LINEAR ESTIMATE FOR THE NUMBER OF ZEROS OF ABELIAN INTEGRALS

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Abstract. We prove a linear in $\deg\omega$ upper bound on the number of real zeros of the Abelian integral $I(t)=\int_{\delta(t)}\omega$, where $\delta(t)\subset\mathbb{R}^2$ is the real oval $x^2y(1-x-y)=t$ and ω is a one-form with polynomial coefficients.

1. Introduction

For the polynomial $H(x,y)=x^2y(1-x-y)$ consider the continuous family $\{\delta(t),t\in\big(0,\frac{1}{64}\big)\}$ of compact connected components of the level curves $\{H=t\}\subset\mathbb{R}^2$. Let $\omega=p(x,y)dx+q(x,y)dy\in\Lambda^1(\mathbb{R}^2)$ be a differential one-form with polynomial coefficients of degree n. Define the complete Abelian integral:

$$I(t)=\oint_{\delta(t)}\omega,\quad t\in\left(0,\frac{1}{64}\right). \tag{1}$$

We provide an explicit answer to the Infinitesimal Hilbert 16th problem for this particular Abelian integral.

Theorem 1.1. *The number of isolated zeros of $I(t)$ on $(0,\frac{1}{64})$ does not exceed $\frac{7}{4}n+9$, where $n=\deg\omega$.*

It is well known that zeros of Abelian integrals correspond to limit cycles appearing in non-conservative perturbations of Hamiltonian, or, more general, integrable systems. Abelian integral (1) is related to perturbations of the integrable quadratic vector field which can be written in the Pfaffian form as follows:

$$\frac{1}{x}dH-\varepsilon\omega=0,\qquad H(x,y)=x^2y(1-x-y). \tag{2}$$

Therefore Theorem 1.1 implies in a standard way the following claim:

Theorem 1.2. *The number of limit cycles appearing in non-conservative perturbation* (2) *and converging, as $\epsilon\to 0$, to a smooth cycle $\delta(t)$, does not exceed $\frac{1}{4}(7n+43)$.*

Indeed, these limit cycles correspond to the isolated zeros of $\int_{\delta(t)}x\omega$, so this upper bound follows from Theorem 1.1 by replacing n by $n+1$.

Our result should be considered in the general context of the Infinitesimal Hilbert 16th problem. So far, the only known general explicit result about the number of zeros of Abelian integrals is the recent result [1] providing double-exponential in $\max(\deg H,\deg\omega)$ upper bound for the number of zeros of Abelian integral. The result of Petrov-Khovanskii, see [13] for an exposition of the result, provides an upper bound which is a linear function of $n=\deg\omega$, but provides no information about the coefficients of this function. It seems reasonable to expect that these two results can be combined together to provide an upper bound which

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would be linear in n and double-exponential in $\deg H$. However, even this upper bound will by far exceed any known examples.

From the other side, for the cases of polynomials H of low degree the situation seems to be better understood. More exact, for a generic polynomial H of third degree Horozov and Iliev [7] were able to provide an explicit upper bound linear in n which seems to be close to the best possible one. The key point was the ellipticity of the level curves of H. This fact allowed to reduce the initial question to the question about number of zeros of polynomial combinations of solutions of some Riccati equation. This last question can be easily dealt with by essentially fewnomials technique (i.e. Rolle lemma). Later, the same approach was applied to the case of elliptic polynomial of fourth degree, see [6].

This result motivated consideration of integrable quadratic vector fields with center whose trajectories are elliptic curves. Gautier in [2] lists all these fields. Based on this list and results of [8], Gautier, Gavrilov and Iliev in [3] proposed a program of studying cyclicity of open nests of cycles defined by such foliations, and, in particular, conjectured that one can provide an effective upper bound for the number of zeros of corresponding Abelian integrals, similar to [7]. Our case is the case (rlv3) in notations of [3]. Many other cases were investigated since then, see [10].

In [5, 11] it was proved that for a certain class of Hamiltonians, which includes this case (rvl3), the number of zeros of the Abelian integrals (1) with $\deg\omega \le n$ can be estimated from above by the maximal number of zeros from a certain explicitly constructed family of algebraic functions. The resulting estimate is sharp for $n = 2$ but our estimate seems to be sharper for $n > 2$.

2. Decomposition in Petrov modules

Define the basic Abelian integrals as

$$I_{i,j}(t) = \frac{1}{i+1}\int_{\delta(t)} x^{i+1}y^j\,dy = \iint_{\Delta(t)} x^i y^j\,dx\wedge dy, \tag{3}$$

where $\Delta(t)$ is the area bounded by the cycle $\delta(t)$. The integrals are well-defined for all (i.e. not only positive) $i, j \in \mathbb{Z}$ due to the following fact:

Remark 2.1. The curve $\delta(t)$ lies in $\{x, y > 0\}$.

Our immediate goal is to construct explicit representation of the Abelian integrals defined in (1) as combinations with polynomial in t coefficients of just three Abelain integrals $J_1(t) = I_{0,0}(t)$, $J_2(t) = I_{2,0}(t)$ and $J_3(t) = I_{3,0}(t)$. In other words, we want to prove that Abelian integrals can be generated, as a $\mathbb{C}[t]$-module, by these 3 basic Abelian integrals.

Let $H(x,y) = \sum_{i,j\in\mathbb{Z}^2} h_{ij}x^iy^j \in \mathbb{R}[x,y]$ be a general Laurent polynomial in two variables and assume that the family $\delta(t) \subset \{H = t\}$ of cycles lies in $\{x, y > 0\}$.

Lemma 2.2. *Abelian integrals $I_{k,l}(t)$ defined by (3) satisfy the following relations:*

$$\begin{aligned} t(k+1)\cdot I_{k,l}(t) &= \sum_{i,j} h_{i,j}(k+i+1)\cdot I_{k+i,l+j}(t),\\ t(l+1)\cdot I_{k,l}(t) &= \sum_{i,j} h_{i,j}(l+j+1)\cdot I_{k+i,l+j}(t),\\ (k+1)\cdot I_{k,l}(t) &= \sum_{i,j} ih_{i,j}\cdot \frac{d}{dt} I_{k+i,l+j}(t),\\ (l+1)\cdot I_{k,l}(t) &= \sum_{i,j} jh_{i,j}\cdot \frac{d}{dt} I_{k+i,l+j}(t), \end{aligned} \tag{4}$$

where $i,j\in\mathbb{Z}$.

Proof. Since $H(x,y)=t$ on the cycle $\delta(t)\subset\mathbb{R}^2$ we have

$$t(k+1)\cdot I_{k,l}(t) = \int\limits_{\delta(t)} H(x,y)x^{k+1}y^l dy = \sum_{i,j} h_{i,j}(k+i+1)I_{k+i,l+j}(t),$$

which is the first equality of (4). The second identity is proved similarly.

By Gelfand-Leray formula we have

$$\frac{d}{dt}\Big(\iint\limits_{\gamma(t)} dH\wedge x^{k+1}y^l dy\Big) = \oint\limits_{\delta(t)} x^{k+1}y^l dy = (k+1)I_{k,l}(t). \tag{5}$$

Replacing H by $\sum_{(i,j)\in\mathbb{Z}^2} h_{ij}x^iy^j$ we get the third identity. The fourth equality is proved similarly. □

For our particular choice $H(x,y)=x^2y(1-x-y)$ we can rewrite the relations of Lemma 2.2 in more convenient form. We have

$$\begin{pmatrix} t(k+1) & -k-3 & k+4 & k+3\\ t(l+1) & -l-2 & l+2 & l+3 \end{pmatrix}\cdot \begin{pmatrix} I_{k,l}\\ I_{k+2,l+1}\\ I_{k+3,l+1}\\ I_{k+2,l+2}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}. \tag{6}$$

Multiplying by $\begin{pmatrix} l+3 & -k-3\\ -l-2 & k+4\end{pmatrix}$ from the left we get an equivalent for $k+l\neq -6$ system of equations:

$$\begin{pmatrix} -2t(l-k) & -k-3 & k+l+6 & 0\\ t(3l-k+2) & -l-2 & 0 & k+l+6 \end{pmatrix}\cdot \begin{pmatrix} I_{k,l}\\ I_{k+2,l+1}\\ I_{k+3,l+1}\\ I_{k+2,l+2}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}. \tag{7}$$

Therefore $I_{k+3,l+1}(t)$ and $I_{k+2,l+2}(t)$ can be represented as $\mu_1^{k,l}(t)I_{k,l}(t)+\mu_2^{k,l}(t)I_{k+2,l+1}(t)$, where $\mu_1^{k,l}$ is a polynomial of degree at most 1 and $\mu_2^{k,l}$ is a constant. We have such representation for any pair (k,l) such that $k+l\neq -6$.

Another corollary of (6):

$$\begin{pmatrix} -2l+k-1 & 3l-k+2 & 2l-2k \end{pmatrix}\cdot \begin{pmatrix} I_{k+2,l+1}\\ I_{k+3,l+1}\\ I_{k+2,l+2}\end{pmatrix} = 0. \tag{8}$$

The equation (8) gives us linear dependence between $I_{k+2,l+1}, I_{k+3,l+1}$ and $I_{k+2,l+2}$ in all cases except $k = l = -1$. But in this case the equation (7) becomes:

$$\begin{pmatrix} 0 & -2 & 4 & 0 \\ 0 & -1 & 0 & 4 \end{pmatrix} \cdot \begin{pmatrix} I_{-1,-1} \\ I_{1,0} \\ I_{2,0} \\ I_{1,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{9}$$

Recall that $J_1(t) = I_{0,0}(t)$, $J_2(t) = I_{2,0}(t)$ and $J_3(t) = I_{3,0}(t)$.

Lemma 2.3. *For any polynomial* 1*-differential form* ω *the Abelian integral* $I(t) = \int_{\delta(t)} \omega$ *can be represented as* $p_1(t)J_1(t) + p_2(t)J_2(t) + p_3(t)J_3(t)$ *for some polynomials* $p_i(t)$ *of degree less than or equal to* $\frac{n}{4}$*, where* $n = \deg \omega$*.*

Remark 2.4. The proof of this result below essentially provides the coefficients $p_i(t)$, i.e. provides an effective decomposition in the Petrov module corresponding to H. This result does not formally follow from the result of Gavrilov [4] since $x^2y(1-x-y)$ is not a semi-weighted homogeneous polynomial. However, in this simple situation it can be obtained by a straightforward computation.

Proof. First of all we will give such a representation for all $I_{k,l}(t)$ if $k+l \leq 3$, $k, l \geq 0$. By (9) we have $I_{1,0} = 2J_2$ and $I_{1,1} = \frac{1}{2}J_2$. Using (8) one can calculate the required representation for $I_{0,1}$ and $I_{2,1}$. This implies similar representation for $I_{0,2}$ and $I_{1,2}$ and then for $I_{0,3}$. Note that for these integrals the coefficients p_i are polynomials of degree 0, i.e. scalar.

For $n = i + j \geq 4$ the proof goes by induction on n. Using (7), we see that the required representation of $I_{k,n-k}$ for $2 \leq k \leq n-1$, together with bounds on the degrees of p_i, follows from the same for $I_{k,l}$ with smaller $k + l$. Using (8) one can obtain that $I_{1,n-1}$ is a linear combination of $I_{1,n-2}$ and $I_{2,n-2}$ (for $k = -1$ and $l = n-3$) because $2l - 2k = 2n - 4 > 0$. Thus we obtain a required representation for $I_{1,n-1}$ and similarly for $I_{0,n}$. Now we use (8) for $k = n-3$ and $l = -1$ to obtain representation of $I_{n,0}$ as a linear combination of $I_{n-1,0}$ and $I_{n-1,1}$. One can easily check that the degrees of p_i are bounded by $n/4$ in all these cases. $\square$

3. Construction of the Picard-Fuchs system.

It is well known that the effective decomposition in Petrov modules allows to explicitly construct the Picard-Fuchs system for the generators of the Petrov module, see e.g. [12]. Here we follow this classical path.

Lemma 3.1. *The column* $J = \begin{pmatrix} J_1(t) \\ J_2(t) \\ J_3(t) \end{pmatrix}$ *satisfies the system*

$$J = \frac{d}{dt}((A + tB)J), \tag{10}$$

where $A = \begin{pmatrix} 0 & -\frac{1}{12} & \frac{1}{12} \\ 0 & -\frac{1}{56} & \frac{1}{56} \\ 0 & -\frac{5}{504} & \frac{5}{504} \end{pmatrix}$ *and* $B = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & \frac{4}{7} & 0 \\ 0 & \frac{2}{21} & \frac{4}{9} \end{pmatrix}$.

Proof. of the lemma 3.1:

By Lemma 2.2 for any k and l (and in particular for $l = 0$ and $k = 0, 1, 3$) we have $(l+1) \cdot I_{k,l}(t) = \sum_{i,j} jh_{i,j} \cdot \frac{d}{dt} I_{k+i,l+j}(t)$.

It implies

$$I_{k,0} = \frac{d}{dt}(I_{k+2,1} - I_{k+3,1} - 2I_{k+2,2}). \tag{11}$$

Using Lemma 2.3 we represent $I_{k,1}$ for $2 \leq k \leq 6$ and $I_{k,2}$ for $k = 2, 4$ and 5 in terms of J_i. After calculation we obtain the result:

$$\begin{aligned}
I_{2,1} &= \frac{1}{2}J_2 - \frac{1}{2}J_3,\\
I_{3,1} &= \frac{1}{4}J_2 - \frac{1}{4}J_3,\\
I_{4,1} &= \frac{1}{7}J_2 - \frac{1}{7}J_3 - \frac{4}{7}tJ_2,\\
I_{5,1} &= \frac{5}{56}J_2 - \frac{5}{56}J_3 - \frac{6}{7}tJ_2,\\
I_{6,1} &= \frac{5}{84}J_2 - \frac{5}{84}J_3 - \frac{4}{7}tJ_2 - \frac{2}{3}tJ_3,\\
I_{2,2} &= \frac{1}{6}J_2 - \frac{1}{6}J_3 - \frac{1}{3}tJ_1,\\
I_{4,2} &= \frac{1}{28}J_2 - \frac{1}{28}J_3 - \frac{1}{7}tJ_2,\\
I_{5,2} &= \frac{5}{252}J_2 - \frac{5}{252}J_3 - \frac{4}{21}tJ_2 + \frac{1}{9}tJ_3,
\end{aligned} \tag{12}$$

The formulas (11) and system (12) together immediately imply the system (10). $\square$

The system (10) can be rewritten as

$$(A + tB) \cdot J' = (I_{3\times 3} - B) \cdot J, \tag{13}$$

where $I_{3\times 3}$ is the 3-dimensional identity operator.

This equation can be rewritten as

$$D(t) \cdot J'(t) = Q(t) \cdot J(t), \tag{14}$$

where

$$Q(t) = \begin{pmatrix} -\frac{1}{2} + 32t & 9 & -10 \\ 0 & \frac{3}{2} + 48t & -\frac{5}{2} \\ 0 & \frac{3}{2} - 24t & -\frac{5}{2} + 80t \end{pmatrix} \text{ and } D(t) = 64t^2 - t.$$

Introducing new variables $X = t^{-\frac{1}{2}}J_1,\ Y = J_2,\ Z = J_3$ we have

$$\begin{cases} D(t)\sqrt{t}X' &= 9Y - 10Z \\ D(t)Y' &= (\frac{3}{2} + 48t)Y - \frac{5}{2}Z \\ D(t)Z' &= (\frac{3}{2} - 24t)Y + (-\frac{5}{2} + 80t)Z \end{cases} \tag{15}$$

4. Proof of Theorem 1.1

Take any Abelian integral $I(t)$. By Lemma 2.3 $I(t) = p_1(t)J_1(t) + p_2(t)J_2(t) + p_3(t)J_3(t)$ where $\deg p_i \leq \frac{n}{4}$. Thus $I(t) = \sqrt{t}p_1X + p_2Y + p_3Z$.

Using (15), we obtain

$$\left(\frac{I}{p_1\sqrt{t}}\right)' = \frac{1}{Dp_1^2t\sqrt{t}} \cdot (\tilde{p}_1Y + \tilde{p}_2Z), \tag{16}$$

where $\tilde{p}_i$ are some polynomials of degree less than or equal to $\frac{n}{2} + 2$.

Recall that $Z = J_3 = I_{0,3}(t) = \iint\limits_{\gamma(t)} y^3 dx \wedge dy$, so Z is positive for $t \in (0, \frac{1}{64})$ by Remark 2.1. Hence the function $w = \frac{Y}{Z}$ is well-defined and by (15) satisfies the Riccati equation

$$Dw' = \left(-\frac{3}{2} + 24t\right) w^2 + (4 - 32t)\, w - \frac{5}{2}. \tag{17}$$

So for the function $S(t) = \tilde{p}_1 w + \tilde{p}_2$ we have

$$DS' = \left(-\frac{3}{2} + 24t\right) \tilde{p}_1 w^2 + (D\tilde{p}_1' + (4-32t)\tilde{p}_1)w + D\tilde{p}_2' - \frac{5}{2}\tilde{p}_1.$$

One can obtain

$$D\tilde{p}_1 S' = \left(-\frac{3}{2} + 24t\right) (S - \tilde{p}_2)^2 + (D\tilde{p}_1' + (4-32t)\tilde{p}_1)(S - \tilde{p}_2) + \left(D\tilde{p}_2' - \frac{5}{2}\tilde{p}_1\right)\tilde{p}_1.$$

Thus the Riccati equation for the function $S(t)$ reads as

$$D\tilde{p}_1 S' = AS^2 + BS + C, \tag{18}$$

where A, B and C are polynomials and $\deg C \leq n + 5$. Now one can introduce new time τ and rewrite (18) as a system

$$\begin{cases} \dot{t} &= D\tilde{p}_1 \\ \dot{S} &= AS^2 + BS + C, \end{cases} \tag{19}$$

where $\dot{\varphi}$ denotes $\frac{d\varphi}{d\tau}$. Denote by Δ_j, $j = 1, \dots, k$ ($k \leq \deg \tilde{p}_1 + 1$) the open intervals into which $(0, \frac{1}{64})$ is split by the zeros of $\tilde{p}_1$. It is clear that in Δ_j between any two zeros of S there is a zero of C. Let λ_j be the number of zeros of C on Δ_j. Thus the number of zeros of S in Δ_j is less than or equal to $\lambda_j + 1$. So the number of zeros of S in $(0, \frac{1}{64})$ is less than or equal to $\sum\limits_{j=1}^{k} (\lambda_j + 1) \leq \deg C + \deg \tilde{p}_1 + 1$. Thus it does not exceed $n + 5 + \frac{n}{2} + 2 + 1 = \frac{3}{2}n + 8$. By (16) we obtain that on $(0, \frac{1}{64})$ the number of zeros of $\left(\frac{I}{p_1\sqrt{t}}\right)'$ does not exceed $\frac{3}{2}n + 8$.

Denote by Ξ_j, $j = 1, \dots, l$ ($l \leq \deg p_1 + 1$) the open intervals into which $(0, \frac{1}{64})$ is split by the zeros of p_1. It is clear that in Ξ_j between any two zeros of I (i.e. zeros of $\frac{I}{p_1\sqrt{t}}$) there is a zero of $\left(\frac{I}{p_1\sqrt{t}}\right)'$. Let l_j be the number of zeros of $\left(\frac{I}{p_1\sqrt{t}}\right)'$ on Ξ_j. Thus the number of zeros of I in Ξ_j is less than or equal to $l_j + 1$. So the number of zeros of I in $(0, \frac{1}{64})$ is less than or equal to $\sum\limits_{j=1}^{l} (l_j + 1)$ and $\sum\limits_{j=1}^{l} l_j \leq \frac{3}{2}n + 8$. Thus the number of zeros of I in $(0, \frac{1}{64})$ is less than or equal to $\frac{3}{2}n + 8 + l \leq \frac{3}{2}n + 8 + \frac{n}{4} + 1 = \frac{7}{4}n + 9$. This proves Theorem 1.1.

References

[1] G. Binyamini, D. Novikov, S. Yakovenko, *On the number of zeros of Abelian integrals*, Inventiones mathematicae, **181** (2), 227-289.

[2] S. Gautier, *Quadratic centers defining elliptic surfaces*, J. of Diff. Eq., **245** (2008), no 12, p. 3545-3569.

[3] S. Gautier, L. Gavrilov, ID. Iliev, *Perturbations of quadratic centers of genus one,* Discrete Contin. Dyn. Syst. **25** (2009), no. 2, 511-535.

[4] L. Gavrilov, *Petrov modules and zeros of Abelian integrals*, Bull. Sci. Math. **122** (1998), no. 8, 571–584.

[5] M. Grau, F. Maosas; J. Villadelprat, *A Chebyshev criterion for abelian integrals.*, Trans. Amer. Math. Soc. **363** (2011), no. 1, 109–129.

[6] F. Girard, M.-A. Jebrane, *Majorations affines du nombre de zeros d'integrales abeliennes pour les hamiltoniens quartiques elliptiques*, Annales de la faculte des sciences de Toulouse, Ser. 6, **7** no. 4 (1998), p. 671-685.

[7] E. Horozov, I. D. Iliev, *Linear estimate for the number of zeros of Abelian integrals with cubic Hamiltonians* Nonlinearity, **11**, (1998) 6, pp. 1521-1537.

[8] I. D. Iliev, *Perturbations of quadratic centers,* Bull. Sci. Math. **122** (1998), no. 2, 107-161.

[9] Yu. Ilyashenko, *Centenial History of Hilbert's 16th problem* Bull. Amer. Math. Soc. (N.S.) **39** (2002), no. 3, 301–354 (electronic).

[10] H. Liang, Y. Zhao, *Limit cycles bifurcated from a class of quadratic reversible center of genus one*, JMAA, **391** (2012) 1, 240-254.

[11] F. Manosas and J. VilladelPrat, *Bounding the number of zeros of certain Abelian integrals*, J. Diff. Eq. **251**, (2011), no. 6, p. 1656-1669.

[12] S. Yakovenko, *Bounded decomposition in the Brieskorn lattice and Pfaffian Picard–Fuchs systems for Abelian integrals*, Bull. Sci. Math **126** (2002), no. 7, 535–554.

[13] H. Żołądek, *The monodromy group*, Instytut Matematyczny Polskiej Akademii Nauk. Monografie Matematyczne (New Series) [Mathematics Institute of the Polish Academy of Sciences. Mathematical Monographs (New Series)], vol. 67, Birkhäuser Verlag, Basel, 2006.

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