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SIMPLE EXPONENTIAL ESTIMATE FOR THE NUMBER OF REAL ZEROS OF COMPLETE ABELIAN INTEGRALS

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1. Abelian integrals and polynomial envelopes of linear ordinary differential equations with meromorphic coefficients

One of the main results of this paper is an upper bound for the total number of real isolated zeros of complete Abelian integrals, exponential in the degree of the form (Theorem 1 below). This result improves a previously obtained in [IY1] double exponential estimate for the number of real isolated zeros on a positive distance from the singular locus. In fact, the theorem on zeros of Abelian integrals is a particular case of a more general result concerning the number of zeros in *polynomial envelopes* of irreducible and essentially irreducible differential operators and equations (see §1.3 below).

The first announcement of these and other results proved below was in [NY]. In §1 all principal results are formulated and all necessary definitions gathered, §2 explains connections between Abelian integrals and polynomial

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envelopes: since the most part of preparatory work was already done elsewhere [IY1], [IY2], [Y], we give only the principal definitions, referring for motivations, examples and detailed explanations to the above mentioned papers. The next section, §3, is the core of the paper. It contains the notion of the Rolle index of a differential operator, and provides tools for estimating this index explicitly. In §4, §5 the other results are proved, and the concluding section §6 deals with possible generalizations.

1.1. Complete Abelian integrals: statement of the problem

Let $H \in \mathbb{R}[x,y]$ be a real polynomial in two variables, $\varSigma_H^{\mathbb{R}} \Subset \mathbb{R}$ the set of real critical values of H, and $t \in \mathbb{R} \smallsetminus \varSigma_H^{\mathbb{R}}$ a regular value. Each (real affine nonsingular) level curve $\varphi_t = \{H(x,y)=t\}$ is a union of connected components, some of them compact *ovals* $\varphi_{t,1},\dots,\varphi_{t,s}$, their number $s = s(t)$ in general depending on t.

If $\omega = P(x,y)\,dx + Q(x,y)\,dy$ is a differential 1-form with real polynomial coefficients, then this form can be integrated along each oval $\varphi_{t,i}$, yielding a real multivalued function

$$I_{H,\omega}\colon \mathbb{R}\smallsetminus\varSigma_H^{\mathbb{R}} \to \mathbb{R}, \qquad t \mapsto I_{H,\omega}(t) = \oint_{\varphi_{t,i}} \omega, \quad i = 1,\dots,s. \tag{1}$$

Obviously, this multivalued function allows for selection of continuous branches over each interval from the domain $\mathbb{R}\smallsetminus\varSigma_H^{\mathbb{R}}$, and one may easily see that in fact each continuous branch is real analytic. The collection of all branches of the function (1) is called the *complete Abelian integral* of the form ω over the level curves of the polynomial H. The problem of finding an upper bound for the number of real isolated zeros of the Abelian integrals was repeatedly posed since early seventies: we refer the reader to the paper [IY1] where the principal references are given.

1.2. Exponential upper bound for the number of zeros of complete Abelian integrals: remarks and discussion

Theorem 1. *Suppose that the polynomial H satisfies the following two properties*:

(1) *its complexification is a Morse function* (*i.e. all critical points of H, including the nonreal ones, are nondegenerate and all complex critical values are pairwise distinct*), *and*

(2) *each complexified level curve* φ_t *after projective compactification intersects transversally the projective line* $\mathbb{C}P^1_\infty \subset \mathbb{C}P^2$ *at infinity.*

Then there exists a constant $c = c(H) < \infty$ *such that any real branch of the complete Abelian integral may have at most* $\exp(c \deg \omega)$ *real isolated zeros on* $\mathbb{R} \smallsetminus \Sigma_H^{\mathbb{R}}$*, where the degree of the form* $\deg \omega$ *is defined as* $\max(\deg P, \deg Q)$.

Remarks. 1. The second assumption on H is equivalent to saying that the principal homogeneous part of H factors as a product of pairwise different linear terms.

2. For any degree m the set of all polynomials of degree $\leqslant m$, satisfying the conditions of the Theorem, is an open dense semialgebraic subset of the linear space of all polynomials of degree $\leqslant m$. Thus Theorem 1 gives an upper bound for a *generic* polynomial.

3. In fact, it follows from the proof of Theorem 1 that either *all branches* of the integral are identically zero, or *all of them* may have only isolated zeros on $\mathbb{R}$.

4. The assertion of the Theorem means that for any choice of the form ω the number of different ovals on the plane $\mathbb{R}^2_{x,y}$, over which the integral of ω may vanish, is at most $\exp(c \deg \omega)$, unless this integral is zero for any oval.

5. In the previous publication [IY1], it was proved that for *almost all* polynomials satisfying the above two conditions and for any *compact* subset $K \Subset \mathbb{R} \smallsetminus \Sigma_H^{\mathbb{R}}$ the number of real isolated zeros of the Abelian integral *on the compact* K can be at most $\exp\exp(c'(H,K) \deg \omega)$, where the constant $c'(H,K)$ depends not only on H, but also on K. Thus Theorem 1 improves the upper estimate, at the same time extending the domain of its validity, as compared to [IY1].

Another important case is that of *hyperelliptic curves*. Recall that a polynomial H is called hyperelliptic, if it has the form

$$H(x,y) = y^2 + p(x), \qquad p \in \mathbb{R}[x], \quad \deg p \geqslant 5. \tag{2}$$

(for $\deg p = 3, 4$ one has the elliptic case, completely studied by G. Petrov). It is known (also due to Petrov) that if p is a Morse polynomial *with all critical points on the real axis*, then the number of real isolated zeros of the corresponding hyperelliptic integral can be at most $O(\deg \omega)$, see [Pe]. We establish a weaker result, but drop away the assumption on critical points.

Theorem 1′. *Assertion of Theorem 1 holds for a hyperelliptic polynomial* (2)*, provided that* $p(x)$ *is a Morse polynomial in one variable.*

1.3. Linear ordinary differential operators and equations. Polynomial and rational envelopes

Consider the field of (complex) rational functions $\Bbbk = \mathbb{C}(t)$ and the (non-commutative) ring $\mathfrak{D} = \Bbbk[\partial]$ of linear ordinary differential operators with rational coefficients and the operation of composition $(L_1, L_2) \mapsto L_1 \circ L_2$, also denoted by $L_1 L_2$. An operator

$$\mathfrak{D} \ni L = \sum_{j=0}^{n} a_j(t)\partial^{n-j}, \quad Lu = a_0\, u^{(n)} + a_1\, u^{(n-1)} + \cdots + a_{n-1}\, u' + a_n\, u, \quad (3)$$

is called *unitary*, if the *principal coefficient* $a_0(t)$ is equal to 1. We say that $L \in \mathfrak{D}$ is *real*, if all coefficients of L belong to the subfield $\mathbb{R}(t) \subset \Bbbk$. We denote by $\operatorname{ord} L$ the order of the operator L, that is, the degree of L in ∂. Note that nonzero rational functions are units (invertible elements) of the ring $\mathfrak{D}$, so we will implicitly consider all multiplicative formulas in $\mathfrak{D}$ modulo such units.

The *singular locus* Σ_L of a unitary operator $L \in \mathfrak{D}$ is the union of the polar loci of all its coefficients. A singular point $t \in \Sigma_L$ is said to be a *regular,* or *Fuchsian singularity*, if each coefficient $a_j(\cdot)$ in (3) has the pole of order at most j at that point (the *Fuchs condition*). It is natural to consider the coefficients as functions on the Riemann sphere $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$, in which case $t = \infty$ can belong to the singular locus Σ_L, and the Fuchs condition at infinity is to be verified in the chart $\tau = 1/t$. An operator is said to be Fuchsian, if the Fuchs condition holds for all singular points on $\mathbb{C}P^1$. A *Fuchsian differential equation* is the one of the form $Lu = 0$, where the operator L becomes Fuchsian after multiplication by an appropriate rational function $0 \neq \varphi \in \Bbbk$.

It is well known that any solution of the equation $Lu = 0$ can be analytically continued along any path γ avoiding the singular locus Σ_L, thus giving rise to an analytic multivalued function ramified over Σ_L. If L is a real operator and $K \subset \mathbb{R} \smallsetminus \Sigma_L$ a real interval, then one may always choose a *fundamental system of solutions* $f_1(t), \dots, f_n(t)$, a string of analytic functions, taking real values on K and linear independent over $\mathbb{C}$. Any other fundamental system of n solutions to the same equation can be obtained from the original one by a nondegenerate linear transformation. All those simple results can be found in many textbooks, for example, in [H].

Let $d \in \mathbb{N}$ be a natural number. We define the *polynomial envelope* of degree d of the equation $Lu = 0$ as the linear space of analytic multivalued functions representable in the form

$$\sum_{j,k=1}^{n} p_{jk}(t)\, f_j^{(k-1)}(t), \qquad p_{jk} \in \mathbb{C}[t], \quad \deg p_{jk} \leqslant d, \tag{4}$$

where $f_1, \dots, f_n$ is a fundamental system of solutions for that equation. Clearly, this definition does not depend neither on the choice of the fundamental system of solutions, nor on the choice of branches of the functions f_j. Sometimes we speak about polynomial envelopes of operators rather than those of differential equations, and use then the notation $\mathfrak{P}_d(L)$ for the corresponding linear space.

In the similar way we may introduce the *rational envelope of degree* d of the same equation as the collection of functions representable in the form similar to (4) but with *rational* rather then polynomial coefficients $p_{jk} \in \Bbbk$ (recall that the degree of a rational function is the total number of its poles, counted with multiplicities, including the pole at $t = \infty$). Note that rational envelopes of any finite degree $d < \infty$ are *not* linear spaces, and in general do depend on the choice of the fundamental solutions f_j and their branches. The notation used for rational envelopes is $\mathfrak{R}_d(L)$, and $\mathfrak{R}(L) = \bigcup_{d \geqslant 0} \mathfrak{R}_d(L)$.

1.4. Irreducible case

Let $a \notin \Sigma_L$ be a nonsingular point, and $\gamma \in \pi_1(\mathbb{C} \smallsetminus \Sigma_L, a)$ any closed loop (more precisely, the homotopy class of some closed loop with the vertex at a). Choose any fundamental system of solutions $\mathbf{f} = (f_1, \dots, f_n)$ considered as a row vector. Then after the analytic continuation over γ the vector function $\mathbf{f}(t)$ undergoes a linear transformation:

$$\Delta_\gamma \mathbf{f} = \mathbf{f} \cdot \mathbf{M}_\gamma \iff \Delta_\gamma f_j = \sum_{k=1}^{n} f_k \mathbf{m}_{\gamma;kj}.$$

Here Δ_γ stands for the operator of analytic continuation along γ, and $\mathbf{M}_\gamma$ is a square $(n \times n)$-matrix with constant complex entries $\mathbf{m}_{\gamma;kj}$. The correspondence $\pi_1(\mathbb{C} \smallsetminus \Sigma_L, a) \to \mathrm{GL}(n, \mathbb{C})$, $\gamma \mapsto \mathbf{M}_\gamma^{-1}$, is a linear representation of the fundamental group, and the image of the fundamental group is called the *monodromy group* of the operator L (or the equation $Lu = 0$).

The operator $L \in \mathfrak{D}$ is called *irreducible*, if the monodromy group of this operator is irreducible, i.e. the operators $\mathbf{M}_\gamma$ have no common invariant nontrivial subspace. For Fuchsian operators (equations) an equivalent algebraic formulation can be given as follows: L is irreducible if and only if it admits no nontrivial factorization $L = L_1 L_2$ in the ring $\mathfrak{D}$ (as usual, modulo the multiplicative subgroup of units $\Bbbk^* \subset \mathfrak{D}$). To avoid confusion, we refer to this property as *indecomposability* of L in $\mathfrak{D}$.

Indeed, if the operator admits a factorization as above, then the fundamental system of solutions to the equation $L_2 u = 0$ would generate an invariant linear subspace for all monodromy operators. Conversely, if the monodromy group is reducible, and $f_1, \dots, f_k$, $0 < k < n$, span the corresponding invariant subspace, then by the classical Riemann theorem one can construct an operator $L_2 \in \mathfrak{D}$ of order k, satisfied by $f_1, \dots, f_k$. But then one can easily show that L is right divisible by L_2, using the division algorithm from [In]: $L = L_1 L_2$. However, if we do not assume the Fuchsian property of L, then only the implication "irreducibility $\implies$ indecomposability" remains, since the coefficients of the operator L_2 will not in general be rational functions.

The principal result concerning zeros in polynomial envelopes, follows.

Theorem 2. *Let $L \in \mathfrak{D}$ be a real irreducible operator, and $K \Subset \mathbb{R} \smallsetminus \Sigma_L$ a compact segment without singular points of L.*

Then there exists a finite number $c = c(L, K) < \infty$ such that any function u from the polynomial envelope of degree d of the operator L, real on K, may have at most $\exp(cd)$ *isolated zeros on that segment*:

$$\limsup_{d \to \infty} \left\{ d^{-1} \cdot \ln N_K(f) : f \in \mathfrak{P}_d(L) \right\} = c(K, L) < \infty.$$

Corollary. *The same exponential upper bound for the number of isolated real zeros holds also for all functions from the rational envelope $\mathfrak{R}_d(L)$ of degree d.*

Indeed, by getting rid of all denominators any function from the rational d-envelope may be transformed into an element from the polynomial envelope of order at most $(n^2 - 1)d$, $n = \operatorname{ord} L$.

Remark. In Theorem 2 no assumption on the nature of the singularities of the equation is made; it is the global condition of irreducibility which is crucial for this result. However, we do not know whether the exponential estimate can be further improved.

Note. A confusion between closely related notions of irreducibility and indecomposability occurred in [NY]. In all formulations of theorems from that paper the irreducibility assumption is to be understood in terms of the monodromy group (and not as indecomposability). However, the difference disappears if only the Fuchsian case is considered.

1.5. Regular singularities at the endpoints and the real spectrum condition

Theorem 2 gives an upper estimate for the number of zeros for polynomial envelopes on a *compact* segment K. We extend now this result for semiintervals with singular endpoints. Let

$$L = t^n\,\partial^n + t^{n-1}\widetilde{a}_1(t)\,\partial^{n-1} + \cdots + t\,\widetilde{a}_{n-1}(t)\partial + \widetilde{a}_n(t) \in \mathfrak{D}$$

be a differential operator with a Fuchsian singularity at $t = 0$. Using the Euler transformation

$$x_1 = u,\ x_2 = t\,\partial x_1,\ x_3 = t\,\partial x_2,\ \ldots,\ x_n = t\,\partial x_{n-1},$$

one may transform the equation $Lu = 0$ into the system of first order linear differential equations

$$t\dot{x} = A(t)x, \qquad x \in \mathbb{C}^n,\ t \in (\mathbb{C}^1, 0),$$

with the matrix function $A(t) = A_0 + t\,A_1 + t^2\,A_2 + \cdots$ holomorphic at $t = 0$ and real on the real axis, if the operator L was real.

Definition. The *spectrum* of the Fuchsian singularity is the spectrum of the associated residue matrix $A(0) = A_0$.

One can easily show that the spectrum of the singularity consists of roots of the so called *indicial equation*

$$\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1) + \widetilde{a}_1(0)\lambda(\lambda-1)\cdots(\lambda-n+2) + \cdots \\ + \widetilde{a}_{n-2}(0)\lambda(\lambda-1) + \widetilde{a}_{n-1}(0)\lambda + \widetilde{a}_n(0) = 0.$$

Moreover, the equation $Lu = 0$ may have a solution of the form $u(t) = t^\lambda\,h(t)$ with $h(t)$ holomorphic at $t = 0$ only if λ is an element from the spectrum (a straightforward computation).

Definition. A point $\alpha \in \Sigma_L$ is the *real Fuchsian singularity* for the real operator $L \in \mathfrak{D}$, if $\alpha \in \mathbb{R}$, the Fuchsian condition holds at α and the spectrum of the singular point entirely belongs to the real axis.

Remark. The above definitions make sense also for the point $t = \infty$ after the change of time $t \mapsto t^{-1}$.

Theorem 3. *If $L \in \mathfrak{D}$ is a real irreducible operator, $\alpha \in \Sigma_L$ a real Fuchsian singularity for L and $K = (\alpha, \beta] \subset \mathbb{R} \smallsetminus \Sigma_L$ a semiinterval without singular points, then the assertion of Theorem 2 remains valid on K.*

Corollary. *If for a real irreducible operator L the locus $\mathbb{R} \cap \Sigma_L$ consists of real Fuchsian singularities only (including the point $t = \infty$), then there exists a constant $C = C(L)$ depending only on L, such that any real branch of any function from the polynomial d-envelope may have at most* $\exp(Cd)$ *isolated zeros.*

Indeed, in this case one may enumerate as $\alpha_1, \dots, \alpha_s$ the points of the locus $\Sigma_L \cap \mathbb{R}$, choose once and forever the points β_i so that $\beta_0 < \alpha_1 < \beta_1 < \cdots < \beta_{s-1} < \alpha_s < \beta_s$ and apply Theorem 2 to each of the semiintervals $(-\infty, \beta_0]$, $[\beta_{i-1}, \alpha_i)$, $(\alpha_i, \beta_i]$ and $[\beta_s, +\infty)$. Then taking the sum of the corresponding upper bounds, one obtains an upper estimate valid for on the whole real axis (one may count or not count the points α_i, this would not affect the exponential bound).

Remark. In Theorem 3 we do not require the singular points *outside the real axis* be real Fuchsian or even just Fuchsian.

1.6. Relaxing the irreducibility condition

If the irreducibility assumption fails, then one can construct an equation with meromorphic coefficients and an arbitrarily rapidly growing number of zeros in polynomial envelopes [IY1]. However, at least partially this condition of irreducibility can be relaxed.

Definition. A linear operator $L \in \mathfrak{D}$ is *essentially irreducible* with the *irreducible core* $L_* \in \mathfrak{D}$, if $L = L_* P_1 P_2 \cdots P_k$, where:

(1) $P_i \in \mathfrak{D}$ are real differential operators of order 1, and
(2) L_* is a real irreducible operator (of any order).

Remarks. 1. An equivalent definition in terms of the monodromy group is as follows: the operator is essentially irreducible, if in the linear n-dimensional (over $\mathbb{C}$) space Λ of solutions of the equation $Lu = 0$ a chain of subspaces (a flag) Λ_i, $i = 0, 1, \dots, k+1$, can be chosen,

$$\{0\} = \Lambda_{k+1} \subset \Lambda_k \subset \Lambda_{k-1} \subset \cdots \subset \Lambda_1 \subset \Lambda_0 = \Lambda,$$

such that:

(1) each subspace Λ_i is invariant by all monodromy operators $\mathbf{M}_\gamma$,
(2) $\dim \Lambda_{i-1} = \dim \Lambda_i + 1$ for all $i = 1, \dots, k$ (we do not require that $\dim \Lambda_0 = \dim \Lambda_1 + 1$),
(3) the induced *quotient representation* in Λ_0/Λ_1 is irreducible (the other induced one-dimensional quotient representations in Λ_{i-1}/Λ_i, $i = 1, \dots, k+1$, are obviously irreducible),
(4) Λ_1 is the null space for a Fuchsian operator.

The equivalence of the two definitions is established using the Riemann theorem: Λ_i is the null space for the composition $P_i P_{i+1} \cdots P_k$.

2. If the initial operator L is Fuchsian, then the last condition in the above list is automatically satisfied. In this case we can also replace irreducibility of L_* by indecomposability, so finally the definition of almost irreducibility can be formulated in terms of *orders* of factors in the indecomposable factorization of L in the ring $\mathfrak{D}$.

3. The essential irreducibility condition satisfied, one may always choose a fundamental system of solutions $f_1, \dots, f_n$ for the equation $Lu = 0$ in such a way that the last $n - i$ functions will constitute a basis for Λ_i. Moreover, for any real interval free from singularities of L, the functions f_i may be chosen real on that interval. It is this form of the essential irreducibility assumption, which will be used below.

4. In [Y] a weaker concept appeared, *almost irreducibility*, which is a particular case of essential irreducibility. The monodromy group is said almost irreducible, if there exists an subspace $\Lambda_{\text{triv}} \subset \Lambda$, on which all monodromy operators are identical (and hence this subspace is invariant), but the quotient representation in $\Lambda/\Lambda_{\text{triv}}$ is already irreducible. Obviously, one can choose any ascending chain of subspaces $\Lambda_i \subset \Lambda_{\text{triv}}$ to show that almost irreducibility implies essential irreducibility.

Theorem 4. *If L is a real essentially irreducible operator and $K \Subset \mathbb{R} \smallsetminus \Sigma_L$ is a compact segment without singular points, then the assertion of Theorem 2 is valid for L.*

Theorem 5. *If $K = (\alpha, \beta] \subset \mathbb{R}$ is a semiinterval with a singular endpoint (like in Theorem 3), and L is an essentially irreducible operator with the* *real Fuchsian singularity at $t = \alpha$, then the assertion of Theorem 2 is valid for L on K.*

2. From Abelian integrals to polynomial envelopes

In this section we reduce the problem on estimating the number of zeros of Abelian integrals to that for polynomial envelopes (the proposition below establishes the implication "Theorem 5 $\implies$ Theorem 1"). Except for Lemma 1, the exposition here reproduces that from [Y] and [IY1].

Proposition. *For any real polynomial H satisfying the assumptions of Theorem 1, one may construct a Fuchsian operator $L = L_H$ depending only on H, with the following properties*:

(1) *L is a real Fuchsian operator*;
(2) *all singular points of L have real spectrum*;
(3) *the monodromy group of L is almost irreducible* (*hence essentially irreducible*);
(4) *for any polynomial form ω the complete Abelian integral of ω over the level curves $H =$* const *belongs to the rational d-envelope of L with $d = \frac{\deg \omega}{\deg H} + O_H(1)$.*

The same result is valid for a hyperelliptic polynomial $H = y^2 + p(x)$ satisfying the assumptions of Theorem $1'$.

The rest of this section contains the proof of this assertion. Starting from §3, we discuss only polynomial envelopes of linear operators.

2.1. The monodromy group of the Gauss–Manin connection

We consider first the case of polynomials described in Theorem 1. A polynomial map $H\colon \mathbb{C}^2 \to \mathbb{C}^1$ defines a topologically locally trivial bundle over the set of (complex) regular values $\mathbb{C} \smallsetminus \Sigma_H^{\mathbb{C}}$, provided that all level curves intersect transversally the infinite line after compactification. The fibers of this bundle are nonsingular affine algebraic curves $\varphi_t = \{H = t\} \subset \mathbb{C}^2$, hence an induced vector bundle with fibers $H_1(\varphi_t, \mathbb{C}) \simeq \mathbb{C}^n$, $n = 2g + s - 1$, is well defined and can be endowed with a locally flat connection, called *Gauss–Manin connection* (here g is the genus of the projective compactification of φ_t, and $s = \deg H$ is the number of points at infinity). The result of the parallel translation in the sense of the Gauss–Manin connection defines a linear representation of the fundamental group $\pi_1(\mathbb{C} \smallsetminus \Sigma_H^{\mathbb{C}}, \cdot)$; after choosing an arbitrary basis $c_1, \dots, c_n$ in some fixed fiber $H_1(\varphi_{t_0}, \mathbb{C})$, the result of continuation of the row vector $\mathbf{c} = (c_1, \dots, c_n)$ along an arbitrary loop γ is the row vector $\mathbf{c} \cdot \mathbf{M}_\gamma$. Since any polynomial form ω restricted on any φ_t is holomorphic, it follows that the row vector function $\boldsymbol{I}(t) = (I_1(t), \dots, I_n(t))$ has the same monodromy group independently of the choice of the form ω: $\Delta_\gamma \boldsymbol{I} = \boldsymbol{I} \cdot \mathbf{M}_\gamma$. Moreover, if we consider the Jacobian matrix $\mathbf{I}(t)$ built from the functions $I_j(t)$ and their derivatives in t up to the order $n-1 = 2g+s-2$, then the monodromy of this function will be the same: $\Delta_\gamma \mathbf{I}(t) = \mathbf{I}(t) \cdot \mathbf{M}_\gamma$. In other words, the monodromy group of any complete Abelian integral is determined by the topology of the Hamiltonian only, if we fix a framing of the bundle.

The representation $\gamma \mapsto \mathbf{M}_\gamma^{-1}$ possesses an additional symmetry due to the fact that H is a polynomial with real coefficients. Choose the base point t_0 on the real axis. Then, since the critical values of H are symmetric with respect to the real axis, the fundamental group $\pi_1(\mathbb{C} \smallsetminus \Sigma_H^{\mathbb{C}}, t_0)$ admits an involution $\gamma \mapsto \overline{\gamma}$ (the mirror symmetry in the real axis), induced by the standard involution $t \mapsto \bar{t}$. At the same time the standard involution $(x, y) \mapsto (\overline{x}, \overline{y})$ induces the involution on the homology level, $\tau \colon H_1(\varphi_t, \mathbb{C}) \to H_1(\varphi_{\bar{t}}, \mathbb{C})$, and without loss of generality we may assume that the basis $c_j \in H_1(\varphi_{t_0}, \mathbb{C})$ was chosen as τ-(anti)real: $\tau(c_j) = \pm c_j$. Then one can easily see that the representation $\gamma \mapsto \mathbf{M}_\gamma^{-1}$ is τ-symmetric: $\mathbf{M}_{\overline{\gamma}} = \overline{\mathbf{M}}_\gamma$.

The reducibility properties of the representation $\gamma \mapsto \mathbf{M}_\gamma^{-1}$ were established in [Y]: it was shown that if the Hamiltonian satisfies the conditions of Theorem 1, then this representation is almost irreducible in the sense explained in §1.6. Moreover, it is known that in this case the monodromy operators corresponding to small loops around each critical value of H, have all eigenvalues equal to 1. This guarantees that the spectrum of all singularities belongs to $\mathbb{Z}$, see below §4.1.

2.2. Lemma on nondegenerate realization

Our local goal is to establish the existence of a differential operator of order exactly equal to n with the prescribed monodromy group.

Lemma 1. *Any τ-symmetric n-dimensional monodromy group is a monodromy group of a real Fuchsian operator of order n: there exists an op-*

erator L and a fundamental system of solutions $\mathbf{f} = (f_1, \dots, f_n)$ *such that* $\Delta_\gamma \mathbf{f} = \mathbf{f} \cdot \mathbf{M}_\gamma$.

Proof. By the classical Plemelj–Röhrl theorem [AI], one may always construct a multivalued almost everywhere nondegenerate analytic matrix function $\mathbf{X}(t)$ ramified over Σ, in such a way that $\Delta_\gamma \mathbf{X}(t) = \mathbf{X}(t)\cdot\mathbf{M}_\gamma$. Moreover, all points of Σ will be regular singularities for $\mathbf{X}(t)$, $\mathbf{X}^{-1}(t)$: $\|\mathbf{X}^{\pm 1}(t)\| = O(|t-\alpha|^{-C})$ for some $C < \infty$, as t tends to a point $\alpha \in \Sigma$, remaining in any sector with the vertex at α.

We construct the vector function $\mathbf{f}$ from the matrix function $\mathbf{X}$ in two steps. First we modify $\mathbf{X}(t)$ to become real-valued on the real segment K containing the real base point t_0. Note that if the monodromy is τ-symmetric, then the matrix function $\mathbf{X}^\dagger(t) = \overline{\mathbf{X}(\bar t)}$, $\bar t = \tau(t)$, will also have the same monodromy, hence the two functions

$$\operatorname{Re}\mathbf{X}(t) = \frac{1}{2}(\mathbf{X}(t) + \mathbf{X}^\dagger(t)), \qquad \operatorname{Im}\mathbf{X}(t) = \frac{1}{2\sqrt{-1}}(\mathbf{X}(t) - \mathbf{X}^\dagger(t)),$$

will also realize the same monodromy. Besides, both functions are real-valued on K and $\mathbf{X} = \operatorname{Re}\mathbf{X} + \sqrt{-1}\operatorname{Im}\mathbf{X}$. Consider now the function $\mathbf{X}_z = \operatorname{Re}\mathbf{X} + z\operatorname{Im}\mathbf{X}$: the determinant of this function is a polynomial in z, which is not identically zero, since $\det\mathbf{X}_z(t)$ is not identically zero for $z = \sqrt{-1}$. Thus there must exist a real z for which the matrix $\mathbf{X}_z$ is nondegenerate almost everywhere on K. Thus without loss of generality one may assume $\mathbf{X}$ being real on K from the very beginning.

We will further modify the matrix function $\mathbf{X}(t)$ in such a way that the first row of this matrix will contain linear independent entries, preserving the monodromy. Without loss of generality we may assume that $\det\mathbf{X}(t_0) \neq 0$, and $\mathbf{X}^{-1}(t) = \mathbf{Z}(t) + O\big(|t-t_0|^n\big)$, where $\mathbf{Z}(t)$ is a polynomial matrix function with real matrix coefficients. Then $\mathbf{Z}(t)\mathbf{X}(t) = \mathbf{E} + O\big(|t-t_0|^n\big)$. Take the row vector function $\mathbf{b}(t) = (1, t-t_0, \dots, (t-t_0)^{n-1})$ and consider the product $\mathbf{f}(t) = \mathbf{b}(t)\cdot\mathbf{Z}(t)\cdot\mathbf{X}(t)$ which has the same monodromy factors $\mathbf{M}_\gamma$, since both $\mathbf{Z}$ and $\mathbf{b}$ are polynomial. Clearly, $f_k(t) = (t-t_0)^{k-1} + O\big(|t-t_0|^n\big)$, hence the functions f_k are linear independent (their Wronski determinant is nonzero at $t = t_0$), real on some segment of the real axis and have at most regular singularities at all points of Σ. Now it follows from the classical Riemann–Fuchs theorem that the tuple $\mathbf{f}$ is a fundamental system of solutions for a certain Fuchsian equation with coefficients real on some segment of the real axis. But being rational, these coefficients must be necessarily real everywhere on $\mathbb{R}$ (outside their polar locus). □

Remark. In [Y] the existence of the tuple $\mathbf{f}$ was proved by finding a polynomial form Ω with linear independent integrals. Such a form was proved to exist for almost all Hamiltonians satisfying the conditions of Theorem 1. In that case one has an explicit estimate for the term $O_H(1)$ in the degree of the envelope.

2.3. Proof of the Proposition

Let $\mathsf{I}(t)$ be the Jacobian matrix for the Abelian integrals of the form ω over the cycles $c_j(t)$ constituting a real basis in the homology space $H_1(\varphi_t, \mathbb{C})$. Fix an arbitrary tuple of linear independent analytic functions $\mathbf{f}(t) = (f_1(t), \dots, f_n(t))$ with the same monodromy matrices M_γ constructed in Lemma 1, and let $\mathsf{F}(t)$ stand for the Jacobian matrix of this tuple. Then one can easily see that the matrix ratio $\mathsf{R}(t) = \mathsf{I}(t) \cdot \mathsf{F}^{-1}(t)$ is a single-valued analytic matrix function. Having only regular singularities, this matrix function must be rational, and the arguments given in [Y] and similar to those from [M] give an upper estimate for the degrees of rational entries of $\mathsf{R}(t)$ in the form $\frac{\deg \omega}{\deg H} + O(1)$. The identity $\mathsf{I}(t) = \mathsf{R}(t)\mathsf{F}(t)$ gives the required representation. $\square$

2.4. The hyperelliptic case

The same arguments establish also the representation of hyperelliptic integrals in the form of rational envelopes. The only thing which needs to be verified is the essential irreducibility condition. In [Y] it is shown that this irreducibility can be deduced from the fact that all vanishing cycles on the fibers φ_t can be chosen and ordered in such a way that the intersection index would be ± 1 for any two subsequent cycles. The latter assumption is completely evident if the polynomial p has all real roots: then its critical points are all real and alternate between maxima and minima, and this natural order transferred onto the vanishing cycles, satisfies the above requirement: each real vanishing cycle (corresponding to a maximum) intersects two neighboring imaginary cycles vanishing at the two minima, with the coefficients ± 1 depending on the orientation of the imaginary cycles.

The general case can be reduced to the above particular one by arguments of connectedness: all (complex) Morse polynomials in one variable constitute a connected subset of the (complex) linear space of polynomials of the given degree in one variable, while the monodromy group (more precisely, the matrices $\mathbf{M}_\gamma$) are locally constant. Thus the assertion of the Proposition holds also for hyperelliptic Morse Hamiltonians.

Remark. In general, the Fuchsian operator will have singularities not only on Σ: if the Jacobian matrix $\mathbf{F}$ degenerates at a certain point t, then the coefficients of the operator L may have a pole at that point. However, *all solutions of the equation* $Lu = 0$ *extend holomorphically at that point*: such singularities are called *apparent*, they have trivial local monodromy and their appearance will not affect our constructions below.

3. Polynomial envelope of an irreducible Fuchsian equation

In this section we prove Theorem 2, the core result of the paper.

3.1. Frobenius–Schlesinger–Polya formula

Let L be an arbitrary linear ordinary differential operator, $\operatorname{ord} L = n$, and $f_1, \dots, f_n$ is a fundamental system of solutions of the equation $Lu = 0$. Introduce the functions $W_k(t)$ as follows: $W_0(t) \equiv 1$, $W_1(t) = f_1(t)$, $W_2(t) = f_1(t)f_2'(t) - f_1'(t)f_2(t)$ and in general $W_k(t)$ for $k = 1, \dots, n$ is the Wronski determinant of the first k functions $f_1, \dots, f_k$:

$$W_k(t) = \det \begin{bmatrix} f_1 & f_2 & \cdots & f_k \\ f_1' & f_2' & \cdots & f_k' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(k-1)} & f_2^{(k-1)} & \cdots & f_k^{(k-1)} \end{bmatrix}$$

It turns out that the differential operator L admits "factorization" in terms of the functions W_k.

Lemma 2 (Frobenius–Schlesinger–Polya)**.** *If L is a unitary operator, $L = \partial^n + \cdots$, then*

$$L = \frac{W_n}{W_{n-1}} \cdot \partial \cdot \frac{W_{n-1}^2}{W_n W_{n-2}} \cdot \partial \cdot \frac{W_{n-2}^2}{W_{n-1} W_{n-3}} \cdot \partial \cdots \partial \cdot \frac{W_1^2}{W_2 W_0} \cdot \partial \cdot \frac{W_0}{W_1} \qquad (5)$$

(*composition of operators*).

Proof. We follow [In, §5.21], adding the proof for the formula given in the footnote.

Introduce the auxiliary differential operators L_k by letting $L_k u$ being equal to the Wronski determinant of the functions $f_1, \dots, f_k, u$ (in the specified order). Clearly, $L_k = W_k \cdot \partial^k + \cdots$. Therefore $L = W_n^{-1} L_n$, since both parts of the equality are unitary differential operators annulated by n linear independent functions $f_1, \dots, f_n$.

First we establish the operatorial identity

$$\partial \cdot \frac{1}{W_k} \cdot L_{k-1} = \frac{W_{k-1}}{W_k^2} \cdot L_k. \tag{6}$$

Indeed, both operators are of order k, and the leading coefficients are the same, being equal to $\frac{W_{k-1}}{W_k}$. Hence the difference between them is an operator of order at most $k-1$. On the other hand, both operators are annulated by the k linear independent functions $f_1, \dots, f_k$: for the first $k-1$ functions this is evident, and on f_k the left hand side part is zero, since $L_{k-1} f_k = W_k$. Thus we conclude that the two parts coincide. The formula (6) remains valid for $k = 1$, if we put formally $L_0 = \mathrm{id}$.

Now we can iterate the formula (6) as follows:

$$\begin{aligned}
L &= W_n^{-1} L_n \\
&= \frac{W_n}{W_{n-1}} \cdot \frac{W_{n-1}}{W_n^2} L_n = \frac{W_n}{W_{n-1}} \cdot \partial \cdot W_n^{-1} \cdot L_{n-1} \\
&= \left(\frac{W_n}{W_{n-1}} \cdot \partial \cdot \frac{W_{n-1}}{W_n} \right) \cdot W_{n-1}^{-1} L_{n-1} \\
&= \left(\frac{W_n}{W_{n-1}} \cdot \partial \cdot \frac{W_{n-1}}{W_n} \right) \cdot \left(\frac{W_{n-1}}{W_{n-2}} \cdot \partial \cdot \frac{W_{n-2}}{W_{n-1}} \right) \cdot W_{n-2}^{-1} L_{n-2} = \cdots \\
&= \left(\frac{W_n}{W_{n-1}} \cdot \partial \cdot \frac{W_{n-1}}{W_n} \right) \cdot \left(\frac{W_{n-1}}{W_{n-2}} \cdot \partial \cdot \frac{W_{n-2}}{W_{n-1}} \right) \cdots \\
&\qquad \cdots \left(\frac{W_2}{W_1} \cdot \partial \cdot \frac{W_1}{W_2} \right) \left(\frac{W_1}{W_0} \cdot \partial \cdot \frac{W_0}{W_1} \right) \cdot W_0^{-1} L_0. \quad \square
\end{aligned}$$

Remark 1. The representation (5) was first established by G. Frobenius [F]; a simplified proof appeared in the paper of G. Pólya [Pó] who refers to the handbook of L. Schlesinger. E. Ince [In] gives an independent proof without referring to Pólya or Schlesinger [Sch]. A lengthy computational proof can be found in [H, Ch. IV, §8(ix)]. Our proof seems to be the shortest of all and the most computation-avoiding.

Remark 2. Decomposition (5) holds even for irreducible operators, since the Wronskians W_k are in general transcendental rather than rational functions.

3.2. Rolle index of a differential operator

If L is a unitary differential operator with the coefficients which are real analytic on some segment $K \subset \mathbb{R}$, then the fundamental system of solutions $f_1, \dots, f_n$ can be also chosen real on K, hence all Wronskians W_k will be real analytic on K as well.

Consider the nonhomogeneous differential equation

$$Lu = h, \qquad h = h(t),\ t \in K \tag{7}$$

with the right hand side part h real analytic on K. Let $f = f(t)$ be any solution of (7). Our goal is now to compare the number of *isolated* zeros of h and f on K. The simplest result in this spirit is the classical Rolle theorem. Denote by $N_K(\varphi)$ the number of real isolated zeros of a function φ real analytic on K, counted with their multiplicities. According to this definition, $N_K(\varphi) = 0$ if $\varphi \equiv 0$ on K.

Rolle theorem. *If $L = \partial$, then for any real segment $K = [\alpha, \beta]$*

$$N_K(u) \leqslant N_K(Lu) + 1. \quad \square$$

Remark. In fact, the Rolle theorem deals with zeros counted without multiplicities, saying that between any two zeros of a differentiable function there must be at least one zero of its derivative. The general case is reduced to this particular one without any difficulties: if $t_0 < t_1 < \cdots < t_s$ are real zeros of u of multiplicities $1+\nu_0, 1+\nu_1, \dots, 1+\nu_s$ ($\nu_j \geqslant 0$), then the derivative ∂u will have zeros of order ν_j at t_j (if $\nu_j > 0$), and also on any of s segments (t_j, t_{j+1}) there must be at least one zero. Thus

$$N_K(u) = \sum_{j=0}^{s}(1+\nu_j) = 1 + \sum_{j=0}^{s}\nu_j + s \leqslant 1 + N_K(\partial u).$$

The case $u = \text{const}$ is trivial.

A simple generalization of this result is due to G. Pólya: it concerns differential equations with the "Property W": the latter means that the equation possesses a fundamental system of solutions such that all Wronskians W_k are nonvanishing on K.

Pólya theorem (1923), see [Pó, Theorem I*]. *If the fundamental system of real analytic solutions $f_1, \dots, f_n$ for the real equation $Lu = 0$ is such that $W_k \neq 0$ on K for all $k = 1, \dots, n = \operatorname{ord} L$, then*

$$N_K(u) \leqslant N_K(Lu) + n.$$

Proof. Assume that $N_K(u) \geqslant N_K(Lu) + n + 1$. By virtue of the decomposition (5), L is the composition of $2n + 1$ operators of differentiation and multiplication by nonvanishing real analytic functions (since the denominators are nonvanishing). By Rolle theorem, each differentiation may decrease the number of real isolated zeros counted with their multiplicities by at most 1, while each multiplication cannot change the number of zeros. Since the number of derivations is n, the number of isolated zeros counted with their multiplicities, will be decreased at most by n, being thus at least $N_K(Lu) + 1$, which contradicts the assumptions. $\square$

If we allow for zeros of the Wronskians, then two additional circumstances must be taken into consideration: the operators of multiplication may cancel zeros corresponding to roots of the denominators, and in general these multiplications will take analytic functions into functions that are only meromorphic (with poles), thus making impossible the straightforward application of the Rolle theorem. However, knowing the number of zeros of Wronskians allows for establishing a result similar to the Pólya theorem.

Theorem 6. *Let L be a unitary differential operator of order n with coefficients that are real analytic on a closed real segment $K \subseteq \mathbb{R} \smallsetminus \Sigma_L$. Then there exists a finite number $\rho = \rho(L, K)$, such that*

$$N_K(u) \leqslant N_K(Lu) + \rho(L, K).$$

for any real analytic function u.

Moreover, if K is an arbitrary connected subset of $\mathbb{R}$, $f_1, \dots, f_n$ is a fundamental system of real analytic on K solutions and W_k are the corresponding Wronskians with

$$N_K(W_k) = \nu_k < \infty, \qquad k = 1, \dots, n,$$

then

$$\rho(L, K) \leqslant (n+2)\nu - 2, \qquad \nu = 1 + \sum_{k=1}^{n} \nu_k.$$

Proof. We start with the second assertion of the theorem. Construct the sequence of operators

$$R_0 = \frac{W_0}{W_1},\ D_1 = \partial \cdot R_0,\ R_1 = \frac{W_1^2}{W_0 W_2} \cdot D_1,\ D_2 = \partial \cdot R_1,\ \dots$$
$$R_{n-1} = \frac{W_{n-1}^2}{W_n W_{n-2}},\ D_n = \partial \cdot R_{n-1},\ R_n = \frac{W_n}{W_{n-1}} \cdot D_n = L.$$

Let u be an analytic on a connected set K. Then any of the functions $R_k u$ or $D_k u$ will have at most $\nu = 1 + \sum_k \nu_k$ intervals of continuity, since the total number of poles of all denominators (without multiplicities) is at most $\sum_k \nu_k$.

Thus any of the d differentiations can decrease the number of isolated zeros (with multiplicities) by at most ν, while each multiplication can eliminate at most $\nu_{k+1} + \nu_{k-1}$ zeros (again counted with multiplicities):

$$N_K(D_k u) \geqslant N_K(R_{k-1} u) - \nu,$$
$$N_K(R_k u) \geqslant N_K(D_k u) - \nu_{k-1} - \nu_{k+1}.$$

Adding these inequalities, we arrive to the final inequality

$$N_K(Lu) \geqslant N_K(u) - n\nu - \nu_n - 2\sum_{k=1}^{n-1} \nu_k \geqslant N_K(u) - (n+2)\nu + 2.$$

Now return to the first part: if K is a compact segment without singularities, then for any choice of a fundamental system of solutions, the Wronskians W_k will be real analytic, hence the number of their zeros will be finite. Thus we have $\nu_k < \infty$, and the above arguments prove finiteness of ρ. $\square$

Note that for $\nu_k = 0$ the assertion of the Theorem coincides with the Pólya inequality. If instead of analytic function u we would start with a meromorphic function with p poles, then the corresponding inequality would take the form

$$N_K(u) \leqslant N_K(Lu) + \nu(n+2) - 2 + np.$$

The inequality obtained in Theorem 6, makes meaningful the following notion of the *Rolle index* of a linear differential operator.

Definition. The Rolle index of a linear operator L on a connected real set K without singularities, is the supremum

$$\sup_u (N_K(u) - N_K(Lu)),$$

taken over all functions real analytic on K and having at most a finite number of isolated zeros.

Remark. The Rolle index is also well defined if the equation $Lu = 0$ has only apparent singularities on K, that is, singular points at which all solutions are in fact analytic (see the last Remark in §2).

We will never deal with the Rolle index itself, but rather with upper estimates for it.

3.3. Generalized Jensen inequality

Theorem 6 reduces the question about the number of isolated zeros of an arbitrary solution for a linear equation to that about the number of zeros of the Wronskians W_k. It turns out that the latter problem admits a natural solution in the complex domain. The result below is a generalization of the classical Jensen formula, which gives an upper estimate for the number of isolated zeros of an analytic function in terms of its growth in the gap between two nested sets.

Let $U \subset \mathbb{C}$ be an open connected and simply connected set (a topological open disk) with a smooth boundary Γ, and $K \Subset U$ a compact subset of U (this means that the distance from K to Γ is strictly positive). For an arbitrary function f holomorphic in a neighborhood of the closure $\overline{U}$ one may define two numbers,

$$M(f) = \max_{t \in \overline{U}} |f(t)|, \qquad m(f) = \max_{t \in K} |f(t)|,$$

which are always related by the inequality $m \leqslant M$, and the equality is possible if and only if $f = \text{const}$.

Lemma 3 [IY2]**.** *There exists a finite constant $\gamma = \gamma(K, U)$ depending only on the relative position of the two sets K and U, such that for any f analytic in a neighborhood of $\overline{U}$, the number $N_K(f)$ of complex isolated zeros of f on K admits an upper estimate*

$$N_K(f) \leqslant \gamma(K, U) \cdot \ln \frac{M(f)}{m(f)}. \quad \square$$

Remark. If $f \equiv 0$, then it is convenient to define $M(f)/m(f) = 1$ as for any other constant: then the above inequality will remain valid in this exceptional case as well.

3.4. Two-sided estimates for the Wronskians

Let $U \subset \mathbb{C}$ be any open connected domain (not necessarily simply connected) and $\mathcal{A}(U)$ the ring of functions analytic (single-valued) in U. Consider a unitary linear operator $L \in \mathfrak{D}_U = \mathcal{A}(U)[\partial]$ of order n with coefficients in $\mathcal{A}(U)$. In the same way as for operators with rational coefficients, the monodromy group of the equation $Lu = 0$ can be introduced and the canonical representation $\pi_1(U, t_0) \to \mathrm{GL}(n, \mathbb{C})$, $\gamma \mapsto \mathsf{M}_\gamma^{-1}$ defined (here t_0 is an arbitrary point from U). If $L \in \mathfrak{D}$ is an operator with rational coefficients, then one may put $U = \mathbb{C} \smallsetminus \Sigma_L$. Denote by $z\colon \mathbb{D} \to U$ the universal covering (in most cases the universal covering space will be the unit disk).

Let $f_1, \dots, f_n$ be a fundamental system of solutions for the equation $Lu = 0$, and $d \in \mathbb{N}$ a natural number. The polynomial envelope of the equation $Lu = 0$ is defined as the linear space of functions spanned by the monomials

$$F_{\alpha jk}(t) = t^\alpha f_j^{(k-1)}, \qquad \alpha = 0, 1, \dots, d,\ j, k = 1, \dots, n.$$

As in the case of $L \in \mathfrak{D}$, this space is well defined and can be considered as a subspace of the space of analytic functions on $\mathbb{D}$ (since solutions are in general multivalued on U).

We arrange the monomials $F_{\alpha jk}$ lexicographically according to the ordering of indices as follows: k is the first letter, α the second and j the third one. Thus all $n^2(d+1)$ monomials will be numbered sequentially, $F_{\alpha jk}$ receiving the number

$$\beta = \beta(\alpha, k, j; d) = (k-1)(d+1)n + \alpha n + j. \tag{8}$$

Let $W_\beta(t)$ be the Wronskian of the first β monomials. From now on we fix L and the system f_j and investigate the behavior of Wronskians in their dependence on the large natural parameter d.

Remark. The notation W_β is somewhat ambiguous, since the meaning of this symbol depends on the choice of d: if we replace d by $d+1$, then all W_β will be changed starting from $\beta = n(d+1)$. Thus a more accurate notation would be $W_{\beta,d}(t)$. However, we will not use this cumbersome construction.

As before, we say that an operator $L \in \mathfrak{D}_U$ is irreducible if the representation $\gamma \mapsto \mathsf{M}_\gamma^{-1}$ is irreducible.

Lemma 4 [IY1]**.** *Let $L \in \mathfrak{D}_U$ be an irreducible differential operator and $D \Subset \mathbb{D}$ an arbitrary compact subset of the universal covering. Denote*

$$A(d, D) = \max_{\beta} \max_{z\in D} |W_\beta(t(z))|, \qquad B(d, D) = \min_{\beta} \max_{z\in D} |W_\beta(t(z))|,$$

where the exterior maximum and minimum are taken over all $\beta = 1, \dots, n^2(d+1)$.

Then

$$A(d, D) \leqslant \exp\exp O_{L,D}(\ln d), \qquad \text{as } d \to \infty,$$

and if D has a nonempty interior, then also

$$B(d, D)^{-1} \leqslant \exp\exp O_{L,D}(d), \qquad \text{as } d \to \infty,$$

where the terms $O_{L,D}(d)$ grow at most linearly in d, with the slope depending only on the choice of the solutions f_j and the compact D. $\square$

Remark. This statement is a quantitative generalization of the fact that solutions of an irreducible equation are linear independent over the field of rational functions. No assumption of realness is required whatsoever.

3.5. Proof of Theorem 2

Let now $L \in \mathfrak{D}$ be as before a real irreducible differential operator with rational coefficients. Take any compact segment $K \Subset \mathbb{R} \smallsetminus \Sigma_L$ free from singularities of the irreducible equation $Lu = 0$. Let $K_* \Subset \mathbb{C}$ be the closure of some open neighborhood of K and U another open connected simply connected set containing K_* strictly inside. From Lemma 4 it follows that each Wronskian $W_\beta(t)$ for all $\beta = 1, \dots, n^2(d+1)$ admits the estimate

$$\begin{aligned}
\max_{t\in\overline{U}} |W_\beta(t)| &\leqslant A(d, \overline{U}) \leqslant \exp\exp O_{L,\overline{U}}(\ln d),\\
\max_{t\in K_*} |W_\beta(t)| &\geqslant B(d, K_*) \geqslant \exp\big(-\exp O_{L,K_*}(d)\big),
\end{aligned}$$

which by Lemma 3 implies that the number of complex isolated zeros of W_β in K_* (and hence the number of real isolated zeros on K) can be at most exponential:

$$N_K(W_\beta) \leqslant \exp O_{L,K,K_*,U}(d) = \exp O_{L,K}(d)$$

(we fix the choice of K_* and U together with that of K).

By Theorem 6, the number of zeros of any linear combination of the monomials $F_{\alpha jk}$ is at most exponential in d as well. But this is exactly the result we need, since the polynomial d-envelope of $Lu = 0$ consists of linear combinations of the monomials. $\square$

Remark. In fact, we proved that for the differential operator L_d annulating the polynomial d-envelope of an irreducible real operator L, the Rolle index grows at most exponentially in d on any real segment K free from singular points of L. Note that the operator L_d may have singularities on K, but all these singularities will be apparent: all solutions of $L_d u = 0$ extend analytically at those points.

4. Singular endpoints

Assume that $L \in \mathfrak{D}$ is a real operator with a real Fuchsian singularity at $t = 0$, and $K = (0, t_*]$ is a real semiinterval without singularities: $K \cap \Sigma_L = \varnothing$. In this section we extend the exponential upper estimate for the number of zeros in polynomial envelopes to cover also such case.

4.1. Local properties of the Wronskians and a special lexicographical ordering

The lexicographical ordering (8) of the monomials $F_{\alpha jk}$ was determined by the ordering of the functions f_j constituting the fundamental system of solutions. For the upper/lower estimates established in Lemma 4, the ordering of the functions f_j was inessential. But if we want to analyze a small neighborhood of the singularity at $t = 0$, then this ordering must be chosen in a specific way.

Lemma 5. *If $t = 0$ is a real Fuchsian singularity for a real differential equation $Lu = 0$, then the monodromy operator Δ_0 corresponding to a small loop around $t = 0$, can be put into the upper triangular form by a real linear transformation*: *this means that a fundamental system of solutions f_j can be chosen so that*

(1) *$f_j(t)$ are real on the segment K, and*
(2) $\Delta_0 f_j = \sum_{i \leqslant j} \mathbf{m}_{ji} f_i$.

(3) $\mathbf{m}_{jj} = \exp 2\pi\sqrt{-1}\lambda_i$, *where* λ_i *are points of the spectrum, repetitions allowed.*

Remark. The matrix $\mathbf{M} = \lfloor \mathbf{m}_{ji} \rfloor$ in general is nonreal. In the simplest example of an equation with all distinct real roots $\lambda_i \neq \lambda_j \in \mathbb{R}$ the monodromy matrix is diagonal: $\mathbf{M} = \operatorname{diag}(\exp 2\pi\sqrt{-1}\,\lambda_i)$. In the general case $\mathbf{M}$ will have the same exponential entries on the diagonal. If the spectrum is not simple, then the Jordanian basis for $\mathbf{M}$ may be nonreal, as the example of the equation $(tu')' = 0$ shows.

This equation has a fundamental system of solutions $f_1 \equiv 1$ and $f_2 = \ln t$, real on $(0, +\infty)$, and the monodromy matrix at the point $t = 0$ is $\left[\begin{smallmatrix} 1 & 2\pi\sqrt{-1} \\ 0 & 1 \end{smallmatrix}\right]$. Clearly, any Jordanian basis for the monodromy will be nonreal.

Proof. The monodromy group of a real equation is τ-symmetric in the sense of §2. Hence for a monodromy matrix $\mathbf{M}$ corresponding to a singularity on the real axis, the identity $\mathbf{M}^{-1} = \overline{\mathbf{M}}$ holds. This means that the spectrum of $\mathbf{M}$ is symmetric with respect to the unit circle: if μ is an eigenvalue, then $\overline{\mu}^{-1}$ also is. If $|\mu| = 1$ is an eigenvalue on the unit circle, then the corresponding eigenvector can be chosen real. Indeed, if $\mathbf{M}\mathbf{z} = \mu\mathbf{z}$, then $\mathbf{M}^{-1}\overline{\mathbf{z}} = \overline{\mathbf{M}}\overline{\mathbf{z}} = \overline{\mu\mathbf{z}} = \mu^{-1}\,\overline{\mathbf{z}}$, hence $\overline{\mathbf{z}}$ is also an eigenvector with the same eigenvalue μ. But then either $\operatorname{Re}\mathbf{z} = \frac{1}{2}(\mathbf{z} + \overline{\mathbf{z}})$ or $\operatorname{Im}\mathbf{z} = \frac{1}{2\sqrt{-1}}(\mathbf{z} - \overline{\mathbf{z}})$ will be a real nonzero eigenvector. This argument shows that in case of a simple spectrum the monodromy matrix $\mathbf{M}$ can be diagonalized by a *real* invertible transformation. By induction one may easily prove that in the case of multiple eigenvalues on the unit circle one may put $\mathbf{M}$ into an upper-triangular (though not Jordanian) form by a real transformation (see the remark above).

Now we show that if the singularity is real Fuchsian, then the monodromy operator has the spectrum on the unit circle. Indeed, if $|\mu| \neq 1$, then there exists $\lambda \notin \mathbb{R}$ such that $\exp 2\pi\sqrt{-1}\lambda = \mu$. Take an eigenfunction (the eigenvector of the monodromy operator) corresponding to μ: this eigenfunction must have a form $t^\lambda\,h(t)$, where $h(t)$ is single-valued (hence meromorphic) in a small punctured neighborhood of $t = 0$. But then, according to §1.5, $\lambda + n$ must belong to the spectrum of the singularity for some integer $n \in \mathbb{Z}$. This contradicts to the assumption that the spectrum is real. Hence $\operatorname{Spec}\mathbf{M}$ belongs to the unit circle, and we can apply the first argument to prove the Lemma. □

If the fundamental system of solutions is chosen according to Lemma 5, then the Wronskians $W_\beta(t)$ associated with the corresponding ordering, will satisfy the following monodromy condition:

$$\Delta_0 W_\beta(t) = \exp(2\pi\sqrt{-1}\,\Lambda_\beta)\,W_\beta(t), \qquad \beta = 1,\dots,n^2(d+1),$$

where $\Lambda_\beta \in \mathbb{R}$ are real numbers. Indeed, the linear subspace spanned by the first β monomials $F_{\alpha jk}$ will be then invariant by Δ_0. More exactly, the monodromy matrix factor for the monomials $F_{\alpha jk}$ after the specified ordering will be the block diagonal matrix of the size $n^2(d{+}1)\times n^2(d{+}1)$ with the upper-triangular block M of the size $n\times n$ occurring $n(d{+}1)$ times on the diagonal. The diagonal entries of the matrix factor M are the exponentials $\exp 2\pi\sqrt{-1}\,\lambda_i$ (see the remark above), while the numbers $\exp 2\pi\sqrt{-1}\,\Lambda_\beta$ are the upper-left $\beta\times\beta$-minors of that large matrix. Moreover, without loss of generality one may assume that $0 \leqslant \Lambda_\beta < 1$, since integer parts of Λ_β are inessential.

Corollary. *The functions*

$$\widetilde{W}_\beta(t) = t^{-\Lambda_\beta}\,W_\beta(t), \qquad 0 \leqslant \Lambda_\beta < 1,$$

are single-valued in a punctured neighborhood of the origin.

4.2. Zeros of Wronskians near Fuchsian singularities

Since f_j have a regular singularity at $t = 0$, the functions $\widetilde{W}_\beta$ may have at most poles at $t = 0$, being real on the real axis. In order to estimate the order of this pole, we introduce the growth exponent for a real function φ as

$$\mathrm{G}_0\,[\varphi] = \limsup_{t\to 0^+} \frac{\ln|\varphi(t)|}{\ln t^{-1}}.$$

Due to the known analytic structure of the fundamental solutions f_j, one has the following estimates:

$$\begin{aligned}
\mathrm{G}_0\left[f_j^{(k-1)}\right] &\leqslant O_L(1),\\
\mathrm{G}_0\left[t^\alpha f_j^{(k-1)}\right] &\leqslant O_L(1) - \alpha,\\
\mathrm{G}_0\left[\partial^\beta\left(t^\alpha\, f_j^{(k-1)}\right)\right] &\leqslant O_L(1) - \alpha + \beta \leqslant O_L(1) + \beta,\\
\mathrm{G}_0\,[W_\beta] &\leqslant \beta\cdot O_L(1) + \frac{1}{2}\beta(\beta-1),\\
\mathrm{G}_0\left[\widetilde{W}_\beta\right] &\leqslant \beta\cdot O_L(1) + \frac{1}{2}\beta(\beta-1).
\end{aligned}$$

These arguments prove the following result.

Lemma 6. *There exists a sequence of real numbers θ_β satisfying the estimate*

$$\theta_\beta \leqslant O_L(d^2) \qquad \forall \beta = 1, 2, \dots, n^2(d+1) \tag{9}$$

such that the functions

$$\widehat{W}_\beta = t^{\theta_\beta}\, W_\beta(t)$$

are holomorphic at $t = 0$. $\square$

Remark. We always choose the branches of W_β and t^θ real on the positive semiaxis.

Remark. The similar estimates can be done in a neighborhood of $t = \infty$ for $\mathrm{G}_\infty[\varphi] = \limsup_{t\to+\infty} \ln|\varphi(t)|/\ln t$:

$$\mathrm{G}_\infty\left[t^\alpha f_j^{(k-1)}\right] \leqslant d \cdot \frac{d}{n} + O_L(1),$$
$$\mathrm{G}_\infty[W_\beta] \leqslant O_L(d^2) \qquad \forall \beta = 1, \dots, n^2(d+1).$$

This proves analyticity of $t^{-\theta_\beta}\, W_\beta(t)$ in a neighborhood of infinity for an appropriate choice of θ_β subject to the same asymptotic estimate.

4.3. Proof of Theorem 3

First consider the case of a finite real singularity; without loss of generality we may put it being at the origin $t = 0$.

Let $K = (0, t_*]$ be the semiinterval, $U \subset \mathbb{C} \smallsetminus (\Sigma_L \smallsetminus 0)$ an open connected simply connected neighborhood of $\overline{K} = [0, t_*]$ and $D \subset U$ a small open disk on a positive distance from $t = 0$ and the boundary of U.

From Lemma 4 it follows that for all $\beta = 1, \dots, n^2(d+1)$

$$\max_{t\in\overline{D}} |W_\beta(t)| \geqslant \exp(-\exp O_{D,L}(d)),$$
$$\max_{t\in\overline{U}} |W_\beta(t)| \leqslant \exp\exp O_{U,L}(\ln d),$$

For any compact subset D of the universal covering over $\mathbb{C} \smallsetminus 0$ the function t^θ (or more precisely, the branch of t_+^θ, real on the positive semiaxis) on D admits the two-sided estimate:

$$\exp(-|\theta| \cdot O_D(1)) \leqslant \min_{t\in D} \left|t_+^\theta\right| \leqslant \max_{t\in D} \left|t_+^\theta\right| \leqslant \exp(|\theta| \cdot O_D(1)). \tag{10}$$

These estimates imply that for all β and for the proper choice of branches

$$\max_{t\in K\cup\overline{D}} |\widehat{W}_\beta(t)| \geqslant \max_{t\in\overline{D}} |\widehat{W}_\beta(t)| \geqslant \exp(-\exp O_{D,L}(d)),$$
$$\max_{t\in\partial U} |\widehat{W}_\beta(t)| \leqslant \exp\exp O_{U,L}(\ln d),$$

But the functions $\widehat{W}_\beta(t)$ are in fact analytic in U. Hence the generalized Jensen lemma can be applied to $\widehat{W}_\beta$, yielding an exponential upper estimate for the number of complex isolated zeros of the latter. But since t^{θ_β} are invertible on the positive semiaxis, the same estimate holds also for the original Wronskians $W_\beta(t)$ on K. Thus one may apply Theorem 6 to the semiinterval K, to prove the simple exponential estimate for the number of isolated zeros in this case. The proof of Theorem 3 in the case of a finite singularity is complete.

If the singularity is at $t=\infty$, then the estimates established in the remark above should be used. □

5. Almost irreducible equations and their polynomial envelopes

In this section we prove that the simple exponential estimate for the number of real isolated zeros holds also for polynomial envelopes of linear equations with essentially irreducible monodromy. The proof is based on the well-known procedure of depression (reduction) of the order of a linear differential equation provided that some of its solutions are known [In].

5.1. Depression of the order of a linear equation

Assume that the operator L admits a nontrivial factorization, $L = L'P$, with $L', P \in \mathfrak{D}$, $\operatorname{ord} P = 1$, $P = \partial + a(t)$, $a \in \Bbbk$. Then the fundamental system of solutions for $Lu = 0$ can be chosen in the form

$$f_1, f_2, \dots, f_n,$$

where f_n is a nontrivial solution for the equation $Pu = 0$.

For an arbitrary function f from the rational envelope $\mathfrak{R}(L)$ we find a differential operator $R = R_f \in \mathfrak{D}$, depending on f, such that $Rf \in \mathfrak{R}(L')$, and estimate the Rolle index of R_f. Let

$$f = \sum_{j,k=1}^{n} r_{jk} f_j^{(k-1)}, \qquad r_{jk} \in \Bbbk,\ \deg r_{jk} \leqslant d,$$

be an arbitrary function from the rational d-envelope of L. Since f_n satisfies a *first* order equation, all derivatives of f_n up to order $n-1$ are proportional to f_n with coefficients from $\Bbbk$ of degrees $O_P(1)$. Thus without loss of generality one may assume that $r_{nk} = 0$ for $k = 2, \dots, n$, and $\deg r_{n1} \leqslant d$. Then the operator $R_0 = P \cdot \widehat{r}_0$, $\widehat{r}_0 = r_{n1}^{-1}$, applied to f, will eliminate the last term while preserving the form of the combination $\sum_{j=1}^{n-1} \sum_{k=1}^{n} r_{jk} f_j^{(k-1)}$: the degrees of the new coefficients will be at most $4d + O_L(1) = O_L(d)$.

The functions $\widetilde{f}_j = P f_j = f_j' + a f_j$ constitute a fundamental system of solutions for the equation $L'u = 0$, since $0 = Lf_j = L'Pf_j = L'\widetilde{f}_j$. Moreover,

$$f_j^{(k-1)} = b_k f_j + \sum_{\ell=1}^{k-1} c_\ell \widetilde{f}_j^{(\ell-1)}, \qquad k = 2, \dots, n, \quad b_k, c_\ell \in \Bbbk,$$

and even more generally, for any operator $A \in \mathfrak{D}$ the operator division with remainder $A = c + BP$ [In] gives

$$Af_j = cf_j + B\widetilde{f}_j, \qquad c \in \Bbbk,\ B \in \mathfrak{D},\ \operatorname{ord} B \leqslant \operatorname{ord} A - 1, \tag{11}$$

The degrees of the coefficients b_k, c_ℓ are at most $O_P(1)$, hence the function $R_0 f$ can be expressed as

$$R_0 f = \sum_{j,k=1}^{n-1} \widetilde{r}_{jk} \widetilde{f}_j^{(k-1)} + \sum_{j=1}^{n-1} q_j f_j,$$

with all rational coefficients of degrees $O_L(d)$.

The first sum is an element from the rational $O_L(d)$-envelope of L', as required. To eliminate the second sum, we apply the operator $R_1 = L \cdot \widehat{r}_1 \in \mathfrak{D}$, $\widehat{r}_1 = q_1^{-1} \in \Bbbk$, $\deg \widehat{r}_1 = O_L(d)$. By (11),

$$R_1 f_1 \equiv 0 \bmod \mathfrak{R}(L'), \qquad R_1 f_j \equiv \widetilde{q}_j f_j \bmod \mathfrak{R}(L') \quad j = 2, \dots, n-1,$$

thus the total number of terms in the second sum is decreased by 1. Iterating the last step $n-1$ times, one may construct operators $R_2, \dots, R_{n-1} \in \mathfrak{D}$ in such way that $R_j = L \cdot \widehat{r}_j$, $\widehat{r}_j \in \Bbbk$, $\deg \widehat{r}_j = O_L(d)$ and

$$R_{n-1} \cdots R_2 \cdot R_1 \sum_{j=1}^{n-1} q_j f_j \equiv 0 \bmod \mathfrak{R}(L').$$

Evidently, the degrees of the coefficients of all combinations and operators will be at most $O(d)$. Finally we put

$$R = \widehat{r}_n \cdot R_{n-1} \cdots R_2 \cdot R_1 \cdot R_0,$$

where the rational coefficient $\widehat{r}_n \in \Bbbk$ is chosen in such a way that R is an operator with polynomial coefficients.

Then $Rf \in \mathfrak{R}(L')$, and

(1) the degrees of the coefficients of Rf represented as an element from $\mathfrak{R}(L')$ will be at most $O_L(d)$,
(2) $R = \widehat{r}_n \cdot L \cdot \widehat{r}_{n-1} \cdot L \cdots \widehat{r}_2 \cdot L \cdot \widehat{r}_1 \cdot P \cdot \widehat{r}_0$, with $\widehat{r}_0 = r_{n1}^{-1}$, $\deg \widehat{r}_j = O_L(d)$.

5.2. Demonstration of Theorems 4 and 5

Let f be an analytic function on a segment K from the polynomial d-envelope of an operator $L \in \mathfrak{D}$, and $L = L'P$, $\operatorname{ord} P = 1$. Consider the operator R constructed in the previous section. This operator transforms f into the function Rf which is also analytic on K (since the coefficients of R are polynomial) and belongs to the rational envelope of L'.

The same arguments that proved Theorem 6, show that the Rolle index of the operator R on K is at most $O_L(d)$ provided that the Rolle index of L is finite on K and independent of d. Indeed, the number of poles of each rational factor is at most $O_L(d)$. Thus if L' is already irreducible, then the simple exponential estimate for the number of zeros of Rf and the upper estimate for the Rolle index of R imply the simple exponential estimate for the number of zeros of f on K.

If $L' = L''P'$, then the above arguments may be iterated, and we conclude by induction that the simple exponential estimate for the number of zeros holds on K for polynomial envelopes of any operator $L = L_* P_1 P_2 \cdots P_m$, if it holds for L_* on K.

In assumptions of Theorem 4 the segment K is compact and on a positive distance from Σ_L, so the Rolle index of L is automatically finite. If K is a semiinterval with a real Fuchsian singularity at the endpoint, then the real spectrum assumption guarantees that the Rolle index of L will be also finite on K despite the presence of the singularity. On the other hand, the real spectrum assumption for L implies that the spectrum of the last irreducible factor L_* will be also real at the endpoint so that the simple exponential estimate holds for the polynomial envelope of L_* on K. □

6. Generalizations

Theorem 2, the principal result which implies all other assertions of this paper, has a global nature and is valid in a much more general settings. Let $U \subseteq \mathbb{C}$ be any open domain symmetric by the involution $\tau\colon t \mapsto \bar{t}$, and $\widetilde{\Bbbk} = \mathcal{A}_\tau(U)$ the field of single-valued analytic functions in U, symmetric by the involution τ. Then we may consider an arbitrary unitary linear differential operator L with coefficients from $\widetilde{\Bbbk}$:

$$L = \sum_{k=0}^{n} a_{n-k}(t)\,\partial^k, \qquad a_i \in \widetilde{\Bbbk},\ a_0 \equiv 1.$$

Then there naturally arises the monodromy representation of the fundamental group of U by $(n\times n)$-matrices $\gamma \mapsto \mathbf{M}_\gamma^{-1}$, and one can define irreducible operators as in §3.4.

Theorem 2′. *If K is a compact subset of $U \cap \mathbb{R}$ and $L \in \widetilde{\Bbbk}[\partial]$ is irreducible, then the assertion of Theorem 2 about the simple exponential bound for the number of real isolated zeros remains valid in this extended setting as well.* □

In other words, coefficients of the operator L may have even essential singularities on the Riemann sphere. In fact, even the assumption that all coefficients are real on the real axis, can be dropped away (however, this would require different type of arguments). Theorem 4 also is valid in this extended setting as well; in both cases one should reproduce literally the same proof.

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