets finitely many translates of Δ, with the number of translates depending only on g (if $\tilde{\phi}_j(\mathcal{C}_j)$ is multivalued then one should take one of its branches). This indeed holds, provided that the fundamental domains Δ are chosen appropriately. It can be deduced, albeit ineffectively, from the definability of theta functions (in both τ and z) on an appropriate fundamental domain **[34]**. In the case $g = 1$, an explicit upper bound for these constants is given in **[24]**.

The appearance of logarithmic factors in (4.3) is the reason that the structure we obtain is not analytically generated. However, the construction does prove the following.

Proposition 28. *There is an analytically generated sharply o-minimal structure* $(\mathcal{S}, \Omega)$ *where every* $u_C \in (\Omega_{\exp})_{\mathcal{F},D}$ *for some uniform* $\mathcal{F} = \mathcal{F}(g)$ *and* $D = D(g)$.

According to Conjecture 25 the structure $\mathcal{S}_{\exp}$ is indeed sharply o-minimal as well, but this remains open.

4.2. Uniformizing maps of abelian varieties

One can essentially repeat the construction above replacing $\mathrm{Jac}(C)$ by an arbitrary (say principally polarized) abelian variety A of genus g. We similarly have a map $u : A \to \Delta$ where $\Delta \subset \mathbb{C}^g$ is a semilinear fundamental domain for the period lattice of A, corresponding to some fixed basis of the holomorphic ones-forms $\omega_1, \ldots, \omega_g$ on A. Propositions 26, 27, and 28 extend to this more general context with essentially the same proof.

4.3. Noetherian functions

We have seen in Sections 4.1 and 4.2 that Abel–Jacobi maps and uniformizing maps of abelian varieties live in a sharply o-minimal structure (in fact, uniformly over all curves or abelian varieties of a given genus). This eventually boils down to the fact that the relevant maps are definable in $\mathbb{R}_{\mathrm{rPfaff}}$. However, we do not believe that all functions arising from geometry are definable in this structure. For instance, we conjecture that the graph of the modular invariant $j(\tau)$ restricted to any nonempty domain is not definable in $\mathbb{R}_{\mathrm{rPfaff}}$. We do not know how to prove this fact, but Freitag **[17]** has recently at least shown that $j(\tau)$ it not itself Pfaffian, on any nonempty domain, as a consequence of the strong minimality of the differential equation satisfied by $j(\tau)$ **[18]**.

One natural extension of the notion of Pfaffian functions are the *Noetherian functions*. Let $B \subset \mathbb{R}^\ell$ be a product of finite intervals. A tuple $f_1, \ldots, f_m : \bar{B} \to \mathbb{R}$ of analytic functions is called a restricted *Noetherian chain* if they satisfy a system of algebraic differential equations of the form

$$\frac{\partial f_i}{\partial x_j} = P(x_1, \dots, x_\ell, f_1, \dots, f_m), \quad \forall i, j. \tag{4.4}$$

We denote the structure generated by the restricted Noetherian functions by $\mathbb{R}_{\text{rNoether}}$. Since all restricted Noetherian functions are restricted analytic, $\mathbb{R}_{\text{rNoether}}$ is o-minimal.

Conjecture 29. *The structure $\mathbb{R}_{\text{rNoether}}$ is sharply o-minimal with respect to some FD-filtration.*

Gabrielov and Khovanskii have considered some local analogs of the theory of fewnomials for *nondegenerate* systems of Noetherian equations in [20], and made some (still local) conjectures about the general case. These conjectures are proved in [7] under a technical condition. However, these results are all local, bounding the number of zeros in some sufficiently small ball.

Despite the general Conjecture 29 being open, an effective Pila–Wilkie counting theorem was obtained in [4] for semi-Noetherian sets.

Theorem 30. *Let A be defined by finitely many restricted Noetherian equalities and inequalities. Then for every $\varepsilon > 0$, we have*

$$\#A^{\text{trans}}(g, H) \leqslant C_{g,A} H^{\varepsilon} \tag{4.5}$$

where $C_{g,A}$ can be computed explicitly from the data defining A.

Of course, provided Conjecture 29 an effective Pila–Wilkie theorem with better bounds (for instance, polynomial in the degree of A) would follow from Theorem 12. More generally, as a consequence of Conjecture 13 we would expect sharper polylogarithmic bounds as well. Some results in this direction are discussed in the following section.

4.4. Bezout-type theorems and point counting with foliations

One can think of the graphs of Noetherian functions equivalently as leafs of algebraic foliations. Partial results in the direction of Conjecture 29 have been obtained in [5] in this language. To state the result we consider an ambient quasi-projective variety $\mathbb{M}$ and a nonsingular m-dimensional foliation $\mathcal{F}$ of $\mathbb{M}$, both defined over $\bar{\mathbb{Q}}$. For $p \in \mathbb{M}$ denote by $\mathcal{L}_p$ the germ of the leaf passing through p. For a pure-dimensional variety $V \subset \mathbb{M}$, denote

$$\Sigma_V := \{p \in \mathbb{M} : \dim(V \cap \mathcal{L}_p) > m - \operatorname{codim}_{\mathbb{M}} V\}. \tag{4.6}$$

If V is defined over $\bar{\mathbb{Q}}$, we denote by δ_V the sum of the degree $\deg V$, the log-height $h(V)$, and the degree of the field of definition of V over $\mathbb{Q}$. Here the log-height is taken, for instance, to be the log-height of the point representing V in an appropriate Chow variety. In terms of this data we have the following Bezout-type theorem.

Theorem 31 ([5, THEOREM 1]). *Let $V \subset \mathbb{M}$ be defined over a number field and suppose* $\operatorname{codim}_M V = m$. *Let K be a compact subset of a leaf of $\mathcal{F}$. Then*

$$\#(K \cap V) \leqslant \operatorname{poly}_K(\delta_V, \log \operatorname{dist}^{-1}(K, \Sigma_V)). \tag{4.7}$$

In fact, the bound in Theorem 31 can be made more explicit giving the precise dependence on $\mathcal{F}$ and on K, and this is important in some applications, but we omit the details for brevity. The same bound without the dependence on $h(V)$ and $\log \operatorname{dist}^{-1}(K, \Sigma_V)$ would be a consequence of Conjecture 29, and establishing such a bound is probably the main step toward proving the conjecture.

As a consequence of Theorem 31 one can deduce some polylogarithmic point-counting results in the spirit of Conjecture 13. We state the simplest result of this type for illustration below.

Theorem 32 ([5, COROLLARY 6]). *Suppose $\mathcal{L}_p$ contains no germs of algebraic curves, for any $p \in \mathbb{M}$. Let K be a compact subset of a leaf of $\mathcal{F}$. Then*

$$\#K(g,h) = \operatorname{poly}_K(g,h). \tag{4.8}$$

Once again, the dependence on K can be made explicit in terms of the foliation $\mathcal{F}$ and this plays a role in some applications. In practice, Theorem 32 and its more refined forms can be used to deduce the conclusion of Conjecture 13 in most arithmetic applications, since the sets appearing in such applications are always defined in terms of leafs of some highly symmetric foliations.

4.5. Q-functions

Many important functions arising from geometry, such as period integrals, are Noetherian. Indeed, such functions arise as horizontal sections of the Gauss–Manin connection and can thus be viewed as solutions of a linear systems of differential equations. However, the structure $\mathbb{R}_{\text{rNoether}}$ only contains the restrictions of such maps to compact domains. If we consider general Noetherian functions on noncompact domains, the result would not even be o-minimal (as illustrated by the sine and cosine functions, for instance). If one is to obtain an o-minimal structure, one must restrict singularities at the boundary.

One candidate class is provided by the notion of *Q-functions* considered in [10, 11]. Let $P \subset \mathbb{C}^n$ be a polydisc, $\Sigma \subset \mathbb{C}^n$ a union of coordinate hyperplanes, and ∇ the connection on $P \times \mathbb{C}^\ell$ given by

$$\nabla v = \mathrm{d}v - A \cdot v \tag{4.9}$$

where A is a matrix of one-forms holomorphic in $\bar{P} \setminus \Sigma$. Suppose that the entries of A are algebraic and defined over $\bar{Q}$, that ∇ has regular singularities along Σ, and that the monodromy of ∇ is quasiunipotent. Finally, let P° be a simply-connected domain obtained by removing from $P \setminus \Sigma$ a branch cut $\{\operatorname{Arg} x_i = \alpha_i\}$ for each of the components $\{x_i = 0\}$ of Σ and for some choice of $\alpha_i \in \mathbb{R} \bmod 2\pi$. Every solution of $\nabla v = 0$ extends as a holomorphic vector-valued function in P°. We call each component of such a function a Q-function. Denote by $\mathbb{R}_{\text{QF}}$ the structure generated by all such Q-functions. This structure contains, as sections of the Gauss–Manin connection, all period integrals of algebraic families.

By the classical theory of regular–singular linear equations, every Q-function is definable in $\mathbb{R}_{\text{an,exp}}$, and $\mathbb{R}_{\text{QF}}$ is thus o-minimal.

Conjecture 33. *The structure $\mathbb{R}_{\mathrm{QF}}$ is sharply o-minimal with respect to some FD-filtration.*

Some initial motivation for Conjecture 33 is provided by the results of [10], which give effective bounds for the number of zeros of Q-functions restricted to any algebraic curve in P. However, treating systems of equations in several variables, and obtaining sharp bounds with respect to degrees, is still widely open.

5. APPLICATIONS IN ARITHMETIC GEOMETRY

In this section we describe some applications of sharply o-minimal structures in arithmetic geometry. For some of these, Theorem 12 suffices, while for others Conjecture 13 is necessary—in some suitable sharply o-minimal structure, such as the one conjectured to exist in Conjecture 33. However, in all cases discussed below one can actually carry out the strategy using known results, mostly Theorem 32 and its generalizations, in place of these general conjectures (though various technical difficulties must be resolved in each case). We thus hope to convince the reader that the strategy laid out below is feasible, on the one hand, and fits coherently into the general framework of sharply o-minimal structures, on the other.

5.1. Geometry governs arithmetic

Geometry governs arithmetic describes a general phenomenon in the interaction between geometry (for instance, algebraic geometry) and arithmetic: namely, that arithmetic problems often admit finitely many solutions unless there is an underlying geometric reason to expect infinitely many. Perhaps the most famous example is given by *Mordell's conjecture*, now Falting's theorem [16]: an algebraic curve $C \subset \mathbb{P}^2$ contains finitely many rational points, unless it is rational or elliptic. The two exceptions in Falting's theorem may be viewed as geometric obstructions to the finitude of rational solutions: the rational parametrization in the former, and the group law in the latter, are geometric mechanisms that can produce infinitely many rational points on the curve.

The Pila–Wilkie theorem itself may be viewed as an instance where geometry (namely the existence of an algebraic part) controls arithmetic (namely the occurrence of many rational points, as a function of height). A general strategy by Pila and Zannier [38] reduces many unlikely intersection questions to the Pila–Wilkie theorem. This has been used to prove the finiteness of solutions, under natural geometric hypotheses, to a large number of Diophantine problems. For instance, the finiteness of torsion points on a subvariety of an abelian variety (Manin–Mumford) [38]; the finiteness of maximal special points on subvariety of a Shimura variety (André–Oort) [35,41]; the finiteness of "torsion values" for sections of families of abelian surfaces (relative Manin–Mumford) [30]; the finiteness of the set of $t \in \mathbb{C}$ for which a Pell equation $P^2 - DQ^2 = 1$ with given $D \in \mathbb{Q}^{\mathrm{alg}}[X,t]$ is solvable in $P,Q \in \mathbb{C}[X]$ [2,31,32]; the finiteness of the set of values $t \in \mathbb{C}$ where an algebraic one-form $f_t = f(t,x)\mathrm{d}x$ is integrable in elementary terms [32]; and various other examples.

5.2. The Pila–Zannier strategy

Below we briefly explain the Pila–Zannier strategy in the Manin–Mumford case. Let A be an abelian variety and $V \subset A$ an algebraic subvariety containing no cosets of abelian subvarieties, both defined over a number field $\mathbb{K}$.

Let $\pi : [0,1]^{2g} \to A$ be the universal covering map of A written in period coordinates, so that rational points with common denominator N in $[0,1]^{2g}$ correspond to N-torsion points in A. One checks that under our assumptions, $X := \pi^{-1}(V)$ has no algebraic part (this can be done with the help of the Pila–Wilkie theorem as well, following a strategy of Pila in [35]). The Pila–Wilkie theorem then implies that the number of torsion points in V is at most $C(X,\varepsilon)N^{\varepsilon}$ where $C(X,\varepsilon)$ is the Pila–Wilkie constant.

On the other hand, there is $c > 0$ such that if $p \in A$ is an N-torsion point then $[\mathbb{Q}(p) : \mathbb{Q}] \gg_A N^c$ by a result of David [15]. Here the implied constant depends effectively on A. This is an example of a Galois lower bound, which in the Pila–Zannier strategy plays the yin to Pila–Wilkie's yang.

Choose $\varepsilon = c/2$ and suppose that V contains an N-torsion point p. Then it contains a fraction of $[\mathbb{K} : \mathbb{Q}]^{-1}$ of its Galois conjugates, and we obtain a contradiction as soon as $N \gg_{A,V} C(X,\varepsilon/2)^{2/c}$. We thus proved a bound for the order of any torsion point in V, and in particular the finiteness of the set of torsion points.

5.3. Point counting and Galois lower bounds

Traditionally in the Pila–Zannier strategy, the Pila–Wilkie theorem is used to obtain an upper bound on the number of special points, while the competing Galois lower bounds are obtained using other methods—usually involving a combination of height estimates and transcendence methods, such as the results of David [15] or Masser–Wüstholz [29].

In [39] Schmidt suggested an alternative approach to proving Galois lower bounds, replacing the more traditional transcendence methods by polylogarithmic counting results as in Conjecture 13. We illustrate again in the Manin–Mumford setting. Let A be an abelian variety over a number field $\mathbb{K}$ and let $p \in A$ be a torsion point. Consider now X given by the graph of the map π defined in the previous section, which is easily seen to contain no algebraic part. The points $p, 2p, \dots, Np$ correspond to N points $x_1, \dots, x_n$ on this graph. Recall that the height of a torsion point in A is $O_A(1)$ (since the Neron–Tate height is zero), and the height of the corresponding point in $[0,1]^{2g}$ is at most N. It follows that $h(x_j) \ll_A \log N$. On the other hand, the field of definition of each x_j is, by the product law of A, contained in $\mathbb{K}(p)$. We thus have

$$N \leqslant \#X\big([\mathbb{K}(p) : \mathbb{Q}], \log N\big) = \mathrm{poly}_A\big([\mathbb{K}(p) : \mathbb{Q}], \log N\big) \tag{5.1}$$

by Conjecture 13, and this readily implies $[\mathbb{Q}(p) : \mathbb{Q}] \gg_A N^c$ for some $c > 0$, giving a new proof of the Galois lower bound for torsion points—and with it a "purely point-counting" proof of Manin–Mumford. This has been carried out in [5] using Theorem 32.

The main novelty of this strategy is that it applies in contexts where we have polylog counting result, and where the more traditional transcendence techniques are not available. In [12] this idea was applied in the context of a general Shimura variety S. It is shown that if the special points $p \in S$ satisfy a discriminant-negligible height bound

$$h(p) \ll_{S,\varepsilon} \operatorname{disc}(p)^{\varepsilon}, \quad \forall \varepsilon > 0 \tag{5.2}$$

where $\operatorname{disc}(p)$ is an appropriately defined discriminant, then they also satisfy a Galois bound $[\mathbb{Q}(p) : \mathbb{Q}] \gg_S \operatorname{disc}(p)^c$ for some $c > 0$. Further, it was already known by the work of many authors based on the strategy of Pila [35] that this implies the André–Oort conjecture for S.

In the case of the Siegel modular variety $S = \mathcal{A}_g$, the height bound (5.2) was proved by Tsimerman [41] as a simple consequence of the recently proven averaged Colmez formula [1,46]. Tsimerman deduces the corresponding Galois bound from this using Masser–Wüstholz' isogeny estimates [29]. However, these estimates are proved using transcendence methods applied to abelian functions, and have no known counterpart applicable when the Shimura variety S does not parameterize abelian varieties (i.e., is not of abelian type). The result of [12] removes this obstruction.

A few months after [12] appeared on the arXiv, Pila–Shankar–Tsimerman have posted a paper [36] (with an appendix by Esnault, Groechenig) establishing the conjecture (5.2) for arbitrary Shimura varieties (by a highly sophisticated reduction to the $\mathcal{A}_g$ case where averaged Colmez applies). In combination with [12] this establishes the André–Oort conjecture for general Shimura varieties (as well as for mixed Shimura varieties by the work of Gao [22]). It is interesting to note that the proof of André–Oort now involves three distinct applications of point-pointing: for functional transcendence, for Galois lower bounds, and for the Pila–Zannier strategy.

5.4. Effectivity and polynomial time computability

In each of the problems listed at the end of Section 5.1, it is natural to ask, when the data defining the problem is given over $\bar{\mathbb{Q}}$, whether one can effectively determine the finite set of solutions; and whether one can compute the set in polynomial time (say, in the degrees and the log-heights of the algebraic data involved, for a fixed dimension). In most cases mentioned above, the use of the Pila–Wilkie theorem is the only source of ineffectivity in the proofs. In fact, for all examples above excluding the André–Oort conjecture, definability of the relevant transcendental sets in an (effective) sharply o-minimal structure is expected to imply the (effective) polynomial time computability of these finite sets. This has been carried out using Theorem 12 for Manin–Mumford [6] and using Theorem 32 for a case of relative Manin–Mumford [5], giving effective polynomial time decidability of these problems. We see no obstacles in similarly applying [5] to the other problems listed above, though this is yet to be verified in each specific case.

In the André–Oort conjecture Siegel's class number bound introduces another source of ineffectivity in the finiteness result. Nevertheless, in [5] Theorem 32 is used to prove the polynomial time decidability of André–Oort for subvarieties of $\mathbb{C}^n$ (i.e., by a polynomial-time algorithm involving a universal, undetermined Siegel constant). This is expected to extend to arbitrary Shimura varieties.

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