On the finite cyclicity of open period annuli

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Abstract

Let Π be an open, relatively compact period annulus of real analytic vector field X_0 on an analytic surface. We prove that the maximal number of limit cycles which bifurcate from Π under a given multi-parameter analytic deformation X_λ of X_0 is finite, provided that X_0 is either Hamiltonian, or generic Darbouxian vector field.

1 Statement of the result

Let S be a real analytic surface without border (compact or not), and X_0 a real analytic vector field on it. An open period annulus of X_0 is an union of period orbits of X_0 which is bi-analytic to the standard annulus $S^1 \times (0, 1)$, the image of each circle $S^1 \times \{u\}$ being a periodic orbit of X_0.

Let X_λ, $\lambda \in (\mathbb{R}^n, 0)$ be an analytic family of analytic vector fields on S, and let Π be an open period annulus of X_0. The *cyclicity* $Cycl(\Pi, X_\lambda)$ of Π with respect to the deformation X_λ is the maximal number of limit cycles of X_λ which tend to Π as λ tends to zero, see Definition 2 bellow. Clearly the vector field X_0 has an analytic first integral f in the period annulus Π which has no critical points. In what follows we shall suppose that the open period annulus Π is relatively compact (i.e. its closure $\bar{\Pi} \subset S$ is compact).

Definition 1. *We shall say that X_0 is a Hamiltonian vector field provided that it has a first integral with isolated critical points in a complex neighborhood of Π. We shall say that X_0 is a generic Darbouxian vector field provided that all singular points of X_0 in a neighborhood of $\bar{\Pi}$ are orbitally analytically equivalent to linear saddles $\dot{x} = \lambda x, \dot{y} = -y$ with $\lambda > 0$.*

Remark 1. *Note that if X_0 is a plane vector field with a first integral H as above, then*

$$X_0 = H_1(H_y \frac{\partial}{\partial x} - H_x \frac{\partial}{\partial y})$$

where H_1 is a non-vanishing real-analytic function in some complex neighborhood of Π. In the case when X_0 is a generic Darbouxian vector field, as we shall see in the next section, it can be covered by a planar Darbouxian vector field with a first integral of the "Darboux type" $H = \prod_{i=1}^{n} P_i^{\lambda_i}$ for some analytic functions P_i in a complex neighborhood of Π.

The main result of the paper are the following

Theorem 1. *The cyclicity $Cycl(\Pi, X_\lambda)$ of the open period annulus Π of a Hamiltonian vector field X_0 is finite.*

Theorem 2. *The cyclicity $Cycl(\Pi, X_\lambda)$ of the open period annulus Π of a generic Darbouxian vector field X_0 is finite.*

The above theorems are a particular case of the Roussarie's conjecture [12, p.23] which claims that the cyclicity $Cycl(\Gamma, X_\lambda)$ of every compact invariant set Γ of X_0 is finite. Indeed, $Cycl(\Pi, X_\lambda) \leq Cycl(\bar{\Pi}, X_\lambda)$. The finite cyclicity of the open period annulus without the assumptions of Theorems 1 and 2 is an open question.

To prove the finite cyclicity we note first that it suffices to show the finite cyclicity of a given one-parameter deformation X_ε. This argument is based on the Hironaka's desingularization theorem, see [11, 2]. Consider the first return map associated to Π and X_ε

$$t \to t + \varepsilon^k M_k(t) + \ldots, t \in (0,1), \varepsilon \sim 0.$$

The cyclicity of the open period annulus Π is finite if and only if the Poincaré-Pontryagin function M_k has a finite number of zeros in $(0,1)$. It has been shown in [4] that M_k allows an integral representation as a linear combination of iterated path integrals along the ovals of Π of length at most k. The finite cyclicity follows then from the non-accumulation of zeros of such iterated integrals at 0 and 1. The proof of this fact will be different in the Hamiltonian and in the generic Darbouxian case.

In the Hamiltonian case we observe that M_k satisfies a Fuchsian equation [3, 4]. We prove in section 4 that the associated monodromy representation is quasi-unipotent, which implies the desired property.

In the Darbouxian case the above argument does not apply (there is no Fuchsian equation satisfied by M_k). We prove the non-oscillation property of an iterated integral by making use of its Mellin transformation, along the lines of [9]. It seems to be difficult to remove the genericity assumption in the Darbouxian case (without this the Hamiltonian case is a sub-case of the Darbouxian one).

The paper is organized as follows. In section 2 we recall the definition of cyclicity and the reduction of multi-parameter to one-parameter deformations. In section 3 we reduce the case of a vector field on a surface to the case of a plane vector field.

Theorem 2 in the Hamiltonian case is proved in section 4 according to the scheme

$$\text{Proposition 4} \Rightarrow \text{Proposition 5} \Rightarrow \text{Proposition 3}$$

$$\{\text{Theorem 5} + \text{Proposition 3}\} \Rightarrow \text{Theorem 6}$$

$$\text{Theorem 6} \Rightarrow \text{Theorem 2 in the Hamiltonian case.}$$

Theorem 2 in the generic Darbouxian case is proved in section 5 of the paper.

2 Cyclicity and non-oscillation of the Poincaré-Pontryagin-Melnikov function

Definition 2. *Let X_λ be a family of analytic real vector fields on a surface S, depending analytically on a parameter $\lambda \in (\mathbb{R}^n, 0)$, and let $K \subset S$ be a compact invariant set of X_{λ_0}. We say that the pair (K, X_{λ_0}) has cyclicity $N = Cycl((K, X_{\lambda_0}), X_\lambda)$ with respect to the deformation X_λ, provided that N is the smallest integer having the property: there exists $\varepsilon_0 > 0$ and a neighborhood V_K of K, such that for every λ, such that $\|\lambda - \lambda_0\| < \varepsilon_0$, the vector field X_λ has no more than N limit cycles contained in V_K. If $\tilde{K}$ is an invariant set of X_{λ_0} (possibly non-compact), then the cyclicity of the pair $(\tilde{K}, X_{\lambda_0})$ with respect to the deformation X_λ is*

$$Cycl((\tilde{K}, X_{\lambda_0}), X_\lambda) = sup\{Cycl((K, X_{\lambda_0}), X_\lambda) : K \subset \tilde{K}, K \text{ is a compact }\}.$$

The above definition implies that when $\tilde{K}$ is an open invariant set, then its cyclicity $Cycl((\tilde{K}, X_{\lambda_0}), X_\lambda)$ is the maximal number of limit cycles which tend to $\tilde{K}$ as λ tends to 0. To simplify the notation, and if there is no danger of confusion, we shall write $Cycl(K, X_\lambda)$ on the place of $Cycl((K, X_{\lambda_0}), X_\lambda)$.

Example 1. *Let $f_\varepsilon(t) = \varepsilon e^{-1/t}(t sin(1/t) - \varepsilon), f_\varepsilon(0) = 0$. One can easily see that $f_\varepsilon(t) = 0$ has finite number of isolated positive zeros for each ε, and this number tends to infinity as $\varepsilon \to 0$. Below we construct a germ X_ε of a vector field having a monodromic planar singular point at the origin, with a return map $x \to x + f_\varepsilon(x)$. Since isolated singular points of the return map correspond to limit cycles, we see that the vector field X_ε has a finite number of limit cycles for each ε, and this number tends to infinity as ε tends to zero. So the cyclicity of the open period annulus $\Pi = \mathbb{R}^2 \setminus \{0\}$ is infinity. Note that, however, the vector field X_ε is not analytic at the origin.*

Here is a construction: on the strip $S = [0, \delta] \times \mathbb{R}$ consider the equivalence relation $(r, \phi) \sim (r + f_\varepsilon(r), \phi - 2\pi)$. Let $p : S \to S/\sim$ be the corresponding projection, and define $\tilde{X}_\varepsilon = p_*(\partial_\phi)$. One can check that for δ small enough thus defined $\tilde{X}_\varepsilon$ is a blow-up of a smooth vector field X_ε defined near the origin, and the return map of X_ε is as prescribed by construction.

Let $\Delta \subset S$ be a cross-section of the period annulus Π which can be identified to the interval $(0, 1)$. Choose a local parameter u on Δ. Let $u \mapsto P(u, \lambda)$ be the first return map and $\delta(u, \lambda) = P(u, \lambda) - u$ the displacement function of X_λ. For every closed interval $[a, b] \subset \Delta$ there exists $\varepsilon_0 > 0$ such that the displacement function $\delta(u, \lambda)$ is well defined and analytic in $\{(u, \lambda) : a - \varepsilon_0 < u < b + \varepsilon_0, \|\lambda\| < \varepsilon_0\}$. For every fixed λ there is a one-to-one correspondance between isolated zeros of $\delta(u, \lambda)$ and limit cycles of the vector field X_λ.

Let $u_0 \in \Delta$ and let us expand

$$\delta(u, \lambda) = \sum_{i=0}^{\infty} a_i(\lambda)(u - u_0)^i.$$

Definition 3 (Bautin ideal [13], [12])**.** *We define the Bautin ideal $\mathcal{I}$ of X_λ to be the ideal generated by the germs $\tilde{a}_i$ of a_i in the local ring $\mathcal{O}_0(\mathbb{R}^n)$ of analytic germs of functions at $0 \in \mathbb{R}^n$.*

This ideal is Noetherian. Let $\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots, \tilde{\varphi}_p$ be a minimal system of its generators, where $p = \dim_{\mathbb{R}} \mathcal{I}/\mathcal{M}\mathcal{I}$, and $\mathcal{M}$ is the maximal ideal of the local ring $\mathcal{O}_0(\mathbb{R}^n)$. Let $\varphi_1, \varphi_2, \ldots, \varphi_p$ be analytic functions representing the generators of the Bautin ideal in a neighborhood of the origin in $\mathbb{R}^n$.

Proposition 1 (Roussarie, [12]). *The Bautin ideal does not depend on the point* $u_0 \in \Delta$. *For every* $[a,b] \subset \Delta$ *there is an open neighborhood* U *of* $[a,b] \times \{0\}$ *in* $\mathbb{R} \times \mathbb{R}^n$ *and analytic functions* $h_i(u,\lambda)$ *in* U*, such that*

$$\delta(u,\lambda) = \sum_{i=0}^{p} \varphi_i(\lambda) h_i(u,\lambda). \tag{1}$$

The real vector space generated by the functions $h_i(u,0), u \in [a,b]$ *is of dimension* p.

Suppose that the Bautin ideal is principal and generated by $\varphi(\lambda)$. Then

$$\delta(u,\lambda) = \varphi(\lambda) h(u,\lambda) \tag{2}$$

where $h(u,0) \not\equiv 0$. The maximal number of the isolated zeros of $h(u,\lambda)$ on a closed interval $[a,b] \subset (0,1)$ for sufficiently small $|\lambda|$ is bounded by the number of the zeros of $h(u,0)$, counted with multiplicity, on $[a,b]$. This follows from the Weierstrass preparation theorem, properly applied, see [2]. Therefore to prove the finite cyclicity of Π it is enough to show that $h(u,0)$ has a finite number of zeros on $(0,1)$. Consider a germ of analytic curve $\xi : \varepsilon \mapsto \lambda(\varepsilon)$, $\lambda(0) = 0$, as well the analytic one-parameter family of vector fields $X_{\lambda(\varepsilon)}$. The Bautin ideal is principal, $\delta(u,\varepsilon) = \varphi(\varepsilon)h(u,\varepsilon)$, and

$$\delta(u,\lambda(\varepsilon)) = \varepsilon^k M_k(u) + \ldots, M_k(u) = c\, h(u,0), c \neq 0$$

where the dots stay for terms containing ε^i, $i \geq k$. M_k is the so called kth order higher Poincaré-Pontryagin-Melnikov function associated to the one-parameter deformation $X_{\lambda(\varepsilon)}$ of the vector field X_0. Therefore, if the cyclicity of the open period annulus is infinite, then M_k has an infinite number of zeros on the interval $(0,1)$

Of course, in general the Bautin ideal is not principal. However, by making use of the Hironaka's theorem, we can always principalize it. More precisely, after several blow up's of the origin of the parameter space, we can replace the Bautin ideal by an ideal sheaf which is principal, see [2] for details. This proves the following

Proposition 2. *If the cyclicity* $Cycl(K, X_\lambda)$ *of the open period annulus* Π *is infinite, then there exists a one parameter deformation* $\lambda = \lambda(\varepsilon)$*, such that the corresponding higher order Poincaré-Pontryagin-Melnikov function* M_k *has an infinite number of zeros on the interval* $(0,1)$.

In the next two sections we shall prove the non-oscillation property of M_k in the Hamiltonian and the Darbouxian case (under the restrictions stated in Theorem 2).

3 Reduction to the case of a plane vector field

Let X_0 be a real analytic vector field on a real analytic surface S. Let Π be an open period annulus of X_0 with compact closure. Let the map $\tau : \Pi \to S^1 \times (0,1)$ be a bi-analytic isomorphism, such that $\delta_t = \tau^{-1}(S^1 \times \{t\})$ is a closed orbit of X_0. We assume that X_0 is either Hamiltonian or generalized Darbouxian in some neighborhood of the closure $\bar{\Pi}$ of Π. Theorems 1 and 2 claims that cyclicity of Π in any family of analytic deformation X_λ of X_0 is finite.

This paragraph is devoted to the reduction of this general situation to the case of a vector field X_0 on $\mathbb{R}^2$ of Hamiltonian or Darboux type near its polycycle. Then Theorem 1 and Theorem 2 follow from Theorem 4 and Theorem 3 below.

First, note that it is enough to prove finite cyclicity of $\tau^{-1}(S^1 \times (0,\varepsilon))$ only. Indeed, finite cyclicity of $\tau^{-1}(S^1 \times [\varepsilon, 1-\varepsilon])$ follows from Gabrielov's theorem, and finite cyclicity of $\tau^{-1}(S^1 \times (1-\varepsilon, 1))$ can be reduced to the above by replacing t by $1-t$.

Consider the Hausdorf limit $\Gamma = \lim_{t\to 0} \tau^{-1}(S^1 \times \{t\})$. It is a connected union of several fixed points $a_1, ..., a_n$ of X_0 (not necessarily pairwise different) and orbits $\Gamma_1, ..., \Gamma_n$ of X_0 such that Γ_i exits from a_i and enters a_{i+1} (where a_{n+1} denotes a_1).

From now on we consider only a sufficiently small neighborhood U of Γ. We assume that $U \cap \Pi = \tau^{-1}(S^1 \times (0,\varepsilon))$, and denote this intersection again by Π. We consider first the Darbouxian case. Note that Γ cannot consist of just one singular point of X_0 by assumption about linearizability of singular points of X_0 in this case.

Lemma 1. *Assume that Theorem 2 holds if U is orientable and all a_i are different. Then Theorem 2 holds in full generality.*

Proof. Assume that for some real analytic surface $\tilde{U}$ there is an analytic mapping $\pi : \tilde{U} \to U$ which is a finite covering on Π. Then the cyclicity of Π for X_λ is the same as cyclicity of $\pi^{-1}(\Pi)$ for the lifting X_λ to $\tilde{U}$. The claim of the Lemma follows from this principle applied to two types of coverings below.

First, taking a double covering of U as $\tilde{U}$, we can assume that U is orientable.

Second, let U be represented as a union of neighborhoods U_i of a_i together with neighborhoods V_i of Γ_i. Glue $\tilde{U}$ as $\tilde{U} = \tilde{U}_1 \cup \tilde{V}_1 \cup ... \cup \tilde{V}_n$, where $\tilde{U}_i$ are bianalytically equivalent to U_i and disjoint, and $\tilde{V}_i$ are bianalytically equivalent to V_i, with natural glueing of $\tilde{U}_i$ to $\tilde{V}_i$, of $\tilde{V}_i$ to $\tilde{U_{i+1}}$ and of of $\tilde{U}_1$ to $\tilde{V}_n$. In other words, $\pi : \tilde{U} \to U$ is one-to-one away from a_i and k_i-to-one in

Figure 1: Proof of Lemma 1.

a neighborhood of a_i if a_i appears k_i times in the list $\{a_1, ..., a_n\}$. Evidently, π is one-to-one on Π, so is bianalytic. $\square$

We will now define a first integral H of X_0 in U. Take any non-singular point $a \in \gamma_1$, and let H be a local first integral of X_0 in a neighborhood U_a of a such that $H(a) = 0$ and $dH(a) \neq 0$. Since U is orientable, Π lies from one side of Γ, and we can assume that intersection of U_a with each cycle δ_t is connected. This allows to extend H to a first integral of X_0 defined on $\Pi \cap U$. Changing sign of H if necessary, we can assume that $H > 0$ on $\Pi \cap U_a$. We define $H(\Gamma) = 0$ by continuity.

Lemma 2. *Extension of H to $\Pi \cap U$ by flow of X_0 can be extended to a multivalued holomorphic function defined in a neighborhood of Γ in a complexification of U.*

Proof. First, H is analytic in some neighborhood of Γ_1, as it is an analytic function extended by analytic flow of X_0. Choose local linearizing coordinates (x, y) near a_2 in such a way that $\Gamma_1 = \{y = 0\}$. By assumption, yx^μ is the local first integral of X_0 near a_2. Therefore $H = f(yx^\mu)$, and, restricting to a transversal $x = x_0 \ll 1$, one can see that f is analytic and invertible. Therefore H can be extended to a neighborhood of a_2.

Moreover, $\left(f^{-1}(H)\right)^{1/\mu}$ is an analytic local first integral near the point $y = 1$ of Γ_2. Therefore it can be extended to a neighborhood of Γ_2 (here we use that U is orientable, so Γ_2 is different from Γ_1), and, as above, to a neighborhood of a_3 (here we use that $a_2 \neq a_3$), and so on. $\square$

Note that from the above construction follows that near each Γ_i the first integral H is equal, up to an invertible function, to x^{λ_i}, where $\{x = 0\}$ is a local equation of γ_i. Also, near any singular point of Γ the first integral H is equal, up to an invertible function, to $x^\lambda y^\mu$.

Corollary 1. *The one-form $\frac{dH}{H}$ is meromorphic one-form in U with logarithmic singularities only.*

Assume that $n \geq 3$. One can easily construct a C^∞ isomorphism of a sufficiently small neighborhood U of Γ with a neighborhood of a regular n-gone in R^2 in such a way that the image of $\Pi \cap U$ will lie inside the n-gone and image of Γ coincides with the n-gone. Due to [?], some neighborhood $U^{\mathbb{C}}$ of U in its complexification is a Stein manifold. This implies that this isomorphism can be chosen bianalytic. Similarly, for $n = 2$ one can map bianalytically a neighborhood of U to a union of two arcs $\{x^2 + (|y| - 1)^2 = 2\} \subset \mathbb{R}^2$, which, for the rest of the paper, will be called "regular 2-gone".

We transfer everything to plane using this isomorphism and will denote the images on plane of the previously defined objects by the same letters. The first integral H takes the form $H = H_1 \prod_{i=1}^n P_i^{\lambda_i}$, where P_i are analytic functions in U with $\{P_i = 0\} = \Gamma_i$, H_1 is an analytic functions non-vanishing in its neighborhood U and $\lambda_i > 0$. Note that $H > 0$ in the part of U lying inside the n-gone. Further we assume that $H_1 \equiv 1$, so $H = \prod P_i^{\lambda_i}$ (one can achieve this by e.g. taking $P_1 H_1^{1/\lambda_1}$ instead of P_1).

The family X_λ becomes a family of planar analytic vector fields defined in a neighborhood U of a regular n-gone $\Gamma \subset \mathbb{R}^2$, and X_0 has a first integral H of Darboux type in U. Let $X_\varepsilon = X_{\lambda(\varepsilon)}$ be a one-parametric deformation of X_0 as in Proposition 2. Define meromorphic forms ω^2, ω_ε as

$$\omega^2(X_0, \cdot) = \frac{dH}{H}, \qquad \omega^2(X_\varepsilon, \cdot) = X_0 + \omega_\varepsilon. \tag{3}$$

According to [4, Theorem 2.1], M_k can be represented as a linear combination of iterated integrals over $\{H = t\}$ of forms which are combinations of Gauss-Manin derivatives of ω_ε.

Recall that the Gauss-Manin derivative of a form η is defined as a form η' such that $d\eta = d(\log H) \wedge \eta'$. In general, η' cannot be uniquely defined from this equation, though its restrictions to $\{H = t\}$ are defined unambiguously. However, since $U^{\mathbb{C}}$ is Stein, in our situation one can choose a meromorphic in U representative of η', with poles on $\breve{\Gamma}$ only (where $\breve{\Gamma}$ is the union of lines containing sides of Γ).

Therefore Theorem 2 follows from the following claim

Theorem 3. *Let $H = \prod_{i=1}^n P_i^{\lambda_i}$ be as above, and let $\gamma(t) \subset \{H = t\}$ be the connected component of its level set lying inside Γ. Zeros of polynomials in iterated integrals $I(t) = \int_{\gamma(t)} \omega_1 ... \omega_k$ corresponding to meromorphic one-forms $\omega_1, ..., \omega_k$ with poles in $\breve{\Gamma}$ cannot accumulate to* 0.

From the above discussion it is clear that Theorem 1 follows on its turn from the following

Theorem 4. *Let*

$$X_0 = H_y \frac{\partial}{\partial x} - H_x \frac{\partial}{\partial y}$$

where H is a real analytic function with isolated singularities in some complex neighborhood of the closed period annulus $\bar{\Pi} = \{\gamma(t) : 0 \leq t \leq 1\}$, where $\gamma(t) \subset \{H = t\}$ is the connected component of the level set of H lying inside Γ. Zeros of the first non-vanishing Poincaré-Pontryagin function M_k, corresponding to a one-parameter analytic deformation X_ε of X_0 cannot accumulate to 0.

4 Non-oscillation in the Hamiltonian case

Here shall prove Theorem 4. This follows from the following two results

Theorem 5 ([4]). *The Poincaré-Pontryagin function M_k satisfies a linear differential equation of a Fuchs type in a suitable complex neighborhood of $0 \in \mathbb{C}$.*

Theorem 6. *The monodromy operator of the above Fuchs equation corresponding to a loop encircling the origin in $\mathbb{C}$ is quasi-unipotent.*

Let us recall that an endomorphism is called unipotent, if all its eigenvalues are equal to 1, and quasi-unipotent if all of them are roots of the unity. The above theorems imply that the Poincaré-Pontryagin-Melnikov function has a representation in a neighborhood of $u = 0$

$$M_k(u) = \sum_{i=0}^{N} \sum_{j=0}^{N} u^{\mu_i} (log(u))^j f_{ij}(u)$$

where $N \in \mathbb{N}$, $\mu_j \in \mathbb{Q}$, and f_{ij} are functions analytic in a neighborhood of $u = 0$. This shows that the zeros of $M_k|_{(0,1)}$ do not accumulate to 0. Of course, similar arguments hold in a neighborhood of $u = 1$, so M_k has a finite number of zeros on $(0, 1)$. This completes the proof of Theorem 2 in the Hamiltonian case. To the end of the section we prove Theorem 6. The open real surface S is analytic and hence possesses a canonical complexification. Similarly, any analytic family of analytic vector fields X_λ is extended to a complex family of vector fields, depending on a complex parameter. In this section, by abuse of notation, the base field will be $\mathbb{C}$. A real object and its complexification will be denoted by the same letter.

Let $U \supset \bar{\Pi}$ be an open complex neighborhood of $\bar{\Pi}$ in which the complexified vector field X_0 has an analytic first integral f with isolated critical points. The restriction of f on the interval $(0,1)$ (after identifying Π to $S^1 \times (0,1)$) is a local variable with finite limits at 0 and 1 Therefore we may suppose that $f(0) = 0$, $f(1) = 1$, and the restriction of f to $(0,1)$ is the canonical local variable on $(0,1) \subset \mathbb{R}$. The function f defines a locally trivial Milnor fibration in a neighborhood of every isolated critical point. There exists a complex neighborhood U of $\bar{\Pi}$ in which F has only isolated critical points. Moreover the compactness of $\bar{\Pi}$ implies that there exists a complex neighborhood $D \subset \mathbb{C}$ of the origin, homeomorphic to a disc, such that the fibration

$$U \cap \{f^{-1}(D \setminus \{0\})\} \xrightarrow{f} D \setminus \{0\} \tag{4}$$

is locally trivial, and the fibers $f^{-1}(t) \cap U$ are open Riemann surfaces homotopy equivalent to a bouquet of a finite number of circles. Consider a one-parameter analytic deformation X_ε of the vector field X_0. As f is a first integral of X_0, then there exists an unique symplectic two-form ω^2, such that

$$\omega^2(X_0, .) = df.$$

Indeed, if in local coordinates

$$X_0 = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$$

then $X_0.df = 0$ implies $(a,b) = \lambda(f_y, -f_x)$, where λ is analytic in U and non-vanishing in Π. It follows that

$$\omega^2 = \frac{dx \wedge dy}{\lambda}.$$

Define a unique meromorphic one-form ω_ε by the formula

$$\omega^2(X_\varepsilon, .) = df + \omega_\varepsilon.$$

The one form ω_ε is meromorphic in U, depends analytically on ε, and $\omega_0 = 0$. Its pole divisor does not depend on ε as in the local variables above it is defined by $\lambda = 0$. Therefore $\omega_\varepsilon = \sum_{i\geq 1} \varepsilon^i \omega_i$ where ω_i are given meromorphic one-forms in U with a common pole divisor which does not intersect the period annulus Π. In the complement of the singular locus of X_ε the vector field X_ε and the one form $df + \omega_\varepsilon$ define the same foliation, and therefore define the same first return map associated to Π. Denote this map by $P(t,\varepsilon)$, where $t \in (0,1)$ is the restriction of f to a cross-section of the period annulus Π (this does not depend on the choice of the cross-section). We have

$$P(t,\varepsilon) = t + \sum_{k\geq 1} \varepsilon^k M_k(t).$$

On each leaf of the foliation defined by X_ε we have $df = -\omega_\varepsilon$ which implies

$$M_1(t) = \int_{\gamma_t} \omega_1$$

where $\{\gamma_t\}_t$ is the family of periodic orbits (with appropriate orientation) of X_0, $\Pi = \cup_{t\in(0,1)}\gamma_t$, [10]. Thus the first Poincaré-Pontryagin-Melnikov function is an Abelian integral and its monodromy representation is straightforward. Namely, the meromorphic one-form ω_1 restricts to a meromorphic one-form on the fibers of the Minlor fibration (4). We may also suppose that $\omega_1|_{f^{-1}(t)}$ has a finite number of poles $\{P_i(t)\}_i$ (after choosing appropriately the domain U). Denote

$$\Gamma_t = U \cap \{f^{-1}(t) \setminus \{P_i(t)\}_i\}$$

The Milnor fibration (4) induces a representation

$$\mathbb{Z} = \pi_1(D \setminus \{0\}, *) \to Aut(H_1(\Gamma_t, \mathbb{Z})) \tag{5}$$

which implies the monodromy representation of M_1. Suppose first that ω_1 is analytic in U. It is well known that the operator of the classical monodromy of an isolated critical point of an analytic function is quasi-unipotent, e.g. [7]. Therefore the representation in $Aut(H_1(U \cap \{f^{-1}(t)\}, \mathbb{Z}))$ of a small loop about 0 in $\pi_1(D \setminus \{0\}, *)$ is quasi-unipotent. More generally, let ω_1 be meromorphic one-form with a finite number of poles on the fibers $U \cap \{f^{-1}(t)\}$. A monodromy operator permutes the poles and hence an appropriate power of it leaves the poles fixed. Therefore this operator is quasi-unipotent too and Theorem 5 is proved in the case $M_1 \neq 0$. Of course, it is well known that an Abelian integral has a finite number of zeros [8, 15].

Let M_k be the first non-zero Poincaré-Pontryagin-Melnikov function. Its "universal" monodromy representation was constructed in [3]. For convenience of the reader we reproduce it here. Recall first that $M_k(t)$ depends on the *free* homotopy class of of the loop γ_t in $\pi_1(\Gamma_t)$ [3, Proposition 1] and that this property does not hold true for the first return map $P(t, \varepsilon)$ (which depends on the homotopy class of γ_t in $\pi_1(\Gamma_t, *)$) . Let $F = \pi_1(\Gamma_t, *)$ be the fundamental group of Γ_t. It is a finitely generated free group. Let $\mathcal{O} \subset \pi_1(\Gamma_t)$ be the orbit of the loop γ_t under the action of $\mathbb{Z}^2 = \pi_1(D \setminus \{0\}, *)$ induced by (4). The set $\mathcal{O}$ generates a normal subgroup of F which we denote by G. The commutator subgroup $(G, F) \subset F$ is the normal sub-group of F generated by commutators $(g, f) = g^{-1}f^{-1}gf$. The Milnor fibration (4) induces a representation

$$\mathbb{Z} = \pi_1(D \setminus \{0\}, *) \to Aut(G/(G, F)). \tag{6}$$

According to [3, Theorem 1], the monodromy representation of M_k is a sub-representation of the monodromy representation dual to (6). Unfortunately the free Abelian group $F/(G,F)$ is not necessarily of finite dimension. To obtain a finite-dimensional representation we use the fundamental fact that M_k has an integral representation as an iterated path integral of length k [4, Theorem 2.1].

To use this, define by induction $F_{i+1} = (F_i, F)$, $F_1 = F$. We will later consider the associated graded group

$$grF = \bigoplus_{i=1}^{\infty} gr^i F, \; gr^i F = F_i / F_{i+1}. \tag{7}$$

It is well know that an iterated integral of length k along a loop contained in F_{k+1} vanishes identically. Therefore, to study the monodromy representation of M_k, we shall truncate with respect to F_{k+1} and obtain a finite-dimensional representation. Namely, for every subgroup $H \subset F$ we denote

$$\tilde{H} = (H \cup F_{k+1})/F_{k+1}.$$

The representation (6) induces a homomorphism

$$\pi_1(\mathbb{C} \setminus D, *) \to Aut(\tilde{G}/(\tilde{G}, \tilde{F})) \tag{8}$$

and the monodromy representation of M_k is a sub-representation of the representation dual to (8) [4]. The Abelian group $\tilde{G}/(\tilde{G}, \tilde{F})$ is, however, finitely generated. Indeed the lower central series of $\tilde{F} = \tilde{F}_1$ is

$$\tilde{F}_1 \supseteq \tilde{F}_2 \supseteq \dots \tilde{F}_k \supseteq \{id\}$$

and hence $\tilde{F}$ is a finitely generated nilpotent group. Each sub-group of such a group is finitely generated too, e.g. [6].

The central result of this section is the following theorem, from which Theorem 6 follows immediately

Proposition 3. *The monodromy representation (8) is quasi-unipotent.*

Indeed, M_k satisfies a Fuchsian equation on D, whose monodromy representation is a sub-representation of the representation dual to (8) [4, Theorem 1.1] and [3, Theorem 1]. To prove Proposition 3 we recall first some basic facts from the theory of free groups, e.g. Serre [14], Hall [6]. The graded group grF (7) associated to the free finitely generated group F is a Lie algebra with a bracket induced by the commutator $(.,.)$ on F. The Milnor fibration (4) induces a representation

$$\mathbb{Z} = \pi_1(D \setminus \{0\}, *) \to Aut_{Lie}(grF) \tag{9}$$

where $Aut_{Lie}(grF)$ is the group of Lie algebra automorphisms of grF. Let l be a generator of $\pi_1(D \setminus \{0\}, *)$. It induces automorphisms $l_* \in Aut_{Lie}(grF)$ and $l_*|_{gr^kF} \in Aut(gr^kF)$. We note that $gr^1F = H_1(\Gamma_t, \mathbb{Z})$ and hence $l_*|_{gr^1F}$ is quasi-unipotent.

Proposition 4. *Let $l_* \in Aut_{Lie}(grF)$ be such that $l_*|_{gr^1F}$ is quasi-unipotent. Then for every $k \geq 1$ the automorphism $l_*|_{gr^kF}$ is quasi-unipotent.*

The proof is by induction. Let $X = \{x_1, x_2, ..., x_\mu\}$ be the free generators of F and consider the free Lie algebra L_X on X. It is a Lie sub-algebra of the associative non-commutative algebra of polynomials in the variables x_i with a Lie bracket $[x, y] = xy - yx$. The canonical map $(x, y) \mapsto [x, y]$ induces an isomorphism of Lie algebras $grF \to L_X$, [14, Theorem 6.1]. Let $L^k_X \subset L_X$ be the graded piece of degree k. We shall show that $l_*|_{L^k_X}$ is quasi-unipotent. The proof is by induction. Suppose that the restriction of l_* on $gr^1F = L^1_X = H_1(\Gamma_t, \mathbb{Z})$ is quasi-unipotent, i.e. for some p, q, the restriction of $(l^p_* - id)^q$ on gr^1F is 0. The operator $Var_* = l^p_* - id$ is a linear automorphism, but not a Lie algebra automorphism. The identity

$$\begin{aligned} Var_*[x,y] &= (l^p_* - id)(xy - yx) = l^p_* x l^p_* y - l^p_* y l^p_* x - xy + yx \\ &= [Var_* x, Var_* y] + [Var_* x, y] + [x, Var_* y] \end{aligned}$$

shows that the restriction of Var^{2q} on L^2_X vanishes identically. Therefore The automorphism l_* restricted to L^2_X or gr^2F is quasi-unipotent. The case $k \geq 3$ is similar. Proposition 4 is proved.□

According to the above Proposition for every $i \in \mathbb{N}$ there are integers m_i, n_i, such that the polynomial $p_i(z) = (z^{m_i} - 1)^{n_i}$ annihilates $l_*|_{gr^kF}$. Proposition 3 will follow on its hand from the following

Proposition 5. *The polynomial $p = \prod_{i=1}^k p_i$ annihilates $l_* \in Aut(\tilde{G}/(\tilde{G}, \tilde{F}))$.*

Proof. Let $l \in \pi_1(D \setminus \{0\}, *)$. It induces an automorphism of the Abelian groups $G/(G,F), G \cap F_i/(G \cap F_i, F), F_i/F_{i+1}$ denoted, by abuse of notation, by l_*. We denote by $p_i(l_*) = (l^{m_i}_* - id)^{n_i}$ the corresponding homomorphisms. It follows from the definitions that the diagram (10) of Abelian groups, is commutative (the vertical arrows are induced by the canonical projections). Therefore if an equivalence classe $[\gamma] \in G/(G,F)$ can be represented by a closed loop $\gamma \in F_i$, then $p_i(l_*)[\gamma]$ can be represented by a closed loop in F_{i+1}. Therefore for every $[\gamma] \in G/(G,F)$, the equivalence class $p(l_*)$ can be represented by a closed loop in F_{k+1}. In other words $p(l_*)$ indices the zero automorphism of $Aut(\tilde{G}/(\tilde{G}, \tilde{F}))$.□

$$
\begin{array}{ccc}
F_i/(F_i,F) & \xrightarrow{p_i(l_*)} & F_i/(F_i,F) \\
\uparrow \pi_2 & & \uparrow \pi_2 \\
G\cap F_i/(G\cap F_i,F) & \xrightarrow{p_i(l_*)} & G\cap F_i/(G\cap F_i,F) \\
\pi_1 \downarrow & & \pi_1 \downarrow \\
G/(G,F) & \xrightarrow{p_i(l_*)} & G/(G,F)
\end{array}
\qquad (10)
$$

5 Non-oscillation in the Darboux case

In this section we prove Theorem 3. First, we consider *elementary iterated integrals* - the iterated integrals over the piece of the cycle lying near the saddles. We give a representation of the Mellin transform of the elementary iterated integral as a converging multiple series. This representation provides an asymptotic series for the elementary iterated integral, with some explicit estimate of the error, see Theorem 7 below.

The general iterated integral of length k turns out to be a polynomial (depending on X_0 and k only) in elementary iterated integrals, by Lemma 3. We give analogue of the estimates of Theorem 7 for such polynomials. This allows to prove a quasianalyticity property: if the asymptotic series corresponding to the iterated integral is zero, then the integral itself is zero. This implies Theorem 3 since the zeros of the partial sums of the asymptotic series do not accumulate to 0, see Corollary 3.

The arguments follow the pattern of [9], so the proofs are replaced by a reference whenever possible.

5.1 Iterated integral as a polynomial in elementary iterated integrals.

Let $\gamma(u)$, $u \in [0,1]$, be a parameterization of the cycle $\gamma_t \subset \{H = t\}$ (we fix some $t > 0$ for a moment). As in [9], the cycle of integration can be split into several pieces γ_j, those lying near the sides of the polycycle, and those near the vertices. We assume that the vector field can be linearized in the charts containing these pieces, and call these pieces elementary. Let $0 = v_0 < v_1 < ... < v_m < 1$ be the parameterization of the ends of these pieces.

The iterated integral in the parameterized form is equal to

$$\int_\Delta g_1(u_1)...g_k(u_k)du_1...du_k,$$

where $\Delta = \{0 \le u_1 \le ... \le u_k \le 1\} \subset \mathbb{R}^k$ is a simplex.

Consider connected components of the complement of Δ to the union of hyperplanes $\cup_{i,j}\{u_j = v_i\}$. Each connected component can be defined as

$$\{0 \le u_1 \le ... \le u_{i_1} < v_1 < u_{i_1+1} \le ... < v_m < u_{i_m+1} \le ... \le u_k \le 1\},$$

i.e. is a product $\Delta_1 \times ... \times \Delta_m$ of several simplices of smaller dimension of the form $\Delta_j = \{v_j < u_{i_j+1} \le ... \le u_{i_{j+1}} < v_{j+1}\}$. Therefore, by Fubini theorem, integral of $g_1(u_1)...g_k(u_k)$ over this connected component is equal to the product of integrals $\int_{\Delta_j} g_{i_j+1}...g_{i_{j+1}} du_{i_j+1}...du_{i_{j+1}}$, i.e. to the product of iterated integrals $\int_{\gamma_j} \omega_{i_j+1}...\omega_{i_j+1}$.

Let us call the iterated integral over an elementary piece γ_j an *elementary iterated integral.* The above arguments show that

Lemma 3. *Iterated integral is a polynomial with integer coefficients in elementary iterated integrals. The polynomial depends on the length of the iterated integral only.*

The above arguments give an explicit form of this polynomial (though we will not need it).

5.2 Mellin transform of elementary iterated integrals

There are two types of elementary pieces: those lying in charts covering sides of the polycycle, and those lying in charts covering saddles. Similarly to [9], the elementary iterated integrals corresponding to the pieces of the first type are just meromorphic functions of the parameter on the transversal, i.e. of t^{1/λ_i}.

From this moment we assume that the elementary piece lies near the saddle $\{P_1 = P_2 = 0\}$. In other words, we assume that $\gamma(t) = \{x^{\lambda_1}y^{\lambda_2} = t\} \cap \{0 \le x, y \le 1\}$.

We give description of iterated integrals in terms of their Mellin transforms. Recall that the Mellin transform of a function $f(t)$ on the interval $[0, 1]$ is defined as $\mathscr{M}f(s) = \int_0^1 t^{s-1}f(t)dt$. To describe the Mellin transform of the elementary iterated integrals over $\gamma(t)$ let us introduce a generalized compensator. We denote in this section by l the length of the elementary iterated integral. For $l \in \mathbb{N}$ and $\alpha = (m_1, n_1, ..., m_l, n_l) \in \mathbb{Z}^{2l}$ we define $\ell^l_\alpha(s; \lambda_1, \lambda_2)$ as

$$\ell^l_\alpha(s; \lambda_1, \lambda_2) = \prod_{j=0}^{l} \left(s + \lambda_1^{-1} \sum_{i=1}^{j} m_i + \lambda_2^{-1} \sum_{i=j+1}^{l} n_i \right)^{-1}. \tag{11}$$

We call $\mathscr{M}^{-1}\ell^l_\alpha(s;\lambda_1,\lambda_2)$ a generalized compensator. Particular case of $l=1$ corresponds to the Ecalle-Roussarie compensator. Generalized compensator is a finite linear combination of monomials of type $t^\mu(\log t)^{l'}$, for $l'\le l$.

We omit λ_1,λ_2 from the notation till the end of the section.

Lemma 4. *After some rescaling of t the Mellin transform of an elementary iterated integral is given by the following formula:*

$$\mathscr{M}\int \omega_1...\omega_l=\sum_\alpha c_\alpha \ell^l_\alpha,\quad \alpha\in\left(\mathbb{Z}_{>-M}\right)^{2l}, \tag{12}$$

where M is an upper bound for the order of poles of ω_i. Moreover, $|c_\alpha|\le C2^{-|\alpha|}$.

This is a straightforward generalization of the construction of [9], which corresponds to $l=1$.

Proof. In the linearizing coordinates the first integral is written as $H=x^{\lambda_1}y^{\lambda_2}$. The Mellin transform of the iterated integral can be computed explicitly for monomial forms $\omega_i=x^{m_i-1}y^{n_i}\,dx$:

$$\mathscr{M}\int \omega_1...\omega_l= \tag{13}$$

$$\int_0^1 t^{s-1}\int_{t^{1/\lambda_1}}^1 x_1^{m_1-1}y_1^{n_1}\int_{x_1}^1 x_2^{m_2-1}y_2^{n_2}\int_{x_2}^1...\int_{x_{l-1}}^1 x_l^{m_l-1}y^{n_l}dx_l...dx_1dt= \tag{14}$$

$$=\int_0^1 t^{\frac{n_1+...+n_l}{\lambda_2}}t^{s-1}\int_{t^{1/\lambda_1}}^1 x_1^{m_1-1-n_1\mu}\int_{x_1}^1...\int_{x_{l-1}}^1 x_l^{m_l-1-n_l\mu}dx_l...dx_1dt= \tag{15}$$

$$=\int_0^1 x_l^{m_l-1-n_l\mu}\int_0^{x_l}x_{l-1}^{m_{l-1}-1-n_{l-1}\mu}...\int_0^{x_1^{\lambda_1}}t^{\frac{n_1+...+n_l}{\lambda_2}+s-1}dt...dx_l= \tag{16}$$

$$=\lambda_1^{-l}\prod_{j=0}^{l}\left(s+\lambda_1^{-1}\sum_{i=1}^{j}m_i+\lambda_2^{-1}\sum_{i=j+1}^{l}n_i\right)^{-1}=\lambda_1^{-l}\ell^l_\alpha. \tag{17}$$

Similar formula holds for $\omega_i=x^{m_i}y^{n_i-1}\,dy$.

After rescaling of H we can assume that the linearizing chart covers the bidisk $\{0\le|x|,|y|\le 2\}$. Then the coefficients of the forms ω_i are meromorphic in the bidisk, with poles on $\{xy=0\}$ of order at most M. So ω_i can be represented as a convergent power series

$$\omega_i=\sum_{m_i,n_i\in\mathbb{Z}_{>-M}}\left(c'_{i,m_i,n_i}x^{m_i-1}y^{n_i}\,dx+c''_{i,m_i,n_i}x^{m_i}y^{n_i-1}\,dy\right),$$

with coefficients $c'_{i,m_i,n_i}, c''_{i,m_i,n_i}$ decreasing as $O(2^{-m_i-n_i})$. Therefore the elementary iterated integral is a converging sum of elementary iterated integrals of monomial forms, with coefficients being products of $c'_{i,m_i,n_i}, c''_{i,m_i,n_i}$, $i = 1, ..., l$ and $m_i, n_i \in \mathbb{N}$. From (13) one gets upper bounds for the elementary iterated integrals of monomial forms, which guarantees that one can perform Mellin transform termwise, and we get the required formula. $\square$

As in [9], one can check that the inverse Mellin transform of Mellin transforms of elementary iterated integrals can be defined as

$$\mathscr{M}^{-1}g = \frac{1}{2\pi i}\int_{\partial\Pi} t^{-s}g(s)ds, \quad \Pi = \{\Re s \le M < +\infty, |\Im s| \le 1\}, \tag{18}$$

where M is sufficiently big. Indeed, $|\ell^l_\alpha(s)| \le 1$ on Π, so (12) converges uniformly on this contour, so one can integrate the series (13) termwise. However, for each term (18) does define the inverse Mellin transform, as each term is just a rational function in s.

Corollary 2. *An elementary iterated integral can be represented as a convergent sum*

$$\int \omega_1 ... \omega_l = \sum_\alpha c_\alpha \mathscr{M}^{-1}\ell^l_\alpha. \tag{19}$$

The following estimate is the keystone of the proof, since it allows to estimate the difference between the elementary iterated integral and the partial sum of its asymptotic series.

Lemma 5. *Let $I =\in \omega_1...\omega_l$ be an elementary iterated integral, and let C be defined as in 4. For any $s \in \mathbb{C}$ denote by $\rho(s)$ the minimal distance from S to the poles of $\mathscr{M}I$.*

Then $|\mathscr{M}I(s)| \le C\rho(s)^{-l}$.

Proof. Indeed, the absolute value of each term in the sum in (12) can be estimated from above as $|c_\alpha|\rho(s)^{-l}$, and the estimate follows from $|c_\alpha| < C2^{-\alpha}$. $\square$

5.3 Asymptotic series of elementary iterated integrals

Inverse Mellin transform of ℓ^l_α is a linear combination of monomials of the type $t^\mu(\log t)^j$, where $\mu \in \lambda_1^{-1}\mathbb{Z} + \lambda_2^{-1}\mathbb{Z}$ and $0 \le j \le l$. Collecting similar terms in the expression for the elementary iterated integral I together, we get a formal series $\hat{I}$ of such terms, possibly divergent:

$$\hat{I} = \sum_{\mu,j} \hat{c}_{\mu,j} t^\mu (\log t)^j, \quad \text{where} \quad \mu \in \lambda_1\mathbb{Z}_{>-M} + \lambda_2\mathbb{Z}_{>-M}, \quad 0 \le j \le l. \tag{20}$$

Theorem 7. *$\hat{I}$ is an asymptotic series of I. Moreover, for each $p \in \mathbb{N}$ there exists $s_p \in [p, p+1]$ such that the partial sums $\hat{I}_p = \sum_{j,\mu<s_p} \hat{c}_{\mu,j} t^\mu (\log t)^j$ of $\hat{I}$ satisfy the following:*

$$|I(t) - \hat{I}_p(t)| \le C s_p^{l^2} t^{s_p}, \quad t \in [0,1] \tag{21}$$

where C depends on I but not on p.

The proof is a generalization of the proof of the corresponding statement from [9].

Proof. Poles of $\mathscr{M}I$ are of the form $-\lambda_1^{-1}\sum_{i=1}^{j} m_i - \lambda_2^{-1}\sum_{i=j+1}^{l} n_i$. Since $\lambda_1, \lambda_2 > 0$, there are $O(p^{l-1})$ poles on the interval $J_p = [-p-1, -p]$, $p \in \mathbb{N}$. Therefore on each interval J_p one can find a point $-s_p$ such that $\rho(-s_p) > O(p^{1-l}) = O(s_p^{1-l})$.

For each $p \in \mathbb{N}$ let us split the contour of integration $\partial\Pi$ into two parts: boundary of $\Pi'_p = \{-s_p \le \Re s \le M, |\Im s| \le 1\}$ and boundary of $\Pi_p = \{\Re s \le -s_p, |\Im s| \le 1\}$. Computing residues, we see that $\frac{1}{2\pi i}\int_{\partial\Pi'_p} t^{-s}\mathscr{M}I ds$ is a partial sum $\hat{I}_p(t)$ of $\hat{I}$ as defined above. Therefore $I(t) - \hat{I}_p(t) = \frac{1}{2\pi i}\int_{\partial\Pi_p} t^{-s}\mathscr{M}I ds$. By Lemma 5, $|\mathscr{M}I(s)| \le O(p^{l^2})$ on $\partial\Pi_p$, and (21) follows. □

5.4 Iterated integrals

Here we extend the Theorem 7 to the algebra $\mathcal{A}$ generated by elementary iterated integrals.

Let $f = P(I_1, ..., I_k) \in \mathcal{A}$ be an element in $\mathcal{A}$, where $P \in \mathbb{C}[u_1, ..., u_k]$ and $I_1, ..., I_k$ are elementary integrals. Substitution of convergent series from (19) instead of $I_1, ..., I_k$ gives a representation of f as a converging multiple sum of products (of length at most k) of generalized compensators. Collecting similar terms, we obtain a formal series $\hat{f}$ similar to (20), probably divergent.

Theorem 8. *For any $p \in \mathbb{N}$ there exists $s_p \in [p, p+1]$ such that the partial sum $\hat{f}_p$ of $\hat{f}$ satisfies the following*

$$|f - \hat{f}_p| \le C s_p^d t^{s_p} \tag{22}$$

for some C, d independent of p.

Before proof of Theorem 8 let us show that it implies Theorem 3.

Corollary 3. *Let $f \in \mathcal{A}$. If $\hat{f} = 0$, then $f \equiv 0$ on $[0,1]$. Also, isolated zeros of f cannot accumulate to 0.*

Proof. To prove the first claim, take a limit as $s_p \to +\infty$ in (22.

Now, if $f \not\equiv 0$, then for some μ we have $|f - t^\mu P(\log t)| = o(t^\mu)$ with some non-zero polynomial P (where $-\mu$ is the rightmost pole of $\mathscr{M} f$). This clearly implies the second claim. □

The proof of Theorem 8 occupies the rest of the paper.

5.4.1 Mellin transform of a product of several generalized compensators

For $V = (v_1, ..., v_n) \in \mathbb{R}^n$ define $\ell_v(s) = \prod_{i=1}^n (s+v_i)^{-1}$. Let $V^j = (v_1^j, .., v_{n_j}^j) \in \mathbb{R}^{n_j}$, $j = 1, .., k$ and define $\Phi(V^1, ..., V^k)(s) = \mathscr{M}\left[\prod(\mathscr{M}^{-1}\ell_{V^j})\right]$.

This is a rational function of s. We want to show that it depends polynomially on $\{V^j\}$. Let K denotes the set of function $\kappa : \{1, ..., k\} \to \mathbb{Z}$ with the condition $\kappa(j) \in \{1, ..., n_j\}$, and define $w_\kappa = v^1_{\kappa(1)} + ... + v^k_{\kappa(k)}$.

Lemma 6. *Let* $S = S(V^1, ..., V^k) = \prod_{\kappa \in \mathrm{K}}(s + w_\kappa)$ *be a polynomial in* $\mathbb{R}[V^1, ..., V^k; s]$. *There exists a polynomial* $R = R_{n_1,...,n_k} \in \mathbb{R}[V^1, ..., V^k; s]$ *such that* $\Phi(V^1, ..., V^k)(s) = RS^{-1}$, $deg_s R < \deg_s S$.

Proof. By continuity of both sides it is enough to prove this for a dense subset of $\prod \mathbb{R}^{n_j}$ consisting of non-resonant tuples $(V^1, ..., V^k)$, namely for those those tuples for which all w_κ are different.

Let $\mathbb{C}_\infty(s)$ be the ring of rational functions in s vanishing at infinity, and define convolution $f_1 * f_2$ for $f_1, f_2 \in \mathbb{C}_\infty(s)$ by extending the rule

$$\frac{1}{s+a} * \frac{1}{s+b} = \frac{1}{s+a+b}$$

by linearity and continuity to the whole $\mathbb{C}_\infty(s)$ (in particular, $(s+a)^{-k} * (s+b)^{-l} = (s+a+b)^{-k-l+1}$). Thus defined convolution is Mellin-dual to the usual product. Therefore $\Phi(V^1, ..., V^k)(s) = \ell_{V^1} * ... * \ell_{V^k}$. Decomposing each factor into simple fractions

$$\ell_{V^j} = \sum_i \frac{Res_{v_i^j} \ell_V^j}{s + v_i^j}, \quad Res_{v_i^j} \ell_V^j = \left(\prod_{i' \neq i} (v_i^j - v_{i'}^j)\right)^{-1}$$

and opening brackets, we see that

$$\Phi(V^1, ..., V^k)(s) = \sum_{\kappa \in \mathrm{K}} \frac{\prod_{j=1}^k Res_{v^j_{\kappa(j)}} \ell_V^j}{s + w_\kappa}.$$

Reducing to a common denominator, we see that $\Phi(V^1, ..., V^k)(s)$ is a rational function in v_i^j, s, with denominator dividing $S\prod_{i,i',j}(v_i^j - v_{i'}^j)$.

We claim that the factors $(v_i^j - v_{i'}^j)$ do not enter denominator of $\Phi(V^1, ..., V^k)(s)$. Indeed, presence of such factor would mean that $\Phi(V^1, ..., V^k)(s)$ becomes unbounded as v_i^j tends to $v_{i'}^j$ for each $s \in \mathbb{C}$, which is not true: for any tuple $(V^1, ..., V^k)$ and every sufficiently big $s \in \mathbb{R}$ the function $\Phi(V^1, ..., V^k)(s)$ is locally bounded near $(V^1, ..., V^k, s)$.

□

5.5 Mellin transform of a product of elementary iterated integrals

Let $I = I_1...I_k$ be a product of several elementary iterated integrals, and let order of I_j be l_j. Then using representation (19) for I_j and opening brackets, we see that

$$\mathscr{M}I = \sum_{\alpha_1,...,\alpha_k} c_{\alpha_1}...c_{\alpha_k}\mathscr{M}\left(\prod_{j=1}^{k}\mathscr{M}^{-1}(\ell_{\alpha_j}^{l_j})\right), \tag{23}$$

where $\alpha_j \in (\mathbb{Z}_{>-M})^{2l_j}$.

Lemma 7. *Let $\rho(s)$ be the distance from s to the set of poles of $\mathscr{M}I$. Then $|\mathscr{M}I(s)| \leq C\rho^{-\prod l_j}(|s|+1)^d$ for some $d > 0$.*

Proof. Let us estimate from above the terms $\mathscr{M}\left(\prod_{j=1}^{k}\mathscr{M}^{-1}(\ell_{\alpha_j}^{l_j}\right)$ from (23). By Lemma 6 it is equal to $R(V^1, ..., V^k; s)/S(V^1, ..., V^k; s)$, where $V^j = (v_1^j, ..., v_{l_j}^j)$ is defined by

$$v_i^j = -\lambda_{j1}^{-1}\sum_{p=1}^{i} m_p^j - \lambda_{j2}^{-1}\sum_{p=i+1}^{l_j} n_p^j, \quad \alpha_j = (m_1^j, ..., n_{l_j}^j) \in \mathbb{Z}_{>-M}^{l_j},$$

as in (11). This means that $V^j = L_j\alpha_j$ for some linear map $L_j : \mathbb{R}^{l_j} \to \mathbb{R}^{l_j}$. Therefore R is a polynomial in $(s; \alpha_1, ..., \alpha_k)$, and

$$|R(s)| \leq \text{const}(1+s)^d(1+\sum|\alpha_j|)^d, \quad \text{for } d = \deg R \geq 0.$$

From the other side, S is a monic polynomial in s of degree $\prod l_j$ with roots in the poles of $\mathscr{M}I$, so $|S(s)| \geq (\rho(s))^{\prod l_j}$. Taken together, this means that

$$\left|\mathscr{M}\left(\prod_{j=1}^{k}\mathscr{M}^{-1}(\ell_{\alpha_j}^{l_j})\right)\right| \leq \text{const}\,(\rho(s))^{-\prod l_j}\,(1+s)^d(1+\sum|\alpha_j|)^d. \tag{24}$$

Now, we know that $|c_{\alpha_j}| \leq C2^{-|\alpha|}$ by Lemma 4, so we estimate $|\mathscr{M}I(s)|$ from above as

$$|\mathscr{M}I(s)| \leq \text{const}\,(\rho(s))^{-\prod l_j}\,(1+s)^d \sum_{\alpha_1,\dots,\alpha_k} 2^{-\sum|\alpha_j|}(1+\sum|\alpha_j|)^d, \qquad (25)$$

which, by convergence of the series, proves the Lemma. □

5.5.1 Proof of Theorem 8

Let I now be a polynomial in several elementary iterated integrals, $I = P(I_1, ..., I_k)$. The set of poles of the Mellin transform $\mathscr{M}I$ of I is the union of sets of poles of Mellin transforms of each monomial of P, so the number of poles of $\mathscr{M}I$ on an interval $J_p = [-p-1, -p]$ counted with multiplicities grows as some power of p.

This means that for each $p \in \mathbb{N}$ one can find $s_p \in J_p$ such that the distance $\rho(s_p)$ from p to the set of poles of $\mathscr{M}I$ will be bigger than $|s_p|^{-d'}$ for some $d' > 0$. Then splitting the contour of integration of the inverse Mellin transform as in Theorem 7, we conclude from Lemma 7 that $|\mathscr{M}I| < C|s_p|^{d''}$ on the $\partial\Pi_p$ for some fixed $d'' > 0$, and the claim follows.

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