BOUNDING THE LENGTH OF ITERATED INTEGRALS OF THE FIRST NONZERO MELNIKOV FUNCTION

PAVAO MARDEŠIĆ, DMITRY NOVIKOV, LAURA ORTIZ-BOBADILLA, AND JESSIE PONTIGO-HERRERA

ABSTRACT. We consider small polynomial deformations of integrable systems of the form $dF = 0$, $F \in \mathbb{C}[x, y]$ and the first nonzero term M_μ of the displacement function $\Delta(t, \epsilon) = \sum_{i=\mu} M_i(t)\epsilon^i$ along a cycle $\gamma(t) \in F^{-1}(t)$. It is known that M_μ is an iterated integral of length at most μ. The bound μ depends on the deformation of dF.

In this paper we give a *universal bound* for the length of the iterated integral expressing the first nonzero term M_μ depending only on the geometry of the unperturbed system $dF = 0$. The result generalizes the result of Gavrilov and Iliev providing a sufficient condition for M_μ to be given by an abelian integral i.e. by an iterated integral of length 1. We conjecture that our bound is optimal.

АННОТАЦИЯ. Для возмущения Гамильтонова векторного поля $dF = 0$, $F \in \mathbb{C}[x, y]$ рассмотрим первый ненулевой член M_μ отображения последования $\Delta(t, \epsilon) = \sum_{i=\mu} M_i(t)\epsilon^i$ соответствующего циклу $\gamma(t) \in F^{-1}(t)$. Известно, что M_μ является итерированным интегралом длины не более чем μ, где μ зависит от возмущения.

Мы доказываем унивесальную оценку сверху на длину итерированного интеграла выражаюшего M_μ, зависящую только от Гамильтониана $dF = 0$. Этот результат обобщает достаточное условие для выразимости M_μ как Абелевого интеграла, доказанное в [11].

1. INTRODUCTION AND MAIN RESULTS

In this paper we study small one-parameter polynomial deformations of planar polynomial Hamiltonian systems. They can be written in the form

$$dF + \sum_{i=1}^{\infty} \epsilon^i \eta_i = 0, \tag{1.1}$$

where $F \in \mathbb{C}[x, y]$ is a polynomial and η_i are polynomial one-forms.

Let $\gamma(t)$ be a continuous family of loops in the first homotopy group $\pi_1(F^{-1}(t), p(t))$ of leaves $\{F = t\}$ of the foliation $F : \mathbb{C}^2 \to \mathbb{C}$, with a base point $p(t)$ chosen on an analytic transversal τ to a generic leaf $\{F = t_0\}$. Consider the *displacement function* Δ of (1.1) along γ, that is the first return map (holonomy) along γ minus

1991 *Mathematics Subject Classification.* 34C07 (primary), 34C05, 34C08 (secondary).

This work was supported by Israel Science Foundation grant 756/12, UNAM PREI Dgapa, Franco-Mexican project LAISLA, Conacyt 219722, Papiit, Dgapa UNAM IN106217, ECOS Nord-Conacyt 249542 and Conacyt 291121.

identity, on the transversal τ parametrized by the values t of the Hamiltonian F,

$$\Delta(t,\epsilon)=\sum_{i=\mu}^{\infty}M_i(t)\epsilon^i,$$

with $M_\mu \not\equiv 0$. The functions M_i are called *Melnikov functions along* $\gamma(t)$ and M_μ is the *first nonzero Melnikov function along* $\gamma(t)$. These functions are sometimes called (principal) Poincaré-Pontryagin functions, as in [10, 19, 20, 21].

For a fixed ϵ, the isolated zeros of $\Delta(\epsilon,\cdot)$ correspond to the *limit cycles* of (1.1). For $\epsilon=0$, there are no isolated zeros, and the Hamiltonian system $dF=0$ has no limit cycles. The real counterpart of these problem, i.e. real perturbations of real planar polynomial Hamiltonian vector field, appear naturally in investigations related to Hilbert 16th problem. In particular, the famous *Arnol'd-Hilbert's problem*, see [1], asks for a bound for the number of limit cycles born for small ϵ.

It is known that near the singular points of Δ there is no one-to-one correspondence between the isolated zeros of Δ and M_μ (*alien cycles* [6]). Still, the first nonzero Melnikov function M_μ carries essential information on the displacement function Δ.

If $\mu=1$, then M_μ is an *abelian integral* and its zeros have been thoroughly studied, see [3]. If $M_1\equiv 0$, one searches for the first nonzero Melnikov function M_μ. The special case of (1.1) of linear deformations

$$dF+\epsilon\eta=0 \tag{1.2}$$

was studied by many authors. Françoise [8] (see also [22]) gave an algorithm for calculating M_μ of (1.2) under a certain condition (called condition (*) of Françoise, cf. Remark 1.13). The algorithm assures that under that assumption, M_μ is always an abelian integral. In [15] and [16] the question of non-abelian character of M_μ was raised if Françoise condition was not verified. Examples where M_μ is not an abelian integral appear in [23], [12] and [20]. Gavrilov [10] showed that in general M_μ is an *iterated integral* of *length* $\ell\le\mu$. Iterated integrals (see Section 5 for precise definitions) are generalizations of abelian integrals. Abelian integrals are iterated integrals of length 1. As length grows, iterated integrals become more complicated and discern finer properties of foliations defined in terms of the fundamental groups of leaves rather than just their homological properties. Considered as functions of t, iterated integrals belong to a wider class of Q-functions defined in [4], see also [2], with complexity explicitly bounded in terms of their length and degrees of F and of the forms η_1.

It happens frequently [20, 21, 18] that M_μ is in fact of length smaller than μ. Of course, iterated integrals of lower length are simpler. Hence, characterizing the length of M_μ as an iterated integral is a natural problem. More important, the order μ of the first nonzero Melnikov function depends essentially on the perturbation, i.e. on the forms η_i. On the contrary, the length of M_μ as an iterated integral often depends only on the topology of the foliation defined by F and the cycle γ. This is known in the generic case, see Corollary 1.12, and also in some non-generic cases, see [18, 20, 21].

In [11], Gavrilov and Iliev gave a sufficient condition for the first nonzero Melnikov function to be an abelian integral, i.e. an iterated integral of length $k=1$.

In this paper, we show that this condition is necessary and generalize their result by giving a sufficient condition for the length of the first nonzero Melnikov function

M_μ of any deformation (1.1) of dF to be at most $k \geq 1$. We define the normal subgroup $\mathcal{O}$ of the fundamental group $\pi_1(F^{-1}(t_0), p(t_0))$ of a non-singular curve $F^{-1}(t_0)$ generated by the orbit of $\gamma(t)$ under the action of the monodromy of the foliation of $\mathbb{C}^2$ defined by F. The upper bound on the length of the iterated integral representing M_μ is given in terms of the position of $\mathcal{O}$ with respect to the filtration of $\pi_1(F^{-1}(t_0), p(t_0))$ by its lower central series.

Remark 1.1. *Let us stress that an important consequence of our Theorem 1.7 is that the bound k on the length of the iterated integrals representing the Melnikov function M_μ is universal, i.e. does not depend on the perturbation, whereas μ certainly depends on η_i. Our bound depends only on the unperturbed system $dF = 0$ and the cycle γ.*

1.1. **First homology of the orbit of a loop γ.** The first nonzero Melnikov function depends only on the class of γ in the free homotopy group of the fiber, see Section 3. In this section we define a group characterizing the underlying topological properties of the fibration defined by F relative to γ. This group is the key ingredient of the main theorem.

Consider the fibration $\mathcal{F}$ defined by $F : \mathbb{C}^2 \setminus F^{-1}(\Sigma) \to \mathbb{C} \setminus \Sigma$, where Σ is a finite set of atypical values of F. Choose a typical value $t_0 \in \mathbb{C} \setminus \Sigma$, $p_0 = p(t_0)$, and let $\gamma(t_0) \in \pi_1\left(F^{-1}(t_0), p_0\right)$.

Choose any path α in $\mathbb{C}^2 \setminus F^{-1}(\Sigma)$ starting at p_0 and ending at p_1. Using Ehresmann connection, we transport γ along α. This gives an isomorphism

$$i_\alpha : \pi_1\left(F^{-1}(t_0), p_0\right) \to \pi_1\left(F^{-1}(F(p_1)), p_1\right) \tag{1.3}$$

and, in particular, an isomorphism of the fundamental groups of connected components of $F^{-1}(t_0)$ if there are several components.

If α is closed, i_α is a well-defined automorphism of $\pi_1\left(F^{-1}(t_0), p_0\right)$, which depends only on the homotopy class of $\alpha \in \pi_1\left(\mathbb{C}^2 \setminus F^{-1}(\Sigma), p_0\right)$. We call it *the monodromy along* α and denote it Mon_α.

Here, and in the sequel, we put

$$\pi_1 = \pi_1\left(F^{-1}(t_0), p_0\right).$$

We define *the orbit of* γ *under monodromy*

$$\mathcal{O}_{p_0}(\gamma) \lhd \pi_1$$

as the smallest normal subgroup of π_1 containing all loops $Mon_\alpha(\gamma)$ for all $\alpha \in \pi_1\left(\mathbb{C}^2 \setminus F^{-1}(\Sigma), p_0\right)$.

Let $K_{p_0}(\gamma) \lhd \mathcal{O}_{p_0}(\gamma)$ be the normal subgroup $K_{p_0}(\gamma) = [\mathcal{O}_{p_0}(\gamma), \pi_1]$ of $\mathcal{O}_{p_0}(\gamma)$.

Consider the lower central sequence of π_1

$$L_1 = \pi_1 \supset L_2 \supset ..., \quad L_{i+1} = [L_i, \pi_1].$$

Remark 1.2. *As the typical level sets $F^{-1}(t)$ are non-compact smooth connected Riemann surfaces, their fundamental groups $\pi_1\left(F^{-1}(t_0), p_0\right)$ are finitely generated free groups. This will be used without explicit reference.*

1.2. **Main results.** We will express the criterion for bounding the length of iterated integrals of the first nonzero Melnikov function in terms of a comparison of K and the groups $\mathcal{O} \cap L_j$, where we denote $\mathcal{O} = \mathcal{O}_{p_0}(\gamma)$ and $K = K_{p_0}(\gamma)$.

Definition 1.3. *The orbit depth κ of γ is defined as*

$$\kappa = \sup \{ j \geq 1 \,|\, \mathcal{O} \cap L_j \not\subseteq K \} \in \mathbb{N} \cup \{+\infty\} \,. \tag{1.4}$$

Lemma 4.1 implies that $\mathcal{O} \cap L_{\kappa'} \subset K$ for any $\kappa' > \kappa$.

Theorem 1.4. *The length of the first nonzero Melnikov function M_μ of any deformation* (1.1) *does not exceed the orbit depth κ of γ.*

Remark 1.5. *Here is a motivation of the result. For $\kappa = +\infty$ the claim is vacuous. However, in many interesting cases $\kappa < \infty$, see e.g. Section9. Then any element of $\mathcal{O}$ lying in $L_{\kappa+1}$ is a product of commutators of monodromy images of $\gamma(t)$ with some elements of π_1. So, by properties of Melnikov function, M_μ necessarily vanishes on these elements, see Section 3. But iterated integrals of length $\leq \kappa$ also vanish on $L_{\kappa+1}$. Moreover, they distinguish elements of $\pi_1/L_{\kappa+1}$: if all iterated integrals of length $\leq \kappa$ vanish on some $\sigma \in \pi_1$, then $\sigma \in L_{\kappa+1}$.*

Note that the first nonzero Melnikov function M_μ is additive on $\mathcal{O}$, so necessarily vanishes on elements whose multiples are in K. This motivates the following

Definition 1.6. *The torsion-free orbit depth k of γ is defined as*

$$k = \sup \left\{ j \geq 1 \,\middle|\, \frac{(\mathcal{O} \cap L_j)\, L_{j+1}}{L_{j+1}} \otimes_{\mathbb{Z}} \mathbb{C} \not\subseteq \frac{(K \cap L_j) L_{j+1}}{L_{j+1}} \otimes_{\mathbb{Z}} \mathbb{C} \right\} \subset \mathbb{N} \cup \{+\infty\} \,. \tag{1.5}$$

Torsion-free orbit depth k of γ does not exceed the orbit depth κ of γ and coincides with it if the abelian group $\mathcal{O}/K$ has no torsion elements. The first claim of the following Theorem is a more precise version of Theorem 1.4.

Theorem 1.7.

(i) *Assume that the typical fiber $\{F = t\}$ is connected. Then the first nonzero Melnikov function M_μ of any deformation* (1.1) *is a combination of base point independent iterated integrals of length not exceeding the torsion-free orbit depth k of γ. The coefficients of the combination are rational functions in t with poles contained in Σ.*

(ii) *If the orbit length κ equals* 1 *or* 2*, then $k = \kappa$ and there exists a deformation* (1.1) *such that the first nonzero Melnikov function M_μ is of length $\ell(M_\mu) = \kappa$.*

(ii') *If the orbit length κ is bigger than* 1*, then there exists a deformation* (1.1) *such that the first nonzero Melnikov function M_μ is of length $\ell(M_\mu) = 2$.*

Corollary 1.8. *If the typical fiber $\{F = t\}$ has d connected components, then the first nonzero Melnikov function M_μ of any deformation* (1.1) *is a combination of base point independent iterated integrals of length not exceeding the torsion-free orbit depth k of γ. The coefficients of the combination are rational functions in $g^{-1}(t)$ for some $g \in \mathbb{C}[t]$, $\deg g \leq d$.*

Remark 1.9.

(i) *Any isomorphism i_α of* (1.3) *respects the lower central sequence filtration as well as $\mathcal{O}$ and K. Therefore the conditions of Theorems 1.4 and 1.7 are independent of the choice of the typical value t_0 and the base point p_0.*

(ii) *It follows from the proof of Theorem 1.7(i) that the forms appearing in the iterated integrals expression for the first nonzero Melnikov function can be* *chosen from a fixed finite set of polynomial one-forms forming a basis of $H^1(F^{-1}(t))$ for any $t \in \mathbb{C} \setminus \Sigma$. Then Theorem 1.7(i) gives the existence of a kind of Petrov module for the first nonzero Melnikov function.*

(iii) *In* [10]*, the author asks for an expression of the first nonzero Melnikov function in terms of base point independent iterated integrals. The proof of Theorem 1.7(ii) indicates how such an expression can be obtained.*

(iv) *We are not aware of examples with $\kappa > k$ or even with non-trivial torsion elements in $H_1(\mathcal{O})$ of* (1.7).

If a path $\sigma \subset F^{-1}(t_0)$ joining two points p_0 and $\tilde{p}_0$ lies on the fiber $F^{-1}(t_0)$, then the isomorphism (1.3) can be written explicitly

$$i_\sigma : \pi_1\left(F^{-1}(t_0), p_0\right) \to \pi_1\left(F^{-1}(t_0), \tilde{p}_0\right), \; i_\sigma(\gamma) = \sigma^{-1}\gamma\sigma. \tag{1.6}$$

Hence, though the isomorphism depends on the choice of σ, different choices of σ lead to conjugate isomorphisms.

Define the commutative group

$$H_{1,p_0}(\mathcal{O}(\gamma)) = \frac{\mathcal{O}_{p_0}(\gamma)}{K_{p_0}(\gamma)}. \tag{1.7}$$

Note that the isomorphism $i_\sigma^* : H_{1,p_0}(\mathcal{O}(\gamma)) \to H_{1,\tilde{p}_0}(\mathcal{O}(i_\sigma(\gamma)))$ induced by (1.6) does not depend on the choice of the path σ joining two points p_0 and $\tilde{p}_0$. This defines an equivalence relation and we denote the corresponding equivalence class of $H_{1,p_0}(\mathcal{O}(\gamma))$ by $H_1(\mathcal{O})(t_0)$. We call it *the first homology group of the orbit of γ.* The definition of $H_1(\mathcal{O})(t_0)$ was given in [10, 11].

It is similar to the first homology group $H_1(F^{-1}(t_0))$, but the numerator is smaller (only the orbit of $\mathcal{O}$), as well as the denominator (only commutators in K instead of $L_2 = [\pi_1, \pi_1]$). Hence, in general, there is neither a natural injection nor surjection between these two groups.

The key object of our studies is the vector space $\mathbb{C}H_1(\mathcal{O})(t_0) = H_1(\mathcal{O})(t_0) \otimes_{\mathbb{Z}} \mathbb{C}$. The first nonzero Melnikov function is independent of p_0 and additive, see Section 3. It is hence well-defined on $H_1(\mathcal{O})(t_0)$ and on $\mathbb{C}H_1(\mathcal{O})(t_0)$.

Theorem 1.10. *For any deformation* (1.1)*, the first nonzero Melnikov function M_μ is an abelian integral if and only if $k = 1$.*

Remark 1.11. *Gavrilov and Iliev* [11] *proved that the equivalent condition of injectivity of the natural mapping*

$$i : H_1(\mathcal{O})(t) \to H_1(F^{-1}(t)). \tag{1.8}$$

implies that the first nonzero Melnikov function M_μ is an abelian integral, i.e. the direct implication of Theorem 1.10. The opposite implication of Theorem 1.10 affirms the hypothetic statement formulated at the end of the open question (3) of [11].

Proof of Theorem 1.10 *assuming Theorem* 1.7.
If $k = 1$, then for any deformation (1.1) M_μ is an abelian integral by Theorem 1.7(ii). If $k \neq 1$, then by Theorem 1.7(iii') there exists a deformation (1.1) such that $\ell(M_\mu) = 2$, so it is not an abelian integral. □

Corollary 1.12.
Consider a polynomial $F \in \mathbb{C}[x, y]$ verifying the following generic properties:

(G1) *All critical points of F are of Morse type and have disctinct critical values.*

(G2) *The fibers intersect the line at infinity transversally.*

Consider any deformation (1.1) *of the foliation $dF = 0$ and the first nonzero Melnikov function $M_\mu(\gamma)$ of the displacement function along a cycle γ of F. Then $M_\mu(\gamma)$ is an abelian integral.*

Proof. Ilyashenko [13], proves that the hypothesis implies that $\mathcal{O} = \pi_1$, so $K = L_2 = L_2 \cap \mathcal{O}$. Hence, $\kappa = 1$ is the smallest $\kappa \geq 1$ verifying (1.4). By [11], for any deformation η, the first nonzero Melnikov function is an iterated integral of length 1, i.e. an abelian integral. □

Remark 1.13. *The above Corollary can be considered as a consequence of results of Ilyashenko* [13] *and Françoise* [8]*. Indeed, Ilyashenko first proves that the above hypothesis implies that $\mathcal{O} = \pi_1$. Next he proves that the following condition is verified.*

(*) *Any form ω such that $\int_\gamma \omega$ vanishes identically is* relatively exact *i.e. can be written as $\omega = gdF + dR$, for some polynomials g and R.*

This condition is now known as the condition (*) of Françoise [8]*. Condition (*) of Françoise assures that Françoise algorithm works. The algorithm itself then produces a polynomial form ω_k such that $M_\mu(\gamma) = \int_\gamma \omega_k$, so $M_\mu(\gamma)$ is an abelian integral.*

1.3. **Exact upper bound.** Theorem 1.7 provides an upper bound on the length of the first non-zero Melnikov function in terms of the topology of the foliation. The exact (i.e. realizable) upper bound can be characterized as follows. By Proposition 3.1 the first non-zero Melnikov functions corresponding to perturbations (1.1) define linear functionals on $\mathbb{C}H_1(\mathcal{O})$. Denote by $\mathcal{K} \subset \mathbb{C}H_1(\mathcal{O})$ the maximal linear subspace such that all these functionals vanish on $\mathcal{K}$. Let the *realizable length k_{re}* be the depth of $\mathcal{K}$, i.e.

$$k_{re} = \sup \left\{ j \geq 1 \;\middle|\; \frac{(\mathcal{O} \cap L_j)\, L_{j+1}}{L_{j+1}} \otimes_{\mathbb{Z}} \mathbb{C} \not\subseteq \frac{(\mathcal{K} \cap L_j) L_{j+1}}{L_{j+1}} \otimes_{\mathbb{Z}} \mathbb{C} \right\} \subset \mathbb{N} \cup \{+\infty\}.$$

Then (almost tautologically) for any $\ell \leq k_{re}$ there exists a perturbation (1.1) such that the length of the first non-zero Melnikov function is at least ℓ. Indeed, let $\sigma \in (\mathcal{O} \cap L_\ell) \setminus \mathcal{K}$, and choose a perturbation (1.1) such that the linear functional on $\mathbb{C}H_1(\mathcal{O})$ corresponding to the first non-zero Melnikov function does not vanish on σ. Then the length of this first non-zero Melnikov function is at least ℓ, as otherwise it would vanish on $\sigma \in L_\ell$.

Unfortunately, $\mathcal{K}$ is hard to compute. Open question (3) in [11] asks if $\mathcal{K} = 0$.

2. Sketch of the proof of Theorem 1.7

The proof consists of several parts. First, we fix a typical fiber $F^{-1}(t_0)$ and study the corresponding first homology group $\mathbb{C}H_1(\mathcal{O})(t_0)$ of the orbit of γ. We show that the first nonzero Melnikov function M_μ defines a linear functional $\mathbf{M}_{\mu,t_0} \in (\mathbb{C}H_1(\mathcal{O})(t_0))^*$, $\mathbf{M}_{\mu,t_0}(\gamma(t_0)) = M_\mu(t_0)$. Moreover, the assumptions of Theorem 1.7 imply that it vanishes on $\mathcal{O} \cap L_{k+1}$.

We want to realize the dual vector space $(\mathbb{C}H_1(\mathcal{O})(t_0))^*$ as the space of linear combinations of iterated integrals.

Let $\mathbb{C}_k$ be the space of jets of degree $\leq k$ of formal power series in non-commuting variables $X_1, ..., X_n$, where $n = \dim H_1(F^{-1}(t))$. Chen ([5] and [9]) constructs an injective *multplicative* homomorphism $II_k : \pi_1/L_{k+1} \to (\mathbb{C}_k)^*$ given by a power series whose coefficients are iterated integrals.

Using Baker-Campbell-Hausdorff formula [14], we prove that $\log \circ II_k$ induces a *linear* isomorphism from $\mathbb{C}H_1(\mathcal{O})(t_0)$ to a factor space $N(t_0)$ of a subspace of $\mathbb{C}_k$.

Next, we prove that for any $\phi \in (N(t_0))^*$ the mapping $\phi \circ \log \circ II_k \in (\mathbb{C}H_1(\mathcal{O})(t_0))^*$ is a combination of base point independent iterated integrals. This proves that $\mathbf{M}_{\mu,t_0}$ is a linear combination of base point independent iterated integrals of length at most k.

Finally, we consider two isomorphic vector bundles $\mathcal{H} = \cup_{t_0 \in \mathbb{C}\setminus\Sigma} \mathbb{C}H_1(\mathcal{O})(t_0)$ and $\mathcal{N} = \cup_{t_0 \in \mathbb{C}\setminus\Sigma} N(t_0)$ and their respective sections $\mathbf{M}_\mu$ and $s = \left((\log \circ II_K)^*\right)^{-1}(\mathbf{M}_\mu)$ defined fiberwise by $\mathbf{M}_{\mu,t_0}$. The above base point independence of iterated integrals implies that the vector bundle $\mathcal{N}$ is well defined. Note that the function $M_\mu(t) = \mathbf{M}_\mu(\gamma(t))$ is multivalued, being evaluation of the univalued section $\mathbf{M}_\mu$ against the multivalued section $\gamma(t)$.

We observe that $\mathcal{N}$ is obtained by operations with regular subbundles of the trivial bundle $\mathbb{C}_k \times (\mathbb{C} \setminus \Sigma)$. Moreover, as the base of $\mathcal{N}^*$ is non-compact, it is trivial [7, Theorem 30.4]. Therefore, there is a basis of sections $\{f_j\}$ of $\mathcal{N}^*$ consisting of linear combinations with rational in t coefficients of linear functionals on $\mathbb{C}_k$ (the latter ones are just coefficients of monomials of a jet). As s is of regular growth, it is a linear combination with rational in t coefficients of f_j.

Pushing back by $(\log \circ II_K)^*$, we see that $\mathbf{M}_\mu = (\log \circ II_K)^*(s)$ is a linear combination with rational in t coefficients of sections $(\log \circ II_K)^*(f_j)$, which are base point independent iterated integrals of length at most k.

Points (ii) and (ii') are proved by direct calculation using Françoise's algorithm of [8, 10], see Section 8.

3. Basic properties of the first nonzero Melnikov function on a fixed fiber.

Let $t_0 \in \mathbb{C} \setminus \Sigma$ be a typical value of F. The following basic properties of the Melnikov functions were observed and proved in [11].

Proposition 3.1. *The first nonzero Melnikov function M_μ is base point independent and defines a linear functional $\mathbf{M}_{\mu,t_0}$ on $\mathbb{C}H_1(\mathcal{O})(t_0)$.*

Let $k \geq 1$ be the torsion-free orbit depth of γ. Then $\mathbf{M}_{\mu,t_0}$ vanishes on $\mathcal{O} \cap L_{k+1}$.

Lemma 3.2. *Let P_1, P_2 be two germs of the form*

$$P_j(t,\epsilon) = t + \sum_{i=\mu_j} M_{j,i}(t)\epsilon^i, \quad M_{j,\mu_j} \not\equiv 0.$$

Then the following holds

(i) $P_1 \circ P_2 = P_2 \circ P_1 \left(\operatorname{mod} \epsilon^{\mu+1}\right)$, *where* $\mu = \max\{\mu_1, \mu_2\}$.

(ii) *In particular,* $P_2^{-1} \circ P_1 \circ P_2$ *and* P_1 *coincide modulo* ϵ^{μ_1+1}.

(iii) *If* $\mu_1 = \mu \leq \mu_2$ *then*

$$P_1 \circ P_2 = t + (M_{1,\mu} + M_{2,\mu})\,\epsilon^\mu + o(\epsilon^\mu).$$

Proof. Direct computation. □

Lemma 3.3. *All Melnikov functions of order $< \mu$ of elements of $\mathcal{O}$ vanish identically.*

Proof. By analytic continuation along α, all Melnikov functions of order $< \mu$ of $Mon_\alpha(\gamma)$ vanish for any α. For their conjugates and products the claim follows from Lemma 3.2 applied to respective first return maps $P_\gamma(\epsilon, t) = t + \Delta(\epsilon, t)$. $\square$

Proof of Proposition 3.1. By Lemma 3.2(ii), M_μ vanishes on K and is base point independent. Therefore, by Lemma 3.2(iii), M_μ defines an additive homomorphism from $H_1(\mathcal{O})(t_0)$ to the linear space of germs of holomorphic functions at t_0, and thus descends to a linear functional on $\mathbb{C}H_1(\mathcal{O})(t_0)$.

If (1.5) holds, then any element of $\mathcal{O} \cap L_{k+1}$ is a product of elements corresponding to torsion elements of $H_1(\mathcal{O})$ and of elements of K. Since $\mathbf{M}_{\mu,t_0}$ is additive, it vanishes on each one of them as well as on their products. $\square$

4. Direct sum decomposition of $\mathbb{C}H_1(\mathcal{O})$

Filtration of π_1 by L_i induces filtrations on both $\mathcal{O}$ and K which define a filtration of $\mathbb{C}H_1(\mathcal{O})$. In this section we calculate it explicitly.

Denote

$$\mathcal{O}_i = \frac{K\,(\mathcal{O} \cap L_i)}{K} \otimes_{\mathbb{Z}} \mathbb{C}. \tag{4.1}$$

Then, by (1.5), $\mathcal{O}_{k+1} = 0$ and we have the flag

$$0 \subset \mathcal{O}_k \subset \mathcal{O}_{k-1} \subset ... \subset \mathcal{O}_1 = \mathbb{C}H_1(\mathcal{O}),$$

and, therefore,

$$\mathbb{C}H_1(\mathcal{O}) \cong \bigoplus_{i=1}^{k} \frac{\mathcal{O}_i}{\mathcal{O}_{i+1}}. \tag{4.2}$$

We have well-defined mappings ϕ_i

$$\mathcal{O}_i \cong \frac{\mathcal{O} \cap L_i}{K \cap L_i} \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\phi_i} \operatorname{Im} \phi_i = \frac{(\mathcal{O} \cap L_i)L_{i+1}}{(K \cap L_i)L_{i+1}} \otimes_{\mathbb{Z}} \mathbb{C}, \tag{4.3}$$

and the epimorphisms ψ_i

$$\frac{L_i}{L_{i+1}} \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\psi_i} \frac{L_i}{(K \cap L_i)\, L_{i+1}} \otimes_{\mathbb{Z}} \mathbb{C}. \tag{4.4}$$

Note that $\operatorname{Im} \phi_i \subseteq \operatorname{Im} \psi_i$.

Lemma 4.1. $\ker \phi_i = \mathcal{O}_{i+1}$. *Therefore* $\frac{\mathcal{O}_i}{\mathcal{O}_{i+1}} \cong \operatorname{Im} \phi_i$.

Proof. As $K \cap L_i \subset \mathcal{O} \cap L_i$, we have

$$(\mathcal{O} \cap L_i) \cap ((K \cap L_i)\, L_{i+1}) = (K \cap L_i)\, (\mathcal{O} \cap L_{i+1})\,,$$

and therefore

$$\ker \phi_i = \frac{(\mathcal{O} \cap L_i) \cap ((K \cap L_i)\, L_{i+1})}{K \cap L_i} = \frac{(K \cap L_i)\, (\mathcal{O} \cap L_{i+1})}{K \cap L_i} \cong \frac{\mathcal{O} \cap L_{i+1}}{K \cap L_{i+1}} = \mathcal{O}_{i+1}.$$

$\square$

Corollary 4.2. *If $\mathcal{O}_k \cong \{0\}$, then $\mathcal{O}_{k'} \cong \{0\}$ for all $k' > k$.*

Indeed, in this case $\ker \phi_k = \{e\}$ is trivial, so ϕ_{k+1} is a mapping from the trivial group, so $\ker \phi_{k+1} = \{e\}$ as well, and so on.

Remark 4.3. *As $K \subset L_2$, the mapping ψ_1 is an isomorphism. Assume that the mapping $\phi_1 : \frac{\mathcal{O}}{K} \to H_1(F^{-1}(t))$ is onto. Then $\pi_1 = \mathcal{O}L_2$ and, taking commutators with π_1, we get $L_2 = KL_3$. Therefore $(\mathcal{O} \cap L_2)L_3 = L_2$ and $(K \cap L_3)L_4 = L_3$ and so on, implying $\operatorname{Im}\phi_i \cong \{0\}$ for all $i > 1$. We conclude that $\mathbb{C}H_1(\mathcal{O}) \cong \operatorname{Im}\phi_1 = H_1(F^{-1}(t))$. This answers the question (4) of* [11].

5. Iterated integrals generate $(\mathbb{C}H_1(\mathcal{O})(t_0))^*$

We construct combinations of iterated integrals forming a basis of $(\mathbb{C}H_1(\mathcal{O})(t_0))^*$.

5.1. **Basic definitions and properties.** Let $\omega_i = \phi_i\, dt$ be one-forms on $[0,1]$. We define the iterated integral as

$$\int_0^1 \omega_1 \dots \omega_n = \int_0^1 \left(\int_0^{t_n} \left(\dots \left(\int_0^{t_3} \left(\int_0^{t_2} \omega_1\right) \omega_2\right) \dots\right) \omega_{n-1}\right) \omega_n, \tag{5.1}$$

or, equivalently,

$$\int_0^1 \omega_1 \dots \omega_n = \int_{\{0 \le t_1 \le \dots \le t_n \le 1\}} \phi_1(t_1) \dots \phi_n(t_n)\, dt_1 \dots dt_n. \tag{5.2}$$

For $n = 1$ we get the usual abelian integral.

Definition 5.1. *We define the formal length of* (5.1) *to be n, and the formal length of a linear combination of iterated integrals to be the maximal formal length of its terms. For function representable as a linear combination of iterated integrals we define its length as the minimal formal length of such representations.*

For a path $\gamma : [0,1] \to F^{-1}(t_0)$ and one-forms $\omega_i \in \Omega^1(F^{-1}(t_0))$ we define $\int_\gamma \omega_1 \dots \omega_n = \int_0^1 \gamma^*\omega_1 \dots \gamma^*\omega_n$. If the forms ω_i are holomorphic, then the homotopic deformation with fixed endpoints of γ does not change the integral.

In general, iterated integrals are not additive for $n > 1$. However, for a product of two paths $\gamma = \gamma_1\gamma_2$ the following shuffling formula holds:

$$\int_\gamma \omega_1 \dots \omega_n = \sum_{i=0}^{n} \int_{\gamma_1} \omega_1 \dots \omega_i \int_{\gamma_2} \omega_{i+1} \dots \omega_n. \tag{5.3}$$

This formula is equivalent to the multiplicativity of the Chen map II defined below. It implies that for $n > 1$ and for a loop γ the iterated integral in general depends on the choice of the base point of γ.

As an important corollary of (5.3), one gets the fundamental fact that iterated integrals of length k vanish on L_{k+1} and are additive on the free abelian group L_k/L_{k+1}. The fundamental theorem of Chen claims that the iterated integral of length k generate $(L_k/L_{k+1} \otimes_{\mathbb{Z}} \mathbb{C})^*$, see [9].

5.2. **Chen's iterated integrals morphism.** Recall the construction of Chen (see [5, 9]). Let $\{\eta_1, ..., \eta_n\}$ be a tuple of polynomial one-forms in $\mathbb{C}^2$ forming a basis of $H^1(F^{-1}(t))$, for all $t \in \mathbb{C} \setminus \Sigma$. Choose a base point $p_0 \in F^{-1}(t_0)$, and let $\{\gamma_i\}_{i=1}^n \subset \pi_1(F^{-1}(t_0), p_0)$ form the basis of $H_1(F^{-1}(t_0))$ dual to the basis $\{\eta_1, ..., \eta_n\}$.

Recall that $\pi_1(F^{-1}(t_0), p_0)$ is a free group generated by $\{\gamma_i\}_{i=1}^n$.

Let $\tilde{X} \xrightarrow{pr} F^{-1}(t_0)$ be the universal cover of $F^{-1}(t_0)$, $pr(\tilde{p}_0) = p_0$. Following Harris [9], we denote by $\mathfrak{g}$ the free Lie algebra generated by formal variables $X_1, ..., X_n$, $\mathfrak{g} \subset U(\mathfrak{g}) = \mathbb{C}[X_1, ..., X_n]$ – a free associative algebra on $X_1, ..., X_n$. Let

$G = \exp \mathfrak{g}$ be the Lie group corresponding to $\mathfrak{g}$ and lying in the formal completion $\hat{U}(\mathfrak{g})$.

Consider the $\mathfrak{g}$-valued form $\sum_{i=1}^n \eta_i \otimes X_i$ on $\tilde{X}$. The horizontal section of the Chen connection $\nabla s = ds - s\sum_{i=1}^n \eta_i \otimes X_i$ on the bundle $\tilde{X} \times G \to \tilde{X}$ passing through the point $(\tilde{p}_0, 1) \in \tilde{X} \times G$ is the graph of Chen's *iterated integrals mapping* $II : \tilde{X} \to G$.

An explicit formula for II is

$$II(\tilde{p}) = 1 + x, \text{ where } x = \sum_{(i_1,\dots,i_\ell)} \left(\int_\gamma \eta_{i_1} \dots \eta_{i_\ell} \right) X_{i_1} \dots X_{i_\ell} \tag{5.4}$$

and γ is a path from $\tilde{p}_0$ to $\tilde{p}$. Restricting to $pr^{-1}(p_0)$, this gives a mapping $II : \pi_1(F^{-1}(t), p_0) \to G$. The invariance of $\sum_{i=1}^n \eta_i \otimes X_i$ with respect to left multiplication by G implies that this mapping is a homomorphism.

Let k be as in Theorem 1.7(ii). Let $\mathfrak{m} \subset \mathbb{C}[X_1, \dots, X_n]$ be the maximal ideal generated by $X_1, \dots, X_n$ and denote $\mathbb{C}_k = \mathbb{C}[X_1, \dots, X_n]/\mathfrak{m}^{k+1}$. Consider the k-th jet $II_k : \pi_1(F^{-1}(t), p_0) \to G_k = G/\mathfrak{m}^{k+1} \subset \mathbb{C}_k$ of II

$$II_k(\gamma) = 1 + x_k, \quad x_k = \sum_{(i_1,\dots,i_\ell), \ell \le k} \left(\int_\gamma \eta_{i_1} \dots \eta_{i_\ell} \right) X_{i_1} \dots X_{i_\ell}. \tag{5.5}$$

Consider moreover the composition

$$\begin{gathered} \log \circ II_k : \pi_1(F^{-1}(t), p_0) \to \mathfrak{g}/\mathfrak{m}^{k+1} \subset \mathbb{C}_k, \\ \log(II_k(\gamma)) = x_k - x_k^2/2 + x_k^3/3 - \dots = \sum_{\|\alpha\| \le k} q_\alpha(\gamma) X^\alpha + \mathfrak{m}^{k+1}, \end{gathered} \tag{5.6}$$

where $q_\alpha(\gamma)$ are some polynomials in the coordinates of $II_k(\gamma)$, i.e. polynomials in $\int_\gamma \eta_{i_1} \dots \eta_{i_\ell}$. Chen's theorem implies that II is an injective homomorphism of groups and that $\log \circ II_k(\pi_1/L_{k+1})$ spans $\mathfrak{g}/\mathfrak{m}^{k+1}$, see [9].

Remark 5.2. *Products of iterated integrals are given by linear combinations of iterated integrals whose length is bounded by the sum of their lengths. Hence, the polynomial $q_\alpha(\gamma)$ can also be written as a linear combination of iterated integrals of length at most $\ell \le k$.*

Define

$$N_0 := \operatorname{Span}\{\log(II_k(K))\} \subset N_1 := \operatorname{Span}\{\log(II_k(\mathcal{O}))\}$$

and consider the quotient $N_{p_0} = \frac{N_1}{N_0}$.

Proposition 5.3. *The mapping*

$$\log \circ II_k : \pi_1/L_{k+1} \to \mathbb{C}_k \tag{5.7}$$

induces a linear isomorphism of $\mathbb{C}H_1(\mathcal{O})$ and N_{p_0}.

Proof. The induced mapping $\log \circ II_k : \mathbb{C}H_1(\mathcal{O}) \to N_{p_0}$ is well-defined and linear by Baker-Campbell-Hausdorff formula. Indeed, if $\sigma_1, \sigma_2 \in \mathcal{O}$, then

$$\log(II_k(\sigma_1\sigma_2)) = \log(II_k(\sigma_1)) + \log(II_k(\sigma_2)) + \sum \log(\delta_\alpha),$$

where δ_α are various group commutators of $II_k(\sigma_1), II_k(\sigma_2)$. But II_k is a homomorphism, so δ_α are images under II_k of various commutators of σ_1, σ_2, i.e. elements of K. Therefore, all $\log(\delta_\alpha)$ are in N_0. Moreover, if $\sigma_2 \in K$, then $\log(II_k(\sigma_2))$ is also in N_0. This proves that the mapping is well-defined and linear. Since it is surjective by definition, we only need to prove injectivity.

Let $\sigma \notin K$ and assume that $\sigma \in \mathcal{O}_i \setminus \mathcal{O}_{i+1}$. Choose $\delta \in \frac{L_i}{L_{i+1}} \otimes_{\mathbb{Z}} \mathbb{C}$ such that $\phi_i(\sigma) = \psi_i(\delta) \neq 0 \in \frac{L_i}{(K\cap L_i)L_{i+1}} \otimes_{\mathbb{Z}} \mathbb{C}$, where ϕ_i, ψ_i are defined in (4.3),(4.4) respectively. In particular, $\delta \notin (K \cap L_i)\, L_{i+1}$.

In the sequel of the proof we work $\pmod{\mathfrak{m}^{i+1}}$ without mentioning this explicitly. By Chen's theorem, the iterated integrals of length i generate $\left(\frac{L_i}{L_{i+1}} \otimes_{\mathbb{Z}} \mathbb{C}\right)^*$, so

$$II_k\left(\frac{L_i}{L_{i+1}} \otimes_{\mathbb{Z}} \mathbb{C}\right) = 1 + P_i,$$

$$II_k\left(\frac{K\cap L_i}{L_{i+1}} \otimes_{\mathbb{Z}} \mathbb{C}\right) = 1 + R_i,$$

where P_i is the set of all homogeneous Lie polynomials in X_j of degree exactly i and $R_i \subset P_i$. Therefore, $II_k(\delta) = 1 + p_i(X)$, $p_i \notin R_i$ and $\log \circ II_k(\delta) = p_i \notin \log(1 + R_i) = R_i$.

Lemma 5.4. $N_0 \cap P_i = R_i$.

Proof. Consider the mapping $\log \circ II_j : L_j/L_{j+1} \otimes_{\mathbb{Z}} \mathbb{C} \to \mathfrak{m}^j/\mathfrak{m}^{j+1}$. Chen's theorem implies that this is an injective linear homomorphism, so $\log \circ II_j(L_j/L_{j+1})$ is a lattice in $\mathfrak{m}^j/\mathfrak{m}^{j+1}$ (recall that $\frac{L_j}{L_{j+1}}$ is a free abelian group) and therefore, $\log \circ II_j(\frac{K\cap L_j}{L_{j+1}})$ is also a lattice.

Now, assume

$$p = \sum \lambda_\alpha v_\alpha \in \mathfrak{m}^i, \tag{5.8}$$

where $v_\alpha = \log \circ II_i(\delta_\alpha)$, for some $\delta_\alpha \in K$, and let $j < i$ be the smallest integer such that not all v_α lie in $\mathfrak{m}^{j+1}$ or, equivalently, not all $\delta_\alpha \in L_{j+1}$. Then, replacing v_α by their linear combinations with *integer coefficients* $w_\beta = \sum_\alpha c^\alpha_\beta v_\alpha$, $c^\alpha_\beta \in \mathbb{Z}$, we can assume that those w_β which are not in $\mathfrak{m}^{j+1}$, are linearly independent over $\mathbb{Z}$ in $\mathfrak{m}^j/\mathfrak{m}^{j+1}$. But then they are linearly independent over $\mathbb{C}$, as they lie in the lattice $\log \circ II_j(\frac{K\cap L_j}{L_{j+1}})$. Therefore all $w_\beta \in \mathfrak{m}^{j+1}$.

But, by Baker-Campbell-Hausdorff formula,

$$w_\beta = \log \circ II_i(\prod_\alpha \delta_\alpha^{c^\alpha_\beta}) + \sum_r \log \circ II_i(\tilde{\delta}_r),$$

where $\tilde{\delta}_r \in K \cap L_{j+1}$ are various commutators of δ_α. So we transformed (5.8) to another representation of p with bigger j. Repeating this several times, we get a representation with $v_\alpha \in\in \mathfrak{m}^i$. This means that $p \in P_i$. $\square$

Hence, $\log \circ II_k(\delta) \neq 0$ in N_{p_0}, thus proving injectivity of $\log \circ II_k : \mathbb{C}H_1(\mathcal{O}) \to N_{p_0}$. $\square$

5.3. Base-point independence of linear functionals on $\mathbb{C}H_1(\mathcal{O})$.

Proposition 5.5.

(i) *The subspaces $N_0 = N_0(p_0), N_1 = N_1(p_0) \subset \mathbb{C}_k$ do not depend on the choice of the base point $p_0 \in F^{-1}(t_0)$.*

(ii) *The map $N_1(p_0) \to N_1(p_1)$ induced by the change of the base point descends to the identity map in the quotient $N(t_0) = N_1/N_0$.*

Lemma 5.6. *For any $\gamma \in \mathcal{O}$ and any, not necessarily closed, path α*

$$\log \circ II(\alpha^{-1}\gamma\alpha) = \log \circ II(\gamma) \pmod{N_0}. \tag{5.9}$$

This will follow from two lemmas

Lemma 5.7. $N_0 = [N_1, \mathfrak{g}]$.

Lemma 5.8. *If $\log \circ II(\alpha) \in \mathfrak{g}$, $\log \circ II(\gamma) \in N_1$, then*

$$\log \circ II(\alpha^{-1}\gamma\alpha) - \log \circ II(\gamma) \in [N_1, \mathfrak{g}].$$

Proof of Lemma 5.6. Chen's construction implies $II(\alpha) \in G$. Hence, $\log \circ II(\alpha) \in \mathfrak{g}$ and Lemma 5.6 follows from the above Lemmas. □

Proof of Lemma 5.7. Let $\gamma \in \mathcal{O}, \sigma \in \pi_1\left(F^{-1}(t), p_0\right)$. Then, as II is a homomorphism, we have

$$\log \circ II([\gamma, \sigma]) = [\log \circ II(\gamma), \log \circ II(\sigma)] + ...,$$

where the dots denote some higher order commutators of $\log \circ II(\gamma)$ and $\log \circ II(\sigma)$.

Standard associated graded algebra arguments show that it is enough to prove that

$$\left\langle \log \circ II(\sigma),\, \sigma \in \pi_1\left(F^{-1}(t), p_0\right)\right\rangle = \mathfrak{g},$$

which is a direct consequence of Chen's theorem. □

Proof of Lemma 5.8. By multiplicativity of II and by the Baker-Campbell-Hausdorff formula,

$$\log \circ II(\alpha^{-1}\gamma\alpha) = \log \circ II(\gamma) + ...,$$

where dots denote higher order commutators of $\log \circ II(\alpha)$ and $\log \circ II(\gamma)$. Each of these commutators necessarily contains at least one $\log \circ II(\gamma) \in N_1$ and at least one of $\log \circ II(\alpha) \in \mathfrak{g}$. But $[N_1, \mathfrak{g}] = N_0 \subset N_1$, so each of these commutators is in $[N_1, \mathfrak{g}] = N_0$. □

Proof of Proposition 5.5. Choosing another base point p_1 results in replacement of

$$N_1(p_0) = \operatorname{Span}\{\log(II_k(\gamma)), \gamma \in \mathcal{O}_{p_0}\}$$

by

$$N_1(p_1) = \operatorname{Span}\{\log(II_k(\alpha^{-1}\gamma\alpha)), \gamma \in \mathcal{O}_{p_0}\},$$

where α is some path joining p_0 and p_1. By Lemma 5.6, the difference is in $N_0(p_0) \subset N_1(p_0)$, so $N_1(p_0) = N_1(p_1)$. Similarly, $N_0(p_0) = N_0(p_1)$.

The second claim then follows directly from Lemma 5.6. □

6. Vector bundles $\mathcal{H}, \mathcal{C}, \mathcal{N}$ and proofs of Theorems 1.4 and 1.7

Here we investigate the dependence of previous constructions on t. The fundamental fact is the following

Lemma 6.1. *Let η_i be polynomial one-forms on $\mathbb{C}^2$, and let $I(t, p_0) = \int_{\gamma(t)} \eta_1 \dots \eta_n$ be an iterated integral of length n over a family of loops $\gamma(t) \subset \pi_1\left(F^{-1}(t), p(t)\right)$. Then $I(t)$ is a multivalued function of regular growth on $(t, p(t)) \in (\mathbb{C} \setminus \Sigma) \times \left(\mathbb{C}^2 \setminus F^{-1}(\Sigma)\right)$.*

This can be proved along the lines of [17]: after a resolution of singularities of the fibration $F : \mathbb{C}P^2 \to \mathbb{C}P^1$, the family $\gamma(t)$ can be cut, by some analytic transversals, into a union of several pieces, each one lying in its own chart. In each chart, F is a monomial, and η_i are meromorphic with poles on the preimage of the union of atypical fibers. As t tends to an atypical value, the iterated integrals along these pieces can be easily polynomially bounded from above, so the total integral $I(t)$ has moderate growth by the shuffling formula (5.3).

We consider several vector bundles over $\mathbb{C} \setminus \Sigma$.

The union of all $\mathbb{C}H_1(\mathcal{O})(t_0)$ over all $t_0 \in \mathbb{C} \setminus \Sigma$ of typical values of F forms a vector bundle $\mathcal{H}$. Taken together, the Melnikov functionals $\mathbf{M}_{\mu,t_0}$, $t_0 \in \mathbb{C} \setminus \Sigma$, define a section $\mathbf{M}_\mu$ of its dual bundle $\mathcal{H}^*$.

The bundle $\mathcal{H}$ inherits the linear Gauss-Manin connection from the homotopy fibration. Evaluation of $\mathbf{M}_\mu$ on its horizontal section $\gamma(t)$ gives the Melnikov function:

$$M_\mu(t) = \mathbf{M}_\mu(\gamma(t)).$$

By Proposition 5.3 we can identify the vector bundle $\mathcal{H}^*$ with the vector bundle $\mathcal{N}^*$, where $\mathcal{N}$ is the vector bundle

$$\mathcal{N} = \left\{\cup_{t\in\mathbb{C}\setminus\Sigma} N(t) \to \mathbb{C} \setminus \Sigma\right\}.$$

The bundle $\mathcal{N}$ is obtained from the trivial bundle

$$\mathcal{C}_k = \{\mathbb{C}_k \times (\mathbb{C} \setminus \Sigma) \to \mathbb{C} \setminus \Sigma\},$$

by taking the subbundle

$$\mathcal{N}_1 = \left\{\cup_{t\in\mathbb{C}\setminus\Sigma} N_1(t) \to \mathbb{C} \setminus \Sigma\right\},$$

and then taking the quotient by the subbundle

$$\mathcal{N}_0 = \left\{\cup_{t\in\mathbb{C}\setminus\Sigma} N_0(t) \to \mathbb{C} \setminus \Sigma\right\}.$$

By Proposition 5.5, all these bundles are well defined, i.e. do not depend on a particular choice of the base points.

Lemma 6.2. *Assume that a subbundle $\mathcal{B}$ of $\mathcal{C}_k$ is spanned by sections with moderate growth as t tends to $\Sigma \cup \{\infty\}$. Then the subbundle $\mathcal{B}^\perp$ of $\mathcal{C}_k^*$ is spanned by sections of the form*

$$\tilde{f}_j = \sum_{\|\alpha\|\le k} b^j_\alpha(t) c_\alpha, \tag{6.1}$$

where $b^j_\alpha(t)$ are rational functions of t, and c_α are the standard linear functionals on $\mathbb{C}_k$ given by taking the coefficients of the monomials X^α of a jet in $\mathbb{C}_k$.

Proof. The vector bundle $\mathcal{B}$ defines a univalued mapping $\mathbb{C} \setminus \Sigma \to \mathrm{Gr}(\mathbb{C}_k, d)$ of the base $\mathbb{C} \setminus \Sigma$ to the Grassmanian of all d-dimensional subspaces of $\mathbb{C}_k$, where d is the dimension of the fibers $B(t)$ of $\mathcal{B}$. This mapping has regular singularities at $\Sigma \cup \{\infty\}$ by assumption, so is a rational map. Equivalently, the coefficients of the equations defining the fiber $B(t)$ are rational functions of t. This is exactly the claim of the Lemma. □

Corollary 6.3. *There exists rational functions $b^j_\alpha(t)$ with poles in Σ such that the sections*

$$f_j = \sum_{\|\alpha\|\le k} b^j_\alpha(t) c_\alpha \tag{6.2}$$

of $\mathcal{C}_k^*$ define a basis of sections of $\mathcal{N}^*$, where c_α are the standard linear functionals on $\mathbb{C}_k$ as in Lemma 6.2.

Proof. Choose a subset of sections $\tilde{f}_j$ of $\mathcal{N}_0^\perp$ as in Lemma 6.2 defining a basis of $(N_1(t)/N_0(t))^*$ at a generic $t \in \mathbb{C} \setminus \Sigma$. Suitable combinations f_j of these sections with rational in t coefficients will then define a basis, for all $t \in \mathbb{C} \setminus \Sigma$. In particular, in representation (6.2) the poles of the coefficients $b_\alpha^j(t)$ will be in Σ. □

We will need later the following Corollary, which is a version of the algebraic de Rham theorem.

Corollary 6.4. *Any cohomology class $\eta \in H^1(F^{-1}(t_0))$ vanishing on $\mathcal{O}L_2/L_2 \subset H^1(F^{-1}(t_0))$ is a restriction of a rational one-form $E \in \Omega^1(\mathbb{C}^2)$ whose integral vanishes identically on γ.*

Proof. For $k = 1$, we have $\log \circ II_1(\gamma) = \sum \int_\gamma \eta_i X_i$, which identifies $\mathbb{C}_1 \cong \mathbb{C} \oplus H_1(F^{-1}(t_0))$ and, therefore, $\mathbb{C}_1^* \cong \mathbb{C} \oplus H^1(F^{-1}(t_0))$. Let $\mathcal{B} = \mathcal{N}_1$ and let

$$f = a_0(t) \cdot \mathbb{1} + \sum a_i(t) c_i,$$

be a linear combination of sections (6.1) such that $f(t_0)$ coincide with $(0, \eta)$. Here $a_i(t)$ are rational functions of t. Then one can take $E = \sum a_i(F)\eta_i$. □

Proof of Theorem 1.7. Now, the section $s = ((\log \circ II_K)^*)^{-1}(\mathbf{M}_\mu)$ also has regular growth, so

$$s = \sum b_j^M(t) f_j,$$

where $b_\alpha^M(t)$ are also rational functions of t.

Therefore

$$M(\gamma(t)) = s(\log \circ II_k(\gamma(t))) = \sum_j b_j^M(t) g_j(\gamma(t)), \tag{6.3}$$

where $g_j(\gamma) = f_j(\log \circ II_k(\gamma))$ are combinations with rational in t coefficients of iterated integrals of γ, by Remark 5.2. Moreover, g_j are base point independent by Proposition 5.5. □

Proof of Theorem 1.4. Theorem 1.4 follows from Theorem 1.7 as the torsion-free orbit length k of γ does not exceed the orbit length κ of γ. □

Proof of Corollary 1.8. If a typical fiber $\{F = t\}$ has d components, then $F = g(H)$, where $H \in \mathbb{C}[x, y]$ and $g \in \mathbb{C}[s], \deg g = d$. Applying Theorem 1.7 to the foliation $\{H = s\}$, we see that M_μ has the form (6.3) with coefficients being rational functions of $s = g^{-1}(t)$, as required. □

7. Alternative proof of Theorem 1.7 using Chen's bordered determinant

In this section we sketch an alternative proof of Theorem 1.7 (i)(ii) inspired by the proof of Gavrilov and Iliev [11] in the abelian integral case. We first recall the main ideas of their proof here.

7.1. **Proof of Gavrilov and Iliev in the abelian integrals case.** In [11] the authors construct a determinant given by the de Rham pairing of a basis of the homology $H_1(\mathcal{O})$ and its dual space $H^1(\mathcal{O})$. The hypothesis of the injectivity of (1.10) assures, by the de Rham theorem, that this dual space is given by integration of polynomial one-forms. The pairing is given by integrating the forms along the corresponding cycles. The determinant is then bordered by adding one row and one column. The row corresponds to the cycle γ along which one considers the dispacement function and its first nonzero Melnikov function. The column corresponds to the first nonzero Melnikov function for the corresponding cycle. We call the above determinant the *bordered de Rham determinant*.

The determinant of the bordered de Rham matrix vanishes due to its size. Developping it with respect to the last row, they obtain an expression for the first nonzero Melnikov function $M_\mu(\gamma)$ as a linear combination of abelian integrals. They study the monodromy of the coefficients and show that they are univalued. It then follows from growth estimates that they are rational functions. They then conclude that the first nonzero Melnikov function is an abelian integral.

7.2. **Our proof in the case of iterated integrals of any order.** In our case, working with simple (abelian) integrals is not enough, since many minors in the bordered de Rham matrix vanish. We hence work with iterated integrals instead of just simple integrals. Instead of the de Rham pairing, we have the Chen pairing given by iterated integration.

The new problem that we encounter is that contrary to the situation with abelian integrals which behave linearly with respect to monodromy, this is not the case with iterated integrals due to the shuffling formula (5.3). We overcome this problem by working in the quotient space by the commutator group K.

7.3. **Bordered Chen pairing matrix.** We first construct the bordered Chen pairing matrix adapted to our use. It is a matrix C_γ, whose rows correspond to cycles forming a basis of $H_1(\mathcal{O})$ and its columns correspond to a basis of a dual space $H^1(\mathcal{O})$. By Proposition 5.3, each functional belonging to $H^1(\mathcal{O})$ can be realized as an iterated integral g_j of some tuple of 1-forms. The pairing is given by iterated integration. By Proposition 5.5 and Proposition 3.1, both the functionals g_j and M_μ are base point independent.

In the chosen basis of $\mathbb{C}H_1(\mathcal{O})$ monodromy acts linearly with corresponding matrix Mon_α.

Next, we border the Chen pairing matrix, thus obtaining the *bordered Chen pairing matrix* $C_{\gamma,M}$. Here the last row corresponds to the cycle γ and the last column corresponds to taking the first nonzero Melnikov function along the corresponding cycle.

We study the determinant of the bordered Chen matrix $\det(C_{\gamma,M})$. Note that on one hand side this determinant vanishes identically, because γ belongs to the orbit $\mathcal{O}$. On the other hand, we develop this determinant with respect to the last column. Denoting $n+1$ the size of the determinant, this gives an expression of the form

$$a_{n+1}(t)M_\mu(\gamma(t)) = \sum_j a_j(t)g_j(\gamma(t)), \tag{7.1}$$

where g_j are iterated integrals of length $\ell \leq k$. The function $a_{n+1}(t)$ is nonzero on $\mathbb{C} \setminus \Sigma$. Monodromy acts linearly on (7.1). This is highly non-trivial. Recall that iterated integrals are in general not linear with respect to product of cycles, instead the shuffling formula (5.3) applies. Here we recover linearity using Baker-Campbell-Hausdorff formula thanks to the vanishing of both g_j and M_μ on K, see Proposition 5.3.

Now, by action of monodromy (7.1) is multiplied by $\det Mon_\alpha$. Hence, the quotients $b_j^M = \frac{a_j}{a_{n+1}}$ are well-defined, univalued and of moderate growth, therefore they are rational functions of t with possible poles in Σ.

8. Proof of Theorem 1.7$(ii), (ii')$.

Assume $k = 1$. Then $\gamma \neq 0$ in $H_1(F^{-1}(t_0))$. Take η such that $M_1 = \int_\gamma \eta \neq 0$, showing that the first nonzero Melnikov function of the deformation $dF + \epsilon\eta = 0$ is the abelian integral M_1.

Assume that $\kappa > 2$. Here we prove that one can find a perturbation ω such that M_2 is the first non-vanishing Melnikov function and it is not an abelian integral.

By hypothesis, there exists a loop $\delta \in (\mathcal{O} \cap L_2) \setminus K$. Recall [8, 10] that

$$M_1(\gamma) = \int_\gamma \omega, \ M_2(\gamma) = \int_\gamma \omega\omega',$$

where ω' is the Gelfand-Leray derivative of ω and the last formula holds if $M_1(\gamma) \equiv 0$.

Lemma 8.1. *There exists ω such that*

$$M_1(\gamma) = \int_\gamma \omega \equiv 0, \qquad \int_\delta \omega\omega' \not\equiv 0.$$

This shows that M_2 is the first nonzero Melnikov function (as otherwise it would vanish on δ). Moreover, M_2 is not an abelian integral as any abelian integral vanishes on $\delta \in L_2$.

Lemma 8.2.

(i) $\frac{L_2}{KL_3}$ *has no torsion elements.*

(ii) $\left(\frac{L_2}{KL_3}\right)^* = \left\{\int \eta_i\eta_j \,|\, \forall\gamma \in \mathcal{O} \ \int_\gamma \eta_i = \int_\gamma \eta_j = 0\right\}.$

Proof. Choose a basis $\mathcal{B} = \{\gamma_i, \sigma_j\}$ of $H_1(F^{-1}(t_0))$ such that $\{\gamma_i\}$ form a basis of $\mathrm{Im}(\phi_1)$. Then $\{[\gamma_i, \gamma_j], [\gamma_i, \sigma_j], [\sigma_i, \sigma_j]\}$ form a basis of the free abelian group L_2/L_3. The first two types of commutators lie in K, so $\frac{L_2}{KL_3}$ is isomorphic to the subgroup of L_2/L_3 generated by $\{[\sigma_i, \sigma_j]\}$, which is hence a free abelian group.

By Chen's theorem, the dual space $(L_2/L_3)^*$ is spanned by iterated integrals of length 2. Let $\mathcal{B}^* = \{\omega_i, \eta_j\}$ be a basis of $H^1(F^{-1}(t_0))$ dual to $\mathcal{B}$. Then

$$\int_{[\gamma_i,\gamma_j]} \omega_{i'}\omega_{j'} = \pm\delta_{ii'}\delta_{jj'}, \int_{[\gamma_i,\gamma_j]} \omega_{i'}\eta_{j'} = \int_{[\gamma_i,\gamma_j]} \eta_{i'}\eta_{j'} = 0,$$
$$\int_{[\gamma_i,\sigma_j]} \omega_{i'}\omega_{j'} = 0, \int_{[\gamma_i,\sigma_j]} \omega_{i'}\eta_{j'} = \pm\delta_{ii'}\delta_{jj'}, \int_{[\gamma_i,\sigma_j]} \eta_{i'}\eta_{j'} = 0,$$
$$\int_{[\sigma_i,\sigma_j]} \omega_{i'}\omega_{j'} = \int_{[\sigma_i,\sigma_j]} \omega_{i'}\eta_{j'} = 0, \int_{[\sigma_i,\sigma_j]} \eta_{i'}\eta_{j'} = \pm\delta_{ii'}\delta_{jj'}.$$

so the iterated integrals $\{\int \eta_i\eta_j, i < j\}$ form a basis of $\left(\frac{L_2}{KL_3}\right)^*$. □

Corollary 8.3. *There exist two forms η_1, η_2, such that $\int_\gamma \eta_i = 0$, but $\int_\delta \eta_1\eta_2 \neq 0$.*

Proof. As $\delta \neq 0$ in $\frac{L_2}{KL_3}$, then the claim follows from Lemma 8.2, $\square$

Proof of Lemma 8.1. Let $\Theta_i \in \Omega^1(\mathbb{C}^2)$, $i = 1, 2$, be polynomial forms such that their restrictions to $F^{-1}(t_0)$ coincide with η_i, and, moreover, $\int_{\gamma(t)} \Theta_i = 0$, for any analytic continuation of γ. Existence of such forms follows from Corollary 6.4.

Take $\omega = \lambda\Theta_1 + \lambda^{-1}(F - t_0)\Theta_2$, $\lambda > 0$. Then

$$\omega' = \lambda\Theta_1' + \lambda^{-1}(F - t_0)\Theta_2' + \lambda^{-1}\Theta_2,$$

and, therefore,

$$\int_\gamma \omega \equiv 0, \qquad \int_{\delta(t_0)} \omega\omega' = \int_{\delta(t_0)} \eta_1\eta_2 + \lambda^2 \int_{\delta(t_0)} \eta_1\eta_1' \neq 0,$$

for sufficiently small λ. $\square$

9. Examples

Example 9.1. *Hamiltonian triangle.*

Let $F = y(x^2 - (y-3)^2)$, and let $\gamma(t)$ be a continuous family of real periodic orbits surrounding the center bounded by the triangle. In this case, according to [11, 20],

$$\mathbb{C}H_1(\mathcal{O}) = \langle \gamma, \delta_1\delta_2\delta_3, [\delta_1, \delta_2] \rangle.$$

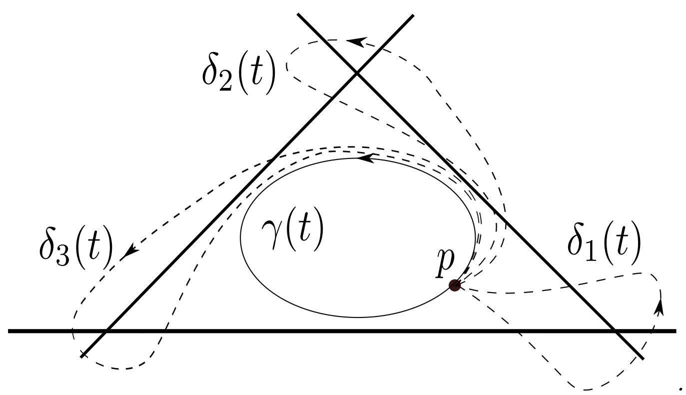

Figure 1. The real loop $\gamma(t)$ and the complex vanishing loops $\delta_i(t)$ as elements of $\pi_1\left(F^{-1}(t), p\right)$.

Hence, $\mathcal{O} \cap L_3 \subset K$ and both k and κ are equal to 2. It follows that the Melnikov function associated to γ and to any polynomial perturbation of $dF = 0$, is an iterated integral of length at most 2.

On the other hand, in [20] *it is shown that for some perturbation of degree 5 the first nonzero Melnikov function M_μ is of length 2 and is not an abelian integral, which illustrates Theorem 1.7(ii).*

Example 9.2. *Product of $(d+1)$ straight lines in general position* [21][18].

Let $F = \prod_{i=1}^{d+1} f_i$, where f_i are real linear functions in two variables, such that the corresponding lines $\{f_i = 0\}$ are in general position in the real plane. Let $\gamma(t)$

be a continuous family of periodic orbits surrounding a center singular point of the Hamiltonian vector field $F_y\frac{\partial}{\partial x} - F_x\frac{\partial}{\partial y}$.

In [21]*, it is shown that the 1-forms* $\{\eta_i = F\frac{df_i}{f_i}\}_{i=1}^d$ *define a basis of the orthogonal complement* $(Im\phi_1)^\perp$ *of the orbit in homology. The space* $(Im\phi_1)^\perp$ *is generated by the so-called residual cycles* $\{\sigma_i\}_{i=1}^d$*, the small cycles winding around the transversal intersections of the level curve with the line at infinity. Moreover, the iterated integrals* $\int \eta_i\eta_j$*, with* $1 \leq i < j \leq d$*, are* $\mathbb{C}[t, 1/t]$*-linear independent on the orbit* $\mathcal{O}$.

By Lemma 8.2 the dual space $(\frac{L_2}{L_3K})^*$ *of* $\frac{L_2}{L_3K}$ *is given by the iterated integrals of length* 2 $\int \eta_i\eta_j$, $1 \leq i < j \leq d$*. If* $\mathrm{Im}(\phi_2) = \frac{\mathcal{O}\cap L_2}{L_3K} \subsetneq \frac{L_2}{L_3K}$ *is a proper linear subspace of* $\frac{L_2}{L_3K}$*, then there should exist a non-trivial linear combination of the iterated integrals* $\int \eta_i\eta_j$ *vanishing identically on* $\mathrm{Im}(\phi_2)$*. This would contradict their linear independence on* $\mathcal{O} \cap L_2$*. Hence* $\mathcal{O} \cap L_2 = L_2$*, which implies* $\mathcal{O} \cap L_3 \subseteq K$*. On the other hand, by computations similar to the triangle case on can check that* $[\sigma_i, \sigma_j] \in \mathcal{O}$ *but not in* K*, so* $\mathcal{O}\cap L_2 \nsubseteq K$*. Therefore* $k = 2$ *in Theorem 1.7, and the Melnikov function associated to* γ *and to any polynomial perturbation of* $dF = 0$*, is an iterated integral of length at most* 2.

Example 9.3. *codim*$(\mathcal{O}) = 1$ *in homology* [19].

Suppose $\pi_1 = \langle\gamma_1, ..., \gamma_n, \sigma\rangle$ *and* $\mathcal{O} = \langle\gamma_1, ..., \gamma_n\rangle$*. Then* $K = L_2$*, so* $L_2 \cap \mathcal{O} \subseteq K$*. Therefore,* k *and* κ*, are both equal to* 1 *and, by Theorem 1.7, the first nonzero Melnikov function* M_μ *is an abelian integral.*

References

[1] Arnold V.I., *Arnold's problems.* Springer-Verlag, Berlin, 2004, xvi+639 p.

[2] Benditkis, S., Novikov, D., *On the number of zeros of Melnikov functions*, Ann. Fac. Sci. Toulouse Math. (6) **20** (2011), no. 3, 465-491.

[3] Binyamini, G., Novikov, D., Yakovenko, S.,*On the number of zeros of Abelian integrals*, Invent. Math. **181** (2010), no. 2, 227-289.

[4] Binyamini, G., Novikov, D., Yakovenko, S., *Quasialgebraic functions*, in: Algebraic methods in dynamical systems, 61-81, Banach Center Publ., 94, Polish Acad. Sci. Inst. Math., Warsaw, 2011.

[5] Chen K.-T. *Iterated path integrals,* Bull. Amer. Math. Soc. **83** (1977), no. 5, 831-879.

[6] Dumortier, F., Roussarie, R., *Abelian integrals and limit cycles*, Journal of Differential Equations **227** (2006) no.1, 116-165.

[7] Forster, O. *Lectures on Riemann surfaces.* Graduate Texts in Mathematics, 81. Springer-Verlag, New York, 1991. viii+254 pp.

[8] Françoise, J.-P., *Successive derivatives of a first return map, application to the study of quadratic vector fields.* Ergodic Theory Dynam. Systems **16** (1996), no. 1, 87-96.

[9] Harris, B. *Iterated integrals and cycles on algebraic manifolds.* Nankai Tracts in Mathematics, 7. World Scientific Publishing Co., Inc., River Edge, NJ, 2004. xii+108 pp.

[10] Gavrilov, L. *Higher order Poincaré-Pontryagin functions and iterated path integrals,* Ann. Fac. Sci. Toulouse Math. (6) **14** (2005), no. 4, 663-682.

[11] Gavrilov, L., Iliev, I. D., *The displacement map associated to polynomial unfoldings of planar Hamiltonian vector fields*, Amer. J. Math. **127** (2005), no. 6, 1153-1190.

[12] Iliev, I. D., *The cyclicity of the period annulus of the quadratic Hamiltonian triangle*, J. Differential Equations **128** (1996), no. 1, 309-326.

[13] Ilyashenko, Yu. S., *The appearance of limit cycles under a perturbation of the equation* $dw/dz = -R_z/R_w$*, where* $R(z, w)$ *is a polynomial.* (Russian) Mat. Sb. (N.S.) **78** (120) 1969 360-373.

[14] Jacobson, N. *Lie algebras.* Republication of the 1962 original. Dover Publications, Inc., New York, 1979. ix+331 pp.

[15] Jebrane, A., Mardešić, P., Pelletier, M., *A generalization of Françoise's algorithm for calculating higher order Melnikov functions*, Bull. Sci. Math. **126** (2002), no. 9, 705-732.
[16] Jebrane, A., Mardešić, P., Pelletier, M., *A note on a generalization of Françoise's algorithm for calculating higher order Melnikov functions*, Bull. Sci. Math. **128** (2004), no. 9, 749-760.
[17] Malgrange, B. *Intégrales asymptotiques et monodromie.* Ann. Sci. École Norm. Sup. (4) **7** (1974), 405-430.
[18] Pontigo Herrera, J. *The first nonzero Melnikov function for a family of good divides*, in preparation.
[19] Pelletier, M., Uribe, M., *Principal Poincaré-Pontryagin function associated to some families of Morse real polynomials*, Nonlinearity **27** (2014), no. 2, 257-269.
[20] Uribe, M. *Principal Poincaré-Pontryagin function of polynomial perturbations of the Hamiltonian triangle*, J. Dyn. Control Syst. **12** (2006), no. 1, 109-134.
[21] Uribe, M., *Principal Poincaré-Pontryagin function associated to polynomial perturbations of a product of* $(d+1)$ *straight lines.* J. Differential Equations **246** (2009), no. 4, 1313-1341.
[22] Yakovenko, S. A geometric proof of the Bautin theorem. *Concerning the Hilbert 16th problem*, 203-219, Amer. Math. Soc. Transl. Ser. 2, 165, Adv. Math. Sci., 23, Amer. Math. Soc., Providence, RI, 1995.
[23] Żołądek, H. *Quadratic systems with center and their perturbations*, J. Differential Equations **109** (1994), no. 2, 223-273.

Université de Bourgogne, Institute de Mathématiques de Bourgogne - UMR 5584 CNRS, Université de Bourgogne, 9 avenue Alain Savary, BP 47870, 21078 Dijon,, FRANCE

E-mail address: mardesic@u-bourgogne.fr

Faculty of Mathematics and Computer Science, Weizmann Institute of Science, Rehovot, 7610001, Israel

E-mail address: dmitry.novikov@weizmann.ac.il

Instituto de Matemáticas, Universidad Nacional Autónoma de México (UNAM), Área de la Investigación Científica, Circuito exterior, Ciudad Universitaria, 04510, Ciudad de México, México

E-mail address: laura@matem.unam.mx

Faculty of Mathematics and Computer Science, Weizmann Institute of Science, Rehovot, 7610001 Israel

E-mail address: jessie-diana.pontigo-herrera@weizmann.ac.il