THE PILA-WILKIE THEOREM FOR SUBANALYTIC FAMILIES: A COMPLEX ANALYTIC APPROACH

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Abstract. We present a complex analytic proof of the Pila-Wilkie theorem for subanalytic sets. In particular, we replace the use of C^r-smooth parametrizations by a variant of Weierstrass division. As a consequence we are able to apply the Bombieri-Pila determinant method directly to analytic families without limiting the order of smoothness by a C^r parametrization. This technique provides the key inductive step for our recent proof (in a closely related preprint) of the Wilkie conjecture for sets definable using restricted elementary functions.

As an illustration of our approach we prove that the rational points of height H in a (compact piece of) a complex-analytic set of dimension k in $\mathbb{C}^m$ are contained in $O(1)$ complex-algebraic hypersurfaces of degree $(\log H)^{k/(m-k)}$. This is a complex-analytic analog of a recent result of Cluckers, Pila and Wilkie for real subanalytic sets.

1. Introduction

1.1. **Statement of the main results.** For a set $A \subset \mathbb{R}^m$ we define the *algebraic part* A^{alg} of A to be the union of all connected semialgebraic subsets of A of positive dimension. We define the *transcendental part* A^{trans} of A to be $A \setminus A^{\text{alg}}$.

Recall that the *height* of a (reduced) rational number $\frac{a}{b} \in \mathbb{Q}$ is defined to be $\max(|a|, |b|)$. For a vector x of rational numbers we denote by $H(x)$ the maximum among the heights of the coordinates. For a set $A \subset \mathbb{C}^m$ we denote the set of $\mathbb{Q}$-points of A by $A(\mathbb{Q}) := A \cap \mathbb{Q}^m$ and denote

$$A(\mathbb{Q}, H) := \{x \in A(\mathbb{Q}) : H(x) \leqslant H\}. \tag{1}$$

For $A \subset \mathbb{R}^{m+n}$ and $y \in \mathbb{R}^n$ we denote the y-fiber of A by

$$A_y \subset \mathbb{R}^m, \qquad A_y := \{x \in \mathbb{R}^m : (x, y) \in A\}. \tag{2}$$

In this paper we develop a new approach to the proof of the following theorem.

Theorem 1. *Let $A \subset \mathbb{R}^{m+n}$ be a bounded subanalytic set and $\varepsilon > 0$. There exists an integer $N(A, \varepsilon)$ such that for any $y \in \mathbb{R}^n$,*

$$\#(A_y)^{\text{trans}}(\mathbb{Q}, H) \leqslant N(A, \varepsilon) H^{\varepsilon}. \tag{3}$$

The result of Theorem 1 is not new. In fact, it was conjectured in [14, Conjecture 1.2] and proved, in a more general form, in the work of Pila and Wilkie [13], where the same result is shown to hold for any A definable in an O-minimal structure. Our goal in the present paper is to develop an alternative complex analytic approach to this theorem. In particular, while the proof of [13] requires the use of

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C^r parametrizations of subanalytic sets, we are able to carry out the arguments completely within the analytic category. The proof of Theorem 1 is given at the end of §5.3. As a first illustration of the advantage of this approach, we prove the following complex analog of a recent result of [5].

Theorem 2 (cf. [5, Theorem 2.3.2])**.** *Let $X \subset \mathbb{C}^{m+n}$ be a locally analytic subset and $K \Subset X$ be a compact subset. Suppose $\dim_{\mathbb{C}} X_y \leqslant k$ for every $y \in \mathbb{C}^n$. Then there exists a constant $C = C(K)$ such that for any $y \in \mathbb{C}^n$, if $H > 2$ then $K_y(\mathbb{Q}, H)$ is contained in the union of at most C complex algebraic hypersurfaces of degree at most $(\log H)^{k/(m-k)}$ in $\mathbb{C}^m$.*

The proof of Theorem 2 is given following Corollary 13. In [5, Theorem 2.3.2] a result similar to Theorem 2 is stated for real globally subanalytic sets. The dimensions are taken over $\mathbb{R}$ and the algebraic hypersurfaces constructured are real. We give the complex version for variation, and also since in most applications the sets involved are in fact complex analytic. Note that in this case the complex analytic version gives more than the real version: the degree of the hypersurface is the same as in [5, Theorem 2.3.2], but intersecting with a complex-algebraic hypersurface imposes two independent real-algebraic conditions. We also remark that the real version [5, Theorem 2.3.2] could also be derived with our method by the same complexification arguments used in our proof of Theorem 1.

Our principal motivation for the complex-analytic approach developed in this paper is the Wilkie conjecture on the improvement of the asymptotic $O(H^\varepsilon)$ to polylogarithmic for certain structures (see §1.2 for details). Theorem 2 can be seen as a natural first inductive step toward this conjecture. However, pursuing this induction further requires analyzing the behavior of the constant $C(A)$ with respect to the complexity of the set A – a problem which requires finer analysis that cannot be carried out in the generality of general analytic sets (as illustrated by counterexamples due to Pila [14, Example 7.5]). In a closely related preprint [1] we build on the ideas developed in this paper to prove the Wilkie conjecture for sets definable using the *restricted* exponential and sine functions – a problem which had previously seemed quite challenging due to the difficulty of computing C^r parametrizations.

In §1.2 we present some motivations for our approach. In §1.3 we briefly review the method of Bombieri-Pila-Wilkie, and in particular explain the point at which C^r-parametrizations are required. In §1.4 we give an outline of the complex-analytic approach developed in this paper and explain how it avoids the use of C^r-parametrizations.

Note. *Shortly after the appearance of the first version of this manuscript (and the closely related* [1]*) on the arXiv, the independently developed preprint* [5] *by Cluckers, Pila and Wilkie appeared. Some of the methods appear to be related: in particular the "quasi-parametrization theorem" 2.2.3 of* [5] *appears similar to our notion of a decomposition datum, and the diophantine application* [5, Theorem 2.3.2] *is also a consequence of our approach. We wish to clarify that Theorem 2 was not included in the original version of this manuscript and appeared (in the real version) originally in* [5]*; in a later version we included the complex Theorem 2 to illustrate how our approach can be used to derive similar results.*

1.2. **Motivation.** There are two directions in which one might hope to improve the Pila-Wilkie estimate $\#A^{\text{trans}}(\mathbb{Q}, H) \leqslant N(A, \varepsilon) H^\varepsilon$:

- Effective estimates: one may hope to obtain effective estimates for the constant $N(A,\varepsilon)$ in terms of the complexity of the equations/formulas used to define A.
- Sharper asymptotics: one may hope to improve the asymptotic dependence on H if A is definable in a suitably tame structure. As a notable example, the Wilkie conjecture states that if A is definable in $\mathbb{R}_{\exp}$ then $\#A^{\text{trans}}(\mathbb{Q},H) = N(A)(\log H)^{\kappa(A)}$.

Both of these directions have been considered in the literature, see e.g. [3, 11, 16, 15]. However, as discussed in §1.3, the proof of the Pila-Wilkie theorem in arbitrary dimensions requires the use of C^r-reparametrizations, whose complexity is difficult to control even in the semialgebraic case. For this reason, most of the work (to our knowledge) has been restricted to A of (real) dimension one or two.

Our primary goal in this paper is to develop an approach that replaces the use of C^r-parametrization by direct considerations on the local complex-analytic geometry of A. In the preprint [1] we use this approach to prove the Wilkie conjecture for sets definable using the *restricted* exponential and sine functions, where Proposition 11 provides the key inductive step. The estimates in [1] are also effective in a suitable sense. We believe that the analytic approach may also play a significant role in advancing toward an effective version of the Pila-Wilkie theorem for Noetherian functions.

Finally, while in this paper we work in the complex analytic setting, our arguments are essentially algebraic – tracing to the Weierstrass preparation and division theorems. One may hope that such an approach could allow a more direct generalization to different algebraic contexts where the analytic notion of C^r-parametrization may be more difficult to recover. In particular we consider it an interesting direction to check whether the method developed in this paper can offer an alternative approach to the work of Cluckers, Comte and Loeser [4] on non-archimedean analogs of the Pila-Wilkie theorem. We remark in this context that in our primary model-theoretic reference [6] the complex-analytic and p-adic contexts are treated in close analogy.

1.3. **Exploring rational points following Bombieri-Pila and Pila-Wilkie.**

1.3.1. *The case of curves.* Let $X \subset \mathbb{R}^2$ be compact irreducible real-analytic curve. Building upon earlier work by Bombieri and Pila [2], Pila [12] considered the problem of estimating $\#X(\mathbb{Q},H)$. More specifically, he showed that if X is transcendental then for every $\varepsilon > 0$ there exists a constant $C(X,\varepsilon)$ such that $\#X(\mathbb{Q},H) \leqslant C(X,\varepsilon)H^\varepsilon$. Bombieri and Pila's method involves constructing a collection of H^ε hypersurfaces $\{H_k\}$ of degree $d = d(\varepsilon)$ such that $X(\mathbb{Q},H)$ is contained in $\cup_k H_k$. We briefly recall the key idea, starting with the notion of an interpolation determinant.

Suppose first that X can be written as the image of an analytic map $\mathbf{f} = (f_1, f_2) : [0,1] \to X$ (the general case will be treated later by subdivision). For simplicity we will suppose that f_1, f_2 extend to holomorphic functions in the disc of radius 2 around the origin, with absolute value bounded by M.

Let $\mathbf{g} := (g_1, \dots, g_\mu)$ be a collection of functions and $\mathbf{p} := (p_1, \dots, p_\mu)$ a collection of points. We define the *interpolation determinant*

$$\Delta(\mathbf{g},\mathbf{p}) := \det(g_i(p_j))_{1\leqslant i,j\leqslant\mu}. \tag{4}$$

Let $d \in \mathbb{N}$ and set $\mu = d(d+1)/2$, the dimension of the space of polynomials in two variables of degree at most d. We define the *polynomial interpolation determinant* of degree d to be

$$\Delta^d(\mathbf{f}, \mathbf{p}) := \Delta(\mathbf{g}, \mathbf{p}), \qquad \mathbf{g} = (f_1^k f_2^l : k, l \in \mathbb{N}, k+l \leqslant d). \tag{5}$$

Note that $\Delta^d(\mathbf{f}, \mathbf{p}) = 0$ if and only if the points $\mathbf{f}(p_1), \ldots, \mathbf{f}(p_\mu)$ all lie on a common algebraic hypersurface of degree at most d. More generally, if $P \subset I$ and $\Delta^d(\mathbf{f}, \mathbf{p}) = 0$ for every $\mathbf{p} \subset P$ then the points $\mathbf{f}(p) : p \in P$ all lie on a common algebraic hypersurface of degree at most d.

Let $H \in \mathbb{N}$. Our goal is to construct a collection of algebraic hypersurfaces H_k whose union contains $X(\mathbb{Q}, H)$. By the above, it will suffice to subdivide I into intervals I_k such that for any $\mathbf{p} \subset I_k$, if $H(\mathbf{p}) \leqslant H$ then $\Delta^d(\mathbf{f}, \mathbf{p}) = 0$. We begin with the following key estimate on the polynomial interpolation determinant.

Lemma 1 (cf. Lemma 9)**.** *Let $I \subset [0,1]$ be an interval of length $\delta < 1/2$, and $\mathbf{p} = (p_1, \ldots, p_\mu) \in I^\mu$. Then*

$$|\Delta^d(\mathbf{f}, \mathbf{p})| \leqslant \mu!(2\mu+2)^\mu M^{d\mu} \delta^{\mu^2/2}. \tag{6}$$

Proof. By translation we may suppose that $I = [-\delta, \delta]$ and that f_1, f_2 are holomorphic in the unit disc $D \subset \mathbb{C}$ with absolute value bounded by M. Denote by $\|\cdot\|$ the maximum norm on the disc of radius δ around the origin.

Every function in $\mathbf{g}$ is holomorphic in D with absolute value bounded by M^d. Consider the Taylor expansions

$$g_i = \sum_{j=0}^{\mu-1} m_j(g_i) + R_\mu(g_i) \qquad m_j(g_i) := c_{i,j} x^j \qquad i = 1, \ldots, \mu. \tag{7}$$

and $R_j(g_i)$ are the Taylor residues. From the Cauchy estimates we have

$$\|m_j(g_i)\| \leqslant M^d \delta^j \qquad \|R_\mu(g_i)\| \leqslant 2M^d \delta^\mu \tag{8}$$

Expand the determinant $\Delta^d(\mathbf{f}, \mathbf{p})$ by linearity using (7), to obtain a sum of $(\mu+1)^\mu$ summands with each g_i replaced by either $m_{j_i}(g_i)$ or $R_\mu(g_i)$. Note that any summand where two different indices j_k, j_l agree vanishes identically since the corresponding functions $m_{j_k}(g_k), m_{j_l}(g_l)$ are linearly dependent. Therefore any non-zero summand must contain a term of order at least one, a term of order at least two, and so on. Then an easy computation using (8) and the Laplace expansion for each determinant gives (6). ☐

Let $I, \mathbf{p}$ be as in Lemma 1 and suppose $\mathbf{f}(p_1), \ldots, \mathbf{f}(p_\mu) \in X(\mathbb{Q}, H)$. Using the bounded heights of $\mathbf{f}(p_j)$ one proves (cf. Lemma 10) that either $\Delta^d(f, \mathbf{p}) = 0$ or

$$|\Delta^d(f, \mathbf{p})| \geqslant H^{-O(d^3)}. \tag{9}$$

Comparing (6) and (9) and recalling that $\mu \sim d^2$ we have either $\Delta^d(f, \mathbf{p}) = 0$ or

$$H^{-O(d^3)} \leqslant |\Delta^d(f, \mathbf{p})| \leqslant 2^{O(d^3)} \delta^{\Omega(d^4)} \tag{10}$$

where we treat M as $O(1)$. Thus if $\delta = H^{-\Omega(1/d)}$ then $\Delta^d(f, \mathbf{p})$ must vanish for any $\mathbf{p}$ as above. Thus as explained above, *all* points $\mathbf{f}(p) \in X(\mathbb{Q}, H)$ with $p \in I$ belong to a single algebraic hypersurface $H_k \subset \mathbb{R}^2$ of degree at most d.

Fix $\varepsilon > 0$ and subdivide I into H^ε subintervals I_k of length $\delta = H^{-\varepsilon}$. Then by the above all points of $X(\mathbb{Q}, H)$ belong to a union of H^ε hypersurfaces $H_k \subset \mathbb{R}^2$ of degree $d = O(1/\varepsilon)$. If X is irreducible and transcendental then it intersects each

H_k properly, and number of intersections between X and H_k is uniformly bounded by some constant $C(X,d)$ depending only on X and d (for instance by Gabrielov's theorem). Thus we have $\#X(\mathbb{Q},H) \leqslant C(X,\varepsilon)H^\varepsilon$.

To handle the case of a general compact irreducible analytic curve $X \subset \mathbb{R}^2$ we note that any such curve can be covered by images of analytic maps $\mathbf{f} : [0,1] \to X$ and the preceding arguments apply.

1.3.2. *Higher dimensions.* It is natural to attempt to generalize the proof of §1.3.1 to sets $X \subset \mathbb{R}^m$ of dimension $\ell > 1$ by induction over ℓ. Namely, the estimates (6) and (9) can be generalized in a relatively straightforward manner, replacing the map $\mathbf{f} : (0,1) \to X$ by an arbitrary analytic map $\mathbf{f} : (0,1)^\ell \to X$ parametrizing an ℓ-dimensional set X. One similarly obtains H^ε hypersurfaces H_k of some fixed degree $d = d(\varepsilon)$ such that

$$X(\mathbb{Q},H) \subset \bigcup_k X \cap H_k. \tag{11}$$

One would then seek to continue treating each intersection $X \cap H_k$ by induction on the dimension. However, at this point a problem arises: even if the original set X was parametrized by an analytic map $\mathbf{f} : (0,1)^\ell \to X$ it is not clear that $X \cap H_k$ could be parametrized in a similar manner. Moreover, if one does obtain a parametrization for each intersection $X \cap H_k$ then the induction constant $C(X \cap H_k, \varepsilon)$ would now depend on the specific parametrizations chosen for $X \cap H_k$, and one must show that these constants are uniformly bounded over all H_k of the given degree d.

In fact, it is not always possible to choose analytic, or even C^∞-smooth, parametrizations for the fibers of a family in a uniform manner – even for semialgebraic families of curves. This fundamental limitation was observed in the work of Yomdin [18]. Consider for example the family of hyperbolas $X_\varepsilon := (-1,1)^2 \cap \{x^2 - y^2 = \varepsilon\}$. If one attempts to write X_ε as a union of images $\operatorname{Im}\phi_j$ for C^∞-smooth functions $\phi_1, \dots, \phi_{N_\varepsilon} : (0,1) \to X_\varepsilon$ with the maximum norms of the derivatives of every order bounded by 1, then N_ε necessarily tends to infinity as $\varepsilon \to 0$. Thus it would not be possible to parametrize all fibers of this family in a uniform manner and apply to them the methods of §1.3.1.

Surprisingly, a theorem due to Yomdin and Gromov [18, 17, 8] states that one can recover the uniformity of N_ε if one replaces the C^∞ condition by C^r-smoothness for a fixed r, at least for semialgebraic families. In [13] Pila and Wilkie generalized this result to the O-minimal setting. Namely, they show [13, Corollary 5.1] that for any set $X \subset (0,1)^m$ of dimension ℓ definable in an O-minimal structure and any $r \in \mathbb{N}$, one can cover X by images of C^r-maps $\phi_1, \dots, \phi_N : (0,1)^\ell \to X$ where $N = N(X,r)$ and every ϕ_j has C^r-norm bounded by 1 in $(0,1)^\ell$. Moreover, [13, Corollary 5.2] if X varies in a definable family (and r is fixed) then N can be taken to be uniformly bounded over the entire family.

One can now check that in the proof sketched in §1.3.1 the analyticity assumption can be replaced by C^r-smoothness (with bounded norms) for sufficiently large $r = r(\varepsilon)$. The intersections $X \cap H_k$ can all be seen as fibers of a single definable family by adding parameters for the coefficients of H_k. One can thus parametrize each intersection $X \cap H_k$ by a uniformly bounded number of C^r maps with unit norms, and the induction step can be carried out as sketched above.

Understanding the behavior of the parametrization complexity $N(X,r)$ in terms of the geometry of the set X and the smoothness order r is a highly non-trivial problem, even in the original context of the Yomdin-Gromov theorem where X is semialgebraic, and certainly in the context of the O-minimal analog. It is this difficulty that prompted us to seek a more direct approach for resolving the problem of uniformity over families.

1.4. **An approach using holomorphic decompositions.** We return to the case of a compact irreducible real-analytic curve $X \subset \mathbb{R}^2$. Let $p \in X$ and consider (X,p) as the germ of a complex-analytic curve. Then by Weierstrass preparation X can be written locally (up to a linear change of coordinate) in the form

$$X = \{h = 0\}, \qquad h(x,y) = y^\nu + a_{\nu-1}(x)y^{d-1} + \cdots + a_0(x) \tag{12}$$

where $a_{\nu-1}, \dots, a_0$ are holomorphic in a neighborhood of p. By Weierstrass division it follows that any F holomorphic in a neighborhood of p can be written in the form

$$F = \sum_{i=0}^{\nu-1} \sum_{j=0}^{\infty} c_{i,j} y^i x^j + Q \tag{13}$$

where Q vanishes identically on (X,p). Moreover, the coefficients $c_{i,j}$ are bounded in terms of the norm of F (cf. Lemma 3). Let $\Delta_p \subset \mathbb{C}^2$ denote a complex polydisc where the decomposition (13) is possible for any holomorphic F. We suppose for simplicity that Δ_p has polyradius 1 (the general case can be treated by rescaling).

The polydiscs Δ_p serve as a replacement for the parametrizations of §1.3.1: we will show that one can construct, in a completely analogous manner, H^ε algebraic hypersurfaces of degree $d = d(\varepsilon)$ containing all points of $(\Delta_p \cap X)(\mathbb{Q}, H)$. Thus, instead of covering X by images of analytic parametrizing maps, we are led to the problem of covering X by such "good neighborhoods" Δ_p.

The key argument is the following analog of Lemma 1. Let f_1, f_2 be two holomorphic functions on the polydisc of radius 2 around p and let their absolute values be bounded by M.

Lemma 2 (cf. Lemma 9)**.** *Let $D \subset \Delta_p$ be a polydisc of polyradius $\delta < 1/2$, and $\mathbf{p} = (p_1, \dots, p_\mu) \in (D \cap X)$. Then*

$$|\Delta^d(\mathbf{f}, \mathbf{p})| \leqslant \mu!(\mu+1)^\mu M^{d\mu} \delta^{\Omega(\mu^2)}. \tag{14}$$

Proof. The proof is essentially the same as that of Lemma 1. We simply replace the Taylor expansion of the function f_1, f_2 by the expansions (13) (and note that Q vanishes on all points of $\mathbf{p}$). In (13) we have at most ν terms of each order k (instead of one term of each order in the case of Taylor expansions), and this only introduces an extra factor depending on ν into the asymptotic $\delta^{\Omega(\mu^2)}$ in (14). □

We now proceed as in §1.3.1 taking $f_1 = x$ and $f_2 = y$. In a similar manner, we can cover $\Delta_p \cap \mathbb{R}^2$ by H^ε polydiscs D_k of polyradius $H^{-\varepsilon/2}$, and for each D_k we find an algebraic hypersurface H_k of degree $d = O(1/\varepsilon)$ such that $(D_k \cap X)(\mathbb{Q}, H) \subset H_k$. Thus we see that $(\Delta_p \cap X)(\mathbb{Q}, H)$ is contained in a union of H^ε algebraic hypersurfaces of degree d. Since X is compact it may be covered by finitely many of the polydiscs Δ_p, and we finally see that $X(\mathbb{Q}, H)$ is contained in a union of $O(H^\varepsilon)$ algebraic hypersurfaces of degree d.

The main advantage of this approach becomes apparent when we consider families of curves. Namely, unlike in the case of analytic parametrizations, the argument above can be made uniform over analytic families. To illustrate this consider again the family of hyperbolas $X_\varepsilon := (-1,1)^2 \cap \{x^2 - y^2 = \varepsilon\}$. The unit polydisc around the origin Δ_0 is a "good neighborhood" in the sense above, *uniformly for every* ε. Indeed, Weierstrass division with respect to $y^2 - x^2 + \varepsilon$ is possible regardless of the value of ε and the norms of the division remain bounded even as $\varepsilon \to 0$. A systematic application of Weierstrass division allows one to generalize this example to an arbitrary family.

The purpose of this paper is to pursue this complex-analytic perspective. In §2 we define the notion of a *decomposition datum* (see Definition 4) generalizing the "good neighborhoods" Δ_p above for complex analytic sets of arbitrary dimension. We then prove in Theorem 3 that one can always cover (a compact piece of) a complex analytic set by finitely many such polydiscs, and that this can be done uniformly over analytic families (with a compact parameter space). In §3 we show that in each such polydisc the rational points of height H can be described in analogy with the Bombieri-Pila method of §1.3.1. In §4 we prove a result analogous to the Pila-Wilkie theorem for complex analytic sets of arbitrary dimension (and their projections) by induction over dimension, in analogy with the Pila-Wilkie method of §1.3.2. Finally in §5 we show that any bounded subanalytic set can be complexified in an appropriate sense, and deduce Theorem 1 from the its complex-analytic version Theorem 4. The key technical tool for this reduction is a quantifier-elimination result of Denef and van den Dries [6].

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2. Uniform decomposition in analytic families

2.1. **Weierstrass division with norm estimates.** If Z is a subset of a complex manifold Ω we denote by $\mathcal{O}(Z)$ the ring of germs of holomorphic functions in a neighborhood of Z. If Z is relatively compact in Ω we denote by $\|\cdot\|_Z$ the maximum norm on $\mathcal{O}(\bar Z)$. We denote by $\mathcal{O}_\Omega$ the structure sheaf of Ω, and if $X \subset \Omega$ is an analytic subset we denote by $\mathcal{I}_X \subset \mathcal{O}_\Omega$ its ideal sheaf and by $\mathcal{I}_{X,p}$ the germ of $\mathcal{I}_X$ at p. Finally for an ideal sheaf $\mathcal{I} \subset \mathcal{O}_\Omega$ we denote by $V(\mathcal{I})$ the analytic set that it defines.

We say that a germ $f \in \mathbb{C}\{z_1,\dots,z_n,w\}$ is regular of order d in w if $f(0,w) = f_1(w)\cdot w^d$ with $f_1(0) \neq 0$. For two polydiscs $\Delta_v \subset \mathbb{C}$ and $\Delta_h \subset \mathbb{C}^n$, we say that $\Delta := \Delta_h \times \Delta_v$ is a *Weierstrass polydisc* for f if $f(z,w)$ has exactly d roots in Δ_v for any fixed $z \in \bar\Delta_h$. In particular, Δ is a Weierstrass polydisc for any sufficiently small Δ_v and sufficiently smaller Δ_h.

Lemma 3. *Let f be regular of order d in w, and $\Delta := \Delta_h \times \Delta_v$ a sufficiently small Weierstrass polydisc for f. Then:*

(1) *The map*
$$\pi : \{f = 0\} \cap \Delta \to \mathbb{C}^n, \qquad \pi(z,w) = z \tag{15}$$
is finite.

(2) *There exists a constant C such that any $g \in \mathcal{O}(\bar\Delta)$ can be decomposed in the form*
$$g = qf + \sum_{k=0}^{d-1} g_j w^j, \qquad g_j = g_j(z) \tag{16}$$

with $\|g_j\|_\Delta, \|q\|_\Delta \leqslant C \cdot \|g\|_\Delta$.

Proof. Since Δ is taken to be sufficiently small we may assume without loss of generality that f is a Weierstrass polynomial of order d in w. Then the first statement is classical and the second is the extended Weierstrass preparation theorem of [9, II.D. Theorem 1]. □

2.2. **Decomposition data.** We denote by $\mathbf{z}$ a fixed system of affine coordinates on $\mathbb{C}^n$. We say that $\mathbf{x}$ is a *standard* coordinate system on $\mathbb{C}^n$ if it is obtained from $\mathbf{z}$ by an affine unitary transformation. Given $\mathbf{x}$, we say that (Δ, Δ') is a pair of polydiscs if $\Delta \subset \Delta'$ are two polydiscs with the same center in the $\mathbf{x}$ coordinates.

For a co-ideal $\mathcal{M} \subset \mathbb{N}^n$ and $k \in \mathbb{N}$ we denote by

$$\mathcal{M}^{\leqslant k} := \{\alpha \in \mathcal{M} : |\alpha| \leqslant k\} \tag{17}$$

and by $H_\mathcal{M}(k) := \#\mathcal{M}^{\leqslant k}$ its Hilbert-Samuel function. The function $H_\mathcal{M}(k)$ is eventually a polynomial in k, and we denote its degree by $\dim \mathcal{M}$.

If (X, p) is the germ of an analytic set in $\mathbb{C}^n$ then there exists a co-ideal $\mathcal{M}$ with $\dim \mathcal{M} = \dim X$ such that every $F \in \mathcal{O}_p$ can be decomposed as

$$F = \sum_{\alpha \in \mathcal{M}} c_\alpha \mathbf{z}^\alpha + Q, \qquad Q \in \mathcal{O}_p \tag{18}$$

where Q vanishes identically on X. For instance one may choose $\mathcal{M}$ to be the complement of the diagram of initial exponents of $\mathcal{I}_{X,p}$, in which case the claim above is a consequence of Hironaka division. The following definition generalizes this notion from the context of germs to the context of a fixed polydisc.

Definition 4. *Let $X \subset \mathbb{C}^n$ be a locally analytic subset, $\mathbf{x}$ a standard coordinate system, (Δ, Δ') a pair of polydiscs centered at the $\mathbf{x}$-origin and $\mathcal{M} \subset \mathbb{N}^n$ a co-ideal. We say that X admits* decomposition *with respect to the* decomposition datum

$$\mathcal{D} := (\mathbf{x}, \Delta, \Delta', \mathcal{M}) \tag{19}$$

if there exists a constant denoted $\|\mathcal{D}\|$ such that for every holomorphic function $F \in \mathcal{O}(\bar{\Delta}')$ there is a decomposition

$$F = \sum_{\alpha \in \mathcal{M}} c_\alpha \mathbf{x}^\alpha + Q, \qquad Q \in \mathcal{O}(\bar{\Delta}) \tag{20}$$

where Q vanishes identically on $X \cap \Delta$ and

$$\|c_\alpha \mathbf{x}^\alpha\|_\Delta \leqslant \|\mathcal{D}\| \cdot \|F\|_{\Delta'} \qquad \forall \alpha \in \mathcal{M}. \tag{21}$$

We define the dimension *of the decomposition datum, denoted* $\dim \mathcal{D}$ *to be* $\dim \mathcal{M}$.

Since $\mathcal{H}_\mathcal{M}(k)$ is eventually a polynomial of degree $\dim \mathcal{M}$, the function $\mathcal{H}_\mathcal{M}(k) - \mathcal{H}_\mathcal{M}(k-1)$ counting monomials of degree k in $\mathcal{M}$ is eventually a polynomial of degree $\dim \mathcal{M} - 1$. If $\dim \mathcal{M} \geqslant 1$ we denote by $e(\mathcal{D})$ the minimal constant satisfying

$$H_\mathcal{M}(k) - H_\mathcal{M}(k-1) \leqslant e(\mathcal{D}) \cdot L(\dim \mathcal{M}, k), \qquad \forall k \in \mathbb{N}. \tag{22}$$

where $L(n, k) := \binom{n+k-1}{n-1}$ denotes the dimension of the space of monomials of degree k in n variables. In the case $\dim \mathcal{D} = 0$ the co-ideal $\mathcal{M}$ is finite and we denote by $e(\mathcal{D})$ its size.

Example 5. *Suppose X admits decomposition with respect to the decomposition datum $\mathcal{D}$, and $\dim \mathcal{D} = 0$. Then $N = \#(X \cap \Delta)$ is finite and satisfies $N \leqslant e(\mathcal{D})$. Indeed, by (20) any polynomial on $\mathbb{C}^n$ can be interpolated on $X \cap \Delta$ by the $e(\mathcal{D})$ monomials of $\mathcal{M}$. Since the linear space of polynomials restricted to $X \cap \Delta$ has dimension N it follows that $N \leqslant e(\mathcal{D})$.*

2.3. **Decomposition data for analytic families.** If $X \subset \mathbb{C}^n$ is a locally analytic subset and $k \in \mathbb{N}$, we denote by $X^{\leqslant k}$ the union of the components of X that have dimension k or less. Note that $X^{\leqslant k}$ is locally analytic as well.

Let $\Omega \subset \mathbb{C}^n$ be an open subset and Λ a complex analytic space. We denote by π_Ω, π_Λ the projections from $\Omega \times \Lambda$ to Ω, Λ respectively. For $X \subset \Omega \times \Lambda$ and $\lambda \in \Lambda$ we denote the λ-fiber of X by

$$X_\lambda \subset \Omega, \qquad X_\lambda := \{p \in \Omega : (p, \lambda) \in X\}. \tag{23}$$

The following theorem is our main result on uniform decomposition in families. It says roughly that if one considers a compact piece of an analytic family X, then each fiber X_λ at every point p admits decomposition with respect to some decomposition datum $\mathcal{D}$ with $\dim \mathcal{D} = \dim X_\lambda$, with the size of the polydisc Δ bounded from below and $\|\mathcal{D}\|, e(\mathcal{D})$ bounded from above *uniformly over the (compact) family.*

Theorem 3. *Let $X \subset \Omega \times \Lambda$ be an analytic subset, $K \Subset \Omega \times \Lambda$ a compact subset and $k \in \mathbb{N}$. There exists a positive radius $r > 0$ and constants $C_D, C_H > 0$ with the following property. For any $(p, \lambda) \in K$ there exists a decomposition datum $\mathcal{D}$ such that:*

1. *$\Delta = \Delta'$ is centered at p, and $B_r(p) \subset \Delta \subset \Omega$.*
2. *$\dim \mathcal{M}_i \leqslant k$, $\|\mathcal{D}\| \leqslant C_D$ and $e(\mathcal{D}) \leqslant C_H$.*
3. *$(X_\lambda)^{\leqslant k}$ admits decomposition with respect to $\mathcal{D}$*

We first consider the problem of constructing decomposition data of dimension k for fibers of a family X, under the assumption that all fibers of X have dimension bounded by k. This basic case essentially reduces to Hironaka division. For completeness we give a proof using Weierstrass division.

Lemma 6. *Let $X \subset \Omega \times \Lambda$ be an analytic subset, $k \in \mathbb{N}$ and suppose $\dim X_\lambda \leqslant k$ for every $\lambda \in \Lambda$. Then for any $p \in \Omega$ and compact $K_\Lambda \Subset \Lambda$ a there exists a finite collection of decomposition data $\{\mathcal{D}_i\}$ such that:*

1. *$\Delta_i = \Delta'_i$ is centered at p and contained in Ω.*
2. *$\dim \mathcal{M}_i \leqslant k$.*
3. *for every $\lambda \in K_\Lambda$ the fiber X_λ admits decomposition with respect to some $\mathcal{D}_i$.*

Proof. Let $\mathbf{z}$ be a standard coordinate system centered at p. We proceed by induction on n. If $n = k$ then the claim holds with any choice of $\mathbf{x}$, $\mathcal{M} = \mathbb{N}^n$ and $\Delta = \Delta'$ any polydisc contained in Ω. The expansion (20) is given by the usual Taylor expansion for F around the origin with $Q \equiv 0$. The inequality (21) is given by the Cauchy estimates.

Suppose $n > k$. By compactness it will suffice to prove the claim in a neighborhood of each $\lambda \in K_\Lambda$. Fix $\lambda_0 \in K_\Lambda$. Since $\dim X_{\lambda_0} < n$ there exists $G \in \mathcal{I}_{X,p}$ such that $G|_{\lambda=\lambda_0} \not\equiv 0$. By a unitary change of the z-coordinates we may suppose that G is regular with respect to z_n, of some order d. Then by Lemma 3 the map

$$\pi_n : V(G) \to \mathbb{C}^{n-1} \times \tilde{\Lambda}, \qquad \pi_n(z_1, \ldots, z_n, \lambda) = (z_1, \ldots, z_{n-1}, \lambda) \tag{24}$$

is finite when restricted to an appropriate polydisc $D = D_z \times D_\lambda$, where $D_z = D_h \times D_v$ and D_h, D_λ are chosen to be sufficiently smaller than D_v. Then $Y := \pi_n(X \cap D)$ is analytic in $D_h \times D_\lambda$ by the proper mapping theorem.

Let $K_{\lambda_0} \subset D_\lambda$ be some compact neighborhood of λ_0. Since $\pi_n : X \to Y$ is finite we have $\dim Y_\lambda \leqslant \dim X_\lambda \leqslant k$ for $\lambda \in K_{\lambda_0}$. Apply the inductive hypothesis with Y for X, K_{λ_0} for K_Λ and D_h for Ω to obtain a finite collection of decomposition data $\{\hat{\mathcal{D}}_i\}$. We let

$$\mathbf{x}_i := (\hat{\mathbf{x}}_i, z_n), \tag{25}$$

$$\Delta_i = \Delta_i' := \hat{\Delta}_i \times D_v, \tag{26}$$

$$\mathcal{M}_i := \hat{\mathcal{M}}_i \times \{0, \ldots, d-1\}. \tag{27}$$

Note that since $\hat{\Delta}_i \subset D_h$ and D_λ are chosen to be sufficiently smaller than D_v, Lemma 3 applies with the polydisc $\Delta_i \times D_\lambda$. Applying the lemma to $F(x, \lambda) \equiv F(x)$ we obtain a decomposition

$$F = \sum_{j=0}^{d-1} z_n^j F_j + QG, \qquad F_j = F_j(z_1, \ldots, z_{n-1}, \lambda) \tag{28}$$

with $\|F_j\|_{\Delta_i \times D_\lambda} = O_{\lambda_0}(\|F\|_{\Delta_i})$.

By construction Y_λ admits decomposition with respect to some $\hat{\mathcal{D}}_i$. Hence we may decompose the functions $F_j(\cdot) \equiv F_j(\cdot, \lambda)$ as

$$F_j = \sum_{\alpha \in \hat{\mathcal{M}}_i} c_{j,\alpha} \hat{\mathbf{x}}_i^\alpha + Q_j, \qquad Q_j \in \mathcal{O}(\overline{\hat{\Delta}_i}) \tag{29}$$

where

(1) $\hat{\mathcal{M}}_i \subset \mathbb{N}^{n-1}$ is a co-ideal and $\dim \hat{\mathcal{M}}_i \leqslant k$.
(2) Q_j vanishes identically on $Y_\lambda \cap \hat{\Delta}_i$.
(3) We have

$$\|c_{j,\alpha} \hat{\mathbf{x}}_i^\alpha\|_{\hat{\Delta}_i} = O_{\lambda_0}(\|F_j\|_{\hat{\Delta}_i}) = O_{\lambda_0}(\|F\|_{\Delta_i}). \tag{30}$$

Plugging (29) into (28) we obtain the decomposition (20). □

To observe the principal limitation of Lemma 6 consider the family $X := \{\lambda_1 x = \lambda_2\} \subset \mathbb{C}_x \times \mathbb{C}^2$. The fiber $X_{(0,0)}$ is one-dimensional while every other fiber is zero-dimensional. We would like to produce decomposition data of dimension zero for the fibers away from the origin, with constants remaining uniformly bounded as we approach the origin. However Lemma 6 only guarantees the existence of decomposition data of dimension one. The following proposition eliminates this limitation, producing for each fiber a decomposition datum of the correct dimension. The idea of the proof is to use blowings-up to avoid the jump in the dimension of the fiber. For instance, the reader may observe that in the preceding example, after blowing up the origin $\{\lambda_1 = \lambda_2 = 0\}$ the strict transform $\tilde{X}$ has only zero-dimensional fibers.

Proposition 7. *Let $X \subset \Omega \times \Lambda$ be an analytic subset and $k \in \mathbb{N}$. Then for any $p \in \Omega$ and compact $K_\Lambda \Subset \Lambda$ a there exists a finite collection of decomposition data $\{\mathcal{D}_i\}$ such that:*

(1) $\Delta_i = \Delta_i'$ *is centered at p and contained in Ω.*
(2) $\dim \mathcal{M}_i \leqslant k$.

(3) *for every $\lambda \in K_\Lambda$ the set $(X_\lambda)^{\leqslant k}$ admits decomposition with respect to some $\mathcal{D}_i$.*

Proof. Let $m := \dim \Lambda$ and $d := \max\{\dim X_\lambda : \lambda \in K_\Lambda\}$. We proceed by induction on (m, d) with the lexicographic order. By compactness it will suffice to prove the claim in a neighborhood of each $\lambda \in K_\Lambda$. Fix $\lambda_0 \in K_\Lambda$. Without loss of generality we may replace X by its germ at (p, λ_0) and Λ, K_Λ by their germs at λ_0. For a sufficiently small germ we have (by semicontinuity of the dimension) $d = \dim X_{\lambda_0}$. If $d \leqslant k$ then the claim follows by Lemma 6, so we assume $d > k$.

We may assume without loss of generality that Λ is smooth. Indeed, otherwise let $\sigma : M \to \Lambda$ be a desingularization [10] of Λ and $\tilde{X} := X \times_\Lambda M$. Note that this does not change the pair (m, d). Every fiber of X is a fiber of $\tilde{X}$, and it suffices to prove the claim for the compact set $\sigma^{-1}(K_\Lambda)$.

We may also assume without loss of generality that $\dim X < m + d$. Indeed, if X has a component X' of dimension $m + d$ then the fibers X'_λ must have pure dimension d so $(X'_\lambda)^{\leqslant k} = \emptyset$. Thus it is enough to prove the claim for the union of the components of X that have dimension strictly smaller than $m + d$.

Since $d = \dim X_{\lambda_0}$ there exists an affine linear projection $\pi_d : \mathbb{C}^n \to \mathbb{C}^d$ such that

$$\pi = \pi_d \times \pi_\Lambda : (X, (p, \lambda_0)) \to (\mathbb{C}^d \times \Lambda, (0, \lambda_0)) \tag{31}$$

is finite and hence $Y = \pi(X)$ is the germ of an analytic subset at $(0, \lambda_0)$. In particular $\dim Y = \dim X < m + d$ so $Y \neq \mathbb{C}^d \times \Lambda$. Then there exists a non-zero $G \in \mathcal{I}_{Y,(0,\lambda_0)}$. Write

$$G = \sum_\alpha c_\alpha(\lambda) w^\alpha, \qquad (w, \lambda) \in \mathbb{C}^d \times \Lambda \tag{32}$$

and let I be the ideal generated by $\{c_\alpha\}$ in $\mathcal{O}_{\Lambda,\lambda_0}$. Then the set $\mathcal{C} := V(I) \subset \Lambda$ is an analytic space of dimension strictly smaller than m, and the claim follows for any $\lambda \in \mathcal{C}$ by induction on m. It remains to construct suitable decomposition data for any $\lambda \notin \mathcal{C}$.

Let $\eta : \tilde{\Lambda} \to \Lambda$ denote the blowing up of I and $X' := X \times_\Lambda \tilde{\Lambda}$. Let

$$\tilde{X} := \mathrm{Clo}[X' \setminus (\mathbb{C}^n \times \eta^{-1}(\mathcal{C}))] \tag{33}$$

be the strict transform of X (where Clo denotes analytic closure). For any $\lambda \in \Lambda \setminus \mathcal{C}$, the fiber X_λ is also a fiber of $\tilde{X}$. Thus it will suffice to prove the claim for the family $\tilde{X}$ and the compact set $\eta^{-1}(K_\Lambda)$. Let $\tilde{\lambda} \in \tilde{\Lambda}$ and we will show that $\dim \tilde{X}_{\tilde{\lambda}} < d$, and the claim thus follows by induction on d.

By definition of the blow-up η, the ideal $I\mathcal{O}_{\tilde{\Lambda},\tilde{\lambda}}$ is principal hence generated by some c_α. Thus we may write $(\mathrm{id} \times \eta)^* G = c_\alpha \tilde{G}$ where $\tilde{G} \in \mathcal{O}_{\mathbb{C}^d \times \tilde{\Lambda},(0,\tilde{\lambda})}$ does not vanish identically on $\mathbb{C}^d \times \{\tilde{\lambda}\}$. Since $I = \langle c_\alpha \rangle$ near $\tilde{\lambda}$ the strict transform satisfies $\tilde{X} \subset V((\pi_d \times \mathrm{id})^* \tilde{G})$ and thus $\pi_d(\tilde{X}_{\tilde{\lambda}}) \subset \mathbb{C}^d \cap \{\tilde{G} = 0\}$. The map $\pi_d|_{\tilde{X}_{\tilde{\lambda}}}$ is finite, being the restriction of a finite map $\pi_d|_{X_\lambda}$ for some $\lambda \in \Lambda$, and we conclude that $\dim \tilde{X}_{\tilde{\lambda}} < d$ as claimed. □

Finally we finish the proof of Theorem 3.

Proof of Theorem 3. By compactness there exists a ball $B \subset \mathbb{C}^n$ such that $p + B \subset \Omega$ for every $(p, \lambda) \in K$. Let $\Lambda' = \Omega \times \Lambda$ and $K_{\Lambda'} = K$. Define

$$X' \subset B \times \Lambda', \qquad X' := \{(q, (p, \lambda)) : q + p \in X_\lambda\}. \tag{34}$$

By definition, $X'_{(p,\lambda)} = (-p) + X_\lambda$ in B. Apply Proposition 7 to $X', K_{\Lambda'}$ with the point $q = 0$ to obtain a finite collection of decomposition data $\{\mathcal{D}_i\}$ with $\dim \mathcal{D}_i \leqslant k$. Let C_D, C_H be the minimum among the corresponding parameters $\|\mathcal{D}_i\|, e(\mathcal{D}_i)$ and choose some $r > 0$ such that $B_r(0) \subset \Delta_i$ for every i.

Now let $(p, \lambda) \in K$. By Proposition 7, $(X'_{(p,\lambda)})^{\leqslant k}$ admits decomposition with respect to some $\mathcal{D}_i$. Define $\mathcal{D}$ as the p-translate of $\mathcal{D}_i$, i.e. $\Delta = p + \Delta_i$ and $\mathbf{x} = p + \mathbf{x}_i$. Then $(X_\lambda)^{\leqslant k} = (p + X'_{(p,\lambda)})^{\leqslant k}$ admits decomposition with respect to $\mathcal{D}$ as claimed. □

3. Interpolation determinants and rational points

Let $A \subset \mathbb{C}^n$ be a ball or polydisc around a point $p \in \mathbb{C}^n$ and $\delta > 0$. We let A^δ denote the δ^{-1}-rescaling of A around p, i.e. $A^\delta := p + \delta^{-1}(A - p)$.

Let $X \subset \mathbb{C}^n$ be an analytic subset and $\mathcal{D}$ a decomposition datum for X, and set $m := \dim \mathcal{M}$. We suppose D is a polydisc in the $\mathbf{x}$ coordinates, centered at p and $D^\delta \subset \Delta$.

3.1. Norm estimates.

Below $\|\cdot\|$ denotes $\|\cdot\|_D$ and $\|\cdot\|_\delta$ denotes $\|\cdot\|_{D^\delta}$. We remark that $\|\mathbf{x}^\alpha\| = \|\mathbf{x}^\alpha\|_\delta \delta^{|\alpha|}$.

Proposition 8. *Let $f \in \mathcal{O}(\bar{\Delta}')$ and denote $M := \|f\|_{\Delta'}$. For every $k \in \mathbb{N}$ we have*

$$f = \sum_{\alpha \in \mathcal{M}^{<k}} m_\alpha(f) + R_k(f) + Q \tag{35}$$

where $Q \in \mathcal{O}(\bar{\Delta})$ vanishes on $X \cap \Delta$ and

$$m_\alpha(f) = c_\alpha \mathbf{x}^\alpha, \qquad R_k(f) = \sum_{\mathcal{M} \ni |\alpha| \geqslant k} c_\alpha \mathbf{x}^\alpha. \tag{36}$$

Moreover,

$$\|m_\alpha(f)\| \leqslant \|\mathcal{D}\| M \delta^{|\alpha|}, \qquad \|R_k(f)\| \leqslant \frac{\|\mathcal{D}\| e(\mathcal{D}) L(m,k)}{(1-\delta)^m} M \delta^k \tag{37}$$

Proof. The decomposition (35) is just (20). Then (21) gives

$$\|m_\alpha(f)\| = \|m_\alpha(f)\|_\delta \delta^{|\alpha|} \leqslant \|m_\alpha(f)\|_\Delta \delta^{|\alpha|} \leqslant \|\mathcal{D}\| M \delta^{|\alpha|} \tag{38}$$

where we used that fact that $D^\delta \subset \Delta$ in the middle inequality. Then

$$\begin{aligned} \|R_k(f)\| \leqslant \sum_{\mathcal{M} \ni |\alpha| \geqslant k} \|\mathcal{D}\| M \delta^{|\alpha|} &\leqslant \|\mathcal{D}\| M e(\mathcal{D}) \sum_{j=0}^\infty L(m, j+k) \delta^{j+k} \\ &\leqslant \|\mathcal{D}\| e(\mathcal{D}) M L(m,k) \delta^k \sum_{j=0}^\infty L(m,j) \delta^j \\ &= \frac{\|\mathcal{D}\| e(\mathcal{D}) L(m,k)}{(1-\delta)^m} M \delta^k. \end{aligned} \tag{39}$$

□

3.2. **Interpolation determinants.** Let $\mathbf{f} := (f_1, \dots, f_\mu)$ be a collection of functions and $\mathbf{p} := (p_1, \dots, p_\mu)$ a collection of points. We define the *interpolation determinant*

$$\Delta(\mathbf{f}, \mathbf{p}) := \det(f_i(p_j))_{1 \leqslant i,j \leqslant \mu}. \tag{40}$$

In the asymptotic notations $\sim_m, O_m, \Omega_m$ below we use the subscript m to indicate that the implied constants depend only on m. The following lemma and its proof are direct analogs of the interpolation determinant estimates of [2]. We remark that in this paper we will not make explicit use of the estimates for the constants C, E in terms of $\|\mathcal{D}\|, e(\mathcal{D})$.

Lemma 9. *Assume $m > 0$. Suppose $f_i \in \mathcal{O}(\bar{\Delta}')$ with $\|f_i\|_{\Delta'} \leqslant M$ and $p_i \in D \cap X$ for $i = 1, \dots, \mu$. Assume $\delta < 1/2$. Then*

$$|\Delta(\mathbf{f}, \mathbf{p})| \leqslant (C\mu^3 M)^\mu \cdot \delta^{E \cdot \mu^{1+1/m}} \tag{41}$$

where

$$C = O_m(\|\mathcal{D}\| \, e(\mathcal{D})^{1/m}), \tag{42}$$

$$E = \Omega_m(e(\mathcal{D})^{-1/m}). \tag{43}$$

Proof. We set

$$k := \max\{j : \sum_{l=0}^{j} e(\mathcal{D}) L(m, l) < \mu\}. \tag{44}$$

Since $\sum_{l=0}^{j} L(m,l) = L(m+1, j)$ is a polynomial of degree m we have $k \sim_m (\mu / e(\mathcal{D}))^{1/m}$.

We consider the expansions (35) for each f_i with k as above,

$$f_i = \sum_{\alpha \in \mathcal{M}^{<k}} m_\alpha(f_i) + R_k(f_i) + Q_i. \tag{45}$$

We note that Q_i vanishes identically on $X \cap \Delta$ and in particular at every p_j. By definition of k, the number of remaining terms in (45) does not exceed μ. We expand $\Delta(\mathbf{f}, \mathbf{p})$ by linearity with respect to each column. We thus obtain a sum of at most μ^μ interpolation determinants Δ_I where each f_i is replaced by either a monomial term $m_\alpha(f_i)$ or a residue term $R_k(f_i)$. By (37) we have for $i = 1, \dots, \mu$ and for every $\alpha \in \mathcal{M}^{<k}$

$$\begin{aligned} \|m_\alpha(f_i)\| &\leqslant C_0 \delta^{|\alpha|} \\ \|R_k(f_i)\| &\leqslant C_0 \delta^k \end{aligned} \qquad \text{where } C_0 := \frac{\|\mathcal{D}\| \, e(\mathcal{D}) L(m,k)}{(1-\delta)^m} M \tag{46}$$

We remark that these are estimates for the maximum norm in D, and in particular they bound the absolute value of $m_\alpha(f_i), R_k(f_i)$ at every point p_j.

Note that if the same index α is repeated in two different columns of Δ_I then these columns are linearly dependent and $\Delta_I \equiv 0$. Thus for every non-zero Δ_I we can have at most

$$H_{\mathcal{M}}(j) - H_{\mathcal{M}}(j-1) \leqslant e(\mathcal{D}) L(m, j) \tag{47}$$

monomial terms of order $|\alpha| = j$. We now expand Δ_I by the Laplace expansion. By definition of k and by (46) we conclude that for each Δ_I we have

$$|\Delta_I| \leqslant \mu! C_0^\mu \delta^S, \qquad S(m,k) := \sum_{l=0}^{k} e(\mathcal{D}) L(m,l) \cdot l. \tag{48}$$

Since $L(m,l)\cdot l$ is a polynomial of degree m in l, we conclude that $S(m,k)\sim_m e(\mathcal{D})k^{m+1}$.

Plugging in $k\sim_m (\mu/e(\mathcal{D}))^{1/m}$ we have

$$C_0 = O_m(\|\mathcal{D}\|\, e(\mathcal{D})^{1/m}\mu^{1-1/m}M), \tag{49}$$

$$S(m,k)\sim_m e(\mathcal{D})^{-1/m}\mu^{1+1/m}. \tag{50}$$

Summing over the (at most) μ^μ determinants Δ_I we obtain (41). □

3.3. **Polynomial interpolation determinants.** Let $d\in\mathbb{N}$ and fix an integer $m<\ell\leqslant n$. Let μ denote the dimension of the space of polynomials of degree at most d in ℓ variables, $\mu=L(\ell+1,d)$. Let $\mathbf{f}:=(f_1,\dots,f_\ell)$ be a collection of functions and $\mathbf{p}:=(p_1,\dots,p_\mu)$ a collection of points. We define the *polynomial interpolation determinant* of degree d to be

$$\Delta^d(\mathbf{f},\mathbf{p}) := \Delta(\mathbf{g},\mathbf{p}), \qquad \mathbf{g}=(\mathbf{f}^\alpha : \alpha\in\mathbb{N}^\ell, |\alpha|\leqslant d). \tag{51}$$

Note that $\Delta^d(\mathbf{f},\mathbf{p})=0$ if and only if there exists a polynomial of degree at most d in ℓ variables vanishing at the points $\mathbf{f}(p_1),\dots,\mathbf{f}(p_\mu)$.

Lemma 10. *Let $H\in\mathbb{N}$ and suppose that*

$$H(f_i(p_j))\leqslant H \qquad \begin{matrix} i=1,\dots,\ell \\ j=1,\dots,\mu. \end{matrix} \tag{52}$$

Then $\Delta^d(\mathbf{f},\mathbf{p})$ either vanishes or satisfies

$$\left|\Delta^d(\mathbf{f},\mathbf{p})\right| \geqslant H^{-\ell\mu d}. \tag{53}$$

Proof. Let $Q_{i,j}$ denote the denominator of $f_i(p_j)$ for $i=1,\dots,\ell$ and $j=1,\dots,\mu$. By assumption $Q_{i,j}\leqslant H$. The row corresponding to p_j in $\Delta^d(\mathbf{f},\mathbf{p})$ consists of rational numbers with common denominator dividing $Q_j:=\prod_i Q_{i,j}^d$. Factoring out Q_j from each row we obtain a matrix with integer entries, whose determinant is either vanishing or at least one in absolute value. In the non-vanishing case we have

$$\left|\Delta^d(\mathbf{f},\mathbf{p})\right| \geqslant \prod_{j=1}^{\mu} Q_j^{-1} \geqslant H^{-\ell\mu d}. \tag{54}$$

□

Comparing Lemmas 9 and 10 we obtain the following.

Proposition 11. *Let $M,H\geqslant 2$, and suppose $f_i\in\mathcal{O}(\bar\Delta')$ with $\|f_i\|_{\Delta'}\leqslant M$. Assume $\delta<1/2$. Let*

$$Y=\mathbf{f}(X\cap D)\subset\mathbb{C}^\ell. \tag{55}$$

There exist a constant $C_1>0$ depending only on ℓ such that if

$$-\log\delta > C_1\frac{d^{-1}\log(\|\mathcal{D}\|\, e(\mathcal{D}))+\log M+\log H}{(d^{\ell-m}/e(\mathcal{D}))^{1/m}} \tag{56}$$

then $Y(\mathbb{Q},H)$ is contained in an algebraic hypersurface of degree at most d in $\mathbb{C}^\ell$. The same conclusion holds for $m=0$ if instead of (56) we assume $d\geqslant e(\mathcal{D})$.

Proof. We consider first the case $m = 0$. In this case according to Example 5 the number of points in $X \cap \Delta$ is bounded by $e(\mathcal{D})$. In particular this bounds the number of points in Y, and all the more in $Y(\mathbb{Q}, H)$. Thus the claim holds with any $d \geqslant e(\mathcal{D})$.

Now assume $m > 0$ and suppose toward contradiction that $Y(\mathbb{Q}, H)$ is not contained in an algebraic hypersurface of degree at most d in $\mathbb{C}^\ell$. Then by standard linear algebra it follows that there exist $\mathbf{p} = (p_1, \dots, p_\mu)$ with $p_1, \dots, p_\mu \in X \cap D$ such that $\{\mathbf{f}(p_j) : j = 1, \dots, \mu\}$ is a subset of $Y(\mathbb{Q}, H)$ and does not lie on the zero locus of a non-zero polynomial of degree d. Then $\left|\Delta^d(\mathbf{f}, \mathbf{p})\right| \neq 0$, and from Lemmas 10 and 9 we have

$$H^{-\ell\mu d} \leqslant \left|\Delta^d(\mathbf{f}, \mathbf{p})\right| \leqslant (C\mu^3 M^d)^\mu \cdot \delta^{E\cdot\mu^{1+1/m}}. \tag{57}$$

Takings logs and using $\mu \sim_\ell d^\ell$ we have

$$\begin{aligned}(d^\ell/e(\mathcal{D}))^{1/m} \log\delta \geqslant \\ -\Omega_\ell\big(\log\|\mathcal{D}\| + m^{-1}\log e(\mathcal{D}) + 3\ell\log d + d\log M + \ell d\log H\big).\end{aligned} \tag{58}$$

Noting that

$$\frac{\log d}{d^{\ell/m}} = O_\ell(1) \tag{59}$$

and collecting all asymptotic constants into C_1 we arrive to contradiction with (56). □

Corollary 12. *Let $\ell = m + 1$ and suppose $f_i \in \mathcal{O}(\bar\Delta')$ with $\|f_i\|_{\Delta'} \leqslant M$. Let*

$$Y = \mathbf{f}(X \cap D) \subset \mathbb{C}^{m+1}. \tag{60}$$

For every $\varepsilon > 0$ there exist two positive constants

$$d = d(m, \varepsilon, e(\mathcal{D})) \tag{61}$$

$$C = C(m, \varepsilon, e(\mathcal{D}), \|\mathcal{D}\|, M) \tag{62}$$

such that if $\delta \leqslant CH^{-\varepsilon}$ then $Y(\mathbb{Q}, H)$ is contained in an algebraic hypersurface of degree at most d in $\mathbb{C}^{m+1}$.

Proof. By Proposition 11 for $m > 0$ it is enough to choose

$$\varepsilon > \frac{C_1}{(d/e(\mathcal{D}))^{1/m}}, \tag{63}$$

$$-\log C > \frac{C_1 d^{-1}\log(\|\mathcal{D}\|\, e(\mathcal{D}) M)}{(d/e(\mathcal{D}))^{1/m}}, \tag{64}$$

and for $m = 0$ it is enough to choose $d = e(\mathcal{D})$ and e.g. $C = 1$. □

Pursuing a different asymptotic direction for the parameters we also obtain the following corollary.

Corollary 13. *Suppose $m > 0$ and $f_i \in \mathcal{O}(\bar\Delta')$ with $\|f_i\|_{\Delta'} \leqslant M$. Let*

$$Y = \mathbf{f}(X \cap D) \subset \mathbb{C}^\ell. \tag{65}$$

There exists a constant

$$C = C(\ell, e(\mathcal{D}), \|\mathcal{D}\|, M) \tag{66}$$

such that if $\delta < C$ then there exists an algebraic hypersurface of degree at most $(\log H)^{m/(\ell-m)}$ in $\mathbb{C}^\ell$ containing $Y(\mathbb{Q}, H)$.

Proof. Choosing $d = (\log H)^{m/(\ell-m)}$ in Proposition 11 we see that the right hand side of (56) is majorated by a constant depending only on $\ell, e(\mathcal{D}), \|\mathcal{D}\|, M$. $\square$

We are now ready to finish the proof of Theorem 2.

Proof for Theorem 2. By Theorem 3 applied to the family K and the dimension k, there exists a positive radius $r > 0$ and constants $C_D, C_H > 0$ with the following property. For any $v = (x, y) \in K$ there exists a decomposition datum $\mathcal{D}_v$ such that:

(1) $\Delta_v = \Delta'_v$ is centered at x, and $B_r(x) \subset \Delta_v$.
(2) $\dim \mathcal{M}_i \leqslant k$, $\|\mathcal{D}\|_v \leqslant C_D$ and $e(\mathcal{D}_v) \leqslant C_H$.
(3) X_y admits decomposition with respect to $\mathcal{D}$

Set $D_v := \Delta_v^{2/C}$, where C is the constant given in Corollary 13 with $\ell = m$ and $\mathbf{f}$ given by the standard coordinates on $\mathbb{C}^m$ (note that C depends on C_D, C_H but not on v). Then by Corollary 13 we see that $(X_y \cap D_v)(\mathbb{Q}, H)$ is contained in an algebraic hypersurface of degree $(\log H)^{k/(m-k)}$. It now remains to cover K by some finite number $C(K)$ of the discs D_v and apply the above to each of them. $\square$

4. Exploring rational points in admissible projections

We begin with a definition.

Definition 14. *Let $X \subset \mathbb{C}^m$ and $W \subset \mathbb{C}^m$ be two sets. We define*

$$X(W) := \{w \in W : W_w \subset X\} \tag{67}$$

to be the set of points of W such that X contains the germ of W around w, i.e. such that w has a neighborhood $U_w \subset \mathbb{C}^m$ such that $W \cap U_w \subset X$.

If $A \subset \mathbb{C}^n$ we denote by $A_{\mathbb{R}} := A \cap \mathbb{R}^n$. We remark that

$$(A(W))_{\mathbb{R}} \subset (A_{\mathbb{R}})(W_{\mathbb{R}}). \tag{68}$$

We will consider Definition 14 in two cases: for $X \subset \mathbb{C}^m$ locally analytic and $W \subset \mathbb{C}^m$ an algebraic variety, and for $X \subset \mathbb{R}^m$ subanalytic and $W \subset \mathbb{R}^m$ a semi-algebraic set.

Our principal motivation for Definition 14 is the following direct consequence (cf. Theorem 6).

Lemma 15. *Let $S \subset \mathbb{R}^m$ be a connected positive-dimensional semialgebraic set and $A \subset \mathbb{R}^m$. Then $A(S) \subset A^{\mathrm{alg}}$.*

We record some simple consequences.

Lemma 16. *Let $A, B, W \subset \mathbb{C}^m$. Then*

$$A(W) \cup B(W) \subset (A \cup B)(W). \tag{69}$$

If $A \subset B$ is relatively open then

$$B(W) \cap A = A(W). \tag{70}$$

4.1. **Projections from admissible graphs.** Let $\Omega_z \subset \mathbb{C}^m, \Omega_w \subset \mathbb{C}^n$ be domains and set $\Omega := \Omega_z \times \Omega_w \subset \mathbb{C}^{m+n}$. Let Λ be an analytic space. We denote by $\pi_z, \pi_w, \pi_\Lambda$ the projections from $\Omega \times \Lambda$ to $\Omega_z, \Omega_w, \Lambda$ respectively. We denote by $\pi : \Omega \times \Lambda \to \Omega_z \times \Lambda$ the projection $\pi = \pi_z \times \pi_\Lambda$.

Let $U \subset \Omega_z \times \Lambda$ be an open subset and $\psi : U \to \Omega_w$ a function, and denote its graph by

$$\Gamma_\psi := \{(z, w, \lambda) \in \Omega \times \Lambda : \psi(z, \lambda) = w\}. \tag{71}$$

We denote by $\tilde{\psi} : U \to \Gamma_\psi$ the map $(z, \lambda) \to (z, \psi(z, \lambda), \lambda)$.

Definition 17. *We say that $\psi : U \to \Omega_w$ is* admissible *if $\Gamma = \Gamma_\psi$ is relatively compact in $\Omega \times \Lambda$, and if there exists an analytic subset $X_\Gamma \subset \Omega \times \Lambda$ which agrees with Γ over U, i.e. $X_\Gamma \cap \pi^{-1}(U) = \Gamma$.*

4.2. **Rational points on admissible projections.** For the remainder of this section we fix an admissible $\phi : U \to \Omega_w$. Our main result in this section is the following theorem.

Theorem 4. *Let $X \subset \Omega \times \Lambda$ be an analytic family. Set*

$$Y := \pi(X \cap \Gamma) \subset \Omega_z \times \Lambda. \tag{72}$$

Let $\varepsilon > 0$. There exist constants $d = d(\Gamma, \varepsilon)$ and $N = N(\Gamma, \varepsilon)$ with the following property. For any $\lambda \in \Lambda$ and any $H \in \mathbb{N}$ there exist at most NH^ε many irreducible algebraic varieties $V_\alpha \subset \mathbb{C}^m$ with $\deg V_\alpha \leqslant d$ such that

$$Y_\lambda(\mathbb{Q}, H) \subset \bigcup_\alpha Y_\lambda(V_\alpha). \tag{73}$$

We begin the proof of Theorem 4 with the following proposition.

Proposition 18. *Let $X \subset \Omega \times \Lambda$ be an analytic family and set*

$$Y := \pi(X \cap \Gamma) \subset \Omega_z \times \Lambda. \tag{74}$$

Let $W \subset \mathbb{C}^m$ be an irreducible algebraic variety.

Let $\varepsilon > 0$. There exist constants $d = d(\Gamma, \varepsilon, \deg W)$ and $N = N(\Gamma, \varepsilon, \deg W)$ with the following property. For any $\lambda \in \Lambda$ and any $H \in \mathbb{N}$ there exist NH^ε hypersurfaces $H_\alpha \subset \mathbb{C}^m$ with $\deg H_\alpha \leqslant d$ such that $W \not\subset H_\alpha$ and

$$(Y_\lambda \cap W)(\mathbb{Q}, H) \subset Y_\lambda(W) \cup \bigcup_\alpha H_\alpha. \tag{75}$$

Proof. Since the statement involves only the intersection $X \cap \Gamma$ we may without loss of generality replace X by $X \cap X_\Gamma$, and assume that $X \cap \pi^{-1}(U) \subset \Gamma$.

Set $k := \dim W$. Let $\mathcal{C}$ denote the projective (hence compact) Chow variety (see [7, Chapter 4]) parametrizing all effective algebraic cycles of dimension k and degree equal to $\deg W$. We denote by $R_V \in \mathcal{C}$ the point corresponding to a cycle V. Then the following family is analytic,

$$X' \subset \Omega \times (\Lambda \times \mathcal{C}), \qquad X' = \{(z, w, \lambda, R_V) : (z, w, \lambda) \in X, z \in \operatorname{supp} V\}. \tag{76}$$

Clearly

$$X'_{(\lambda, \mathcal{R}_W)} = X_\lambda \cap (W \times \Omega_w). \tag{77}$$

We apply Theorem 3 to X' with the compact set $\bar{\Gamma} \times \mathcal{C}$ and consider the conclusion for points of the form (p, λ, R_W) for $(p, \lambda) \in \bar{\Gamma}$. We conclude that there exist $r, C_D, C_H > 0$ depending only on $\Gamma, \deg W$ such that for any $(p, \lambda) \in \bar{\Gamma}$ there exists a decomposition datum $\mathcal{D}$ satisfying

(1) $\Delta = \Delta'$ is centered at p, and $B_r(p) \subset \Delta \subset \Omega$.
(2) $\dim \mathcal{M}_i < k$, $\|\mathcal{D}\| \leqslant C_D$ and $e(\mathcal{D}) \leqslant C_H$.
(3) The set

$$Z_\lambda = (X_\lambda \cap (W \times \Omega_w))^{<k}. \tag{78}$$

admits decomposition with respect to $\mathcal{D}$.

Fix $\lambda \in \Lambda$, let $q \in Y_\lambda \cap W$ and suppose $q \notin \operatorname{Sing} W$ and $q \notin Y_\lambda(W)$. Then the germ of W at q is smooth k-dimensional and not contained in Y_λ. Equivalently its image $\tilde\psi(W \times \{\lambda\}) \subset \Gamma$ is the germ of a smooth k-dimensional analytic set at $\tilde\psi(q,\lambda)$ which is not contained in X_λ. Since we assume $X \subset \Gamma$ in a neighborhood of $\tilde\psi(q,\lambda)$ we conclude that the dimension of

$$X_\lambda \cap (W \times \Omega_w) = X_\lambda \cap \Gamma_\lambda \cap (W \times \Omega_w) = X_\lambda \cap \tilde\psi(W \times \{\lambda\}) \tag{79}$$

at $\tilde\psi(q,\lambda)$ is strictly smaller than k, i.e. $(q, \psi(q,\lambda)) \in Z_\lambda$ and thus $q \in \pi(Z_\lambda)$. In conclusion,

$$Y_\lambda \cap W \subset Y_\lambda(W) \cup \operatorname{Sing} W \cup \pi(Z_\lambda). \tag{80}$$

Fix a hypersurface $H_0 \subset \mathbb{C}^m$ containing $\operatorname{Sing} W$ and not containing W. It is clear that one can choose H_0 of some degree d_0 depending only on $\deg W$.

Since $\dim W = k$, one can choose a subset of k coordinates on $\mathbb{C}^m$, say $\mathbf{f} = (z_1, \dots, z_k)$, such that $\mathbf{f} : W \to \mathbb{C}^k$ is dominant. In particular, no non-zero polynomial in the coordinates $\mathbf{f}$ vanishes on W. Since $\bar\Gamma$ is compact, the coordinates $\mathbf{f}$ are certainly bounded (in absolute value) in the r-neighborhood of $\bar\Gamma$ by some number M. Fix some $\varepsilon' > 0$ whose value will be determined later, and let

$$d = d(m, \varepsilon', C_H) \tag{81}$$

$$C = C(m, \varepsilon', C_H, C_D, M) \tag{82}$$

be the two constants of Corollary 12.

Let $\delta = CH^{-\varepsilon'}$. Let $p \in (\bar\Gamma)_\lambda$ and denote $D_p = D_{\delta r}(p) \subset \Omega$. We apply Corollary 12 to $\mathbf{f}$ and the set Z_λ and conclude that there exists a polynomial $P_p(\mathbf{f})$ of degree at most d such that

$$(\pi_z(D_p \cap Z_\lambda))(\mathbb{Q}, H) \subset \{P_p = 0\}. \tag{83}$$

We let $H_p := \{P_p = 0\} \subset \mathbb{C}^m$.

Finally it remains to cover the compact set $(\bar\Gamma)_\lambda$ by the polydiscs $\{D_p : p \in S\}$ for some finite set $S \subset (\bar\Gamma)_\lambda$ and take

$$\{H_\alpha\} = \{H_0\} \cup \{H_p : p \in S\}. \tag{84}$$

Then (80) and the choice of H_0, H_p gives

$$(Y_\lambda \cap W)(\mathbb{Q}, H) \subset Y_\lambda(W) \cup \bigcup_\alpha H_\alpha. \tag{85}$$

as claimed.

Since each D_p has radius at least $CrH^{-\varepsilon'}$ and $\bar\Gamma$ is compact, it is easy to see that one can choose a covering of size at most $NH^{\varepsilon'(n+m)}$, where $N = N(Cr, \Gamma, \varepsilon')$. Finally taking $\varepsilon' = \varepsilon/(n+m)$ we obtain the statement of the proposition. □

The following Lemma gives an inductive proof of Theorem 4, which is obtained for the case $W = \mathbb{C}^m$.

Lemma 19. *Let $X \subset \Omega \times \Lambda$ be an analytic family and set*

$$Y := \pi(X \cap \Gamma) \subset \Omega_z \times \Lambda. \tag{86}$$

Let $W \subset \mathbb{C}^n$ be an irreducible algebraic variety.

Let $\varepsilon > 0$. There exist constants $d = d(\Gamma, \varepsilon, \deg W)$ and $N = N(\Gamma, \varepsilon, \deg W)$ with the following property. For any $\lambda \in \Lambda$ and any $H \in \mathbb{N}$ there exist at most NH^ε many irreducible algebraic varieties $V_\alpha \subset \mathbb{C}^m$ with $\deg V_\alpha \leqslant d$ such that

$$(Y_\lambda \cap W)(\mathbb{Q}, H) \subset \bigcup_\alpha Y_\lambda(V_\alpha). \tag{87}$$

Proof. We proceed by induction on $\dim W$. Apply Proposition 18 to obtain a family of at most $N'H^{\varepsilon/2}$ hypersurfaces $\{H_{\alpha'}\} \subset \mathbb{C}^m$ of degree d'. Let $\{W_\alpha\}$ denote the union over α' of the sets of irreducible components of $W \cap H_{\alpha'}$. Then

$$\#\{W_\alpha\} \leqslant d'N'H^{\varepsilon/2}, \qquad \dim W_\alpha = \dim W - 1, \qquad \deg W_\alpha \leqslant d' \cdot \deg W \tag{88}$$

and

$$(Y_\lambda \cap W)(\mathbb{Q}, H) \subset Y_\lambda(W) \cup \bigcup_\alpha W_\alpha. \tag{89}$$

Apply the inductive hypothesis to each W_α to obtain collections $W_{\alpha,\beta}$, of size at most $N''H^{\varepsilon/2}$ for each α, such that

$$(Y_\lambda \cap W_\alpha)(\mathbb{Q}, H) \subset \bigcup_\beta W_{\alpha,\beta}. \tag{90}$$

Finally we take $\{V_\alpha\}$ to be the union of the sets $\{W\}$ and $\{W_{\alpha,\beta}\}$. The size of $\{V_\alpha\}$ is bounded by $1 + d'N'N''H^\varepsilon$ as claimed and (87) is satisfied by (89) and (90). □

5. Subanalytic sets and L^D_{an}

Let $I = [-1, 1]$. For $m \geqslant 0$ we let $\mathbb{R}\{X_1, \dots, X_m\}$ denote the ring of power series converging in a neighborhood of I^m. To each $f \in \mathbb{R}\{X_1, \dots, X_m\}$ we naturally associate the map $f : I^m \to \mathbb{R}$.

We recall the language L^D_{an} of [6]. The language includes a countable set of variables $\{X_1, X_2, \dots\}$, a relation symbol $<$, a binary operation symbol D, and an m-ary operation symbol f for every $f \in \mathbb{R}\{X_1, \dots, X_m\}$ satisfying $f(I^m) \subset I$. We view I as an L^D_{an}-structure by interpreting $<$ and f in the obvious way and interpreting D as *restricted division*, namely

$$D(x, y) = \begin{cases} x/y & |x| \leqslant |y| \text{ and } y \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{91}$$

We denote by L_{an} the language obtained from L^D_{an} by omitting D.

For every L^D_{an}-term $t(X_1, \dots, X_m)$ we have an associated map $t : I^m \to I$ which we denote $x \to t(x)$. If t is an L_{an}-term then this map is real analytic in I^m.

For an L^D_{an}-formula $\phi(X_1, \dots, X_m)$ we write $\phi(I^m)$ for the set of points $x \in I^m$ satisfying ϕ. If $A \subset I^m$ we write $\phi(A) := \phi(I^m) \cap A$. We will use the following key result of [6].

Theorem 5. *I has elimination of quantifiers in L^D_{an}. As a consequence, a set $A \subset I^m$ is subanalytic in $\mathbb{R}^m$ if an only if it is defined by a quantifier-free formula ϕ of L^D_{an}.*

5.1. **Admissible formulas.** Let $U \subset \mathring{I}^m$ be an open subset. We define the notion of an L^D_{an}-term *admissible* in U by recursion as follows: a variable X_j is always admissible in U; a term $f(t_1, \dots, t_m)$ is admissible in U if and only if the terms $t_1, \dots, t_m$ are admissible in U; and a term $D(t_1, t_2)$ is admissible in U if t_1, t_2 are admissible in U and if

$$|t_1(x)| \leqslant |t_2(x)| \text{ and } t_2(x) \neq 0 \tag{92}$$

for every $x \in U$. An easy induction gives the following.

Lemma 20. *If t is admissible in U then the map $t : U \to I$ is real analytic.*

We will say that an L^D_{an}-formula ϕ is admissible in U if all terms appearing in ϕ are admissible in U. Here and below, when speaking about "terms appearing in ϕ" we consider not only the top-level terms appearing in the relations, but also every sub-term appearing in the construction tree of each term. The following proposition shows that when considering definable subsets of I one can essentially reduce to admissible formulas.

Proposition 21. *Let $U \subset \mathring{I}^m$ be an open subset and $\phi(X_1, \dots, X_m)$ a quantifier-free L^D_{an}-formula. There exist open subsets $U_1, \dots, U_k \subset U$ and quantifier-free L^D_{an}-formulas $\phi_1, \dots, \phi_k$ such that ϕ_j is admissible in U_j and*

$$\phi(U) = \bigcup_{j=1}^{k} \phi_j(U_j). \tag{93}$$

Proof. We prove the claim by induction on the number N of U-inadmissible terms in ϕ. Clearly if this number is zero we are done. Otherwise let t be some minimal U-inadmissible term in ϕ, i.e. such that all sub-terms appearing in t are U-admissible. Express ϕ in the form

$$\phi(X_1, \dots, X_m) = \phi'(X_1, \dots, X_m, t) \tag{94}$$

for a quantifier-free L^D_{an}-formula ϕ' of $m+1$ free variables. By the definition of admissibility it is clear that $t = D(t_1, t_2)$, and by minimality t_1, t_2 are admissible in U. We let

$$\phi_1 \equiv \phi \qquad\qquad U_1 = \{x \in U : |t_1(x)| < |t_2(x)|\} \tag{95}$$

and

$$\phi_2 = (|t_1| = |t_2|) \wedge (t_2 \neq 0) \wedge \phi'(X_1, \dots, X_m, 1) \tag{96}$$

$$\phi_3 = (|t_1| > |t_2| \vee (t_2 = 0)) \wedge \phi'(X_1, \dots, X_m, 0). \tag{97}$$

with $U_2 = U_3 = U$. Note that for readability we use the absolute value as a shorthand above, but it is clear that the relations can be expressed in terms of the unary minus operation corresponding to the function $- : I \to I, x \to (-x)$.

Since t_1, t_2 are admissible in U they define real analytic (in particular continuous) functions there, and the relation defining U_1 is indeed open. Moreover in U_1 the term t is admissible by definition and hence the number of U_1-inadmissible terms in ϕ_1 is strictly smaller than N. Similarly, the new relations introduced in ϕ_2 (resp. ϕ_3) are admissible in U, and since the inadmissible t is replaced by the admissible term 1 (resp. 0) the number of U_2 (resp. U_3) inadmissible terms is strictly smaller than N. It is an easy exercise to check that

$$\phi(U) = \bigcup_{j=1}^{3} \phi_j(U_j). \tag{98}$$

The proof is now concluded by applying the inductive hypothesis to each pair ϕ_j, U_j for $j = 1, 2, 3$. ☐

5.2. **Basic formulas and equations.** We say that ϕ is a *basic D-formula* if it has the form

$$\big(\wedge_{j=1}^{k} t_j(X_1,\ldots,X_m) = 0\big) \wedge \big(\wedge_{j=1}^{k'} s_j(X_1,\ldots,X_m) > 0\big) \tag{99}$$

where t_j, s_j are L_{an}^D-terms. It is easy to check the following.

Lemma 22. *Every quantifier free L_{an}^D-formula ϕ is equivalent in the structure I to a finite disjunction of basic formulas. If ϕ is U-admissible then so are the basic formulas in the disjunction.*

We say that ϕ is a *basic D-equation* if $k' = 0$, i.e. if it involves only equalities. If ϕ is a basic D-formula we denote by $\tilde\phi$ the basic D-equation obtained by removing all inequalities.

Let ϕ be a U-admissible basic D-formula for some $U \subset I^m$. Then $\tilde\phi$ is U-admissible as well. Moreover since all the terms s_j evaluate to continuous functions in U the strict inequalities of ϕ are open in U and we have the following.

Lemma 23. *Suppose ϕ is U-admissible basic D-formula. Then $\phi(U)$ is relatively open in $\tilde\phi(U)$.*

The set defined by an admissible D-equation can be described in terms of admissible projections in the sense of §4.1.

Proposition 24. *Let $U \subset \mathring{I}^{m+l}$ and ϕ be a U-admissible D-equation,*

$$\phi = (t_1 = 0) \wedge \cdots \wedge (t_k = 0). \tag{100}$$

In the notations of §4.1, there exist

1. *Complex domains*
$$\Omega_z \subset \mathbb{C}^m, \qquad \Omega_w \subset \mathbb{C}^N, \qquad \Lambda \subset \mathbb{C}^l \tag{101}$$
with $N \in \mathbb{N}$ and $I^{m+l} \subset \Omega_z \times \Lambda$.
2. *An open complex neighborhood $U \subset U_{\mathbb{C}} \subset \Omega_z \times \Lambda$.*
3. *An analytic map $\psi : U_{\mathbb{C}} \to \Omega_w$.*
4. *An analytic set $X \subset \Omega \times \Lambda$.*

such that ψ is admissible and $Y := \pi(X \cap \Gamma_\psi)$ satisfies $Y_{\mathbb{R}} = \phi(U)$.

Proof. Let $\{s_1,\ldots,s_N\}$ denote all terms of the type $s_j = D(s_{j,1}, s_{j,2})$ appearing in ϕ. As a notational convenience we write $X = X_1,\ldots,X_{m+n}$ and $W = W_1,\ldots,W_N$ for variables on $\mathbb{R}^{n+m}$ and $\mathbb{R}^N$.

For every term $t(X)$ appearing in ϕ we define an L_{an} term $t'(X,W)$ by recursion as follows: if t is a variable then $t' := t$; if $t = f(t_1,\ldots,t_k)$ then $t' = f(t'_1,\ldots,t'_k)$; finally if $t = s_j$ then $t' = W_j$. As L_{an}-terms, every term t' corresponds to a real-analytic function $t' : I^{m+N+n} \to I$. We let $\Omega_z \times \Omega_w \times \Lambda$ denote some complex neighborhood of I^{m+N+n} to which every term t' admits analytic continuation.

By Lemma 20, all terms appearing in ϕ evaluate to real analytic maps from U to I. We define a map $\psi : U \to \Omega_w$ by

$$\psi(x) = (s_1(x), \dots, s_N(x)). \tag{102}$$

We note that $\psi(U) \subset I^N$ and, by definition of U-admissibility, all the terms $s_{j,2}$ evaluate to non-vanishing functions on U. Let U_w be a relatively compact neighborhood of I^N in Ω_w. Then there exists a relatively compact open neighborhood $U_{\mathbb{C}} \subset \Omega_z \times \Lambda$ of U such that

(1) $(U_{\mathbb{C}})_{\mathbb{R}} = U$ and $\psi(U_{\mathbb{C}}) \subset U_w$.
(2) All terms appearing in ϕ admit analytic continuation to $U_{\mathbb{C}}$.
(3) All the terms $s_{j,2}$ evaluate to non-vanishing maps on $U_{\mathbb{C}}$.

Henceforth we view $U_{\mathbb{C}}$ as the domain of ψ. The graph $\Gamma = \Gamma_\psi$ is relatively compact in $\Omega \times \Lambda$, being contained in the product of the relatively compact sets $U_{\mathbb{C}} \subset \Omega_z \times \Lambda$ and $U_w \subset \Omega_w$.

By construction of t' it is clear that

$$t(z, \lambda) = t'(z, \psi(z, \lambda), \lambda) \qquad \text{for } (t, \lambda) \in U_{\mathbb{C}}. \tag{103}$$

We define the analytic subset $X_\Gamma \subset \Omega \times \Lambda$ by

$$X_\Gamma = \{s'_{j,2} W_j = s'_{j,1} : j = 1, \dots, N\}. \tag{104}$$

It is easy to check by induction that X_Γ agrees with Γ over $U_{\mathbb{C}}$ (using (103) and the fact that $s_{j,2}$ evaluate to non-vanishing maps on $U_{\mathbb{C}}$).

Finally we define $X \subset \Omega \times \Lambda$ by

$$X = \{t'_1 = \cdots = t'_k = 0\} \tag{105}$$

and set $Y := \pi(X \cap \Gamma) \subset \Omega_z \times \Lambda$. Then (103) at the points of $U = (U_{\mathbb{C}})_{\mathbb{R}}$ gives $Y_{\mathbb{R}} = \phi(U)$. $\square$

5.3. **Estimate for subanalytic sets.** To state the general form of our main result we introduce the notion of *complexity* of a semi-algebraic set. We say that a semialgebraic set $S \subset \mathbb{R}^m$ has complexity (m, s, d) if it defined by a semialgebraic formula involving s different relations $P_j > 0$ or $P_j = 0$, where the polynomials P_j have degrees bounded by d.

Theorem 6. *Let $A \subset \mathbb{R}^{m+n}$ be a bounded subanalytic set and $\varepsilon > 0$. There exist constants $d = d(A, \varepsilon)$ and $N = N(A, \varepsilon)$ with the following property. For any $y \in I^n$ and any $H \in \mathbb{N}$ there exist at most NH^ε many smooth connected semialgebraic sets $S_\alpha \subset \mathbb{C}^m$ with complexity (m, d, d) such that*

$$A_y(\mathbb{Q}, H) \subset \bigcup_\alpha A_y(S_\alpha). \tag{106}$$

Proof. Rescaling A by a sufficiently large integer, $\tilde{A} := \frac{1}{M} A$ we may assume that $\tilde{A} \subset \mathring{I}^{n+m}$. This clearly does not affect the heights of points by more than a constant factor, and it will suffice to prove the claim for $\tilde{A}$ and then rescale the varieties $\tilde{V}_\alpha$ back to $V_\alpha := MV_\alpha$. We thus assume without loss of generality that $A \subset \mathring{I}^{n+m}$.

By Theorem 5 we may write $A = \phi(\mathring{I}^{m+n})$ for some quantifier-free L_{an}^D-formula ϕ. By Proposition 21 and Lemma 22 we may write

$$A = \bigcup_{j=1}^{k} \bigcup_{i=1}^{n_j} \phi_{ji}(U_j) \tag{107}$$

where ϕ_{ji} is a U_j-admissible basic D-formula. By the first part of Lemma 16 it is clear that it will suffice to prove the claim with A replaced by each $\phi_{ij}(U_j)$. We thus assume without loss of generality that ϕ is already a U-admissible basic D-formula and prove the claim for $A = \phi(U)$.

Recall that $\tilde{\phi}$ is a U-admissible D-equation. We write $B = \tilde{\phi}(\mathring{I}^{m+n})$. Applying Proposition 24 to $\tilde{\phi}$ and using Theorem 4 we construct a locally analytic set $Y \subset \mathbb{C}^{m+n}$ such that $Y_{\mathbb{R}} = B$, and for any $y \in I^n$ there exist at most $N'H^\varepsilon$ many irreducible algebraic varieties $V_\alpha \subset \mathbb{C}^m$ with $\deg V_\alpha \leqslant d'$ such that

$$Y_y(\mathbb{Q}, H) \subset \bigcup_\alpha Y_y(V_\alpha) \tag{108}$$

with d, N as in Theorem 4. By (68) and $Y_{\mathbb{R}} = B$ we have

$$B_y(\mathbb{Q}, H) \subset \bigcup_\alpha B_y(\mathcal{V}_\alpha), \qquad \mathcal{V}_\alpha = (V_\alpha)_{\mathbb{R}}. \tag{109}$$

Finally, we recall that A is relatively open in B by Lemma 23. Then the same is true for the fiber $A_y \subset B_y$, and

$$\begin{aligned} A_y(\mathbb{Q}, H) &\subset A_y \cap (B_y(\mathbb{Q}, H)) \subset A_y \cap \bigcup_\alpha B_y(\mathcal{V}_\alpha) = \bigcup_\alpha (A_y \cap B_y(\mathcal{V}_\alpha)) \\ &= \bigcup_\alpha A_y(\mathcal{V}_\alpha) \end{aligned} \tag{110}$$

where the last equality is given by the second part of Lemma 16.

If we write each real-algebraic variety $\mathcal{V}_\alpha$ as a union of smooth connected strata $\mathcal{V}_\alpha = \cup_j S_{\alpha,j}$ then we have

$$A_y(\mathbb{Q}, H) \subset \bigcup_\alpha A(\mathcal{V}_\alpha) \subset \bigcup_{\alpha,j} A(S_{\alpha,j}). \tag{111}$$

It remains to note that since $\deg V_\alpha \leqslant d'$, the number and complexity of the strata $S_{\alpha,j}$ is bounded by some number d depending only on d'. This follows from general uniformity properties in the algebraic category, and in fact one may derive explicit (and polynomial in d') estimates for d (for details see [1, Proposition 37]). Taking $N = N'd$ finishes the proof. □

We can now finish the proof of Theorem 1.

Proof of Theorem 1. By Theorem 6 for any $y \in I^n$ and any $H \in \mathbb{N}$ there exist at most NH^ε many smooth connected semialgebraic sets $S_\alpha \subset \mathbb{C}^m$ such that

$$A_y(\mathbb{Q}, H) \subset \bigcup_\alpha A_y(S_\alpha). \tag{112}$$

By Lemma 15, for any positive dimensional S_α we have $A(S_\alpha) \subset A_y^{\mathrm{alg}}$. Thus $\#A_y^{\mathrm{trans}}(\mathbb{Q}, H)$ is bounded by the number of zero-dimensional strata, i.e. by NH^ε. □

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