MULTIPLICITIES OF NOETHERIAN DEFORMATIONS

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ABSTRACT. The *Noetherian class* is a wide class of functions defined in terms of polynomial partial differential equations. It includes functions appearing naturally in various branches of mathematics (exponential, elliptic, modular, etc.). A conjecture by Khovanskii states that the *local* geometry of sets defined using Noetherian equations admits effective estimates analogous to the effective *global* bounds of algebraic geometry.

We make a major step in the development of the theory of Noetherian functions by providing an effective upper bound for the local number of isolated solutions of a Noetherian system of equations depending on a parameter ε, which remains valid even when the system degenerates at $\varepsilon = 0$. An estimate of this sort has played the key role in the development of the theory of Pfaffian functions, and is expected to lead to similar results in the Noetherian setting. We illustrate this by deducing from our main result an effective form of the Łojasiewicz inequality for Noetherian functions.

1. INTRODUCTION

One of the cornerstones of algebraic geometry is the Bezout theorem: a system of polynomial equations in a complex projective space always admits a specified number of solutions depending on their degrees. This statement has profound implications for the algebraic category: essentially every geometric and topological property of an algebraic variety can be estimated in terms of the degrees of the equations defining it.

Moving beyond the algebraic category, Khovanskii has defined the class of real *Pfaffian functions.* In his theory of Fewnomials [18], Khovanskii has shown that the number of solutions of a system of real Pfaffian equations admits an effective upper bound in terms of the degrees of the equations. This fundamental result has been the basis of many subsequent works, showing that the geometry of real Pfaffian sets is tame and admits effective estimates in terms of degrees [27, 25, 16].

The real Pfaffian class consists of functions satisfying a system of differential equations with a certain triangularity condition. It is surprisingly general, and includes many important transcendental functions — most notably the real exponential. On the other hand, not all differential systems appearing naturally in mathematics are real Pfaffian. We mention a few key examples:

- Exponential maps of (complex) commutative algebraic groups, for instance complex exponentials and elliptic functions. Solutions of equations involving such functions have been studied in the context of diophantine approximation, [19, 20, 28].
- Abelian integrals and iterated Abelian integrals. Solutions of equations involving such functions have been studied in relations to perturbations of Hamiltonian systems and their limit cycles [2, 7].
- Functions of modular type, for instance Klein's modular invariant j and Ramanujan's functions P, Q, R. Solutions of equations involving such functions have been studied in transcendental number theory, [22, 1].
- Hamiltonian flow maps in completely integrable systems.

At the time of the development of the theory of Fewnomials, Khovanskii also considered the more general notion of *Noetherian functions*, excluding the triangularity condition from the definition of the Pfaffian functions. One may loosely say that a collection of functions is Noetherian if each of their derivatives can be expressed algebraically in terms of the functions themselves (see §1.1 for two equivalent precise definitions). All functions listed above, while not real-Pfaffian, do form Noetherian systems.

Noetherian functions do not satisfy global estimates similar to those obtained in the theory of Fewnomials. However, Khovanskii has conjectured that a local analog of these estimates continues to hold in the class of Noetherian functions, which would form a type of local analog of the Pfaffian class. As in the Pfaffian case, the key step is to establish an upper bound for the number of solutions of a system of equations involving Noetherian functions in terms of degrees — this time in a suitably defined local sense.

In this paper we make a main step in the development of the theory of Noetherian functions by establishing an upper bound for the local number of solutions of a system of Noetherian equations depending on a Noetherian parameter ε. The novelty of our result is that the estimate is valid even if the system degenerates and has non-isolated solutions when $\varepsilon = 0$. A result of this type has been established by Gabrielov in the (complex) Pfaffian case, and subsequently used to derive estimates for many topological and geometric properties in the Pfaffian category (see [16] for a survey). As an illustrative example, we use our estimate to establish an effective Łojasiewicz inequality in the Noetherian category following Gabrielov. We expect many other results of [16] to follow similarly.

1.1. **Noetherian functions.** We begin by defining our principal object of investigation, namely the rings of Noetherian functions.

Definition 1 (Noetherian functions, algebraic [26])**.** *A ring S of analytic functions in a domain $U \subset \mathbb{C}^n$ is called a* ring of Noetherian functions *if it is generated over the polynomial ring $\mathbb{C}_n$ by functions $\phi_1, \ldots, \phi_m \in S$ and closed under differentiation with respect to each variable x_i.*

Any element of $\psi \in S$ is called a Noetherian function*. We will say that it has degree d with respect to $\phi_1, \ldots, \phi_m$ if d is the minimal degree of a polynomial $P \in \mathbb{C}_{n+m}$ such that $\psi = P(x_1, \ldots, x_n, \phi_1, \ldots, \phi_m)$. When there is no risk of confusion we will simply call this the degree of ψ.*

Alternatively, equip $\mathbb{C}^{n+m}$ with the affine coordinates $(x_1, \ldots, x_n, f_1, \ldots, f_m)$ and denote by $\mathbb{C}_{n+m}$ the ring of polynomials in these variables. Consider a distribution $\Xi \subset T\mathbb{C}^{n+m}$ defined by

$$\Xi = \langle V_1, \ldots, V_n \rangle, \quad V_i = \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} g_{ij} \frac{\partial}{\partial f_j} \quad \text{for } i = 1, \ldots, n. \tag{1}$$

where $g_{ij} \in \mathbb{C}_{n+m}$. We call $\delta = \max_{ij} \deg g_{ij}$ the *degree of the chain.* For any point where (1) is integrable, we will denote by Λ_p the germ of an integral manifold through p.

Definition 2 (Noetherian functions, geometric [14])**.** *A tuple $\phi_1, \ldots, \phi_m$ of analytic functions on a domain $U \subset \mathbb{C}^n$ is called a* Noetherian chain *if its graph forms an integral manifold of the distribution Ξ (for some choice of the coefficients $g_{ij} \in \mathbb{C}_{n+m}$). In other words, if the following system of differential equations is satisfied*

$$\frac{\partial \phi_j}{\partial x_i} = g_{ij}(x_1, \ldots, x_n, \phi_1, \ldots, \phi_m). \tag{2}$$

Any function $\psi = P(x_1, \ldots, x_n, \phi_1, \ldots, \phi_m)$ *where* $P \in \mathbb{C}_{n+m}$ *is called a Noetherian function. We will say that it has degree* d *with respect to* $\phi_1, \ldots, \phi_m$ *if* d *is the minimal degree of such a polynomial* $P \in \mathbb{C}_{n+m}$. *When there is no risk of confusion we will simply call this the degree of* ψ.

It is straightforward to check that the generators of a ring of Noetherian functions forms a Noetherian chain and vice versa.

Remark 3. *Note that by the above, if* Λ *is an integral manifold of* (1) *then the projection onto the* x*-variables gives a system of coordinates on* Λ*, and with respect to these coordinates the Noetherian functions are simply the restrictions of polynomials* $P \in \mathbb{C}_{n+m}$ *to* Λ. *This viewpoint is particularly convenient for our purposes and will be used frequently.*

We define a *Noetherian set* to be the common zero locus of any collection of Noetherian functions with a common domain of definition U.

As solutions of non-linear differential equations, Noetherian functions do not admit good global behavior — in fact, their domains of definition may include movable singularities and natural boundaries. It is natural therefore to begin the study of these functions with their local properties: what can be said about germs of Noetherian functions and sets in terms of the system (1) that defines them?

In order to be more concrete, consider any system (1) with the parameters m, n, δ and an n-tuple $\psi_1, \ldots, \psi_n$ of Noetherian functions of degrees bounded by d. Let $\mathcal{N}(m, n, \delta, d; 0)$ denote the maximal possible multiplicity of a common zero of the equations $\psi_1 = \cdots = \psi_n = 0$, assuming that the zero is isolated. More generally, consider any family $\psi_1^\varepsilon, \ldots, \psi_n^\varepsilon$ of Noetherian functions of degrees bounded by d (for each fixed ε) and depending analytically on ε. Let $\mathcal{N}(m, n, \delta, d)$ denote the maximal possible number of isolated zeros (counted with multiplicities) of the equations $\psi_1^\varepsilon = \cdots = \psi_n^\varepsilon = 0$ which converge to a given point point as $\varepsilon \to 0$. Crucially, here it is not assumed that the given point is an isolated solution of the limiting system at $\varepsilon = 0$.

It is a general principle that estimates for $\mathcal{N}(m, n, \delta, d)$ imply estimates for the local topological and analytic structure of functions and sets, by a combination of topological arguments (e.g. Morse theory) and analytic-geometric arguments (e.g. the study of polar curves). The more restrictive $\mathcal{N}(m, n, \delta, d; 0)$ does not control the topological and analytic-geometric structure to the same extent, but it is a convenient first approximation which is often easier to study and still provides useful information. Instances of the problem of estimating $\mathcal{N}(m, n, \delta, d; 0)$ and $\mathcal{N}(m, n, \delta, d)$ have been studied by authors in various areas of mathematics. Below we present an outline of some of the main contributions.

1.2. **Historical sketch.** The case $n = 1$ is special and has attracted considerable attention. In this case the problem reduces to the study of polynomial functions on the trajectory of a (non-singular) polynomial vector field at a point. Moreover, by an argument due to Khovanskii [12], in this case the quantities $\mathcal{N}(m, n, \delta, d; 0)$ and $\mathcal{N}(m, n, \delta, d)$ coincide. Therefore the problem is to estimate the multiplicity of the restriction of a polynomial to the trajectory of a non-singular vector field at a point.

Brownawell and Masser [8, 9] and Nesterenko [21] have studied the problem of estimating $\mathcal{N}(m, 1, \delta, d)$ with motivations in transcendental number theory. These authors have also considered the case where one takes into account the multiplicity at several (rather than just one) points. Nesterenko also considered singular systems of differential equations in [22] and established an estimate which is sharp with respect to d up to a multiplicative constant and doubly-exponential with respect to m. This estimate was improved to single-exponential in m in [4].

Risler [24, 13] has considered the problem in the context of control theory and specifically non-holonomic systems. Gabrielov [12] refined this work and found a surprising reformulation in terms of the Milnor fibers of certain deformations, leading to an estimate for $\mathcal{N}(m, 1, \delta, d)$ which is simply-exponential in m and polynomial in d and δ. Gabrielov's approach was refined in [3] to give a sharp asymptotic with respect to d.

Novikov and Yakovenko [23] have considered the problem with motivations in the study of abelian integrals, and have produced a bound which is valid not only for local multiplicity but for the number of zeros (counted with multiplicities) in a small ball of a specified radius. Yomdin [29] has obtained a similar result motivated by the study of cyclicities in dynamical systems. For both of these works, the principal difficulty is to obtain an estimate in an interval whose radius does not degenerate along deformations where the polynomial becomes identically vanishing on the trajectory.

The case $n > 1$ is considerably more involved, because germs of functions and sets in several variables can degenerate in much more complicated ways. Khovanskii [18] has studied systems (1) satisfying an additional triangularity condition and defined over the reals, leading to the influential theory of Pfaffian functions. Khovanskii has shown that the global geometry of Pfaffian sets (e.g. number of connected components, sum of Betti numbers) can be estimated in terms of the degrees of the coefficients of the system (1).

Gabrielov [10] has established a local complex analog of Khovanskii's estimates for the number of solutions of a system of Pfaffian equations, namely an upper bound for $\mathcal{N}(m, n, \delta, d)$ (for the Pfaffian case). Gabrielov used this result to produce an effective form of the Łojasiewicz inequality for Pfaffian functions, and later in a series of joint works with Vorobjov (for example see [16, 15, 11]) established many results on the complexity of various geometric constructions within the Pfaffian class (e.g. stratification, cellular decomposition, closure and frontier).

In the early 1980s Khovanskii conjectured that the quantity $\mathcal{N}(m, n, \delta, d)$ admits an effective upper bound (see [14] for the history and another equivalent form of this conjecture). The conjecture in this generality has remained unsolved.

Building upon the one-dimensional approach of Gabrielov [12], Gabrielov and Khovanskii [14] have established an estimate for $\mathcal{N}(m, n, \delta, d; 0)$ using a topological deformation technique. In our previous work [5] we have established a weaker estimate for this quantity using algebraic techniques, and extended these local estimates to an estimate on the number of zeros (counted with multiplicities) in a sufficiently small ball, provided that the ball is not too close to a non-isolated solution of the system. However, neither of these approaches give an estimate for the more delicate quantity $\mathcal{N}(m, n, \delta, d)$.

In [6] we began the investigation of non-isolated solutions of systems of Noetherian equations for $n = 2$. We have obtained an explicit upper bound for an appropriately defined "non-isolated intersection multiplicity". However, this bound is not sufficient if one is interested in estimating the number of solutions born from a deformation of a non-isolated solution.

1.3. **Statement of our result.** Consider a system (1) and the corresponding ring of Noetherian functions S defined in a domain $U \subset \mathbb{C}^n$, and let $p \in U$.

Let $\rho \in S$ and let $X \subset U$ be a germ of a Noetherian set

$$X = \{x \in U : \psi_1(x) = \cdots = \psi_{n-1}(x) = 0\} \tag{3}$$

at the point p, where $\psi_i \in S$ for $i = 1, \ldots, k$.

Definition 4. *The* deformation multiplicity*, or* deflicity *of X with respect to ρ is the number of isolated points in $\rho^{-1}(y) \cap X$ (counted with multiplicities) which converge to p as $y \to \rho(p)$.*

We let $\mathcal{M}(m,n,\delta,d)$ denote the maximum possible deformation multiplicity for any system (1) with parameters m,n,δ and any Noetherian set defined as above with $\deg\psi_i \leqslant d$ and $\deg\rho \leqslant d$.

Remark 5. *This notion is closely related to the quantity $\mathcal{N}(m,n,\delta,d)$ defined in section 1.1. Indeed, let a family $\psi_1^\varepsilon,\dots,\psi_n^\varepsilon$ be given where $\psi_i^\varepsilon \in S$ and $\deg\psi_i^\varepsilon \leqslant d$. Let us assume further that the dependence on ε is Noetherian, i.e. that these functions may be viewed as Noetherian functions in the variables $x_1,\dots,x_n,\varepsilon$ and have degrees bounded by d in this sense as well. Then the number of isolated zeros of $\psi_1^\varepsilon = \cdots = \psi_n^\varepsilon = 0$ which converge to p as $\varepsilon \to 0$ coincides with the deflicity of the set*

$$X = \{(x,\varepsilon) \in U : \psi_1(x,\varepsilon) = \cdots = \psi_n(x,\varepsilon) = 0\} \tag{4}$$

with respect to $\rho \equiv \varepsilon$.

We can now state our main result.

Theorem 1. *The quantity $\mathcal{M}(m,n,\delta,d)$ admits an effective upper bound,*

$$\mathcal{M}(m,n,\delta,d) \leqslant (\max\{d,\delta\}(m+n))^{16(m+n)^{20n+3}} = (\max\{d,\delta\}(m+n))^{(m+n)^{O(n)}} \tag{5}$$

Recall that according to Khovanskii's conjecture, the quantity $\mathcal{N}(m,n,\delta,d)$ admits an effective upper bound. Our result applies only under the additional condition that the dependence of the deformation on ε is Noetherian (the reader may consult Remark 17 for a discussion of the point in our argument where this extra condition is needed). However, in most applications one needs to deal only with explicitly presented deformations which do depend in a Noetherian manner on ε. One can therefore expect that Theorem 1 will suffice for the investigation of many local topological and analytic-geometric properties of Noetherian sets and functions.

Remark 6. *One can also consider systems* (1) *involving rational (rather than polynomial) coefficients, as well as rational P,R. As long as one considers points p away from the polar locus of these functions, Theorem 1 remains valid and can be proved in the same way. On the other hand, when p belongs to the singular locus of the system, the situation is considerably more involved.*

1.4. **Application: Łojasiewicz inequality.** As an example of an application of Theorem 1, we prove an effective version of Łojasiewicz inequalities for Noetherian functions. Let $\Xi_{\mathbb{R}} \subset T\mathbb{R}^{n+m}$ be a real distribution spanned by real vector fields V_i as in (1), with $g_{ij} \in \mathbb{R}_{n+m}$ of degree $\leqslant \delta$, where $\mathbb{R}_{n+m} = \mathbb{R}[x_1,...,x_n,f_1,...,f_m]$, and let $\Lambda_p \subset \mathbb{R}^{n+m}$ be a germ of its integral manifold through p. The restrictions of real polynomials of degree d to Λ_p are called real Noetherian functions of degree d.

Theorem 2. *Let f,g be real Noetherian functions of n variables of degree d defined by the same Noetherian chain, $f(p)=0$, and assume that $\{f=0\} \subset \{g=0\}$ near p. Then there exists a constant $0 < k \leqslant (\max\{d,\delta\}(m+n))^{(m+n)^{O(n)}}$ such that $|f| > |g|^k$ near p.*

The proof below is essentially the proof from [10], with Theorem 1 replacing [10, Theorem 2.1]. Denote the complexifications of f,g by the same letters. They are evidently Noetherian functions, defined by the (complexification of the) same Noetherian chain. Denote by $\Delta \subset \mathbb{C}^2$ the polar curve of f with respect to g, i.e. the set of critical values of the mapping $(f,g) : \Lambda_p \to \mathbb{C}^2$.

Lemma 7. *Let $f = \sum c_i g^{\lambda_i}$ be the Puiseux expansion of an irreducible component $\Delta' \neq \{f = 0\}$ of Δ. Let s be the least common denominator of λ_i, and let $r = s\lambda_1$. Then*

$$r, s \leqslant (2(d+\delta-1)(m+n))^{16(m+n)^{20n+3}}$$

Proof. Let $\Sigma' \subset \Lambda_p$ be an irreducible component of the critical set of the mapping (f, g) whose image is Δ'. Then r is at most the number of points in $\{f(x) = \epsilon, x \in \Sigma'\}$ converging to p as $\epsilon \to 0$, and s is at most the number of points in $\{g(x) = \epsilon, x \in \Sigma'\}$ converging to p as $\epsilon \to 0$. We can assume that these points are isolated: after cutting by a linear space L' of suitable dimension, the polar curve of the restriction of f to L' with respect to the restriction of g to L' will have Δ' as its irreducible component and the problem is reduced to a similar one in smaller dimension. Therefore it is enough to bound from above the deflicity of $X = \{dg \wedge df = 0\}$ with respect to f or g. But X is a Noetherian set of degree $2(d+\delta-1)$, so Theorem 1 implies the claim of the Lemma. □

Proof of Theorem 2. Let Σ be the union of points of minima of restrictions of $|f|$ to the intersection of level curves of $|g|$ with a small ball around p. We can assume that the closure of Σ contains p, otherwise the problem is reduced to a similar one of lower dimension. Therefore the image of Σ under (f, g) belongs to a polar curve of f relative to g, and not to $\{f = 0\}$ by the condition. Therefore Theorem 2 follows from the previous Lemma. □

1.5. **Acknowledgments.** We would like to express our gratitude to A. Gabrielov, A. Khovanskii, P. Milman and S. Yakovenko for valuable discussions and to the anonymous referee for numerous corrections and helpful remarks.

2. Background

2.1. **Noetherian functions: integrability and multiplicity estimates.** A system of the form (1) does not necessarily admit an integral manifold through every point of $\mathbb{C}^{n+m}$, since we do not assume that the vector fields V_i commute globally. The *integrability locus* $\mathcal{IL} \subset \mathbb{C}^{n+m}$ of (1) is defined to be the union of all the integral manifolds of the system. In [14] it was shown that $\mathcal{IL}$ is algebraic for any system (1), and moreover that the following theorem holds.

Theorem 3 ([14, Theorem 4])**.** *Let the system* (1) *have parameters m, n, δ. Then $\mathcal{IL}$ can be defined as the zero locus of a set of polynomials of degrees not exceeding*

$$d_{\mathcal{IL}} = \frac{(m+1)(\delta-1)}{2}[2\delta(n+m+2)-2m-2]^{2m+2} + \delta(n+2) - 1 \tag{6}$$

We now state the main result of [14], namely an upper bound for the multiplicity of an isolated common zero of a tuple of Noetherian functions.

Theorem 4 ([14, Theorem 1])**.** *Let the system* (1) *have parameters m, n, δ and let $\psi_1, \ldots, \psi_n$ be Noetherian functions of degrees at most d with respect to this system. Then the multiplicity of any isolated solution of the equations $\psi_1 = \cdots = \psi_n = 0$ does not exceed the maximum of the following two numbers:*

$$\tfrac{1}{2}Q\left((m+1)(\delta-1)[2\delta(n+m+2)-2m-2]^{2m+2} + 2\delta(n+2) - 2\right)^{2(m+n)}$$

$$\tfrac{1}{2}Q\left(2(Q+n)^n(d+Q(\delta-1))\right)^{2(m+n)}, \quad \textit{where} \quad Q = e\, n\left(\tfrac{e(n+m)}{\sqrt{n}}\right)^{\ln n + 1}\left(\tfrac{n}{e^2}\right)^n$$

2.2. **Multiplicity operators.** We shall make extensive use of the notion of multiplicity operators introduced in [5]. The multiplicity operators are a collection of partial differential operators which can be used to study the (local structure of) solutions of n equations in n variables. They may be thought of as playing a role similar to the usual derivatives in the case of functions of a single variable. For the convenience of the reader we recall in this section the basic definitions and properties of multiplicity operators that shall be used in the sequel.

Let $F = (F_1, \dots, F_n) : B^n(0,1) \to \mathbb{C}^n$ be a holomorphic mapping extendable to a neighborhood of the unit ball $B^n(0,1)$. We will denote by $\|\cdot\|$ the maximum norm on the unit ball. As a notational convenience, all holomorphic functions considered in this subsection are assumed to be holomorphic in the unit disc.

For every $k \in \mathbb{N}$ there is a finite family of polynomial differential operators $\{M_B^\alpha\}$ called the *basic multiplicity operators*. Any operator $M^{(k)}$ in the convex hull of this set is called a *multiplicity operator* of order k. Every multiplicity operator of order k is a polynomial differential operator of order k and degree $\dim J_{n,k} - k$, where $J_{n,k}$ denotes the space of k-jets of functions in n variables. We recall that $\dim J_{n,k} = \binom{n+k}{k}$.

Example 8. *In the familiar case $n = 1$ the basic multiplicity operators of order k are given (up to normalization by some universal constants) by the first k derivatives. We will outline proofs for the various results presented below in this more familiar special case to help the reader appreciate the general statements.*

We will use the convention that when applying a multiplicity operator $M^{(k)}$ to a polynomial $P \in \mathbb{C}_{n+m}$, the derivatives of the variables $f_1, \dots, f_m$ with respect to $x_1, \dots, x_n$ shall be given by the Noetherian system (1). In this way $M^{(k)}(P) \in \mathbb{C}_{n+m}$ and moreover, if $\deg P = d$ then $\deg M^{(k)}(P) \leqslant d_M(n, \delta, d, k)$, where

$$d_M(n, \delta, d, k) = \binom{n+k}{k}(d + k\delta) - k \tag{7}$$

At points where (1) is integrable, this agrees of course with applying $M^{(k)}$ to P viewed as a Noetherian function of $x_1, \dots, x_n$ by restriction of P to the integral manifold of (1) through the point. At points where (1) is not integrable $M^{(k)}(P)$ has no intrinsic meaning and will in general depend on the order in which we choose to evaluate repeated derivatives. We fix this order arbitrarily. In fact we will only consider $M^{(k)}(P)$ on points where the system is integrable, so this arbitrary choice will make no difference in our arguments.

We will sometimes need to evaluate multiplicity operators with respect to a linear subspace of the x-coordinates. If $\mathfrak{T}$ is such a linear subspace and $M^{(k)}$ a multiplicity operator of dimension $\dim \mathfrak{T}$, we will denote by $M_{\mathfrak{T}}^{(k)}$ the operator given by

$$[M_{\mathfrak{T}}^{(k)}(F)](p) = [M^{(k)}(F|_{\mathfrak{T}_p})](p), \qquad \mathfrak{T}_p := p + \mathfrak{T}, \tag{8}$$

viewed again as an operator acting on $\mathbb{C}_{n+m}$. Also, to simplify the notation we denote by $M_p^{(k)}$ (resp. $M_{\mathfrak{T},p}^{(k)}$) the differential functional obtained by applying $M^{(k)}$ (resp. $M_{\mathfrak{T}}^{(k)}$), followed by evaluation at the point p.

The following proposition is the basic property of the multiplicity operators.

Proposition 9 ([5, Proposition 5])**.** *We have* $\operatorname{mult} F_p > k$ *if and only if* $M_p^{(k)}(F) = 0$ *for all multiplicity operators of order* k.

In the case $n = 1$, Proposition 9 corresponds to the familiar fact that p is a zero of multiplicity k of F if and only if $F(p) = F'(p) = \cdots = F^{(k)}(p) = 0$.

We now state a result relating the multiplicity operators of F to its set of common zeros.

Theorem 5 ([5, Theorem 1]). *Assume that $\|F\| \leqslant 1$ and that F has $k+1$ zeros (counted with multiplicities) in the polydisc D_r^n. Then for every k-th multiplicity operator $M^{(k)}$,*

$$C_{n,k}^Z \, |M_0^{(k)}(F))| < r \tag{9}$$

where $C_{n,k}^Z$ is a universal constant.

We outline the proof of Theorem 5 in the case $n = 1$. We assume for simplicity that $r < 1/2$. Then by standard arguments F can be decomposed in the form

$$F(z) = (z - z_0) \cdots (z - z_k) U(z) \tag{10}$$

where $\|U\| \leqslant 2^{k+1}$ and $z_0, \ldots, z_k$ are the $k+1$ zeros of F in the disc of radius r. It is now straightforward to verify that the first k derivatives of F are smaller than $(C_{1,k}^Z)^{-1} r$ for an appropriate constant $C_{1,k}^Z$.

We now state some results relating the multiplicity operators of F to its growth around the origin and its behavior under perturbation.

Theorem 6 ([5, Theorem 2]). *Let $\|F\| \leqslant 1$ and $M^{(k)}$ a k-th multiplicity operator. There exist positive universal constants $A_{n,k}, B_{n,k}$ with the following property:*

For every $r < |M_0^{(k)}(F)|$ there exists $A_{n,k} r < \tilde{r} < r$ such that

$$\|F(z)\| \geqslant B_{n,k} \, |M_0^{(k)}(F)| \, \tilde{r}^k \text{ for every } \|z\| = \tilde{r}. \tag{11}$$

We outline the proof of Theorem 6 in the case $n = 1$. If $P_k(z)$ denotes the k-th Taylor polynomial of F then

$$F(z) = P_k(z) + O(z^{k+1}), \qquad \|P_k\| = \Omega(|M_0^{(k)}(F)|). \tag{12}$$

Using polynomial inequalities, for instance the Cartan lemma, one can show that $|P_k(z)| = \Omega(|M_0^{(k)}(F)| \, z^k)$ for z outside a set of small discs around the zeros of P_k. Subsequently for $|z|$ sufficiently smaller than $|M_0^{(k)}(F)|$ we see that $|P_k(z)|$ dominates $O(z^{k+1})$, allowing one to derive (11) from (12).

Corollary 10 ([5, Corollary 15]). *Let $\|F\|, \|G\| \leqslant 1$ and $M^{(k)}$ be a k-th multiplicity operator. There exist universal constants $A''_{n,k}, B''_{n,k}$ with the following property:*

For every $\|G(0)\| < r < B''_{n,k} \, |M_0^{(k)}(F)|$ there exists $A''_{n,k} r < \tilde{r} < r$ such that

$$\|F(z)\| > \left\|G^{k+1}(z)\right\| \text{ for every } \|z\| = \tilde{r} \tag{13}$$

where G^{k+1} is taken component-wise.

In particular, Corollary 10 in combination with the Rouché principle implies that if F, G are as in the corollary and E is a holomorphic function satisfying $\|E\| < 1$ then

$$\#\{z : F(z) = 0, \|z\| < \tilde{r}\} = \#\{z : F(z) + E(z) G^{k+1}(z) = 0, \|z\| < \tilde{r}\}. \tag{14}$$

Finally, we state a result relating to multiplicity operators of F to its rate of growth along analytic curves.

Theorem 7 ([5, Theorem 3]). *Let $M^{(k)}$ be a multiplicity operator of order k and $\gamma \subset (\mathbb{C}^n, 0)$ a germ of an analytic curve. Then*

$$\operatorname{ord}_\gamma M^{(k)}(F) \geqslant \min\{\operatorname{ord}_\gamma f_i : i = 1, \ldots, n\} - k \tag{15}$$

In the case $n = 1$, Theorem 7 corresponds to the familiar fact that taking a k-th derivative of a function can decrease the order of its zero by at most k.

3. Structure of the proof

Consider a system (1) with parameters n, m, δ and the corresponding ring of Noetherian functions S. By Remark 3 we view S as the restriction of the ring $\mathbb{C}_{n+m}$ to an integral manifold Λ_p of (1).

Let $\rho \in S$ with $\deg \rho \leqslant d$ and let $X \subset \Lambda_p$ be a germ of a Noetherian set

$$X = \{x \in \Lambda : \psi_1(x) = \cdots = \psi_{n-1}(x) = 0\} \tag{16}$$

at the point p, where $\psi_i \in S$ and $\deg \psi_i \leqslant d$ for $i = 1, \ldots, n-1$.

By definition, $\rho = R|_{\Lambda_p}$ and $\psi_i = P_i|_{\Lambda_p}$ where $R, P_1, \ldots, P_{n-1}$ are elements of $\mathbb{C}_{n+m}$ of degrees bounded by d. To simplify the notation we will let P denote the tuple $P_1, \ldots, P_{n-1}$. When the point p and the integral manifold Λ_p are clear from the context, we will refer to the deflicity of X with respect to ρ at p simply as the deflicity of P, R.

3.1. **Analytic expression for the deflicity.** Since Noetherian functions are analytic, the germ X admits a decomposition into irreducible analytic components. Let $\{\gamma_i\}$ denote the components of X which are curves and such that $\rho|_{\gamma_i} \not\equiv \text{const}$ (where each curve is counted with an associated multiplicity m_i). We will call these curves the *good curves* of X with respect to ρ. All other components (whether they are curves on which ρ is constant or higher-dimensional components) will be called *bad components*.

To motivate this definition, note that since ρ is an analytic function, it cannot admit isolated zeros on components of X that have dimension greater than one, and therefore the bad components do not contribute any isolated zeros in the definition of the deflicity of X with respect to ρ. On the other hand, the number of solutions of $\rho = y$ on a good curve γ_i converging to p as $y \to \rho(p)$ is equal by definition to $\operatorname{mult}_{\gamma_i}(\rho - \rho(p))$. Since each good curve is transversal to $\rho^{-1}(\varepsilon)$ for all sufficiently small ε, each solution of $\rho = \varepsilon$ on γ_i should be counted with multiplicity m_i. To conclude, we have following proposition.

Proposition 11. *The deflicity of X with respect to ρ at the point p is given by*

$$\sum_i m_i \operatorname{mult}_{\gamma_i}(\rho - \rho(p)) \tag{17}$$

where the sum is taken over the good components γ_i.

3.2. **The set of non-isolated intersections.** Recall that $P = (P_1, \ldots, P_{n-1})$ and R denote polynomials of degree bounded by d.

Definition 12. *The set of non-isolated intersection of P with respect to R, denoted by $\mathcal{NI}(P; R)$, is the set of all points $q \in \mathbb{C}^{n+m}$ such that q belongs to a bad component of the set $\{P|_{\Lambda_q} = 0\}$ with respect to $R|_{\Lambda_q}$.*

The union of the bad component on each particular integral manifold Λ_q is analytic, but generally not algebraic. However, the following proposition shows that all of these sets together do form an algebraic set.

Proposition 13. *The set $\mathcal{NI}(P; R)$ is algebraic. Moreover, it can be defined by equations of degrees not exceeding*

$$d_{\mathcal{NI}} = \max[d_{\mathcal{IL}}, d_M(n, \delta, d, k)] \qquad \text{where } k = \mathcal{N}(m, n, \delta, d; 0) \tag{18}$$

where $d_{\mathcal{IL}}, d_M$ are as given in (6), (7) *respectively.*

Proof. A point $q \in \mathbb{C}^{n+m}$ belongs to $\mathcal{NI}(P; R)$ if (1) is integrable at the point, and the system of equations

$$P|_{\Lambda_q} = (R - R(q))|_{\Lambda_q} = 0 \tag{19}$$

admits a non-isolated solution through q. We can write down equations of degree $d_{\mathcal{IL}}$ to guarantee integrability by Theorem 3.

Since the maximal possible multiplicity of an isolated solution is bounded by k, a solution is non-isolated if and only if it has multiplicity greater than this number. By Proposition 9 this is equivalent to the vanishing of $M^{(k)}(P, R - R(q))$ for every multiplicity operator $M^{(k)}$ of order k. It remains to note that these expressions evaluate to polynomials (as functions of q) of the required degrees. □

The real analytic function $\operatorname{dist}(\cdot, \mathcal{NI}(P;R))$ will play a key role in our arguments. We will refer to it as the *critical distance*.

3.3. **The inductive step.** We will prove our main result, Theorem 1, by induction on the dimension of the set $\mathcal{NI}(P;R)$. To facilitate this induction we let $\mathcal{M}(m,n,\delta,d;e)$ denote the maximum possible deformation multiplicity at p for any system (1) with parameters m,n,δ and any pair X,ρ defined as in the beginning of §3 with $\deg P \leqslant d$ and $\deg R \leqslant d$, and such that the dimension of $\mathcal{NI}(P;R)$ near p is bounded by e.

The case $e = 0$ corresponds to systems where the intersection at p is in fact isolated. In this case the deformation multiplicity is given by the usual multiplicity of p as a solution of the equations $P = R - R(p) = 0$. Thus $\mathcal{M}(m,n,\delta,d;0)$ is bounded by Theorem 4.

Theorem 1 now follows by an easy induction from the following.

Theorem 8. *The quantity* $\mathcal{M}(m,n,\delta,d;e)$ *satisfies*

$$\mathcal{M}(m,n,\delta,d;e) \leqslant \mathcal{M}(m,n,\delta,d',e-1) \tag{20}$$

where

$$d' = (Ad_E + B)(k+1) + n + m, \tag{21}$$

A, B are defined in Proposition 16, d_E in Proposition 25 and $k = \mathcal{N}(m,n,\delta,d;0)$ as in Proposition 15. More explicit,

$$d' \leqslant (\max\{d,\delta\})^{32(m+n)^4} (m+n)^{40(m+n)^5}.$$

We now present a schematic proof of this statement assuming the validity of Propositions 15, 16, 18 and 25. The reader is advised to review this proof before reading the details of the aforementioned propositions in order to gain perspective on the context in which they are used. The propositions are presented in the three subsections of §4 and in §5 respectively, and each can be read independently of the others.

Proof. Consider a system (1) with parameters (n,m,δ), an integral manifold Λ_p, and a pair P,R of degree d such that $\dim \mathcal{NI}(P;R) = e$. Let $\{\gamma_i\}$ denote the good curves of P,R on Λ_p and let $\{\mathcal{NC}_i\}$ denote the irreducible components of $\mathcal{NI}(P;R)$. We assume that $R|_{\Lambda_p} \not\equiv \text{const}$ (otherwise there is nothing to prove). We will establish the claim by constructing a new pair P', R of degree d' such that

(i). The deflicity of P', R is no smaller than of P, R on Λ_p.
(ii). We have $\dim \mathcal{NI}(P';R) < \dim \mathcal{NI}(P;R) = e$.

Proposition 15 establishes a condition under which a perturbation of P does not decrease the deflicity. Roughly, the order of the perturbations along each good curve γ_i must be greater than a certain prescribed order ν_i. Under this condition, we show by a Rouché principle argument that the perturbation cannot decrease the number roots converging to p.

Proposition 16 relates the orders ν_i to the critical distance. Namely, we give two explicit constants A, B such that for any good curve γ_i the order of the critical distance function on

γ_i is at least $\frac{\nu_i - B}{A}$. Thus, to satisfy the requirements of Proposition 15, it essentially suffices to find a function E minorizing the critical distance.

The construction of E is carried out in Proposition 25, where we construct a polynomial $E \in \mathbb{C}_{n+m}$ of an explicitly bounded degree d_E which minorizes the critical distance along each good curve and does not vanish identically on any of the $\mathcal{NC}_i$. We choose a generic affine-linear functional $\ell \in (\mathbb{C}^{n+m})^*$ which vanishes at p, and does not vanish identically on any of the $\mathcal{NC}_i$ and define

$$E' = E^A \cdot \ell^B \tag{22}$$

where A, B are the coefficients given in Proposition 16. Then E' satisfies the asymptotic conditions of Proposition 15.

Our second goal is condition (ii). We achieve this condition in two steps. Namely, it would clearly suffice to construct a perturbation P' satisfying the following two conditions:

(A). $\mathcal{NI}(P'; R) \subseteq \mathcal{NI}(P; R)$.
(B). None of the components $\mathcal{NC}_i$ are contained in $\mathcal{NI}(P'; R)$.

We establish condition (A) by a Sard-type argument, essentially using the fact that the occurrence of a non-isolated intersection is a condition of infinite codimension. Specifically, if we choose $Q_1, \ldots, Q_{n-1}$ to be sufficiently generic polynomials of degree $n+m$ and define

$$P'_j = P_j + Q_j (E')^{k+1} \tag{23}$$

then according to Proposition 18 condition (A) is satisfied. Moreover, P' clearly satisfies the asymptotic conditions of Proposition 15 and hence condition (i).

It remains to establish condition (B). Consider a component $\mathcal{NC}_i$. Recall that E' does not vanish at generic points of $\mathcal{NC}_i$ by construction. And Q_1, being chosen to be sufficiently generic, certainly does not either. Since P_1 vanishes at every point of $\mathcal{NC}_i$ by definition, we see that P'_1 does not vanish at generic points of $\mathcal{NC}_i$, and hence condition (B) is satisfied. This concludes the proof. □

Proof of Theorem 1. We claim that in notations of Theorem 8,

$$d' \leqslant (\max\{d, \delta\})^{32(m+n)^4} (m+n)^{40(m+n)^5}. \tag{24}$$

Indeed, rough estimate for $\mathcal{N}(m, n, \delta, d; 0)$ in Theorem 4 gives

$$\mathcal{N}(m, n, \delta, d; 0) < (\delta + d)^{8(m+n)^2} (m+n)^{8(m+n)^3}. \tag{25}$$

Let denote the right hand side by C. We have $B, (k+1) \leqslant C$ as defined in Proposition 16. Then $d_M(n, \delta, d, B) < C^{n+1}(d+\delta)$ by (7), and $d_{\mathcal{JL}} < \delta^{2(m+n)}(m+n)^{6(m+n)}$, by (6). Therefore, according to Proposition 13, $d_{\mathcal{NI}} < C^{m+n}$ and $A < C^{(m+n)^2}$.

From (45) we have $d_H < C^{m+n}$. Therefore, from (57), $d_E < C^{2(m+n)^2}$. Therefore

$$d' < \left(C^{3(m+n)^2} + C\right) C + m + n < C^{4(m+n)^2} \leqslant (\max\{d, \delta\})^{32(m+n)^4} (m+n)^{40(m+n)^5}.$$

Now, applying (24) at most n times, we get

$$\mathcal{M}(m, n, \delta, d; e) \leqslant \mathcal{M}(m, n, \delta, \tilde{d}, 0), \quad \tilde{d} \leqslant (\max\{d, \delta\})^{(m+n)^{8n}} (m+n)^{(m+n)^{20n}},$$

so, by (25), we get the required bound. □

4. Good perturbations

Let $P = (P_1, \dots, P_{n-1})$ and R be polynomials of degree bounded by d, and let Λ_p be the germ of an integral manifold of (1) at the point p. Let $\{\gamma_i\}$ denote the set of good curves of P, R through the point p with associated multiplicities m_i. Let $\ell \in (\mathbb{C}^n)^*$ be a generic linear functional on the x-plane which is non-vanishing on vectors tangent to each γ_i at p, and let $\mathcal{T} = \ker \ell$. At generic point q of each γ_i, the multiplicity m_i is equal to the multiplicity of the isolated intersection $P = 0$ with the $(n-1)$-dimensional $\mathcal{T}$-plane. In particular, it is bounded by $k = \mathcal{N}(m, n, \delta, d; 0)$.

Let $M := M_{\mathcal{T}}^{(k)}$ be a generic k-th order multiplicity operator through the $\mathcal{T}$ plane. In particular, we require that M does not vanish identically on any of the γ_i (which is certainly possible by Proposition 9) and moreover that the order of M along each γ_i is the minimal possible. It is easy to verify that these conditions are satisfied by $\mathcal{T}, M$ outside a proper algebraic subset of the class of $(n-1)$-planes and multiplicity operators of order k (since having order greater than a given value is given by the vanishing of certain algebraic expressions depending on the coefficients of $\mathcal{T}, M$).

Remark 14. *In fact, since the multiplicity is taken through the $n-1$-dimensional plane $\mathcal{T}$, and the restriction of P to $\mathcal{T}$ may be though of as Noetherian function in the ambient space $\mathbb{C}^{n+m-1}$ cut out by $\ell = 0$, we could have chosen $k = \mathcal{N}(m, n-1, \delta, d; 0)$. However, since this does not affect our overall estimates and complicates the notation we use the weaker bound above.*

4.1. **Deflicity preserving perturbations.** Let E and $Q_1, \dots, Q_{n-1}$ be holomorphic functions on Λ_p.

Proposition 15. *Suppose that for every γ_i*

$$\operatorname{ord}_{\gamma_i} E > \max[\operatorname{ord}_{\gamma_i} M, \operatorname{ord}_{\gamma_i}(R - R(p))] \tag{26}$$

and let

$$P'_j = P_j + Q_j E^{k+1} \quad j = 1, \dots, n-1. \tag{27}$$

Then the deflicity of P', R is no smaller than the deflicity of P, R.

Proof. We assume for simplicity of the notation that $\ell(p) = R(p) = 0$. We may also assume that P, Q, E, R are all normalized to have norm bounded by 1 in some fixed ball in Λ_p.

Recall that ℓ is transversal to each γ_i at p. We say that the *pro-branches* of γ_i with respect to ℓ are the irreducible components γ_{ij} of the set $\gamma_i \cap \ell^{-1}(\mathbb{R}_{\geqslant 0})$. Each pro-branch is an irreducible germ of a real analytic curve. For $s \in (R_{\geqslant 0}, 0)$ there exists a unique point $\gamma_{ij}(s) \in \gamma_{ij} \cap \ell^{-1}(s)$, defining an analytic parametrization of γ_{ij}.

From Proposition 11 and standard theory of holomorphic curves we see that the deflicity of P, R is equal to

$$\sum_{\gamma_i} \operatorname{mult}_{\gamma_i} R = \sum_{\gamma_{ij}} \operatorname{ord}_{\gamma_{ij}} R \tag{28}$$

Here, if a curve γ_i appears with multiplicity m_i then we consider each of its pro-branches as appearing m_i times in the set γ_{ij}. Similarly, if we let γ'_i denote the good curves corresponding to P', R and γ'_{ij} their pro-branches (taking multiplicities into account), then the deflicity of P', R is equal to

$$\sum_{\gamma'_i} \operatorname{mult}_{\gamma'_i} R = \sum_{\gamma'_{ij}} \operatorname{ord}_{\gamma'_{ij}} R \tag{29}$$

We will show that if E satisfies the growth condition (26) then there is an injective map ι from the set $\{\gamma_{ij}\}$ to the set $\{\gamma'_{ij}\}$ such that

$$\operatorname{ord}_{\gamma_{ij}} R = \operatorname{ord}_{\iota(\gamma_{ij})} R. \tag{30}$$

Therefore the right-hand side of (29) is no smaller than the right-hand side of (28), and the conclusion of the proposition follows.

Introduce an equivalence relation $\sim$ on $\{\gamma_{ij}\}$ by letting $\gamma_{ij} \sim \gamma_{i'j'}$ if and only if

$$\operatorname{ord}_s \operatorname{dist}(\gamma_{ij}(s), \gamma_{i'j'}(s)) > \max[\operatorname{ord}_{\gamma_{ij}} M, \operatorname{ord}_{\gamma_{ij}} R] \tag{31}$$

In other words, two pro-branches are equivalent if and only if the distance between them is smaller by an order of magnitude than M and R (evaluated at one of them). It is easy to check that this is indeed an equivalence relation. We call the equivalence classes C_α of $\sim$ *clusters*.

Choose a representative γ_α for each cluster C_α. By (31) we can choose an order $\nu_\alpha \in \mathbb{Q}$ such that

$$\min_{\gamma_{ij}\sim\gamma_\alpha} [\operatorname{ord}_s \operatorname{dist}(\gamma_\alpha(s), \gamma_{ij}(s))] > \nu_\alpha > \max[\operatorname{ord}_{\gamma_\alpha} M, \operatorname{ord}_{\gamma_\alpha} R], \tag{32}$$

and by (26) we can also require that

$$\operatorname{ord}_{\gamma_\alpha} E > \nu_\alpha. \tag{33}$$

Consider the ball $B_\alpha(s) \subset \{\ell = s\}$ with center at $\gamma_\alpha(s)$ and radius s^{ν_α}. By (32) and the definition of $\sim$ it follows that for s sufficiently small, $B_\alpha(s)$ meets the pro-branches γ_{ij} in the cluster C_α and only them.

Given (32) and (33), Corollary 10 and the subsequent (14) apply with $r = s^{\nu_\alpha}/A''_{n,k}$ for s sufficiently small. It follows that the number of zeros of $P' = 0$ in the ball $B'_\alpha(s) \subset \{\ell = s\}$ with center at $\gamma_\alpha(s)$ and radius $s^{\nu_\alpha}/A''_{n,k}$ is at least the size of the cluster C_α.

By (32) the balls $B'_\alpha(s)$ are disjoint for different α and sufficiently small s. Consider now the pro-branches γ'_{ij}. We will say that γ'_{ij} *lies in* B'_α if $\gamma'_{ij}(s) \in B'_\alpha(s)$ for sufficiently small s. By the above, at least $\#C_\alpha$ of the pro-branches must lie in B'_α. Let ι be an arbitrary injection from the pro-branches γ_{ij} belonging to C_α to the pro-branches γ'_{ij} lying in B'_α.

The construction will be finished if we prove that ι satisfies (30). But this is clear, since by (32) the radius of $B'_\alpha(s)$ is an order of magnitude smaller than R on γ_α and therefore any pro-branch γ'_{ij} lying in B'_α must satisfy

$$\operatorname{ord}_{\gamma'_{ij}} R = \operatorname{ord}_{\gamma_\alpha} R. \tag{34}$$

□

4.2. **Minorizing the growth conditions by the critical distance.** Proposition 15 allows us to construct a perturbation of P, R which does not decrease the deflicity, given a function E which is small on the good curves γ_i compared to R and M.

In this subsection we show that R and M are minorized by the critical distance, and hence in order to apply Proposition 15 it suffices to minorize the critical distance. More precisely,

Proposition 16. *For every good curve γ_i we have*

$$\max[\operatorname{ord}_{\gamma_i} M, \operatorname{ord}_{\gamma_i}(R - R(p))] \leqslant A \operatorname{ord}_{\gamma_i} \operatorname{dist}(\cdot, \mathcal{NJ}(P, R - R(p))) + B \tag{35}$$

where

$$\begin{aligned} A &:= \max[d_{\mathcal{NJ}}, d_M(n, \delta, d, B)]^{n+m} \\ B &:= \mathcal{N}(m, n, \delta, d; 0) \end{aligned}$$

Proof. We start with the estimate for the function $R - R(p)$. Let $k = \mathcal{N}(m, n, \delta, d; 0)$ and let I_R generated by the polynomials $M^{(k)}(P, R - R(p))$ (where $M^{(k)}$ ranges over all multiplicity operators of order k) and the polynomials defining the integrability locus $\mathcal{IL}$ provided by Theorem 3. We denote the set of all of these generators by $\{G_i\}$.

Arguing in the same manner as in the proof of Proposition 13, we see that the zero locus of I_R is contained in $\mathcal{NI}(P; R)$ (in fact, it consists of those points in $\mathcal{NI}(P; R)$ where $R = R(p)$).

By the effective Łojasiewicz inequality [17], there exists a constant $C > 0$ such that

$$\max_i |G_i(\cdot)| \geqslant C \operatorname{dist}(\cdot, Z(I_R))^A \geqslant C \operatorname{dist}(\cdot, \mathcal{NI}(P; R))^A. \tag{36}$$

Let γ_i be a good curve. Then $\gamma_i \subset \mathcal{IL}$ and the polynomials defining $\mathcal{IL}$ vanish identically on it. Therefore along γ_i the maximum must be attained for one of the generators given by the multiplicity operator $M^{(k)}$, and hence

$$\operatorname{ord}_{\gamma_i} M^{(k)}(P, R - R(p)) \leqslant A \operatorname{ord}_{\gamma_i} \operatorname{dist}(\cdot, \mathcal{NI}(P; R)). \tag{37}$$

Since $P_1, \dots, P_{n-1}$ vanish identically on γ_i, the conclusion now follows by application of Theorem 7.

Proceeding now to the estimate for M, recall that we have $M = M_{\mathcal{T}}^{(k)}(P)$ for a generic multiplicity operator $M_{\mathcal{T}}^{(k)}$ where the direction $\mathcal{T}$ was chosen to be transversal to the good curves on Λ_p. However, in principle $\mathcal{T}$ may be parallel to good curves on other integral manifolds (i.e., its translate may contain them).

To avoid this problem we let $\mathcal{T}_1, \dots, \mathcal{T}_n$ be a tuple of linearly independent $(n-1)$-dimensional subspaces of the x-plane, each satisfying the same genericity condition as $\mathcal{T}$. For any integral manifold Λ and any point $q \in \{P|_\Lambda = 0\}$, if $q \notin \mathcal{NI}(P; R)$ then the zero locus of $P|_\Lambda$ near q must be a curve, and at least one of $\mathcal{T}_1, \dots, \mathcal{T}_n$ must not be parallel to this curve. Then there exists a multiplicity operator of order k in the $\mathcal{T}_i$ direction which is not vanishing at q.

Let I_M denote the ideal generated by the polynomials defining the integrability locus and all multiplicity operators of order k through any of spaces $\mathcal{T}_1, \dots, \mathcal{T}_n$ of P. By the above, the zero locus of I_M is contained in $\mathcal{NI}(P; R)$. Moreover, the integrability conditions vanish identically on γ_i, and all other generators of I_M have orders no smaller than the order of $M_{\mathcal{T}}^{(k)}(P)$ (since our original choice of $\mathcal{T}$ and $M_{\mathcal{T}}^{(k)}$ was generic). We can now complete the proof in a manner analogous to the argument we used for I_R. We leave the details for the reader. ☐

Remark 17. *Proposition 16 is the only step where we essentially need the assumption that the deformations under consideration are Noetherian with respect to the deformation parameter. Namely, in order to apply the effective Łojasiewicz inequality we require that the entire deformation space be algebraic, rather than the weaker condition that each fiber $R^{-1}(\varepsilon)$ be separately algebraic as provided in the Gabrielov-Khovanskii conjecture.*

4.3. **Sard-type claim for generic perturbations.** We are interested in applying Proposition 15 in order to produce a perturbation of P, R which does not decrease deflicity, and which reduces the set of non-isolated intersections $\mathcal{NI}(P; R)$. The first step is to show that our perturbation does not create new non-isolated intersections. We will show that for a sufficiently generic choice of the coefficients Q_j this will be the case.

Let E be a Noetherian function. Let $\beta \in \mathbb{N}$ and denote by $\mathcal{P}_\beta$ the space of polynomials of degree bounded by β in n variables. Finally let $(Q_1, \dots, Q_n) \in \mathcal{P}_\beta^n$, and set $P_i' = P_i + Q_i E$ for $i = 1, \dots, n$.

Proposition 18. *Assume $R|_{\Lambda_p} \not\equiv \text{const}$. There exists a neighborhood $U_p \subset \mathbb{C}^{n+m}$ of p such that for a generic tuple $Q = (Q_1, \dots, Q_n) \in \mathcal{P}^n_{n+m}$ (outside a proper algebraic set),*

$$\mathcal{NI}(P'; R) \cap U_p \subseteq \mathcal{NI}(P; R) \cap U_p \tag{38}$$

where P'_i is defined as above.

Proof. Fix a Euclidean ball U_p centered around p such that R is not constant on any integral manifold of (1) in U_p. Then for every point in $q \in U_p \cap \mathcal{IL}$ either Lemma 19 or Lemma 20 below is applicable, depending on whether $E(q) = 0$, with parameters $\beta = n + m$ and $l = n$. In both cases it follows that there exists a set $B(q) \subset \mathcal{P}^n_{n+m}$ of codimension $n + m + 1$, such that if $Q \notin B(q)$ then $q \notin \mathcal{NI}(P; R)$ implies $q \notin \mathcal{NI}(P'; R)$. Moreover, as noted in the proof of Lemmas 19 and 20, the graph of the relation $Q \in B(q)$ in each lemma is algebraic in q, so that the set

$$G = \{(Q, q) \in \mathcal{P}^n_{n+m} \times U_p : Q \in B(q)\}, \tag{39}$$

is real algebraic constructible. Since $\dim U_p = n + m$ it follows that the projection of G to $\mathcal{P}^n_{n+m}$ has codimension at least 1. Any Q outside this projection will satisfy (38). $\square$

It remains to state and prove the following two lemmas.

Lemma 19. *Suppose that $q \in \mathcal{IL}, E(q) \neq 0$ and suppose further that $R|_{\Lambda_q} \not\equiv R(q)$. Let $0 \leqslant l \leqslant n$.*

For $(Q_1, \dots, Q_l)$ outside a set B_l of codimension $\beta + 1$ in $\mathcal{P}^l_\beta$, the set

$$Z_l := \{P'_1 = \dots = P'_l = 0, R = R(q)\} \cap \Lambda_q \tag{40}$$

is empty or has pure codimension $l + 1$ (in a neighborhood of q).

Proof. Let B_l be the set of Q_i violating the condition. It is defined by the conditions $M^{(k)}_q(F^l_1, \dots, F^l_n) = 0$ for all multiplicity operators $M^{(k)}$ of every order k, where $\left(F^l_i\right)^n_{i=1}$ are n-tuples consisting of $P'_1|_{\Lambda_q}, \cdots, P'_l|_{\Lambda_q}, R|_{\Lambda_q} - R(q)$ and $n - l - 1$ generic linear functions vanishing at q. These expressions are polynomial in the coefficients of $Q_1, \dots, Q_l$, so the set B_l is algebraic and its dimension is well defined. We note that the conditions are algebraic with respect to q as well.

We prove the claim by induction. The case $l = 0$ corresponds precisely to our assumption $R|_{\Lambda_q} \not\equiv R(q)$.

Suppose that the claim is proved for $l-1$. Consider the projection $\pi : \mathcal{P}^l_\beta \to \mathcal{P}^{l-1}_\beta$ forgetting the last coordinate. The set $\pi^{-1}(B_{l-1})$ has codimension at least $\beta+1$ by induction. The claim will follow if we show that the fiber of each point outside B_{l-1} intersects B_l in codimension at least $\beta + 1$.

Let $(Q_1, \dots, Q_{l-1}) \notin B_{l-1}$. Then Z_{l-1} is either empty or has pure codimension l. If it is empty, there is nothing to prove. Otherwise $(Q_1, \dots, Q_l)$ will belong to the fiber if and only if P'_l vanishes identically on some irreducible component of Z_{l-1}. Since there are finitely many such components, it will suffice to check that identical vanishing on each of them has codimension at least $\beta + 1$. Let Z'_{l-1} be one such component.

Since $E(q) \neq 0$, we have $P'_l|_{Z'_{l-1}} \equiv 0$ if and only if $Q_l \equiv -P_l/E$ identically on Z'_{l-1}. This is an affine-linear condition on Q_l of codimension at least $\dim \mathcal{P}_\beta|_{Z'_{l-1}}$. It remains only to note that this dimension is at least $\beta + 1$: for instance if x_j is a coordinate not identically vanishing on Z'_{l-1} then clearly $1, x_j, \dots, x_j^\beta$ are linearly independent as functions defined on Z'_{l-1}. $\square$

Lemma 20. *Suppose that $q \in \mathcal{IL}, q \notin \mathcal{NI}(P;R)$ and $E(q)=0$. Let $0 \leqslant l \leqslant n$.*

For $(Q_1,\dots,Q_l)$ outside a set B_l of codimension $\beta+1$ in $\mathcal{P}_\beta^l$, the set

$$Z_l := \{P_1' = \cdots = P_l' = P_{l+1} = \cdots = P_n = 0, R = R(q)\} \cap \Lambda_q \tag{41}$$

is zero-dimensional in a neighborhood of q (that is, contains only, possibly, q).

Proof. Let B_l be the set of Q_i violating the condition. We can check that B_l is algebraic as in the proof of Lemma 19.

We prove the claim by induction. The case $l=0$ corresponds precisely to our assumption that $q \notin \mathcal{NI}(P;R)$.

Suppose that the claim is proved for $l-1$. Consider the projection $\pi : \mathcal{P}_\beta^l \to \mathcal{P}_\beta^{l-1}$ forgetting the last coordinate. The set $\pi^{-1}(B_{l-1})$ has codimension at least $\beta+1$ by induction. The claim will follow if we show that the fiber of each point outside B_{l-1} intersects B_l in codimension at least $\beta+1$.

Let $(Q_1,\dots,Q_{l-1}) \notin B_{l-1}$. Then the set Z_{l-1} is zero dimensional (in a neighborhood of q). If it is in fact empty, then one of the equations defining it is non-vanishing at q. In this case, since $E(q)=0$ by assumption, Z_l is empty as well.

We therefore must consider the case that $Z_{l-1} = \{q\}$ (in a neighborhood of q). In this case, the equations

$$\{P_1' = \cdots = P_{l-1}' = P_{l+1} = \cdots = P_n = 0, R = R(q)\} \cap \Lambda_q \tag{42}$$

define a curve $\gamma \subset \Lambda_q$. Thus $(Q_1,\dots,Q_l)$ will belong to B_l if and only if P_l' vanishes identically on an irreducible component of this curve. Since there are finitely many such components, it will suffice to check that identical vanishing on each of them has codimension at least $\beta+1$. Let γ' be one such component.

We have $P_l'|_{\gamma'} \equiv 0$ if and only if $EQ_l \equiv -P_l$ identically on γ'. This is an affine-linear condition on Q_l of codimension at least $\dim \mathcal{P}_\beta|_{\gamma'}$: if $E \not\equiv 0$ on γ' then this is clear, and otherwise the condition is never satisfied because S_{l-1} is a regular sequence and hence P_l does not vanish identically on γ. The conclusion now follows as in the proof of Lemma 19. □

5. Minorizing the critical distance

Let $P=(P_1,\dots,P_{n-1})$ and R be polynomials of degree bounded by d, and let Λ_p be the germ of an integral manifold of (1) at the point p. Let $\{\gamma_i\}$ denote the set of good curves of P,R through the point p with associated multiplicities m_i.

Our goal in this section is to construct a Noetherian function E of bounded degree such that E minorizes the critical distance on good curves, and does not vanish identically on any of the top-dimensional components of $\mathcal{NI}(P;R)$. The main step in the construction is the following Lemma.

We introduce some notations to facilitate our proof. If $\mathcal{T}$ is a linear subspace of the x-coordinates, then for every point $q \in \mathcal{IL}$ we will denote by $\mathcal{T}_q \subset \Lambda_q$ the integral submanifold through q of the sub-distribution of (1) corresponding to $\mathcal{T}$. Similarly $B_\mathcal{T}(q,r) \subset \mathcal{T}_q$ will denote the ball of radius r around q in $\mathcal{T}_q$.

The x-plane provides natural coordinates on the integral manifolds of (1) in a neighborhood U_p of p. In particular for any two integral manifolds $\Lambda_{q_1}, \Lambda_{q_2} \subset U_p$ we have a map $\tau_{q_2}^{q_1} : \Lambda_{q_1} \to \Lambda_{q_2}$ mapping each point in Λ_{q_1} to the point with the same x-coordinates in Λ_{q_2}. If we choose U_p small enough, then by the analytic dependence of flows on initial conditions, for every $q \in U_p$ we have

$$\operatorname{dist}(q, \tau_{q_2}^{q_1} q) \leqslant 2 \operatorname{dist}(q_1, q_2). \tag{43}$$

Lemma 21. *Let $\mathcal{NC}$ be an irreducible component of $\mathcal{NI}(P;R)$. There exists a polynomial $H \in \mathbb{C}_{n+m}$ such that*

(1) *For every good curve γ we have*

$$\operatorname{ord}_\gamma H \geqslant \operatorname{ord}_\gamma \operatorname{dist}(\cdot, \mathcal{NC}) \tag{44}$$

(2) *H does not vanish identically on $\mathcal{NC}$.*
(3) *The degree of H is bounded by*

$$d_H := (k+1) d_M(n, \delta, d, k) \qquad \text{where } k = \mathcal{N}(m, n, \delta, d; 0) \tag{45}$$

Proof. For any $q \in \mathcal{NC}$ the set $\mathcal{NI}(P;R) \cap \Lambda_q$ consists of those components of $\{P = 0\} \cap \Lambda_q$ which have dimension greater than 1, or where R is constant. In particular, $\mathcal{NC} \cap \Lambda_q$ is a union of such components. Assume now that $q' \in \mathcal{NC}$ is generic. Then locally near q' we will have $\mathcal{NC} \cap \Lambda_{q'} = \{P = 0\} \cap \Lambda_{q'}$. Thus if $\operatorname{codim}_{\Lambda_{q'}}(\mathcal{NC} \cap \Lambda_{q'}) = l$, then letting $\Phi = (P_1, \ldots, P_l)$ (up to a reordering of P_j), we have $\mathcal{NC} \cap \Lambda_{q'} = \{\Phi = 0\} \cap \Lambda_{q'}$

Let now $\mathcal{T}$ be a generic l-dimensional subspace of the x coordinates. Denote by λ the multiplicity of the isolated zero $\{\Phi|_{\mathcal{T}_{q'}} = 0\}$. In particular, $\lambda \leqslant k$. By Proposition 9 there exists a multiplicity operator $M^{(\lambda)}$ of order λ such that $M^{(\lambda)}_{\mathcal{T},q'}(\Phi) \neq 0$. Let $M = M^{(\lambda)}_{\mathcal{T}}(\Phi)$.

Claim 22. *At any point $q \in \mathcal{NC}$ we have $\operatorname{mult}_q \Phi|_{\mathcal{T}_q} \geqslant \lambda$. Moreover, if $M(q) \neq 0$ then*

(1) $\operatorname{mult}_q \Phi|_{\mathcal{T}_q} = \lambda$.
(2) $\mathcal{NC} \cap \Lambda_q = \{\Phi = 0\} \cap \Lambda_q$ *in a neighborhood of q.*

Proof of the claim. The first claim $\operatorname{mult}_q \Phi|_{\mathcal{T}_q} \geqslant \lambda$ holds because λ was chosen as the generic (and hence minimal) multiplicity of a zero of $\Phi|_{\mathcal{T}_{q'}}$ at a point of $\mathcal{NC}$. Assume now that $M(q) \neq 0$. Then $\operatorname{mult}_q \Phi|_{\mathcal{T}_q}$ cannot exceed λ by Proposition 9. In particular it is finite, so the set $\{\Phi = 0\} \cap \Lambda_q$ has codimension l near q. Since l was chosen to be the generic (hence maximal) codimension of $\mathcal{NC}$ intersected with any integral manifold, the codimension of $\mathcal{NC} \cap \Lambda_q$ is at most l. Since $\mathcal{NC} \cap \Lambda_q \subset \{\Phi = 0\} \cap \Lambda_q$ we see that $\mathcal{NC} \cap \Lambda_q$ must in fact be a union of irreducible components of $\{\Phi = 0\} \cap \Lambda_q$. Suppose toward contradiction that this set has another component $\mathcal{C}$ through q.

Since $M(q) \neq 0$, the set $\{\Phi = 0\} \cap \Lambda_q$ has an isolated intersection with $\mathcal{T}_q$ at q. Consider generic $q' \in \Lambda_q$ arbitrarily close to q. Then $\mathcal{T}_{q'}$ will meet $\mathcal{C}$ at some point q_1 close to q, and the set $\mathcal{NC} \cap \Lambda_q$ at some other point q_2 close to q. Both points q_1, q_2 correspond to zeros of $\Phi|_{\mathcal{T}_{q'}}$ and by the first part of the claim $\operatorname{mult}_{q_2} \Phi|_{\mathcal{T}_{q_2}} \geqslant \lambda$. As $q' \to q$ both $q_1, q_2 \to q$ and hence $\operatorname{mult}_q \Phi|_{\mathcal{T}_q} > \lambda$ contradicting what was already proved. $\square$

Let γ be any good curve, and denote by $\gamma(t)$ a pro-branch with $\gamma(0) = p$. Since $\{\Phi = 0\} \cap \Lambda_p$ contains γ it does not equal $\mathcal{NC} \cap \Lambda_p$ around p, and Claim 22 implies that $M(p) = 0$. Since $M(\gamma(t))$ is analytic we can fix $t_0 > 0$ such that its modulus is monotone for $t \leqslant t_0$. We claim that

$$\operatorname{ord}_\gamma \operatorname{dist}(\cdot, \mathcal{NC}) \leqslant (\lambda + 1) \operatorname{ord}_\gamma M. \tag{46}$$

Suppose to the contrary that for some $\varepsilon > 0$ and for arbitrarily small t there exist points $y(t) \in \mathcal{NC}$ such that

$$\rho(t) < |M(\gamma(t))|^{\lambda+1+\varepsilon}, \qquad \rho(t) := \operatorname{dist}(\gamma(t), y(t)). \tag{47}$$

Then we will produce a sequence of points in $\mathcal{NC}$ converging to $\gamma(t_0)$, which is impossible since $\mathcal{NC}$ is closed and γ is a good curve. More specifically, we analytically continue the point $y(t)$ to a curve in $\mathcal{NC} \cap \Lambda_{y(t)}$ using (two applications of) Lemma 23 and show that the endpoint of this curve converges to $\gamma(t_0)$ as $t \to 0$.

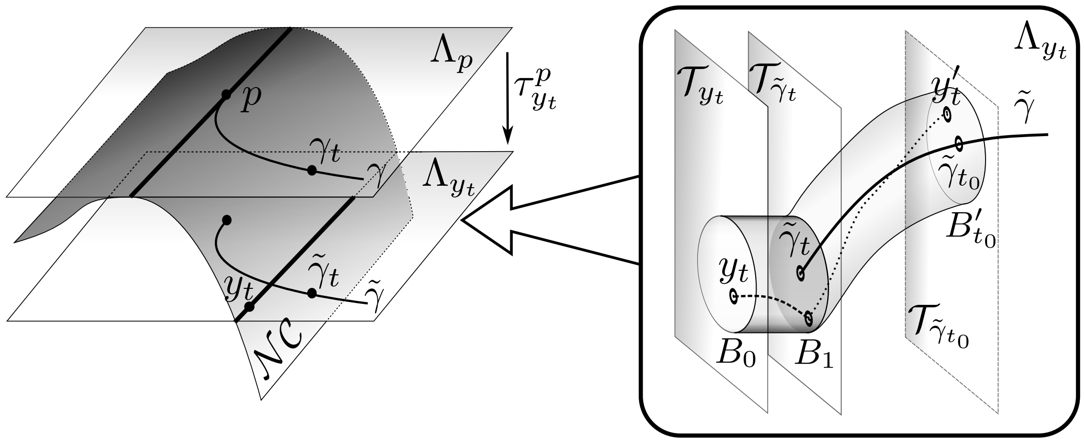

FIGURE 1. On the left, the entire ambient space; On the right, the leaf $\Lambda_{y(t)}$. The two dotted lines correspond to the two steps of analytic continuation. In the interest of space we render $\tau^p_{y(t)}(X)$ as $\tilde X$ and $X(t)$ as X_t.

In what follows whenever we use asymptotic class notation (O, o, Ω, Θ) with complex-valued functions we implicitly interpret it as applying to their modulus. We stress that the asymptotic constants are understood to be independent of t, s. Since M is holomorphic in U_p it is Lipschitz there, and it follows that for any two points $p_1, p_2 \in U_p$ we have

$$M(p_1) - M(p_2) = O(\operatorname{dist}(p_1, p_2)) \tag{48}$$

and similarly from Φ. For instance, from (47) it follows that

$$\rho(t) = O(M(y(t))^{\lambda+1+\varepsilon}). \tag{49}$$

We can thus choose positive $r(t) \in \mathbb{R}_+$ satisfying

$$r(t) = o(M(y(t))), \tag{50}$$

$$\rho(t) = o(r^\lambda(t) M(y(t))). \tag{51}$$

Consider the family of balls $B_s = B_{\mathcal{T}}(c_s, r(t)) \subset \Lambda_{y(t)}$ connecting B_0 and B_1,

$$B_0 = B_{\mathcal{T}}(y(t), r(t)), \qquad B_1 = B_{\mathcal{T}}(\tau^p_{y(t)}\gamma(t), r(t)) \tag{52}$$

by a linear motion of their centers c_s in the x-coordinates. Denote $\Phi_s := \Phi|_{B_s}$. Note that by (43) and the triangle inequality we have

$$\operatorname{dist}(\gamma(t), c_s) = O(\rho(t)). \tag{53}$$

We claim that Lemma 23 applies to Φ_s. Indeed,

(i) Φ_s admits at most λ zeros for every $s \in [0,1]$. Indeed, by (53) we see that $\operatorname{dist}(c_s, y(t)) = O(\rho(t))$. Then (48) and (49) give $M(c_s) = \Theta(M(y(t)))$. On the other hand, if Φ_s has more than λ zeros in B_s then by Theorem 5 we have $M(c_s) = O(r(t))$. This contradicts (50).

(ii) Φ_s admits at least one zero for every $s \in [0,1]$. Indeed, by Theorem 6 there exists a radius $A_{n,\lambda} r(t) < \tilde r(t) < r(t)$ such that the minimum of $\|\Phi_0\|$ over $\partial B_{\mathcal{T}}(y(t), \tilde r(t))$ is $\Omega(M(y(t))\tilde r^\lambda(t))$, which by (51) is asymptotically larger than $\rho(t)$. On the other hand, by (43) we may view Φ_s as a perturbation of size $O(\rho(t))$ of Φ_0. By the

Rouché principle, such a perturbation does not change the number of zeros of Φ_0 in $B_{\mathcal{T}}(y(t),\tilde r(t))$. Since $y(t)$ is such a zero, Φ_s must have a zero as well.

(iii) The zero $y(t)$ of Φ_0 has multiplicity λ and does not bifurcate for small s. Indeed, $y(t)\in\mathcal{NC}$ and $M(y(t))\neq 0$ by construction, so by Claim 22 it must be a root of multiplicity λ of Φ_0. Moreover, $\mathcal{NC}\cap\Lambda_{y(t)}=\{\Phi=0\}\cap\Lambda_{y(t)}$ locally near $y(t)$. It follows that $y(t)$ cannot bifurcate for small values of s: it must remain an element of $\mathcal{NC}\cap\Lambda_{y(t)}$, and λ is the minimal possible multiplicity for such a root.

By Lemma 23 we conclude that Φ_s has a zero $y_s(t)$ of multiplicity λ. Moreover by (iii) above $y_s(t)\in\mathcal{NC}\cap\Lambda_{y(t)}$ for small s. By analyticity the same must hold for every $s\in[0,1]$. Thus $y_1(t)$ is a zero of Φ_1 and $y_1(t)\in\mathcal{NC}\cap\Lambda_{y(t)}$.

Now consider the family of balls $B'_s=B_{\mathcal{T}}(c'_s,r(t))\subset\Lambda_{y(t)}$ for $s\in[t,t_0]$ where

$$c'_s=\tau^p_{y(t)}\gamma(s). \tag{54}$$

Note that $B'_t=B_1$. As before we set $\Phi'_s:=\Phi|_{B'_s}$ and claim that Lemma 23 applies to Φ'_s. Indeed,

(i') Φ'_s admits at most λ zeros for every $s\in[0,1]$. Indeed, by (43) we see that $\operatorname{dist}(c'_s,\gamma(s))=O(\rho(t))$. By (48) we have

$$M(c'_s)=M(\gamma(s))+O(\rho(t))=\Omega(M(\gamma(t)))+O(\rho(t))=\Omega(M(y(t))) \tag{55}$$

where we used the monotonicity of $M|_\gamma$ in the second equality. On the other hand, if Φ_s has more than λ zeros in B'_s then by Theorem 5 we have $M(c'_s)=O(r(t))$. This contradicts (50).

(ii') Φ'_s admits at least one zero for every $s\in[0,1]$. Indeed, let $\tilde B_s:=B_{\mathcal{T}}(\gamma(s),r(t))$ and $\tilde\Phi_s=\Phi|_{\tilde B_s}$. By (43) we may view $\tilde\Phi_s$ as a perturbation of size $O(\rho(t))$ of Φ_s. Arguing as in the previous item (ii) and using (55), we see that this perturbation does not change the number of zeros in an appropriately chosen ball around the center. Since the center $\gamma(s)$ is a zero of $\tilde\Phi_s$, it follows that Φ_s must have a zero as well.

(iii') The zero $y_1(t)$ of Φ'_t has multiplicity λ and does not bifurcate for small s. Indeed, $\operatorname{dist}(y(t),y_1(t))=O(r(t))$ and it follows using (50) that $M(y_1(t))=\Theta(M(y(t)))$. Since the latter is non-zero by construction we conclude that $M(y_1(t))\neq 0$. Moreover $y_1(t)\in\mathcal{NC}$ by construction. The proof is now concluded in the same way as the previous item (iii).

We thus apply Lemma 23 and conclude in the same way as before that Φ'_{t_0} has a zero $y'(t)$, and moreover that $y'(t)\in\mathcal{NC}\cap B'_{t_0}$. As t tends to zero the center of B'_{t_0} tends to $\gamma(t_0)$ and its radius tends to zero, hence $y'(t)$ tends to $\gamma(t_0)$. As $y'(t)\in\mathcal{NC}$ we obtain the desired contradiction. Therefore (46) is proved, and taking $H=M^{\lambda+1}$ concludes the proof. □

The proof of Lemma 21 will be completed once we prove the following lemma.

Lemma 23. *Let $U\subset\mathbb{C}^l$ be an open domain, and $\Phi_s:U\to\mathbb{C}^l$ be an analytic family of holomorphic mappings $s\in[0,1]$. Let $\lambda\in\mathbb{N}$ and assume that*

(i) *Φ_s has at most at most λ zeros in U, counting multiplicities, for all s.*
(ii) *Φ_s has at least one zero in U for all s.*
(iii) *Φ_0 has a zero y_0 of multiplicity λ in U, while lies on a germ of a curve y_s of zeros of multiplicity λ of Φ_s (i.e. y_0 doesn't bifurcate into several zeros for small values of s).*

Then y_s can be analytically extended to a curve of zeros of multiplicity λ of Φ lying in U for all $s\in[0,1]$.

Proof of the Lemma. Indeed, the curve y_s can be analytically extended in s as long as it doesn't leave U, as the non-bifurcating condition is analytic. Suppose toward contradiction that y_s leaves U, and let S denote the infimum of the set $\{s : y_s \notin U\}$. Then by (ii) the map Φ_S must have some other zero $y' \in U$, and since U is open y' may be continued to a zero y'_s of Φ_s for s close to S. But then for s slightly smaller than S we have a zero y_s of multiplicity λ (by (iii)) and a zero y'_s, contradicting (i). □

We require one more standard fact, whose proof we include for the convenience of the reader.

Fact 24. *Let $V \subset \mathbb{C}^N$ be an affine variety of degree D. Then V is set-theoretically cut out by polynomials of degree D.*

Proof. Let $x \in \mathbb{C}^N \setminus V$. We will find a polynomial of degree bounded by D that does not vanish at x. If V is a hypersurface then it is the zero locus of a polynomial of degree D and the claim is obvious. Otherwise choose a generic projection $\pi : \mathbb{C}^N \to \mathbb{C}^{\dim V+1}$, such that $\pi(x) \notin \pi(V)$. Then $\pi(V)$ is a hypersurface of degree D, and the previous argument produces a polynomial of degree D which vanishes on $\pi^{-1}\pi(V)$. Since x is not contained in this set, the proof is completed. □

Finally we can present the construction of the function E.

Proposition 25. *There exists a polynomial $E \in \mathbb{C}_{n+m}$ such that*

(1) *For every good curve γ we have*

$$\operatorname{ord}_\gamma E \geqslant \operatorname{ord}_\gamma \operatorname{dist}(\cdot, \mathcal{NI}(P;R)) \tag{56}$$

(2) *H does not vanish identically on any irreducible component of $\mathcal{NI}(P;R)$.*

(3) *The degree of E is bounded by*

$$d_E := d_{\mathcal{NI}}^{2(n+m)} + d_H \tag{57}$$

Proof. Let $\mathcal{NI}(P;R) = \cup_{i=1,\dots,s}\mathcal{NC}_i$ be the irreducible decomposition of $\mathcal{NI}(P;R)$. By Proposition 13 the set $\mathcal{NI}(P;R)$ can be defined by polynomial equations of degree $d_{\mathcal{NI}}$. Therefore, $s \leqslant d_{\mathcal{NI}}^{n+m}$ and any irreducible component $\mathcal{NC}_i$ of this set has degree bounded by $d_{\mathcal{NI}}^{n+m}$. Choose a polynomial Q_i of this degree which vanishes on $\mathcal{NC}_i$ and not on any $\mathcal{NC}_j$ for $j \neq i$. Also construct for each $\mathcal{NC}_i$ the polynomial H_i provided by Lemma 21.

Let

$$E = \sum_{i=1}^{s} H_i \prod_{j\neq i} Q_j. \tag{58}$$

Let γ be a good curve, and suppose that $\operatorname{dist}(\cdot, \mathcal{NI}(P;R))|_\gamma$ attains its minimum on the component $\mathcal{NC}_i$. Then

$$\operatorname{ord}_\gamma H_i, \operatorname{ord}_\gamma Q_i \geqslant \operatorname{ord}_\gamma \operatorname{dist}(\cdot, \mathcal{NI}(P;R)) \tag{59}$$

and since each summand in (58) is a product containing either H_i or Q_i,

$$\operatorname{ord}_\gamma E \geqslant \operatorname{ord}_\gamma \operatorname{dist}(\cdot, \mathcal{NI}(P;R)). \tag{60}$$

Moreover, for each component $\mathcal{NC}_i$, all summands other than the i-th vanish identically on $\mathcal{NC}_i$, whereas the i-th summand does not. Therefore E does not vanish identically on any component $\mathcal{NC}_i$, and the proposition is proved. □

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