COMPLEX CELLULAR STRUCTURES

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Abstract. We introduce the notion of a *complex cell*, a complexification of the cells/cylinders used in real tame geometry. Complex cells are equipped with a natural notion of holomorphic extension, and the hyperbolic geometry of a cell within its extension provides the class of complex cells with a rich geometric function theory absent in the real case. We use this to prove a complex analog of the cellular decomposition theorem of real tame geometry. In the algebraic case we show that the complexity of such decompositions depends polynomially on the degrees of the equations involved.

Using this theory, we sharpen the Yomdin-Gromov algebraic lemma on C^r-smooth parametrizations of semialgebraic sets: we show that the number of C^r charts can be taken to be polynomial in the smoothness order r and in the complexity of the set. The algebraic lemma was initially invented in the work of Yomdin and Gromov to produce estimates for the topological entropy of C^∞ maps. Combined with work of Burguet, Liao and Yang, our refined version establishes an optimal sharpening of these estimates for *analytic* maps, in the form of tight bounds on the tail entropy and volume growth. This settles a conjecture of Yomdin who proved the same result in dimension two in 1991. A self-contained proof of these estimates using the refined algebraic lemma is given in an appendix by Yomdin.

The algebraic lemma has more recently been used in the study of rational points on algebraic and transcendental varieties. We use the theory of complex cells in these two directions. In the algebraic context we prove a sharpening of a result of Heath-Brown on interpolating rational points in algebraic varieties. In the transcendental context we prove an interpolation result for (unrestricted) logarithmic images of subanalytic sets.

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1. Introduction

In this section we aim to provide an intuitive motivation for the formal notion of complex cells introduced in §2. We discuss the smooth parametrization problem, particularly the Yomdin-Gromov algebraic lemma and its sharpenings established in this paper. We then discuss the hyperbolic obstruction to a Gromov-Yomdin type lemma in the holomorphic category and motivate the notion of a complex cell as a way of overcoming this obstruction. Finally we briefly discuss the applications of our parametrization results in smooth dynamics and in diophantine geometry.

1.1. **Smooth parametrizations.**

1.1.1. *The Yomdin-Gromov Algebraic Lemma.* In [50, 49] Yomdin gave a proof of Shub's entropy conjecture for C^∞-maps. A key step in this proof was the construction of C^r-smooth parametrizations of semialgebraic sets with the number of charts depending only on the combinatorial data. More precisely, for a C^r-smooth function $f : U \to \mathbb{R}^n$ on a domain $U \subset \mathbb{R}^m$ we denote by $\|f\|$ the maximum norm on U and

$$\|f\|_r := \max_{|\boldsymbol{\alpha}|\leqslant r} \frac{\|D^{\boldsymbol{\alpha}} f\|}{\boldsymbol{\alpha}!}. \tag{1}$$

The original parametrization theorem of [50, 49] involved a technical condition on the removal of a small piece from the semialgebraic set being parameterized. The formulation was further sharpened by Gromov [21], who proved the result in the following clean formulation known as the Yomdin-Gromov Algebraic Lemma (see also [13] for a detailed proof).

Theorem ([21, 3.3. Algebraic Lemma])**.** *Let $X \subset [0,1]^n$ be a semialgebraic set of dimension μ defined by conditions $p_j(\mathbf{x}) = 0$ or $p_j(\mathbf{x}) < 0$ where p_j are polynomials and $\sum \deg p_j = \beta$. Let $r \in \mathbb{N}$. There exists a constant $C = C(n, \mu, r, \beta)$ and semialgebraic maps $\phi_1, \dots, \phi_C : (0,1)^\mu \to X$ of such that their images cover X and $\|p_j\|_r \leqslant 1$ for $j = 1, \dots, C$.*

Understanding the behavior of the constant $C(n, \mu, r, \beta)$ has been the key difficulty in establishing a conjectural sharpening of Yomdin's results for analytic maps, as we discuss in §1.3. More precisely, what is needed is an estimate $C(n, \mu, r, \beta) = \operatorname{poly}_n(r, \beta)$. Such an estimate is our first main result.

Theorem 1. *In the Yomdin-Gromov algebraic lemma one may take $C(n, \mu, r, \beta) = \operatorname{poly}_n(\beta) \cdot r^\mu$. Moreover the maps ϕ_j can be chosen to be semialgebraic of complexity $\operatorname{poly}_\ell(\beta, r)$.*

1.1.2. *Pila-Wilkie's generalization to o-minimal structures.* Parametrizations by C^r-smooth functions have also been used to great effect in the seemingly unrelated study of rational points on algebraic and transcendental sets. The algebraic lemma was first introduced to this subject by Pila and Wilkie [37], who proved its far-reaching generalization for arbitrary o-minimal structures.

Theorem ([37, Theorem 2.3])**.** *Let $\{X_p \subset [0,1]^n\}$ be a family of sets definable in an o-minimal structure, with $\dim X_p \leqslant \mu$. There exists a constant $C = C(X,r)$ such that for any p there exist definable maps $\phi_1,\ldots,\phi_C : (0,1)^\mu \to X_p$ such that their images cover X and $\|\phi_j\|_r \leqslant 1$ for $j = 1,\ldots,C$.*

In the diophantine direction, understanding the behavior of $C(X,r)$ with respect to the geometry of the family X and the smoothness order r is also a question of great importance as we discuss in §1.4. In the recent paper [17] Cluckers, Pila and Wilkie made significant progress in this direction by proving that for subanalytic sets (and also sets in the larger structure $\mathbb{R}_{\text{an}}^{\text{pow}}$) one has $C(X,r) = \operatorname{poly}_X(r)$. For subanalytic sets we obtain a similar result with a precise control over the degree.

Theorem 2. *In the Pila-Wilkie algebraic lemma for $\mathbb{R}_{an}$ one may take $C(X,r) = O_X(r^\mu)$ where $\mu := \max_p(\dim X_p)$.*

1.1.3. *Mild parametrizations.* It is natural to inquire whether one can in fact replace the C^r-charts in the algebraic lemma with C^∞ charts with appropriate control over the derivatives. In [39] Pila introduced a notion of this type called *mild parametrization* and investigated its diophantine applications related to the Wilkie conjecture.

Definition 1 (Mild parametrization)**.** *A smooth map $f : U \to [0,1]^n$ on a domain $U \subset \mathbb{R}^m$ is said to be (A,C)-mild if*

$$\|D^{\boldsymbol{\alpha}} f\| \leqslant \boldsymbol{\alpha}!(A|\boldsymbol{\alpha}|^C)^{|\boldsymbol{\alpha}|}, \qquad \forall \boldsymbol{\alpha} \in \mathbb{N}^m. \tag{2}$$

In [26] it is shown that every subanalytic set $X \subset [0,1]^n$ admits $(A,0)$-mild (i.e. analytic) parametrizations, but this result is not uniform over families and its diophantine applications are restricted to the case of curves [26] and surfaces [27]. We prove the following uniform version.

Theorem 3. *Let $\{X_p \subset [0,1]^n\}$ be a subanalytic family of sets, with $\dim X_p \leqslant \mu$. There exist constants $A = A(X)$ and $C = C(X)$ such that for any p there exist $(A,2)$-mild maps $\phi_1,\ldots,\phi_C : (0,1)^\mu \to X_p$ whose images cover X. If $\{X_p\}$ is semialgebraic as in the Yomdin-Gromov algebraic lemma then one may take $A, C = \operatorname{poly}_n(\beta)$.*

The construction of a mild parametrization is a-priori more delicate than its C^r counterpart since it requires one to control derivatives of all orders at once. In fact, Thomas [47] has shown that there exist o-minimal structures without definable mild parametrizations. In our results there is also a key difference between the C^r algebraic lemma and its mild counterpart. Namely, in the C^r version the parameterizing maps are themselves subanalytic, and can be chosen to depend subanalytically on the parameters of the family. We make no such guarantee in the mild version. In fact, we show that any parameterization of a family of hyperbolas by a definable family of $\mathbb{R}_{\text{an}}$ maps must have unbounded C^r-norms for some finite r. More precisely we have the following sharp estimate.

Proposition 2. *Let $F := (x,y) : [0,1]^2 \to [0,1]^2$ be an $\mathbb{R}_{an}$-definable map satisfying $x(e,t) \cdot y(e,t) = e$. Suppose that for each r there exists a constant M_r such that*

$$|\partial_t^r x(e,t)| < M_r \qquad\qquad |\partial_t^r y(e,t)| < M_r \tag{3}$$

whenever the derivatives are defined. Then the Euclidean length of $\log F(e_0,[0,1])$ is bounded by a constant independent of $e_0 \neq 0$ (here $\log$ is applied coordinate-wise).

This result is easily seen to be essentially tight: one can indeed construct parameterizations F as in Proposition 2 to cover any definable family of intervals of constant logarithmic length in the hyperbolas. Somewhat surprisingly, our proof goes via reduction using complex cells to the following simple statement from geometric function theory, which is proved using a p-valent version of the Koebe 1/4-theorem.

Lemma 3. *Let $\xi : D(2) \to \mathbb{C} \setminus \{0\}$ be holomorphic and suppose that ξ is p-valent. Then the length of $\log \xi([-1, 1])$ is bounded by a constant $8\pi p(p + 1)$.*

The proof of Proposition 2 and Lemma 3 is given in §10.4.

1.2. **Holomorphic parametrizations.** In the preceding section we discussed the problem of constructing smooth parametrizations for semialgebraic and subanalytic sets. It is natural to wonder whether such constructions can be analytically continued to give holomorphic parametrizations in some suitable sense. In this section we demonstrate an obstruction of a hyperbolic-geometric nature to a naive formulation, and motivate the notion of a complex cell as a possible way of overcoming this obstruction.

1.2.1. *Parametrizations of analytic curves and questions of uniformity.* Let $D \subset \mathbb{C}$ denote the unit disc and $D^{1/2} \subset \mathbb{C}$ denote the disc of radius two. If X is a holomorphic curve and $K \subset X$ is compact then one can always cover K by images of analytic charts $f_j : D \to X$. To avoid bad behavior near the boundary, we can also require that each f_j extend as a holomorphic map to $D^{1/2}$ (this is an arbitrary choice which could be replaced by any fixed neighborhood of $\bar{D}$). Parameterizations of this type are useful in analysis: they allow one to study holomorphic structures on X in terms of their pullback to $D \subset D^{1/2}$, where a rich geometric function theory is available. Moreover, the theory of resolution of singularities shows that this type of parametrization can be extended to parametrizations of analytic sets of any dimension. A notion of this type was introduced by Yomdin [51] under the name *analytic complexity unit (acu)*. A similar notion was considered under the name "doubling coverings" in [19].

Ideally one would like to prove that holomorphic parameterizations of the type described above can be made with a uniform number of charts when X is replaced by an analytic family X_ε of curves (or higher dimensional sets) and K by its compact subfamily. Indeed, this would be an appropriate analog of the Yomdin-Gromov algebraic lemma in the holomorphic category. Unfortunately this is impossible, even for simple families of algebraic curves. Consider the family X_ε of hyperbolas restricted to the polydisc of radius 2 and K_ε their restriction to a polydisc of radius 1,

$$X_\varepsilon := \{(x, y) : xy = \varepsilon,\ x, y \in D^{1/2}\} \qquad K_\varepsilon := X_\varepsilon \cap (D \times D). \tag{4}$$

The projection to the x-axis gives biholomorphisms $K_\varepsilon \simeq A_\varepsilon := A(\varepsilon, 1)$ and $X_\varepsilon \simeq A_\varepsilon^{1/2} := A(\varepsilon/2, 2)$ where $A(r_1, r_2) := \{r_1 < |z| < r_2\}$. Let $f : D^{1/2} \to X_\varepsilon \simeq A_\varepsilon^{1/2}$. If we equip $D^{1/2}$ and $A_\varepsilon^{1/2}$ with their respective hyperbolic metric then the Schwarz-Pick lemma implies that f is distance-contracting (see §5 for a brief reminder on this subject). In particular the diameter of $f(D)$ in $A_\varepsilon^{1/2}$ is bounded by the diameter of D in $D^{1/2}$, which is a constant independent on ε. On the other hand, the diameter of $K_\varepsilon \simeq A_\varepsilon$ in $A_\varepsilon^{1/2}$ tends to infinity as $\varepsilon \to 0$ (in fact with order $\log |\log \varepsilon|$, see Remark 41). It follows that to cover K_ε by images of maps as above at least

$\log|\log\varepsilon|$ charts are required, and this is easily seen to be a tight asymptotic. We note that Yomdin [52] obtains a similar result with $|\log\varepsilon|$ which holds under the assumption that the maps are p-valent for some fixed p.

1.2.2. *A uniform parameterization result with discs and annuli.* It turns out that in the one-dimensional case, the hyperbolic restriction described above is essentially the only obstacle to uniform parametrization. Namely, suppose that in addition to covering K_ε by images $f(D)$ of holomorphic maps $f : D^{1/2} \to X_\varepsilon$ we allow also images $f(A)$ of holomorphic maps $f : A^{1/2} \to X_\varepsilon$ for any annulus A. Note that the diameter A in A^δ is not uniformly bounded: it depends on the conformal modulus. With this added freedom one can indeed construct a parameterization with a uniformly bounded number of charts for a family of curves. We proceed to explain the elementary geometric considerations that lead to this result.

To simplify our presentation we suppose that $X_\varepsilon \subset \mathbb{C}^2 \times \mathbb{C}_\varepsilon$ is a family of algebraic plane curves which project properly under $\pi : (x, y) \to x$ (although a similar local argument works in the analytic case without the assumption of properness as well). We will parameterize the fibers of $K_\varepsilon := X_\varepsilon \cap (D \times D)$. Fix $\varepsilon \in \mathbb{C}$. Let $\Sigma \subset \mathbb{C}$ denote the finite set of critical values of $\pi|_{X_\varepsilon}$. Denote by ν the order of the monodromy group of π. Note that $\#\Sigma$ and ν are bounded in terms of $\deg X_\varepsilon$ and in particular uniformly in ε. Suppose $\mathcal{C} \subset \mathbb{C}$ is one of the following three types:

(1) A disc $p + D(r)$ such that $p + D(2r)$ does not meet Σ.
(2) A disc $p+D(r)$ such that $p \in \Sigma$ and $p+D(2^\nu r)$ does not meet any additional points in Σ.
(3) An annulus $p + A(r_1, r_2)$ such that $p + A(r_1/2^\nu, 2^\nu r_2)$ does not meet Σ.

In case 1 one can construct a map $f : D \to p + D(r) \to X_\varepsilon$ by lifting π. In cases 2,3 the lift might be multivalued with a monodromy of finite order ν (uniformly bounded in ε), but precomposing with the map $z \to z^\nu$ we similarly obtain univalued charts $f : D \to X_\varepsilon$ or $f : A_r \to X_\varepsilon$. Moreover the assumption on $\mathcal{C}$ ensure that f extends holomorphically to $D^{1/2}$ or $A_r^{1/2}$. Taking the $\deg X_\varepsilon$ different lifts of π we thus obtain charts covering $X_\varepsilon \cap \pi^{-1}(\mathcal{C})$.

It remains to show that D can be covered by finitely many domains $\mathcal{C}$ as above, with their number depending only on the number of points in Σ (but not on their positions). This is a simple exercise in plane geometry which we urge the reader to attempt for themselves. It is also instructive to check that a similar statement would not hold had we not allowed annuli as in item 3 above. In Figure 1 we illustrate such a decomposition for the critical points $\Sigma = \{\pm\sqrt{\varepsilon}\}$ of the hyperbola $y^2 = x^2 - \varepsilon$ (this is just our original example $xy = \varepsilon$ rotated to satisfy our assumption of proper projection to the x-axis).

Decompositions of a similar type appeared under the name "Swiss-cheese decompositions" in [23] where they were used in the study of Green functions on Riemann surfaces. They also appeared in [8] where they were used in the study of collisions of singular points of Fuchsian differential equations.

1.2.3. *Parametrizing higher dimensional sets.* It is natural to ask whether a similar "uniform parametrization" statement might hold in higher dimensions. A naive attempt might be to use products of discs and annuli as the domains of definition. However, since complex annuli (unlike discs) have conformal moduli, it turns out to be more natural to allow domains consisting of families of annuli with varying

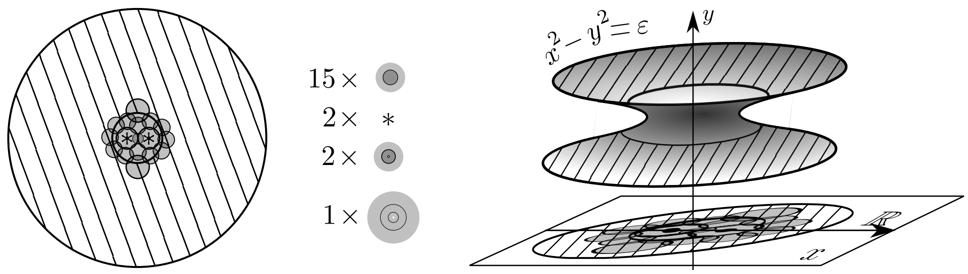

FIGURE 1. Covering of the hyperbola with discs and annuli, where $*$ corresponds to points of $\Sigma = \{\pm\sqrt{\varepsilon}\}$.

moduli. As an illustrative example, in the two-dimensional case we allow domains of the form

$$\mathcal{C} = D_\circ(1) \odot A(\mathbf{z}_1, 2) := \{(\mathbf{z}_1, \mathbf{z}_2) : 0 < |\mathbf{z}_1| < 1,\ |\mathbf{z}_1| < |\mathbf{z}_2| < 2\}. \tag{5}$$

The notation $\odot$ above is an example of a more general notation introduced in §2, and the domain above is an example of a *complex cell*. The definition closely resembles the notion of cells/cylinders in semialgebraic/subanalytic geometry, with the order relation $x < y$ replaced by the complex $|x| < |y|$.

To generalize the parametrization result from curves to higher dimensional sets we require not only an analog of the domains D or A, but also an analog of their extensions $D^{1/2}, A^{1/2}$. Correspondingly our complex cells $\mathcal{C}$ are endowed with a natural notion of δ-extension $\mathcal{C} \subset \mathcal{C}^\delta$. In the example above

$$\mathcal{C}^\delta = D^\delta(1) \odot A^\delta(\mathbf{z}_1, 2) = D(\delta^{-1}) \odot A(\delta\mathbf{z}_1, 2\delta^{-1}) := \\ \{(\mathbf{z}_1, \mathbf{z}_2) : |\mathbf{z}_1| < \delta^{-1},\ \delta|\mathbf{z}_1| < |\mathbf{z}_2| < 2\delta^{-1}\}. \tag{6}$$

This is defined for any $1/2 < \delta < 1$: we require the original fiber $A(\mathbf{z}_1, 2)$ to remain an annulus over the extension $D^\delta(1)$, and for $\delta < 1/2$ it would become an empty set.

Our main result, Theorem 8, shows in particular that one can uniformly uniformize any family of analytic sets if one is permitted to use general complex cells as the domains for the charts. More generally, Theorem 8 provides a cellular analog of the constructions of local resolution of singularities (LRS), see e.g. [5]. Loosely speaking it allows one to transform a collection of analytic functions into normal crossings uniformly over families - using complex cells as the domains of the charts. In the algebraic category, Theorem 8 gives effective control (polynomial in the degrees) on the number and complexity of the charts in terms of the complexity of the functions being transformed into normal crossings. This is where the most important step toward improving the asymptotics of the Yomdin-Gromov algebraic lemma takes place, and the proof makes extensive use of the hyperbolic properties of a cell $\mathcal{C}$ viewed as a subset of it extension $\mathcal{C}^\delta$.

1.2.4. *Sectorial parametrizations.* In the preceding sections we discussed the problem of establishing an analog of the algebraic lemma with holomorphic functions admitting extensions to whole discs, and showed that (unless one also allows annuli and more general complex cells) such an analog is impossible for hyperbolic reasons. A more modest goal might be to establish an analog of the algebraic lemma where the parametrizing maps admits analytic continuation to some suitable large domain (though not a full disc). If the domain is sufficiently large, one could then hope to control the derivatives using complex-analytic methods, for instance the Cauchy estimates.

It turns out that the correct domains for analytic continuation are complex sectors. More specifically, let $S(\varepsilon)$ denote the sector $S(\varepsilon) = \{\operatorname{Arg} z < \varepsilon, |z| < 2\}$, and for $B = (0,1)^\mu$ let $B(\varepsilon)$ denote the direct product of μ copies of $S(\varepsilon)$. All of our refinements of the algebraic lemma follow from the following statement about parametrizations with complex-analytic continuation to sectors.

Theorem 4. *Let $\{X_p \subset [0,1]^n\}$ be a subanalytic family of sets, with $\dim X_p \leqslant \mu$. Set $B = (0,1)^\mu$. There constants $C = C(X)$ and $\varepsilon = \varepsilon(X)$ such that for any p there exist maps $\phi_1, \ldots, \phi_C : B \to X_p$ whose images cover X_p. Moreover each ϕ_j extends holomorphically to $B(\varepsilon)$ and has unit C^1-norm there. If $\{X_p\}$ is semialgebraic as in the Yomdin-Gromov algebraic lemma then one may take $C, \varepsilon^{-1} = \operatorname{poly}_n(\beta)$.*

This theorem is proved (in a more general form) in §10.1. We start by constructing a complex-cellular parameterization for X_p, and then show how to construct maps from B into a complex cell which extend to $B(\varepsilon)$ with bounded C^1-norms. The C^r and mild statements of the algebraic lemma are deduced from the sectorial version by a simple (and self-contained) argument based on the Cauchy estimates in §10.3.

1.3. **Applications in dynamics.** The Yomdin-Gromov algebraic lemma was first invented for the purpose of studying the properties of the topological entropy of smooth maps. We recall the basic notions of topological entropy theory, Shub's entropy conjecture and Yomdin's result for C^∞-maps. We then discuss a refinement of these results in the analytic category culminating in Theorem 5 confirming in arbitrary dimension a conjecture of Yomdin that was previously known only in dimension two. Theorem 5 follows immediately from the refined algebraic lemma in combination with the work of Burguet, Liao and Yang [14]. A self-contained proof of Theorem 5 using the refined algebraic lemma is given in §A (by Yomdin). Below we mostly follow the notations and presentation of [14] in our survey.

1.3.1. *Topological entropy.* Let M be a compact metric space with metric $d : M^2 \to \mathbb{R}_{\geqslant 0}$ and let $f : M \to M$ be a continuous map. For $n \in \mathbb{N}$ define the n-th iterated metric by

$$d_n : M \times M \to \mathbb{R}_{\geqslant 0}, \qquad d_n(x,y) = \max_{i=0,\ldots,n-1} d(f^{\circ i}(x), f^{\circ i}(y)). \tag{7}$$

Recall if X is a metric space and $\Lambda \subset X$ we say that a set $K \subset X$ is ε-spanning for Λ if the ε-neighborhood of K contains Λ. For any subset $\Lambda \subset M$ and $\varepsilon > 0$ and $n \in \mathbb{N}$ we let $r_n(f, \Lambda, \varepsilon)$ denote the minimal cardinality of an ε-spanning set of Λ with respect to the d_n-metric.

The ε-*topological* and *topological* entropies of Λ are defined by

$$h(f,\Lambda,\varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log r_n(f,\Lambda,\varepsilon), \qquad h(f,\Lambda) = \lim_{\varepsilon\to 0} f(f,\Lambda,\varepsilon). \tag{8}$$

We set $h(f,\varepsilon) := h(f,M,\varepsilon)$ and $h(f) := h(f,M)$. This latter quantity is called the *topological entropy* of f. As its name implies, the topological entropy is independent of the choice of metric d. It was first defined by Adler, Konheim and McAndrew

[1] in a purely topological fashion, and later shown by Bowen [10] to be equivalent to the metric definition presented above.

1.3.2. *Tail entropy.* For $x \in M$ we define the *infinite dynamical ball* $B_\infty(x,\varepsilon)$ by

$$B_\infty(x,\varepsilon) := \{y \in M : d_n(x,y) < \varepsilon \quad \forall n \in \mathbb{N}\}. \tag{9}$$

Following Bowen [11], the ε-*tail* and *tail* entropies of f are defined by

$$h^*(f,\varepsilon) = \sup_{x\in M} h(f, B_\infty(x,\varepsilon)), \qquad h^*(f) = \lim_{\varepsilon\to 0} h^*(f,\varepsilon). \tag{10}$$

Bowen [11] has shown that the ε-tail entropy bounds the difference between the ε-entropy and the entropy, that is

$$|h(f) - h(f,\varepsilon)| < h^*(f,\varepsilon). \tag{11}$$

If $h^*(f,\varepsilon) = 0$ for some $\varepsilon > 0$ then the system (f, M) is called entropy-expansive (h-expansive); if $h^*(f) = 0$ then it is called *asymptotically entropy expansive* (asymptotically h-expansive). This notion of tail entropy was first introduced (with a different definition) by Misiurewicz [32] under the name *conditional entropy.* Misiurewicz showed that for asymptotically h-expansive systems the measure-theoretical entropy is upper-semicontinuous, and such systems therefore always admit an invariant measure of maximal entropy.

1.3.3. *Yomdin's results for smooth maps.* Assume from now on that M is a compact C^∞-smooth manifold equipped with a Riemannian metric. In [50] Yomdin proved Shub's entropy conjecture for C^∞ maps. This conjecture states that the logarithm of the spectral radius of $\operatorname{Spec} f$ of $f_* : H_*(M,\mathbb{R}) \to H_*(M,\mathbb{R})$ is a lower bound for $h(f)$. In fact it is relatively easy to show (by comparing volumes) that

$$\log \operatorname{Spec} f < h(f) + \max_{k=1,\dots,\dim M} v_k^*(f) \tag{12}$$

where $v_k^*(f)$ is the k-dimensional *local volume growth*, a quantity related to $h^*(f)$ whose precise definition we postpone to §A. Yomdin's fundamental result was that if f is a C^r-map then

$$v_k^*(f) \leqslant \frac{kR(f)}{r}, \qquad R(f) = \lim_{n\to\infty} \frac{1}{n} \sup_{x\in M} \|D_x f^{\circ n}\| \tag{13}$$

For C^∞-maps this implies $v_k^*(f) = 0$, and in combination with (12) yields Shub's entropy conjecture. Buzzi [16] later observed that Yomdin's argument also implied the same estimate $h^*(f) \leqslant (\dim M/r)R(f)$ for the tail entropy. For C^∞-maps this implies $h^*(f) = 0$, i.e. that C^∞-maps are asymptotically h-expansive.

1.3.4. *Controlling $h^*(f,\varepsilon)$ for analytic maps.* Bounding $h^*(f,\varepsilon)$ explicitly as a function of ε is an important problem with consequences for the computation of $h(f)$ (for instance using (11)) and for the study of the semicontinuity properties of the entropy. However, for arbitrary C^∞ maps $f : M \to M$ the rate of convergence of $h^*(f,\varepsilon)$ to zero as $\varepsilon \to 0$ can be arbitrarily slow, as shown by the examples of Burguet, Liao and Yang [14, Theorem L].

In [51] Yomdin considered the case of a real-analytic map $f : M \to M$ of a compact analytic surface M. In this context, using a holomorphic variant of the algebraic lemma based on the Bernstein inequality for polynomials, he was able to prove that

$$v_1^*(f,\varepsilon) \leqslant C(f) \cdot \frac{\log|\log\varepsilon|}{|\log\varepsilon|}. \tag{14}$$

Yomdin conjectured [51, Conjecture 6.1] that a similar estimate should hold (for v_k^*) for M of arbitrary dimension, but the limitation of the holomorphic parametrization technique to one dimension prevented such a generalization (see [52] for some results in this direction in dimension two). In [14, Theorem N] it is shown that the bound in Yomdin's conjecture is essentially sharp (for class slightly larger than the analytic class).

In [14] Burguet, Liao and Yang revisited the problem of the rate of convergence of the tail entropy for analytic, and more general, maps. The precise statement of their main result is technical and depends on the behavior of the algebraic lemma's constant $C(n,\mu,r,\beta)$. However, under the hypothesis $C(n,\mu,r,\beta) = \mathrm{poly}_n(r,\beta)$ the results of [14, Corollary B] take a particularly simple form: namely, they imply the generalization of Yomdin's result (both for v_k^* and h^*) to arbitrary dimension (and also for a class of functions somewhat larger than the analytic class). It is also shown in [14] that $C(n,1,r,\beta) = \mathrm{poly}_n(r,\beta)$ and this recovers Yomdin's result for analytic surfaces.

To summarize, following the work of [14] it has been clear that proving the $C(n,\mu,r,\beta) = \mathrm{poly}_n(r,\beta)$ is the missing step to establishing Yomdin's conjecture. Combined with the refined algebraic lemma proved in this paper this is now a theorem.

Theorem 5. *Let M be a compact analytic manifold and $f : M \to M$ be an analytic map. Then*

$$h^*(f,\varepsilon) \leqslant C(f) \cdot \frac{\log|\log\varepsilon|}{|\log\varepsilon|}, \qquad v_k^*(f,\varepsilon) \leqslant C(f) \cdot \frac{\log|\log\varepsilon|}{|\log\varepsilon|} \tag{15}$$

for $k = 1,\ldots,\dim M$.

Proof. We indicate the proof with the notations of [14]. Applying the refined algebraic lemma to the graph of the polynomial map $P : [0,1]^l \to [0,1]^l$ in [14, Section 2.1.2] one obtains the estimate $C_{r,l,m} = \mathrm{poly}_{m,l}(r)$. This implies that the sequence $M_k := k^{k^2}$ is (l,m)-admissible for any $l,m \in \mathbb{N}$. Then [14, Corollary B] applies to any map f which satisfies $\|f\|_r < Cr^{r^2}$ for some constant C and every $r \geqslant 0$. In particular, it applies to analytic maps. We also have

$$G_M(t) = \sup_k\{k\log k < t\} = \Theta\left(\frac{t}{\log t}\right). \tag{16}$$

Thus (15) follows from the conclusion of [14, Corollary B]. □

1.4. **Applications in diophantine geometry.** For $p \in \mathbb{P}^\ell(\mathbb{Q})$ we define $H(p)$ to be $\max_i |\mathbf{p}_i|$ where $\mathbf{p} \in \mathbb{Z}^{\ell+1}$ is a projective representative of p with

$$\gcd(\mathbf{p}_0,\ldots,\mathbf{p}_\ell) = 1. \tag{17}$$

For $\mathbf{x} \in \mathbb{A}^\ell(\mathbb{Q})$ we define its height to be the height of $\iota(\mathbf{x})$ for the standard embedding

$$\iota : \mathbb{A}^\ell \to \mathbb{P}^\ell, \qquad \iota(\mathbf{x}_{1..\ell}) = (1 : \mathbf{x}_1 : \cdots : \mathbf{x}_\ell). \tag{18}$$

For a set $X \subset \mathbb{P}^\ell(\mathbb{R})$ we denote

$$X(\mathbb{Q}, H) := \{\mathbf{x} \in X \cap \mathbb{P}(\mathbb{Q})^\ell : H(x) \leqslant H\}, \tag{19}$$

and similarly for $X \subset \mathbb{R}^\ell$.

In [9] Bombieri and Pila introduced the interpolation determinant method for estimating the quantity $\#X(\mathbb{Q}, H)$ as a function of H when X is the graph of a C^r (or C^∞) smooth function $f : [0, 1] \to \mathbb{R}$. It turns out that two very different asymptotic behaviors are obtained depending on whether the graph of f belongs to an algebraic plane curve. The algebraic lemma has been used in both of these directions, to generalize from graphs of functions to more general sets. In §B.1 we give a complex-cellular analog of the Bombieri-Pila determinant method and use it to deduce some applications for both the algebraic and transcendental contexts. We briefly describe the main results below.

1.4.1. *A result for algebraic varieties.* We prove the following result.

Theorem 6. *Let $X \subset \mathbb{P}(\mathbb{C})^\ell$ be an irreducible algebraic variety of dimension m and degree d. Then $X(\mathbb{Q}, H)$ is contained in N hypersurfaces of degree k, none of which contain X, where*

$$N = \operatorname{poly}_\ell(d, k) \cdot H^{(m+1)d^{-1/m}(1+\operatorname{poly}_\ell(d)/k)}. \tag{20}$$

We make note of two particular choices for k, namely

$$k = \operatorname{poly}_\ell(d)/\varepsilon \qquad \Longrightarrow \qquad N = \operatorname{poly}_\ell(d, 1/\varepsilon) \cdot H^{(m+1)d^{-1/m}+\varepsilon} \tag{21}$$

$$k = \operatorname{poly}_\ell(d) \cdot \log H \qquad \Longrightarrow \qquad N = \operatorname{poly}_\ell(d, \log H) \cdot H^{(m+1)d^{-1/m}}. \tag{22}$$

The first choice improves the dependence on the degree d to polynomial in Heath-Brown's result [25] for hypersurfaces and Broberg's result [12, Theorem 1] (see also Marmon [30]). The second choice replaces an H^ε factor by a power of $\log H$, similar to the various results established for curves by Pila [40] and Salberger [45]. We remark that in the case of curves, Walsh [48] recently proved a result eliminating the $\log H$ factor altogether and it would be interesting to study whether this can be generalized to arbitrary dimension. We briefly survey the history of these results in §B.2.1 and prove Theorem 6 in §B.2.2.

1.4.2. *The Pila-Wilkie theorem.* Let $f : I \to \mathbb{R}^2$ and suppose that $X_f := f(I)$ is transcendental. Bombieri and Pila [9] and Pila [36] used the determinant method to show that $\#X(\mathbb{Q}, H) = O_{X,\varepsilon}(H^\varepsilon)$ for any $\varepsilon > 0$. To generalize this result to higher dimensions, let $X \subset \mathbb{R}^\ell$ and denote by X^{alg} the union of all positive dimensional connected semialgebraic sets contained in X and by $X^{\mathrm{trans}} := X \setminus X^{\mathrm{alg}}$. Pila and Wilkie [37] proved that for any X definable in an o-minimal expansion of $\mathbb{R}$ we have

$$\#X^{\mathrm{trans}}(\mathbb{Q}, H) = O_{X,\varepsilon}(H^\varepsilon) \qquad \forall \varepsilon > 0. \tag{23}$$

Moreover, if X varies over a definable family then the asymptotic constants can be taken uniform over the family. The o-minimal version of the Yomdin-Gromov algebraic lemma plays the central role in the proof of the Pila-Wilkie theorem.

1.4.3. *The Wilkie conjecture.* Wilkie [37] has conjectured that for sets definable in $\mathbb{R}_{\exp}$ the asymptotic in the Pila-Wilkie theorem can be improved to $\operatorname{poly}_X(\log H)$, and it is natural to make similar conjectures for other "natural" geometric structures (although such a result does not hold for general subanalytic curves, see [38, Example 7.5]). One of the key obstacles to proving the Wilkie conjecture, at least following the Pila-Wilkie strategy, is to obtain an estimate on $C(X,r)$ which is polynomial in r and the complexity of X. Some one-dimensional and some restricted two-dimensional cases of the Wilkie conjecture have been established using methods involving Pfaffian functions and mild parametrizations [41, 27, 42, 15]. In [6] we developed a complex-analytic approach to the Pila-Wilkie theorem in the subanalytic case, and in [7] we used this approach to prove the Wilkie conjecture for the structure $\mathbb{R}^{\mathrm{RE}}$ generated by the exponential and trigonometric functions *restricted* to some finite interval.

1.4.4. *Applications to unlikely intersections in diophantine geometry.* The Pila-Wilkie theorem has been applied to great effect in the study of unlikely intersections in diophantine geometry. We briefly mention Pila-Zannier's proof the the Manin-Mumford conjecture [44], Pila's proof of the André-Oort conjecture for modular curves [43] and Masser-Zannier's work on simultaneous torsion points in elliptic families [31]. We refer the reader to [53, 46] for a survey. Some of the most striking diophantine applications, particularly around the study of modular curves and Shimura varieties, require the generality of the Pila-Wilkie theorem for $\mathbb{R}_{\mathrm{an},\exp}$, i.e. beyond the subanalytic category. Obtaining polylogarithmic asymptotics for such unrestricted sets is the natural challenge going beyond the scope of [7].

1.4.5. *Interpolating rational points on log-sets.* For definiteness let $\log_e : \mathbb{C}\setminus\{0\} \to \mathbb{C}$ denote the principal branch of the standard complex logarithm. Below log can denote $\lambda \log_e(\cdot)$ for any $\lambda \in \mathbb{C}\setminus\{0\}$, i.e. logarithm with respect to any (fixed) base. We extend log as a function of several variables coordinate-wise.

Definition 4 (Log-set)**.** *If $A \subset (\mathbb{C}\setminus\{0\})^\ell$ is a bounded subanalytic set (where $\mathbb{C}^\ell$ is identified with $\mathbb{R}^{2\ell}$) then we call $\log A$ a log-set.*

Any bounded subanalytic set is a log-set, but log-sets are of course more general as they involve application of an *unrestricted* logarithm. In §B.3.1 we show that log-sets appear naturally in the application of the Pila-Wilkie theorem to the Andrè-Oort conjecture for modular curves in Pila's work [43]. They are therefore perhaps the most natural candidate to consider looking for results in the direction of the Wilkie conjecture for unrestricted sets (with an eye to the diophantine applications). We prove the following result in this direction.

Proposition 5. *Let $\{X_\lambda \subset [0,1]^\ell\}$ be an $\mathbb{R}_{an}$-definable family let n denote the maximal dimension of the fibers of X. Fix $k\in\mathbb{N}$. There exists an $\mathbb{R}_{an}$-definable family $\{Y_{\lambda,\mu} \subset X_\lambda\}$ with maximal fiber dimension strictly smaller than n, such that for any parameter λ there is a collection of $N = H^{O_X(k^{-1/n})}$ parameters $\{\mu_j\}$ satisfying*

$$(\log X_\lambda)^{\mathrm{trans}}(\mathbb{Q},H) \subset \bigcup_{j=1}^N \log Y_{\lambda,\mu_j} \tag{24}$$

Each $Y_{\lambda,\mu}$ is defined by intersecting X_λ with a collection of polynomial equations of degree k in the variables $\log \mathbf{x}_1,\dots,\log \mathbf{x}_\ell$.

Proposition 5 is reminiscent of the main inductive step in the proof of the Pila-Wilkie theorem. In particular, fixing k such that $N = H^\varepsilon$ and using induction over the fiber dimension, the proposition immediately yields a proof of the Pila-Wilkie theorem for log-sets. On the other hand, setting $k = (\log H)^n$ yields an interpolation result using a constant number of hypersurfaces of polylogarithmic degree and can be seen as the first inductive step toward the Wilkie conjecture. This is similar to the main result of [17] for sets definable in $\mathbb{R}^{\mathrm{pow}}_{\mathrm{an}}$.

The proof of Proposition 5 is given in §B.3.3. It relies heavily on the theory of complex cells, and is effective in the following sense: *a generalization of the polynomial complexity of the algebraic CPT to a reduct of $\mathbb{R}_{an}$ which includes the* restricted *logarithm would automatically yield a proof of the Wilkie conjecture for log-sets* $\log A$ *with A definable in the reduct.* While we do not prove such a version of the CPT in this paper (and there appear to be some technical obstacles to doing so), this seems to offer a plausible approach to proving the Wilkie conjecture for a large class of unrestricted sets that are important in applications.

Even for the case that k is a constant independent of H, Proposition 5 yields additional information compared to the standard proof of the Pila-Wilkie theorem. Namely, in the standard proof the sets $Y_{\lambda,\mu}$ are also defined by intersecting X_λ with a collection of polynomial equations of degree k in the variables $\log \mathbf{x}_1, \ldots, \log \mathbf{x}_\ell$. In general such sets would not be subanalytic. In our version certain cancellations in the logarithmic terms allow one to find suitable equations which are *subanalytic* on X_λ, thus proving that $Y_{\lambda,\mu}$ are also subanalytic. This eventually implies that the part of X_λ^{alg} responsible for the presence of many points of height H is also a union of log-sets.

2. Complex cellular structures

2.1. **Discs and annuli in the complex plane.** For $r \in \mathbb{C}$ (resp. $r_1, r_2 \in \mathbb{C}$) with $|r| > 0$ (resp. $|r_2| > |r_1| > 0$) we denote

$$D(r) := \{|z| < |r|\} \qquad D_\circ(r) := \{0 < |z| < |r|\} \tag{25}$$

$$A(r_1, r_2) := \{|r_1| < |z| < |r_2|\} \qquad * := \{0\}. \tag{26}$$

We also set $S(r) := \partial D(r)$. For any $0 < \delta < 1$ we define the δ-extensions, denoted by superscript δ, by

$$D^\delta(r) := D(\delta^{-1} r) \qquad D_\circ^\delta(r) := D_\circ(\delta^{-1} r) \tag{27}$$

$$A^\delta(r_1, r_2) := A(\delta r_1, \delta^{-1} r_2) \qquad *^\delta := *. \tag{28}$$

We also set $S^\delta(r) = A(\delta r, \delta^{-1} r)$. The notion of δ-extension is naturally associated with the Euclidean geometry of the complex plane. However, in many cases it is more convenient to use a different normalization associated with the hyperbolic geometry of our domains. For any $0 < \rho < \infty$ we define the $\{\rho\}$-extension $\mathcal{F}^{\{\rho\}}$ of $\mathcal{F}$ to be $\mathcal{F}^\delta$ where δ satisfies the equations

$$\begin{aligned} \rho &= \frac{2\pi\delta}{1-\delta^2} \qquad \text{for } \mathcal{F} \text{ of type } D, \\ \rho &= \frac{\pi^2}{2|\log\delta|} \qquad \text{for } \mathcal{F} \text{ of type } D_\circ, A. \end{aligned} \tag{29}$$

The motivation for this notation comes from the following fact, describing the hyperbolic-metric properties of a fiber $\mathcal{F}$ within its $\{\rho\}$-extension. For a reminder on the hyperbolic metric associated to a planar domain see §5. Fact 6 is proved by explicit computation in §5.6.

Fact 6. *Let $\mathcal{F}$ be a fiber of type $A, D, D_\circ$ and let S be a component of the boundary of $\mathcal{F}$ in $\mathcal{F}^{\{\rho\}}$. Then the length of S in $\mathcal{F}^{\{\rho\}}$ is at most ρ.*

2.2. **Complex cells.**

2.2.1. *The general setting.* We introduce a notation that will be used throughout the paper. Let $\mathcal{X}, \mathcal{Y}$ be sets and $\mathcal{F} : \mathcal{X} \to 2^{\mathcal{Y}}$ be a map taking points of $\mathcal{X}$ to subsets of $\mathcal{Y}$. Then we denote

$$\mathcal{X} \odot \mathcal{F} := \{(x, y) : x \in \mathcal{X}, y \in \mathcal{F}(x)\}. \tag{30}$$

In this paper $\mathcal{X}$ will be taken to be a subset of $\mathbb{C}^n$ and $\mathcal{Y}$ will be $\mathbb{C}$. If $r : \mathcal{X} \to \mathbb{C} \setminus \{0\}$ then for the purpose of this notation we understand $D(r)$ to denote the map assigning to each $x \in \mathcal{X}$ the disc $D(r(x))$, and similarly for $D_\circ, A$.

We now introduce the central notion of this paper, namely the notion of a complex cell of *length* $\ell \in \mathbb{Z}_{\geqslant 0}$ and *type* $\mathcal{T}(\mathcal{C}) \subset \{*, D, D_\circ, A\}^\ell$. If U is a complex manifold we denote by $\mathcal{O}(U)$ the space of holomorphic functions on U, and by $\mathcal{O}_b(U) \subset \mathcal{O}(U)$ the subspace of bounded functions.

Definition 7 (Complex cells)**.** *To base our induction, a complex cell $\mathcal{C}$ of* length *zero is the singleton $\mathbb{C}^0$. The* type *of $\mathcal{C}$ is the empty word. A complex cell of length $\ell + 1$ has the form $\mathcal{C}_{1..\ell} \odot \mathcal{F}$ where the* base *$\mathcal{C}_{1..\ell}$ is a cell of length ℓ, and the* fiber *$\mathcal{F}$ is one of $*, D(r), D_\circ(r), A(r_1, r_2)$ where $r \in \mathcal{O}_b(\mathcal{C}_{1..\ell})$ satisfies $|r(\mathbf{z}_{1..\ell})| > 0$ for $\mathbf{z}_{1..\ell} \in \mathcal{C}_{1..\ell}$; and $r_1, r_2 \in \mathcal{O}_b(\mathcal{C}_{1..\ell})$ satisfy $0 < |r_1(\mathbf{z}_{1..\ell})| < |r_2(\mathbf{z}_{1..\ell})|$ for $\mathbf{z}_{1..\ell} \in \mathcal{C}_{1..\ell}$. The type $\mathcal{T}(\mathcal{C})$ is $\mathcal{T}(\mathcal{C}_{1..\ell})$ followed by the type of the fiber.*

Next, we define the notion of a $\boldsymbol{\delta}$-extension (resp. $\{\boldsymbol{\rho}\}$-extension) of a cell of length ℓ where $\boldsymbol{\delta} \in (0, 1)^\ell$ (resp. $\boldsymbol{\rho} \in (0, \infty)^\ell$).

Definition 8. *The cell of length zero is defined to be its own $\boldsymbol{\delta}$-extension. A cell $\mathcal{C}$ of length $\ell + 1$ admits a $\boldsymbol{\delta}$-extension $\mathcal{C}^{\boldsymbol{\delta}} := \mathcal{C}_{1..\ell}^{\boldsymbol{\delta}_{1..\ell}} \odot \mathcal{F}^{\boldsymbol{\delta}_{\ell+1}}$ if $\mathcal{C}_{1..\ell}$ admits a $\boldsymbol{\delta}_{1..\ell}$-extension, and if the function r (resp. r_1, r_2) involved in $\mathcal{F}$ admits holomorphic continuation to $\mathcal{C}_{1..\ell}^{\boldsymbol{\delta}_{1..\ell}}$ and satisfies $|r(\mathbf{z}_{1..\ell})| > 0$ (resp. $0 < |r_1(\mathbf{z}_{1..\ell})| < |r_2(\mathbf{z}_{1..\ell})|$) in this larger domain. The $\{\boldsymbol{\rho}\}$-extension $\mathcal{C}^{\{\boldsymbol{\rho}\}}$ is defined in an analogous manner.*

As a shorthand, when we speak of a complex cell $\mathcal{C}^{\boldsymbol{\delta}}$ (resp. $\mathcal{C}^{\{\boldsymbol{\rho}\}}$) we imply that $\mathcal{C}$ is a complex cell admitting a $\boldsymbol{\delta}$ (resp $\{\boldsymbol{\rho}\}$) extension. We will usually speak about δ-extensions where $\delta \in (0, 1)$ by identifying δ with $\boldsymbol{\delta} := (\delta, \ldots, \delta)$ and similarly with $\rho \in (0, \infty)$.

Remark 9. *It is sometimes convenient to consider repeated extensions. We denote $\mathcal{C}^{\{\rho_1\}\{\rho_2\}} := (\mathcal{C}^{\{\rho_1\}})^{\{\rho_2\}}$. A simple computation shows that $\mathcal{C}^{\{\rho_1\}\{\rho_2\}} \subset \mathcal{C}^\rho$ where $\rho^{-1} = \operatorname{poly}(\rho_1^{-1}, \rho_2^{-1})$.*

The *dimension* of a cell denoted $\dim \mathcal{C}$ is its length minus the number of $*$s in its type. This clearly agrees with the dimension of $\mathcal{C}$ as a complex manifold.

2.2.2. *The real setting.* We introduce the notions of a *real* (complex) cell $\mathcal{C}$, the *real part* of a real cell $\mathcal{C}$, and a *real* holomorphic function on a complex cell. Below we let $\mathbb{R}_+$ denote the set of positive real numbers.

Definition 10 (Real structure on complex cells)**.** *The cell of length zero is real and equals its real part. A cell* $\mathcal{C} := \mathcal{C}_{1..\ell} \odot \mathcal{F}$ *is real if* $\mathcal{C}_{1..\ell}$ *is real and the radii involved in* $\mathcal{F}$ *can be chosen to be real on* $\mathcal{C}_{1..\ell}$*; The real part* $\mathbb{R}\mathcal{C}$ *(resp. positive real part* $\mathbb{R}_+\mathcal{C}$*) of* $\mathcal{C}$ *is defined to be* $\mathcal{C} \cap \mathbb{R}^\ell$ *(resp.* $\mathcal{C} \cap \mathbb{R}^\ell_+$*); A holomorphic function on* $\mathcal{C}$ *is said to be real if it is real on* $\mathbb{R}\mathcal{C}$*.*

A simple induction shows that a real cell is invariant under the conjugation $\mathbf{z} \to \bar{\mathbf{z}}$. Note that he positive real part of a real cell is connected, while the real part is disconnected if $\mathcal{C}$ has fibers of type $D_\circ, A$. We remark that for a function $f : \mathcal{C} \to \mathbb{R}$ to be real it is enough to require that it is real on $\mathbb{R}_+\mathcal{C}$. Indeed, in this case $f(z) = \overline{f(\bar{z})}$ on $\mathbb{R}_+\mathcal{C}$ and it follows by holomorphicity the equality holds over $\mathcal{C}$.

2.2.3. *Algebraicity.* For $X \subset \mathbb{C}^\ell$ a pure-dimensional algebraic variety we define the *degree* $\deg X$ to be the number of intersections between X and a generic affine-linear hyperplane of complementary dimension (this is the same as the degree of the projective closure of X in $\mathbb{C}P^\ell$). We extend this by linearity to arbitrary varieties. We remind the reader that $\deg(X \times Y) = \deg X \cdot \deg Y$ and by the Bezout theorem $\deg(X \cap Y) \leqslant \deg X \cdot \deg Y$.

For a domain $U \subset \mathbb{C}^\ell$, we say that a holomorphic function $f : U \to \mathbb{C}$ is *algebraic* if its graph $G_f \subset \mathbb{C}^\ell \times \mathbb{C}$ is an analytic component of $(U \times \mathbb{C}) \cap X$, where $X \subset \mathbb{C}^\ell \times \mathbb{C}$ is an algebraic variety. The minimal degree β of a variety X satisfying this condition is called the *degree*, or *complexity*, of f. For $F : U \to \mathbb{C}^k$ we say that F is algebraic if each of its components is, and we define its complexity to be the maximum among the complexities of the components. As an easy consequence of the Bezout theorem we have for any pair of composable algebraic maps F, G the estimate

$$\deg F \circ G \leqslant \operatorname{poly}_{\ell,k}(\deg F, \deg G). \tag{31}$$

We define the notion of an *algebraic* complex cell of complexity β by induction as follows: a cell of length 0 is algebraic of complexity 1; A cell $\mathcal{C} = \mathcal{C}_{1..\ell} \odot \mathcal{F}$ is algebraic if $\mathcal{C}_{1..\ell}$ is algebraic and the radii involved in $\mathcal{F}$ are algebraic, and the complexity of $\mathcal{C}$ is the maximum among the complexity of $\mathcal{C}_{1..\ell}$ and the complexities of the radii defining $\mathcal{F}$.

2.3. **Cellular maps.** We equip the category of complex cells with *cellular maps* defined as follows.

Definition 11 (Cellular map)**.** *Let* $\mathcal{C}, \hat{\mathcal{C}}$ *be two cells of length* ℓ*. We say that a holomorphic map* $f : \mathcal{C} \to \hat{\mathcal{C}}$ *is* cellular *if it takes the form* $\mathbf{w}_j = \phi_j(\mathbf{z}_{1..j})$ *where* $\phi_j \in \mathcal{O}_b(\mathcal{C}_{1..j})$ *for* $j = 1, \ldots, \ell$ *and moreover* ϕ_j *is a monic polynomial in* $\mathbf{z}_j$*. We say that a cellular map* f *is* prepared *(resp.* a translate*) in* $\mathbf{z}_j$ *if* $\phi_j(\mathbf{z}_{1..j}) = \mathbf{z}_j^q + \tilde{\phi}_j(\mathbf{z}_{1..j-1})$ *for some* $q \in \mathbb{N}_{\geqslant 1}$ *(resp.* $q = 1$*) and* $\tilde{\phi}_j$ *is holomorphic on* $\mathcal{C}_{1..j-1}$*. We say that* f *is* prepared *(resp. a translate) if it is prepared (resp. a translate) in* $\mathbf{z}_1, \ldots, \mathbf{z}_\ell$*. We say that* f *is* real *if* $\mathcal{C}, \hat{\mathcal{C}}$ *are real and the components of* f *are real.*

Cellular maps preserve the length and dimension of cells. The composition of two cellular maps is cellular, but note that the composition of two prepared cellular maps is not necessarily prepared. We will often be interested in covering a cell by cellular images of other cells. Toward this end we introduce the following definition.

Definition 12. *Let* $\mathcal{C}^{\boldsymbol{\delta}}$ *be a cell and* $\{f_j : \mathcal{C}_j^{\boldsymbol{\delta}'} \to \mathcal{C}^{\boldsymbol{\delta}}\}$ *be a finite collection of cellular maps. We say that this collection is a* $(\boldsymbol{\delta}', \boldsymbol{\delta})$-cellular cover *of* $\mathcal{C}$ *if* $\mathcal{C} \subset \cup_j (f_j(\mathcal{C}_j))$. *If* $(\boldsymbol{\delta}', \boldsymbol{\delta})$ *are clear from the context we will speak simply of cellular covers.*

The number of maps f_j in a cellular cover is called the *size* of the cover. If the maps f_j are all algebraic of complexity β we say that the cover has complexity β.

The real analog of a cellular cover is defined as follows.

Definition 13. *Let* $\mathcal{C}^{\boldsymbol{\delta}}$ *be a real cell and* $\{f_j : \mathcal{C}_j^{\boldsymbol{\delta}'} \to \mathcal{C}^{\boldsymbol{\delta}}\}$ *be a finite collection of real cellular maps. We say that this collection is a* real $(\boldsymbol{\delta}', \boldsymbol{\delta})$-cellular cover *of* $\mathcal{C}$ *if* $\mathbb{R}_+\mathcal{C} \subset \cup_j (f_j(\mathbb{R}_+\mathcal{C}_j))$. *If* $(\boldsymbol{\delta}', \boldsymbol{\delta})$ *are clear from the context we will speak simply of real cellular covers.*

The restriction to positive real parts in the definition of real cellular covers is a notational convenience. One can cover the remaining components of $\mathbb{R}\mathcal{C}$, for instance using the signed covering maps introduced in §3.1.

Remark 14. *We remark that if* $\{f_j : \mathcal{C}_j^{\boldsymbol{\delta}'} \to \mathcal{C}^{\boldsymbol{\delta}}\}$ *is a cellular cover of* $\mathcal{C}$ *and* $\{f_{jk} : \mathcal{C}_{jk}^{\boldsymbol{\delta}''} \to \mathcal{C}_j^{\boldsymbol{\delta}'}\}$ *is a cellular cover of* $\mathcal{C}_j$ *then* $\{f_j \circ f_{jk}\}$ *is a* $(\boldsymbol{\delta}'', \boldsymbol{\delta})$*-cover of* $\mathcal{C}$. *We will often use this basic principle without further reference.*

The following theorem implies in particular, by the remark above, that a cellular cover can always be replaced by a cellular cover consisting of prepared maps.

Theorem 7 (Cellular Preparation Theorem, CPrT)**.** *Let* $f : \mathcal{C}^{\{\rho\}} \to \hat{\mathcal{C}}$ *be a (real) cellular map. Then there exists a (real) cellular cover* $\{g_j : \mathcal{C}_j^{\{\rho\}} \to \mathcal{C}^{\{\rho\}}\}$ *of size* $\operatorname{poly}_f(\rho)$ *such that each* $f \circ g_j$ *is prepared.*

If $\mathcal{C}, \hat{\mathcal{C}}, f$ *vary in a definable family then the size of the cover is* $\operatorname{poly}(\rho)$ *uniformly over the family, and the maps* g_j *can be chosen from a single definable family. If* $\mathcal{C}, \hat{\mathcal{C}}, f$ *are algebraic of complexity* β *then the cover has size* $\operatorname{poly}(\beta, \rho)$ *and complexity* $\operatorname{poly}(\beta)$.

In this paper when we say that a family of sets or functions is *definable* we mean that it is definable in $\mathbb{R}_{\mathrm{an}}$. The reader unfamiliar with this terminology can think instead of a family whose total space is a bounded subanalytic set.

2.4. **The Cellular Parameterization Theorem.** Recall that in semialgebraic geometry, a cell is said to be *compatible* with a function if the function vanishes either identically or nowhere on the cell. We introduce a complex analog below.

Definition 15. *For* $\mathcal{C}$ *a complex cell and* $F \in \mathcal{O}_b(\mathcal{C})$ *we say that* F *is* compatible *with* $\mathcal{C}$ *if* F *vanishes either identically or nowhere on* $\mathcal{C}$. *For* $f : \hat{\mathcal{C}} \to \mathcal{C}$ *a cellular map we say that* f *is compatible with* F *if* f^*F *is compatible with* $\hat{\mathcal{C}}$.

The following is our main result.

Theorem 8 (Cellular Parameterization Theorem, CPT)**.** *Let* $\rho, \sigma \in (0, \infty)$. *Let* $\mathcal{C}^{\{\rho\}}$ *be a (real) cell and* $F_1, \ldots, F_M \in \mathcal{O}_b(\mathcal{C}^{\{\rho\}})$ *(real) holomorphic functions. Then there exists a (real) cellular cover* $\{f_j : \mathcal{C}_j^{\{\sigma\}} \to \mathcal{C}^{\{\rho\}}\}$ *of size* $\operatorname{poly}_{\mathcal{C},\{F_j\}}(\rho, 1/\sigma)$, *such that each* f_j *is prepared and compatible with each* F_k.

If $\mathcal{C}, F_1, \ldots, F_M$ *vary in a definable family then the cover has size* $\operatorname{poly}(\rho, 1/\sigma)$ *uniformly over the family and the* f_j *can be chosen from a single definable family. If* $\mathcal{C}, F_1, \ldots, F_M$ *are algebraic of complexity* β *then the cover has size* $\operatorname{poly}_\ell(\beta, M, \rho, 1/\sigma)$ *and complexity* $\operatorname{poly}(M, \beta)$.

In §3.2 we show that the cellular structure of the maps essentially implies automatic uniformity over families in the statements of the CPT and CPrT. We state the family versions of these theorems for convenience of use, to avoid having to make such reductions on numerous occasions.

2.5. **Topology and hyperbolic geometry of complex cells.** A complex cell $\mathcal{C}$ is homotopically equivalent to a product of points (for fibers $*, D$) and circles (for fibers $D_\circ, A$). Thus $\pi_1(\mathcal{C}) \simeq \prod G_i$ where G_i is trivial for $*, D$ and $\mathbb{Z}$ for $D_\circ, A$. We let γ_i denote the generator of G_i chosen with positive complex orientation for G_i non-trivial and $\gamma_i = e$ otherwise.

Definition 16. *Let $f : \mathcal{C} \to \mathbb{C} \setminus \{0\}$ be continuous. We define the monomial associated to f to be $\mathbf{z}^{\boldsymbol{\alpha}(f)}$ where*

$$\boldsymbol{\alpha}_i(f) = f_*\gamma_i \in \mathbb{Z} \simeq \pi_1(\mathbb{C} \setminus \{0\}). \tag{32}$$

It is easy to verify that $f \mapsto \boldsymbol{\alpha}(f)$ is a group homomorphism from the multiplicative group of continuous maps $f : \mathcal{C} \to \mathbb{C} \setminus \{0\}$ to the multiplicative group of monomials, which sends each monomial to itself.

For any hyperbolic Riemann surface X we denote by $\operatorname{dist}(\cdot, \cdot; X)$ hyperbolic distance on X (see §5 for a reminder on this topic). We use the same notation when $X = \mathbb{C}$ to denote the usual Euclidean distance, and when $X = \mathbb{C}P^1$ to denote the Fubini-Study metric. For $x \in X$ and $r > 0$ we denote by $B(x, r; X)$ the r-ball around x in X. For $A \subset X$ we denote by $B(A, r; X)$ the union of r-balls around all points of A.

The following lemma shows that a holomorphic function with a non-vanishing bounded $\{\rho\}$-extension is equivalent to its associated monomial up to a unit in a strong sense.

Lemma 17 (Monomialization lemma)**.** *Let $0 < \rho < \infty$ and let $f : \mathcal{C}^{\{\rho\}} \to \mathbb{C} \setminus \{0\}$ be a bounded holomorphic map. Then $f = \mathbf{z}^{\boldsymbol{\alpha}(f)} \cdot U(\mathbf{z})$ and*

$$\operatorname{diam}(\operatorname{Re} \log U(\mathcal{C}); \mathbb{R}) < O_f(1) \cdot \rho, \qquad \operatorname{diam}(\operatorname{Im} \log U(\mathcal{C}); \mathbb{R}) < O_f(1). \tag{33}$$

If $\mathcal{C}$ and f vary in a definable family then $|\boldsymbol{\alpha}(f)|$ and the asymptotic constants in (33) are uniformly bounded over the family. If $\mathcal{C}, f$ are algebraic of complexity β then $|\boldsymbol{\alpha}(f)| = \operatorname{poly}_\ell(\beta)$ and

$$\operatorname{diam}(\operatorname{Re} \log U(\mathcal{C}); \mathbb{R}) < \operatorname{poly}_\ell(\beta) \cdot \rho, \quad \operatorname{diam}(\operatorname{Im} \log U(\mathcal{C}); \mathbb{R}) < \operatorname{poly}_\ell(\beta). \tag{34}$$

In light of the monomialization lemma, the CPT can be viewed as a cellular analog of the monomialization of functions/ideals in the theory of resolution of singularities. Indeed, if $\mathcal{C}^{\{\rho\}}$ is a cell compatible with F then either F vanishes identically or $F : \mathcal{C}^\delta \to \mathbb{C} \setminus \{0\}$, in which case F is equivalent to $\mathbf{z}^{\boldsymbol{\alpha}(f)}$ up to a unit on $\mathcal{C}$; hence the cells constructed in the CPT may be viewed as "cellular charts" where $F_1, \ldots, F_M$ are monomialized.

We now give several results on the geometry of holomorphic maps from cells to hyperbolic Riemann surfaces. We begin with the following *domination lemma*, which is valid for arbitrary cellular extensions. In this paper we never use this lemma directly: instead, we use the finer *fundamental lemmas* stated later, which are valid only for extensions with a sufficiently small ρ. However we still state and prove the domination lemma to stress another line of close analogy between complex cells and resolution of singularities.

Lemma 18 (Domination Lemma)**.** *Let $\mathcal{C}^{\{\rho\}}$ be a complex cell and suppose*[1] *that $\rho > 2e$. Let $f : \mathcal{C}^{\{\rho\}} \to \mathbb{C} \setminus \{0,1\}$ be holomorphic. Then on $\mathcal{C}$ one of the following holds:*

$$|f| = O_\ell(\log\log\rho), \qquad |1/f| = O_\ell(\log\log\rho). \tag{35}$$

The domination lemma can be seen as a cellular analog of a standard argument from the theory of resolution of singularities [5, Lemma 4.7]: *suppose f, g and $f-g$ are monomials (up to a unit). Then either f divides g or g divides f.* This allows one to principalize an ideal by monomializing its generators and their pairwise differences. To see the analogy with the domination lemma, suppose $\mathcal{C}^{\{\rho\}}$ is a cell compatible with $f, g, f-g$. Then $f/g : \mathcal{C}^{\{\rho\}} \to \mathbb{C} \setminus \{0,1\}$ and the domination lemma applies to show that either f/g or g/f is bounded from above (i.e. one divides the other in $\mathcal{O}_b(\mathcal{C})$). Moreover, the bound depends only on ℓ and ρ.

Remark 19. *The domination lemma for one-dimensional discs immediately implies the Little Picard Theorem. Indeed, suppose $f : \mathbb{C} \to \mathbb{C} \setminus \{0,1\}$ is entire. Applying the domination lemma to $f|_{D(r)}$ for every $r > 0$ we see that f is bounded away from 0 or ∞ uniformly in $\mathbb{C}$ and therefore constant by Liouville's theorem.*

Similarly, the domination lemma for one-dimensional annuli implies the Great Picard Theorem. Indeed, suppose $f : D_\circ(1) \to \mathbb{C} \setminus \{0,1\}$ has an essential singularity at 0. Applying the domination lemma to $f|_{A(\varepsilon,1/2)}$ for every $0 < \varepsilon < 1/2$ we see that f is bounded away from 0 or ∞ uniformly in $D_\circ(1/2)$ which is impossible, for instance by the removable singularity theorem.

Next, we state the more refined *fundamental lemmas* on maps $f : \mathcal{C}^{\{\rho\}} \to X$ into a hyperbolic Riemann surface X. By contrast with the domination lemma, these lemma yield more precise asymptotic estimates on the image of a complex cell as the extension ρ tends to zero. The proofs of the fundamental lemmas rely on the (hyperbolic) geometry and topology of X. Rather than formulate the most general possible form, we prefer to give three separate formulations for the most useful cases $X = \mathbb{D}, \mathbb{D} \setminus \{0\}, \mathbb{C} \setminus \{0,1\}$.

Lemma (Fundamental Lemma for $\mathbb{D}$)**.** *Let $\mathcal{C}^{\{\rho\}}$ be a complex cell. Let $f : \mathcal{C}^{\{\rho\}} \to \mathbb{D}$ be holomorphic. Then*

$$\operatorname{diam}(f(\mathcal{C}); \mathbb{D}) = O_\ell(\rho). \tag{36}$$

Lemma (Fundamental Lemma for $\mathbb{D} \setminus \{0\}$)**.** *Let $\mathcal{C}^{\{\rho\}}$ be a complex cell. Let $f : \mathcal{C}^{\{\rho\}} \to \mathbb{D} \setminus \{0\}$ be holomorphic. Then one of the following holds:*

$$f(\mathcal{C}) \subset B(0, e^{-\Omega_\ell(1/\rho)}; \mathbb{C}), \qquad \operatorname{diam}(f(\mathcal{C}); \mathbb{D} \setminus \{0\}) = O_\ell(\rho). \tag{37}$$

In particular, one of the following holds:

$$\log|f(\mathcal{C})| \subset (-\infty, -\Omega_\ell(1/\rho)), \qquad \operatorname{diam}(\log|\log|f(\mathcal{C})||; \mathbb{R}) = O_\ell(\rho). \tag{38}$$

Lemma (Fundamental Lemma for $\mathbb{C} \setminus \{0,1\}$)**.** *Let $\mathcal{C}^{\{\rho\}}$ be a complex cell and $0 < \rho < 1$. Let $f : \mathcal{C}^{\{\rho\}} \to \mathbb{C} \setminus \{0,1\}$ be holomorphic. Then one of the following holds:*

$$f(\mathcal{C}) \subset B(\{0,1,\infty\}, e^{-\Omega_\ell(1/\rho)}; \mathbb{C}P^1), \qquad \operatorname{diam}(f(\mathcal{C}); \mathbb{C} \setminus \{0,1\}) = O_\ell(\rho). \tag{39}$$

The proofs of the monomialization, domination and fundamental lemmas are given in §5.

[1]Note that if $\mathcal{C}$ admits a $\{\rho\}$-extension it also admits a ρ'-extension for any $\rho' > \rho$, so the restriction on ρ is only relevant for the asymptotics in (35).

2.6. **Overview of the proof of the CPT.** In this section we aim to give an intuitive overview of the proof of the CPT. We will consider only the algebraic case, which requires the most delicate arguments to control the complexity with respect to β. We will construct a covering using general cellular maps instead of prepared maps (one can later use the CPrT to obtain a prepared cover).

2.6.1. *Reduction to a single F.* We first note that the CPT can be easily reduced to the case of one function F. Indeed, suppose we wish to perform a cellular decomposition for the functions $F_1,\ldots,F_M$. Let F be the product of all the F_j, except those that vanish identically on $\mathcal{C}$. We construct a cellular decomposition $f_j:\mathcal{C}_j^{\{\sigma\}}\to\mathcal{C}^{\{\rho\}}$ compatible with F. Each of the maps f_j whose image is disjoint from the zeros of F is already compatible with each F_k. Each of the remaining cells $\mathcal{C}_j$ has dimension strictly smaller than $\mathcal{C}$ since cellular maps preserve dimension. Thus we can proceed by induction over the dimension.

2.6.2. *Reduction to arbitrary ρ,σ.* We will allow ourselves to assume that ρ is as small as we wish as long as $1/\rho=\operatorname{poly}_\ell(\beta)$. We will also allow ourselves to assume that σ is as large as we wish as long as $\sigma=\operatorname{poly}_\ell(\beta)$. Both of these assumptions are justified by Theorem 9 which shows that cells with a given extension can be "refined" into cells with a finer extension. This theorem generalizes the idea of covering a disc or annulus by several smaller discs or annuli. The proof of this theorem is independent of the CPT and the reader can safely skip the details for now.

2.6.3. *Reduction to a proper covering map.* We proceed with the case of a single $F\in\mathbb{C}[\mathbf{z}_{1..\ell+1}]$. Let $\mathcal{C}:=\mathcal{C}_{1..\ell}\odot\mathcal{F}$. We begin the proof by induction over ℓ. Let $D\in\mathbb{C}[\mathbf{z}_{1..\ell}]$ denote the discriminant of F and construct a cellular decomposition $f_j:\mathcal{C}_j^{\{\rho\}}\to\mathcal{C}_{1..\ell}^{\{\rho\}}$ compatible with D. Any of the cells f_j with $f_j(\mathcal{C}_j)$ contained in the zeros of D has dimension strictly smaller than $\mathcal{C}_{1..\ell-1}$, and we can therefore handle the cell $\mathcal{C}_j\odot(f_j^*\mathcal{F})$ with the function f_j^*F by induction over dimension. The images of these cells under (f_j,id) cover the part of $\mathcal{C}$ lying over the zeros of D.

It remains to cover the part lying over the images $f_j(\mathcal{C}_j)$ that are disjoint from the zeros of D. For each such cell we can reduce again to the cell $\mathcal{C}_j\odot(f_j^*\mathcal{F})$ and the function f_j^*F. We return to the original notation, replacing $\mathcal{C}\odot\mathcal{F},F$ by $\mathcal{C}_j\odot(f_j^*\mathcal{F}),f_j^*F$. What we have gained is that the projection

$$\pi:(\mathcal{C}\odot\mathcal{F})^{\{\rho\}}\cap\{F=0\}\to\mathcal{C},\qquad \pi(\mathbf{z}_{1..\ell+1})=\mathbf{z}_{1..\ell} \tag{40}$$

is now a proper covering map. We denote the degree of π by ν, and note that $\nu=\operatorname{poly}_\ell(\beta)$.

2.6.4. *Covering the zeros.* The projection π admits ν possibly multivalued sections $y_j:\mathcal{C}\to\mathbb{C}$. If we set $\hat{\mathcal{C}}:=\mathcal{C}_{\times\nu!}$ then y_j can be lifted to a univalued section $y_j:\hat{\mathcal{C}}\to\mathbb{C}$. However, the covering of order $\nu!$ has complexity exponential in ν and therefore in β, and thus cannot be used explicitly in our construction. In fact, one can show that y_j already lifts to a univalued function on $y_j:\mathcal{C}_{\times\nu_j}\to\mathbb{C}$ where $\nu_j\leqslant\nu$ is the size of the $\pi_1(\mathcal{C})$-orbit of y_j, see Lemma 47. This is a simple statement about permutation groups, using crucially the fact that $\pi_1(\mathcal{C})$ is abelian. We denote $\hat{\mathcal{C}}_j:=\mathcal{C}_{\times\nu_j}$.

Using the maps y_j we construct cellular maps $(R_{\nu_j},\mathrm{id}):\hat{\mathcal{C}}_j\odot *\to\mathcal{C}\odot\mathcal{F}$ whose images are contained in the zeros of F and hence compatible with F. Moreover

Proposition 22 shows that these maps admit $\{\rho \cdot \nu_j\}$-extensions, and choosing $\rho \cdot \nu_j < \sigma$ gives the required extension property.

2.6.5. *Covering the complement of the zeros.* We now make a simplifying assumption, namely we replace $\mathcal{F}$ by $\mathbb{C}$. This simplifies the presentation by allowing us to avoid minor technicalities around $\partial\mathcal{F}$. Since $\mathbb{C}$ is unbounded we will formally have to allow unbounded cells in the covering that we construct, but the difference will be minor. The goal is therefore to cover each fiber $\mathbb{C}\setminus\{y_1,\dots,y_\nu\}$ by a finite collection of discs, punctured discs and annuli admitting $\{\sigma\}$-extensions. The challenge is that this covering should depend holomorphically on the base point $\mathbf{z}_{1..\ell} \in \mathcal{C}$, and should be of size $\text{poly}_\ell(\beta)$.

2.6.6. *The affine invariants $s_{i,j,k}$.* To achieve our goal we will study the relative positions of the points $y_j \in \mathbb{C}$. Toward this end we introduce the following notation. Let y_i, y_j, y_k be three distinct sections and denote $\hat{\mathcal{C}}_{i,j,k} := \mathcal{C}_{\times\nu_i\nu_j\nu_k}$. We define a map $s_{i,j,k}$ as follows,

$$s_{i,j,k} : \hat{\mathcal{C}}_{i,j,k} \to \mathbb{C}\setminus\{0,1\}, \qquad s_{i,j,k} = \frac{y_i - y_j}{y_i - y_k}. \tag{41}$$

By the fundamental lemma for maps into $\mathbb{C}\setminus\{0,1\}$ one of the following holds:

$$\begin{gathered} s_{i,j,k}(\hat{\mathcal{C}}_{i,j,k}) \subset B(\{0,1,\infty\}, e^{-\Omega_\ell(1/(\nu^3\rho))}; \mathbb{C}P^1), \\ \operatorname{diam}(s_{i,j,k}(\hat{\mathcal{C}}_{i,j,k}); \mathbb{C}\setminus\{0,1\}) = O_\ell(\nu^3\rho). \end{gathered} \tag{42}$$

We remark that equivalently (42) holds if we replace $\hat{\mathcal{C}}_{i,j,k}$ by $\hat{\mathcal{C}}$, since $s_{i,j,k}$ on $\hat{\mathcal{C}}$ factors through $\hat{\mathcal{C}}_{i,j,k}$. We note that $s_{i,j,k}$ is invariant under affine transformations of $\mathbb{C}$, and is in fact the only affine invariant of (y_i, y_j, y_k). We think of (42) as implying that the relative positions of any three sections move very little for $\rho \ll 1$. This is the crucial ingredient from hyperbolic geometry that will enable us to carry out the decomposition into discs and annuli uniformly over the base $\hat{\mathcal{C}}_{i,j,k}$.

Remark 20 (Fulton-MacPherson compactification)**.** *In* [20] *Fulton and MacPherson construct a compactification $X[\nu]$ for the configuration space $X^\nu\setminus\Delta$ of ν distinct labeled points in an algebraic variety X (where Δ is the union of all diagonals). In the case $X = \mathbb{C}$, the construction using* screens *is as follows. Write $y_1,\dots,y_\nu$ for the coordinates on $\mathbb{C}^\nu\setminus\Delta$. For every subset $S \subset \{1,\dots,\nu\}$ define the map*

$$\iota_S : \mathbb{C}^\nu\setminus\Delta \to \mathbb{P}(\mathbb{C}^{|S|}/\mathbb{C}), \qquad i_S(y_1,\dots,y_\nu) = (y_\alpha)_{\alpha\in S} \tag{43}$$

where $\mathbb{C} \subset \mathbb{C}^{|S|}$ is the diagonally embedded subspace. The map ι_S remembers the relative positions of $\{y_\alpha\}_{\alpha\in S}$ up to translation (corresponding to the diagonally embedded $\mathbb{C}$) and homothety (corresponding to the projectivization). Consider the map

$$\iota : \mathbb{C}^\nu\setminus\Delta \to (\mathbb{P}^1)^{\binom{\nu}{3}}, \qquad \iota = \prod_{|S|=3} \iota_S. \tag{44}$$

Then $\mathbb{C}[\nu]$ is the Zariski closure of the graph of ι. It is an important fact that only the screens ι_S for S of size three are required to fully determine the compactification: if one defines ι by a product over all $S \subset \{1,\dots,\nu\}$ one obtains an isomorphic compactification.

In our setting, we have a natural map $\mathbf{y} = (y_1,\dots,y_\nu) : \hat{\mathcal{C}} \to \mathbb{C}^\nu\setminus\Delta$. For an appropriate choice of the a rational coordinate on each $\mathbb{P}^1$ factor in (44)*, the*

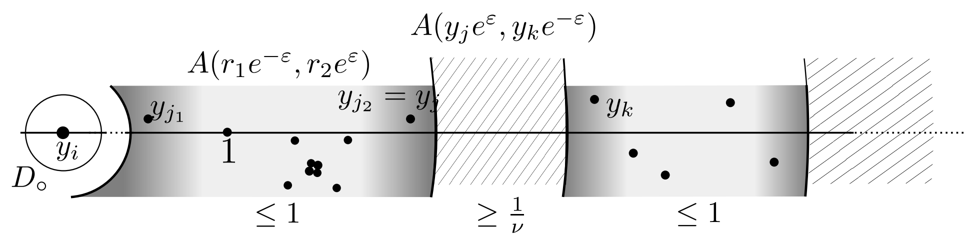

FIGURE 2. Clustering around y_i; distances are in log-scale.

coordinates of ι can be identified with our maps $s_{i,j,k}$ which are indeed invariant under translation and homothety. The equations (42) *imply in particular that the image $\mathbf{y}(\hat{\mathcal{C}})$ is of small diameter, not only in the metric induced from $\mathbb{C}^\nu$, but also in the finer metric induced from the compactification $\mathbb{C}[\nu]$. It is crucial here that one only needs to use the screens S of size three. If we were to write equations similar to* (42) *for screens of arbitrary size the factor ν^3 would be replaced by $\nu!$, which is exponential in β and hence too large for our purposes.*

As we proceed to describe the cellular cover for $\mathbb{C} \setminus \{y_1, \dots, y_\nu\}$ we will make no explicit reference to the Fulton-MacPherson compactification, working instead directly with (42)*. However the reader may find it helpful to interpret these inequalities as estimates on the metric induced from $\mathbb{C}[\nu]$.*

2.6.7. *Clustering around a center y_i.* We fix a section y_i. We will cluster the remaining sections into annuli based on their relative distance from y_i. Since our constructions are invariant under affine transformations we may assume after an appropriate transformation that $y_i = 0$.

Pick an arbitrary base point $p \in \hat{\mathcal{C}}$. We will say that y_j is close to y_k if they satisfy $\log|y_k(p)/y_j(p)| < 1/\nu$, and define the *clusters* around y_i to be the equivalence classes of the transitive closure of this relation. Then for any y_j, y_k, if they are in the same cluster we have $\log|y_k(p)/y_j(p)| < 1$ and otherwise we have $\log|y_k(p)/y_j(p)| \geqslant 1/\nu$.

2.6.8. *The Voronoi cells associated to y_i.* We will use the clusters around y_i to construct a collection of *Voronoi cells* mapping into $\mathcal{C} \odot \mathbb{C}$. Their defining property, which motivates our choice of naming, is the following: the Voronoi cells associated to y_i cover every point $z \in \mathbb{C} \setminus \{y_1, \dots, y_\nu\}$ such that $\operatorname{dist}(z, y_i) \leqslant 2\operatorname{dist}(z, y_j)$ for any $j \neq i$. In other words, for every section y_i we will construct cells that cover every point in $\mathbb{C}$ except those that are (twice) closer to some other section y_j. Clearly then the union of these cells for every y_i will cover $\mathbb{C} \setminus \{y_1, \dots, y_\nu\}$, which is our goal.

We begin by covering the empty area between two clusters. This is fairly straightforward. Suppose that y_j is a section with $|y_j(p)|$ maximal within its cluster, and y_k is a section with $|y_k(p)|$ minimal within the next cluster. Then the annulus $A(y_je^{\varepsilon}, y_ke^{-\varepsilon})$ for say $\varepsilon = 1/\nu^2$ does not meet any of the sections y_j over p, and assuming ρ is sufficiently small this remains true uniformly over $\hat{\mathcal{C}}$ by (42). We similarly cover the area between $y_i = 0$ and the first cluster by a punctured disc, and outside the last cluster by $A(\cdot, \infty)$.

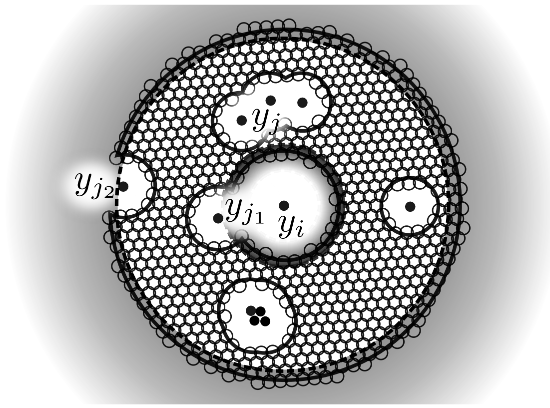

FIGURE 3. Voronoi cells in a cluster

2.6.9. *Voronoi cells in a cluster.* It remains to cover the part of $\mathbb{C}$ that lives outside the annuli above, i.e. near one of the clusters. Choose one of the clusters and let y_j be a section in the cluster. Since our constructions are invariant under affine transformations we may assume after an appropriate transformation that $y_i = 0$ and also $y_j = 1$. With this choice of coordinates $s_{i,k,j} = y_k$ and (42) implies that y_k does not move much over the cell $\hat{\mathcal{C}}$. We will construct our cells over the base $\hat{\mathcal{C}}_{i,j} := \mathcal{C}_{\times \nu_i \nu_j}$. Note that the complexity of this cell is $\operatorname{poly}_\ell(\beta)$ as required.

Suppose y_{j_1} (resp. y_{j_2}) is the section with $r_1 := |y_{j_1}(p)|$ minimal (resp. $r_2 := |y_{j_2}(p)|$ maximal) within its cluster. Then $1/e < r_1 < r_2 < e$. Moreover the sections in the cluster uniformly remain within $\mathcal{A} := A(r_1 e^{-\varepsilon}, r_2 e^{\varepsilon})$ while the sections belonging to other clusters remain outside $\mathcal{A}^{\Theta(1-1/\nu)}$.

Let U be the set obtained from $\mathcal{A}$ by removing discs of radius $1/10$ around each of the points $y_k(p)$ for y_k belonging to the cluster. Note that any point in these discs is much closer to $y_k(p)$ than to $y_i(p) = 0$, so we not need to cover them with the Voronoi cells of y_i. Moreover, the same remains true uniformly over $\hat{\mathcal{C}}$ since by (42) the points y_k move very little. To finish the construction we must therefore cover U using discs whose $\{\sigma\}$-extensions remain inside $\mathcal{A}^{\Theta(1-1/\nu)}$ and away from the 1/20-balls around the points $y_k(p)$ (this way even as p varies over $\hat{\mathcal{C}}$ the discs and their extensions will not meet them). This is clearly possible to achieve with $\operatorname{poly}_\ell(\nu) = \operatorname{poly}_\ell(\beta)$ discs as required.

3. GENERALITIES ON COMPLEX CELLS

In this section we cover some generalities on complex cells, their subanalytic structure and some uniformity results for families.

3.1. **The ν-cover of a cell.** A key difference between the classical notion of real cellular decompositions and the complex counterpart is the presence of a non-trivial fundamental group, and with it the existence of non-trivial covering maps. Recall that for a cell $\mathcal{C}$ of length ℓ we identify $\pi_1(\mathcal{C}) \simeq \prod G_i$ where G_i is trivial for $*, D$ and $\mathbb{Z}$ for $D_\circ, A$.

Definition 21 (The $\boldsymbol{\nu}$-cover of a cell)**.** *Let $\mathcal{C}$ be a cell of length ℓ and let $\boldsymbol{\nu} = (\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_\ell) \in \pi_1(\mathcal{C})$ be such that $\boldsymbol{\nu}_j | \boldsymbol{\nu}_k$ whenever $j > k$ and $G_j = G_k = \mathbb{Z}$. We* *define the* $\boldsymbol{\nu}$-cover $\mathcal{C}_{\times\boldsymbol{\nu}}$ *of* $\mathcal{C}$ *and the associated* cellular map $R_{\boldsymbol{\nu}}:\mathcal{C}_{\times\boldsymbol{\nu}}\to\mathcal{C}$ *by induction on* ℓ. *For* $\ell=0$ *we let* $\mathcal{C}_{\times\boldsymbol{\nu}}:=\mathcal{C}$ *and* $R_{\boldsymbol{\nu}}:=\mathrm{id}$. *For* $\mathcal{C}=\mathcal{C}_{1..\ell-1}\odot\mathcal{F}$ *we let*

$$\mathcal{C}_{\times\boldsymbol{\nu}_\ell}:=(\mathcal{C}_{1..\ell-1})_{\times\boldsymbol{\nu}_{1..\ell-1}}\odot(R^*_{\boldsymbol{\nu}}\mathcal{F}_{\times\boldsymbol{\nu}_\ell}) \tag{45}$$

where $D(r)_{\times\boldsymbol{\nu}_\ell}:=D(r^{1/\boldsymbol{\nu}_\ell})$ *and similarly for* $D_\circ, A$. *We set* $R_{\boldsymbol{\nu}}(\mathbf{z}_{1..\ell})_k:=\mathbf{z}^{\boldsymbol{\nu}}$.

Note that the $D_\circ(r)_{\times\boldsymbol{\nu}_\ell}$ fiber (and similarly with A) does not conform, a-priori, with the definition of a complex cell since $r^{1/\boldsymbol{\nu}_\ell}$ may in general be multivalued (with cyclic monodromy of order dividing $\boldsymbol{\nu}_\ell$). However the divisibility conditions on $\boldsymbol{\nu}$ guarantee that $R_{\boldsymbol{\nu}}:(\mathcal{C}_{1..\ell-1})_{\times\boldsymbol{\nu}_{1..\ell-1}}\to\mathcal{C}_{1..\ell-1}$ maps $\pi_1(\mathcal{C}_{1..\ell-1})_{\times\boldsymbol{\nu}}$ into $\boldsymbol{\nu}_\ell\pi_1(\mathcal{C}_{1..\ell-1})$ and the pullback $R^*_{\boldsymbol{\nu}_\ell}(D_\circ(r))_{\times\boldsymbol{\nu}_\ell}$ is indeed univalued and well defined up to a root of unity. We will usually consider the ν-cover with $\nu\in\mathbb{N}$, meaning that we take $\boldsymbol{\nu}$ with $\boldsymbol{\nu}_i=\nu$ when $G_i=\mathbb{Z}$ and $\boldsymbol{\nu}_i=1$ otherwise.

A minor technicality arises in the real setting. Namely, the real part $\mathbb{R}\mathcal{C}_{\times\boldsymbol{\nu}}$ does always not cover $\mathbb{R}\mathcal{C}$: if $\boldsymbol{\nu}$ is even then only the positive real part is covered. To cover the remaining components of the real part we introduce the notion of a *signed* cover. Namely, for a sequence $\boldsymbol{\sigma}\in\{\pm1\}^\ell$ we define $\mathcal{C}_{\times\boldsymbol{\nu},\boldsymbol{\sigma}}$ and $R_{\boldsymbol{\nu},\boldsymbol{\sigma}}$ by induction as above but taking $R_{\boldsymbol{\nu},\boldsymbol{\sigma}}(\mathbf{z}_{1..\ell+1})_k:=\boldsymbol{\sigma}_k\cdot\mathbf{z}_k^{\boldsymbol{\nu}}$. It is then clear that $\mathbb{R}\mathcal{C}_{\times\boldsymbol{\nu},\boldsymbol{\sigma}}$ cover $\mathbb{R}\mathcal{C}$ when $\boldsymbol{\sigma}$ ranges over all possible signs.

The pullback to a $\boldsymbol{\nu}$-cover will be used in our treatment to resolve the ramification of multivalued cellular maps. We record a simple proposition concerning the interaction between extensions and ν-covers.

Proposition 22. *Let* $\mathcal{C}$ *be a complex cell and* $\nu\in\mathbb{N}$.

1. *If* $\mathcal{C}$ *admits a* δ*-extension then* $\mathcal{C}_{\times\nu}$ *admits a* $\delta^{1/\nu}$*-extension and the covering map* R_ν *extends to* $R_\nu:(\mathcal{C}_{\times\nu})^{\delta^{1/\nu}}\to\mathcal{C}^\delta$.
2. *If* $\mathcal{C}$ *admits a* $\{\rho\}$*-extension then* $\mathcal{C}_{\times\nu}$ *admits an* $\{\nu\rho\}$*-extension and the covering map* R_ν *extends to* $R_\nu:(\mathcal{C}_{\times\nu})^{\{\nu\rho\}}\to\mathcal{C}^{\{\rho\}}$.

If $\mathcal{C}$ *is algebraic of complexity* β *then* $C_{\times\nu}$ *is algebraic of complexity* $\mathrm{poly}_\ell(\beta,\nu)$.

We leave the simple inductive proof to the reader. We remark that for the part 2 it is crucial that covering maps are defined to be identical on fibers of type D.

3.2. **Uniformity in families.** In this section we show how the cellular structure of our maps implies automatic uniformity over families, for instance in the statement of the CPT.

For $p\in\mathcal{C}_{1..j}$ we denote

$$\mathcal{C}_p=\{\mathbf{z}_{j+1..\ell}:(p,\mathbf{z}_{j+1..\ell})\in\mathcal{C}\}. \tag{46}$$

If $\mathcal{C}$ has type $\mathcal{F}_1\odot\cdots\odot\mathcal{F}_\ell$ then $\mathcal{C}_p$ is a cell of type $\mathcal{F}_{j+1}\odot\cdots\odot\mathcal{F}_\ell$. If $\mathcal{C}$ admits a δ-extension then so does $\mathcal{C}_p$.

By definition cellular map $f:\mathcal{C}\to\hat{\mathcal{C}}$ induces a cellular map $f_{1..j}:\mathcal{C}_{1..j}\to\hat{\mathcal{C}}_{1..j}$ for $j=1,\dots,\ell$ by restriction to the first j coordinates. The following proposition follows directly from the definitions.

Proposition 23. *Let* $f:\mathcal{C}\to\hat{\mathcal{C}}$ *be a cellular map between cells of length* ℓ *and* $j=1,\dots,\ell$. *Then the number of points in* $f^{-1}_{1..j}(p)$ *for* $p\in\hat{\mathcal{C}}_{1..j}$ *is bounded by a constant* $\nu(f,j)$ *independent of* p. *More explicitly, one may take*

$$v(f,j)=\prod_{k=1}^{j}\deg_{\mathbf{z}_k}\phi_k \tag{47}$$

in the notations of Definition 11.

The following remark illustrates how the cellular structure of the maps in the CPT automatically implies uniformity over families.

Remark 24. *Suppose $\ell = n + m$ and we view $\mathcal{C}$ in the statement of the CPT as a family of m-dimensional cells $\mathcal{C}_p$ parameterized over $p \in \mathcal{C}_{1..n}$. Let $f_j : \mathcal{C}_j^{\{\sigma\}} \to \mathcal{C}^{\{\rho\}}$ be the maps constructed in the CPT. By Proposition 23 the sets*

$$P_j = \{p_{j,k}\} := (f_j)^{-1}(p) \tag{48}$$

are finite with the number of points uniformly bounded over p. For each $p_{j,k}$ restriction of f_j to the fiber gives a prepared cellular map $f_{j,k} : (\mathcal{C}_j)_{p_{j,k}}^{\{\sigma\}} \to \mathcal{C}_p^{\{\rho\}}$ which is compatible with the restrictions of $F_1, \ldots, F_m$ to $\mathcal{C}_p$, and such that $f_{j,k}((\mathcal{C}_j)_{p_{j,k}})$ cover $\mathcal{C}_p$. In other words we obtain a cellular decomposition as in the CPT for the fibers $\mathcal{C}_p$ with the number of cells uniformly bounded over p.

3.3. **Laurent expansion in a complex cell.** Let $\mathcal{C} = \mathcal{F}_1 \odot \cdots \odot \mathcal{F}_\ell$ be a complex cell. In this section we show that any holomorphic function on $\mathcal{C}$ can be expanded into a series analogous to the classical Laurent series. We begin by introducing the notion of *normalized monomials*. Let $\boldsymbol{\alpha} \in \mathbb{Z}^\ell$. We define the $\mathcal{C}$-normalized monomial $\mathbf{z}^{[\boldsymbol{\alpha}]} := \mathbf{z}_1^{[\boldsymbol{\alpha}_1]} \cdots \mathbf{z}_\ell^{[\boldsymbol{\alpha}_\ell]}$ where: if $\mathcal{F}_j = D(r_j)$ or $\mathcal{F}_j = D_\circ(r_j)$ then

$$\mathbf{z}_j^{[\boldsymbol{\alpha}_j]} = \begin{cases} (\mathbf{z}_j/r_j)^{\boldsymbol{\alpha}_j} & \boldsymbol{\alpha}_j \geqslant 0 \\ 0 & \text{otherwise;} \end{cases} \tag{49}$$

if $\mathcal{F}_j = A(r_{1,j}, r_{2,j})$ then

$$\mathbf{z}_j^{[\boldsymbol{\alpha}_j]} = \begin{cases} (\mathbf{z}_j/r_{1,j})^{\boldsymbol{\alpha}_j} & \boldsymbol{\alpha}_j \geqslant 0 \\ (\mathbf{z}_j/r_{2,j})^{\boldsymbol{\alpha}_j} & \text{otherwise;} \end{cases} \tag{50}$$

and if $\mathcal{F}_j = *$ then

$$\mathbf{z}_j^{[\boldsymbol{\alpha}_j]} = \begin{cases} 1 & \boldsymbol{\alpha}_j = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{51}$$

The normalization is such that $\mathbf{z}^{[\boldsymbol{\alpha}]}$ is bounded by 1 in $\mathcal{C}$ and achieves the bound only when $\boldsymbol{\alpha} = 0$.

Below we write $\operatorname{pos} \boldsymbol{\delta}$ for the vector whose i-th coordinate is $|\boldsymbol{\delta}_i|$.

Proposition 25. *Let $f \in \mathcal{O}_b(\mathcal{C})$. Then f has a series expansion, absolutely convergent on compacts in $\mathcal{C}$, as follows:*

$$f(\mathbf{z}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^\ell} c_{\boldsymbol{\alpha}} \mathbf{z}^{[\boldsymbol{\alpha}]} \tag{52}$$

where $|c_{\boldsymbol{\alpha}}| \leqslant \|f\|$. If $\mathcal{C}$ admits a $\boldsymbol{\delta}$-extension and $f \in \mathcal{O}_b(\mathcal{C}^{\boldsymbol{\delta}})$ then also $|c_{\boldsymbol{\alpha}}| < \boldsymbol{\delta}^{\operatorname{pos} \boldsymbol{\alpha}} \|f\|_{\mathcal{C}^\delta}$.

Proof. We proceed by induction on ℓ. The case $\mathcal{F}_\ell = *$ reduces immediately to the claim for $\mathcal{C}_{1..\ell-1}$. Otherwise the standard formula for Laurent expansions in $\mathcal{F}_\ell$ gives

$$f(\mathbf{z}) = \sum_{\boldsymbol{\alpha}_\ell = -\infty}^{\infty} c_{\boldsymbol{\alpha}_\ell}(\mathbf{z}_{1..\ell-1}) \mathbf{z}_\ell^{[\boldsymbol{\alpha}_\ell]} \tag{53}$$

where

$$c_{\boldsymbol{\alpha}_\ell}(\mathbf{z}_{1..\ell-1}) = \frac{\mathbf{z}_\ell^{\boldsymbol{\alpha}_\ell}}{\mathbf{z}_\ell^{[\boldsymbol{\alpha}_\ell]}}(2\pi i)^{-1} \oint \frac{f(\mathbf{z}_{1..\ell-1},\zeta)}{\zeta^{\boldsymbol{\alpha}_\ell+1}}\,\mathrm{d}\zeta \tag{54}$$

and the integral is over a simple positively oriented curve in $\mathcal{F}_\ell$. The Cauchy estimate now implies $\|c_{\boldsymbol{\alpha}_\ell}\|_{\mathcal{C}_{1..\ell-1}} \leqslant \|f\|$ and we proceed to expand $c_{\boldsymbol{\alpha}_\ell}$ by induction on ℓ to obtain a series (52) with $|c_{\boldsymbol{\alpha}}| < \|f\|$. To see that this multivariate series is absolutely convergent on compacts note that for every $\mathbf{z} \in \mathcal{C}$ and $\boldsymbol{\alpha} \in \mathbb{Z}^\ell$ we have

$$|\mathbf{z}^{[\boldsymbol{\alpha}]}| \leqslant \rho^{|\boldsymbol{\alpha}|} \quad \text{where} \quad \rho := \max_{\boldsymbol{\sigma}\in\{-1,1\}^\ell} \mathbf{z}^{[\boldsymbol{\sigma}]} < 1 \tag{55}$$

and (52) is thus majorated by $\sum_{\boldsymbol{\alpha}}(\rho+\varepsilon)^{|\boldsymbol{\alpha}|}$ in a neighborhood of $\mathbf{z}$. For the final claim we write a Laurent expansion in $\mathcal{C}^{\boldsymbol{\delta}}$ and rewrite the $\mathcal{C}^{\boldsymbol{\delta}}$-normalized monomials as $\mathcal{C}$-normalized monomials, gaining an extra factor of $\boldsymbol{\delta}^{\operatorname{pos}\boldsymbol{\alpha}}$. □

Let $\mathcal{P}_\ell := D(1)^{\times\ell}$ denote the standard unit polydisc. We write simply $\mathcal{P}$ when ℓ is clear from the context.

Corollary 26. *Let $f \in \mathcal{O}_b(\mathcal{C})$. Then there is a decomposition*

$$f(\mathbf{z}) = \sum_{\boldsymbol{\sigma}\in\{-1,1\}^\ell} f_{\boldsymbol{\sigma}}(\mathbf{z}^{[\boldsymbol{\sigma}]}) \tag{56}$$

where $f_{\boldsymbol{\sigma}} \in \mathcal{O}(\mathcal{P})$. If $\mathcal{C}$ admits a δ-extension and $f \in \mathcal{O}_b(\mathcal{C}^\delta)$ then $f_\sigma \in \mathcal{O}(\mathcal{P}^\delta)$, and moreover for any $\delta < \varepsilon < 1$ we have

$$\|f_{\boldsymbol{\sigma}}\|_{\mathcal{P}^\varepsilon} \leqslant (1-\delta/\varepsilon)^{-\ell}\,\|f\|_{\mathcal{C}^\delta}\,. \tag{57}$$

Proof. For the first statement we simply collect all summands with index $\boldsymbol{\alpha}$ of sign $\boldsymbol{\sigma}$ in (52) into $f_{\boldsymbol{\sigma}}$ (if $\boldsymbol{\alpha}_j = 0$ we arbitrarily treat it as positive for this purpose). For the second part, we have

$$\|f_{1,\dots,1}(\mathbf{w})\|_{\mathcal{P}^\varepsilon} \leqslant \sum_{\boldsymbol{\alpha}\in(\mathbb{Z}_{\geqslant 0})^\ell} |c_{\boldsymbol{\alpha}}|\,\|\mathbf{w}^{\boldsymbol{\alpha}}\|_{\mathcal{P}^\varepsilon} \leqslant \sum_{\boldsymbol{\alpha}\in(\mathbb{Z}_{\geqslant 0})^\ell} (\delta/\varepsilon)^{|\boldsymbol{\alpha}|} = (1-\delta/\varepsilon)^{-\ell}, \tag{58}$$

and similarly for the other choices of the signs. □

3.4. **Subanalyticity.** We say that a function is subanalytic on a domain $U \subset \mathbb{C}^n$ if it is defined there and its graph over U forms a subanalytic set. Our goal in this section is to prove the following proposition.

Proposition 27. *Let $0 < \delta < 1$. A complex cell $\mathcal{C}$ admitting a δ-extension is a subanalytic set. If $f \in \mathcal{O}_b(\mathcal{C}^\delta)$ then f is subanalytic on $\mathcal{C}$.*

Proof. We prove both claims by induction on the length of $\mathcal{C}$, the case of length zero being vacuous. Let $\mathcal{C} = \mathcal{C}_{1..\ell} \odot \mathcal{F}$. Then the radii $r(z)$ or $r_1(z), r_2(z)$ of $\mathcal{F}$ are subanalytic by induction and it easily follows that $\mathcal{C}$ is subanalytic. Now let $f \in \mathcal{O}_b(\mathcal{C}^\delta)$. By Corollary 26 we may write f as a sum of $2^{\ell+1}$ summands $f_{\boldsymbol{\sigma}}(\mathbf{z}^{[\boldsymbol{\sigma}]})$. In the definition of $\mathbf{z}^{[\boldsymbol{\sigma}]}$ each of the radii involved are subanalytic on $\mathcal{C}_{1..\ell}$ by induction, and each of the divisions involved are "restricted" in the sense of [18], i.e. we always have $|\mathbf{z}^{[\boldsymbol{\sigma}]}| \leqslant 1$ for $\mathbf{z} \in \mathcal{C}$. It then follows, for instance using the subanalytic language of [18], that the graph of each $f_{\boldsymbol{\sigma}}$ and hence of f is indeed subanalytic over $\mathcal{C}$. □

4. Semialgebraic and subanalytic sets

In our terminology, a cell in an o-minimal structure (below *classical cell*) is the $\odot$-product of a sequence of intervals (either closed or open on each side) with the two endpoints given by continuous definable functions. The image $f(\mathbb{R}_+\mathcal{C})$ of a real prepared cellular map is itself a cell in this classical sense. More explicitly if $f(\mathbf{z}_{1..\ell})_j = \mathbf{z}_j^{q_j} + \phi_j(\mathbf{z}_{1..j-1})$ and $\mathcal{C} = \mathcal{F}_1 \odot \cdots \odot \mathcal{F}_\ell$ then

$$f(\mathbb{R}_+\mathcal{C}) = \mathcal{I}_1 \odot \cdots \odot \mathcal{I}_\ell, \quad I_j := \begin{cases} \{\phi_j\} & \mathcal{F}_j = * \\ (\phi_j, \phi_j + |r^{q_j}|) & \mathcal{F}_j = D(r), D_\circ(r) \\ (\phi_j + |r_1^{q_j}|, \phi_j + |r_2^{q_j}|) & \mathcal{F}_j = A(r_1, r_2). \end{cases} \tag{59}$$

Furthermore, we note that f restricts to a real-analytic diffeomorphism from $\mathbb{R}_+\mathcal{C}$ onto its image.

A classical cell is called compatible with a continuous function F if $\operatorname{sign} F$ is constant on the cell. Since $\mathbb{R}_+\mathcal{C}$ is connected, we see that $f(\mathbb{R}_+\mathcal{C})$ is compatible with F in this classical sense if and only if f is compatible with F in our sense.

4.1. Cellular parametrizations for semialgebraic and subanalytic sets.

Definition 28. *A semialgebraic set $S \subset \mathbb{R}^n$ has* complexity (ℓ, β) *if $S = \pi_{1..n}(\tilde{S})$ where $\tilde{S} \subset \mathbb{R}^\ell$ is given by*

$$\tilde{S} = \{\operatorname{sign} P_1(\mathbf{x}_{1..\ell}) = \sigma_1, \ldots, \operatorname{sign} P_N = \sigma_N\}, \qquad \sigma_1, \ldots, \sigma_N \in \{-1, 0, 1\} \tag{60}$$

where $P_1, \ldots, P_N \in \mathbb{R}[\mathbf{x}_{1..\ell}]$ have degrees at most β and $N \leqslant \beta$. If $\ell = n$ we will say simply that S has complexity β.

The following semialgebraic parametrization result is a simple consequence of the CPT.

Corollary 29. *Let $\rho, \sigma \in \mathbb{R}_+$ and let $S \subset (0,1)^n$ be semialgebraic of complexity (ℓ, β). Then there exist* $\operatorname{poly}_\ell(\beta, \rho, 1/\sigma)$ *real cellular maps $f_j : \mathcal{C}_j^{\{\sigma\}} \to \mathcal{P}_n^{\{\rho\}}$, each of complexity* $\operatorname{poly}_\ell(\beta)$*, such that $f_j(\mathbb{R}_+\mathcal{C}_j^{\{\sigma\}}) \subset S$ and $\cup_j f_j(\mathbb{R}_+\mathcal{C}_j) = S$.*

Proof. We first reduce to the case $\tilde{S} \subset (0,1)^\ell$ in the notations of Definition 28. We may split $\tilde{S}$ to a union of sets and prove the result for each of their projections separately, so we reduce to the case $\tilde{S} \subset (0,1)^n \times I_{n+1} \times \cdots \times I_\ell$ where each I_j is one of $\{0\}, (0,\infty), (-\infty, 0)$. In the case of $\{0\}$ we can forget the x_j variable, and in the case $(-\infty, 0)$ we can replace x_j by $-x_j$, reducing to the case $\tilde{S} \subset (0,1)^n \times (0,\infty)^m$. In the same manner but using $x_j \to 1/(1+x_j)$ (which increases the degrees at most polynomially) we reduce to the case $\tilde{S} \subset (0,1)^\ell$.

Now apply the real CPT to the cell $\mathcal{P}_\ell^{\{\rho\}}$ with the collection of polynomials defining $\tilde{S}$. We obtain a collection of real prepared maps $\tilde{f}_j : \tilde{\mathcal{C}}_j^{\{\sigma\}} \to \mathcal{P}_\ell^{\{\rho\}}$ with the required complexity estimates such that $\tilde{f}_j(\mathbb{R}_+\tilde{\mathcal{C}}_j^{\{\sigma\}}) \subset \tilde{S}$ and $\cup_j \tilde{f}_j(\mathbb{R}_+\tilde{\mathcal{C}}_j) = \tilde{S}$. The cellular structure of $\tilde{f}_j$ implies that if we now take $\mathcal{C}_j := (\tilde{C}_j)_{1..n}$ and $f_j := (\tilde{f}_j)_{1..n}$ we have indeed $f_j(\mathbb{R}_+\mathcal{C}_j^{\{\sigma\}}) \subset S$ and $\cup_j f_j(\mathbb{R}_+\mathcal{C}_j) = S$. □

In an analogous manner one obtains the following subanalytic version.

Corollary 30. *Let $\rho, \sigma \in \mathbb{R}_+$ and let $S \subset (0,1)^n$ be subanalytic. Then there exist* $\operatorname{poly}_S(\rho, 1/\sigma)$ *real cellular maps $f_j : \mathcal{C}_j^{\{\sigma\}} \to \mathcal{P}_n^{\{\rho\}}$ such that $f_j(\mathbb{R}_+\mathcal{C}_j^{\{\sigma\}}) \subset S$ and $\cup_j f_j(\mathbb{R}_+\mathcal{C}_j) = S$.*

Remark 31. *We remark that from the proof it is clear that in Corollaries 29 and 30 one can also require the maps f_j to be compatible with an additional collection of functions $F_j \in \mathcal{O}_b(\mathcal{P}_n^{\{\rho\}})$. We also remark that by rescaling one can clearly replace the domain $(0,1)^n$ in Corollaries 29 and 30 by any other bounded semialgebraic/subanalytic ambient set. We will sometimes use $[0,1]^n$.*

4.2. **Preparation theorems.** The real CPT implies the preparation theorem for subanalytic functions of Parusinski [35] and Lion-Rolin [29], as we illustrate below. We illustrate the algebraic case here (where we get more effective information) but the subanalytic case follows in a similar manner. Let $F : (-1,1)^n \to \mathbb{R}$ be a bounded semialgebraic function and let $G_F \subset \mathbb{R}^n_x \times \mathbb{R}_y$ be its graph. We aim to cover $(0,1)^n$ by cylinders where F admits a simple expansion.

We apply Corollary 29 for the set G_F, requiring also that the maps f_j be compatible with y (see Remark 31). We obtain maps $f_j : \mathcal{C}_j^{\{\sigma\}} \to \mathcal{P}_n^{1/2} \times \mathcal{P}^{1/2}$ admitting such that $f_j(\mathbb{R}_+\mathcal{C}_j^{\{\sigma\}}) \subset G_F$ and $f_j(\mathbb{R}_+\mathcal{C}_j)$ cover G_F, and moreover each map is compatible with y.

Let $f : \mathcal{C}^{\{\sigma\}} \to \mathcal{P}_n^{1/2} \times \mathcal{P}^{1/2}$ be one of the maps f_j. Since G_F is a graph the type of $\mathcal{C}$ must end with $*$. It follows from the monomialization lemma that on each $\mathcal{C}^{\{\sigma\}}$ we have either $f^*y \equiv 0$ or

$$f^*y = \mathbf{z}^{\boldsymbol{\alpha}(j)} U_j(\mathbf{z}) \tag{61}$$

where U is a holomorphic map bounded away from zero and infinity on $\mathcal{C}$. To rewrite this expansion in the $\mathbf{x}$-coordinates recall that

$$\mathbf{z}_j = (\mathbf{x}_j - \phi_j(\mathbf{z}_{1..j-1}))^{1/\nu_j} \tag{62}$$

where we restrict $\mathbf{z}_j$ to the positive real part $\mathbb{R}_+\mathcal{C}_j$ and take the positive branch. Since $F(x) \equiv y$ on G_F we have on the cylinder $f(\mathbb{R}_+\mathcal{C}_j)$ the expansion

$$F(\mathbf{x}) = \mathbf{z}^{\boldsymbol{\alpha}} U(\mathbf{z}) \tag{63}$$

where $\mathbf{z}$ is given by (62). In other words we have obtained cylinders where F expands as a monomial with fractional powers times a unit. This implies the preparation theorem of [29] (in the bounded semialgebraic case). Indeed, in [29] the unit is required to be bounded away from zero and infinity and satisfy an analytic expansion of the form $U(\mathbf{z}) = V(\psi(\mathbf{x}_{1..n-1}, \mathbf{x}_n))$ where

$$\begin{multline}\psi(\mathbf{x}_{1..n-1}, \mathbf{x}_n) = (\psi_1(\mathbf{x}_{1..n-1}), \dots, \psi_s(\mathbf{x}_{1..n-1}), \\ \mathbf{x}_n^{1/p}/a_1(\mathbf{x}_{1..n-1}), b_1(\mathbf{x}_{1..n-1})/\mathbf{x}_n^{1/p}), \tag{64}\end{multline}$$

with p a positive integer and ψ_i, a_1, b_1 are bounded subanalytic functions, and V is a non-zero analytic function on the compact closure of the image of ψ. In our case this expansion is the Laurent expansion of $U_j(\mathbf{z})$ with respect to $\mathbf{z}_n$. We of course obtain a wealth of additional information on the number of cylinders, their complexity, and effective estimates on the monomial $\mathbf{z}^\alpha$ and the unit $U(\mathbf{z})$ from the algebraic monomialization lemma.

We remark that, as usual in the passage from the real analytic to the holomorphic category, the definitions involving existence of converging expansions are replaced by purely local counterparts. For instance, while the notion of a function *reducible* in a cylinder requires a careful inductive definition in [29], any holomorphic function compatible with a complex cell is *automatically* reducible there. Similarly, while the definition of of a unit involves a delicate analytic expansion (64) in [29], a unit in a complex cell is a holomorphic function satisfying a purely topological definition (the associated monomial being equal to zero), which in turn implies the real condition.

5. The principal lemmas

5.1. **Hyperbolic geometry of complex cells.** For the proofs of the domination, fundamental and monomialization lemmas we will require some basic notions of hyperbolic geometry. Recall that the upper half-plane $\mathbb{H}$ admits a unique hyperbolic metric of constant curvature -4 given by $|\mathrm{d}z|/2y$. A Riemann surface X is called *hyperbolic* if its universal cover is the upper half-plane $\mathbb{H}$. In this case U inherits from $\mathbb{H}$ a unique metric of constant curvature -4 which we denote by $\mathrm{dist}(\cdot,\cdot;U)$ (we sometimes omit U from this notation if it is clear from the context). In particular a domain $U\subset\mathbb{C}$ is hyperbolic if its complement contains at least two points. The Schwarz-Pick lemma states that any holomorphic map between hyperbolic Riemann surfaces is non-expanding in their respective hyperbolic metrics.

5.2. **Maps from cells into hyperbolic Riemann surfaces.** We will require the following notion of *skeleton* of a cell.

Definition 32 (Skeleton of a cell)**.** *If $\mathcal{C}$ is a cell whose type does not include $D_\circ$ then we define its* skeleton $\mathcal{S}(\mathcal{C})$ *as follows: the skeleton of the cell of length zero is the singleton $\mathbb{C}^0$; The skeleton of $\mathcal{C}_{1..\ell}\odot\mathcal{F}$ is $\mathcal{S}(\mathcal{C}_{1..\ell})\odot\partial\mathcal{F}$ where*

$$\partial * := * \qquad \partial D(r) := S(r) \qquad \partial A(r_1,r_2) := S(r_1)\cup S(r_2). \tag{65}$$

Each connected component of $\mathcal{S}(\mathcal{C})$ is a product of ℓ circles and points, and the number of connection components is equal to 2^α, where α is the number of symbols A in the type of $\mathcal{C}$.

Let $\mathcal{C}$ be a complex cell whose type does not contain $D_\circ$. For the results discussed in this section any coordinate with type $*$ can be removed without loss of generality, so we assume that the type does not contain $*$. Then $\dim\mathcal{C}=\ell(\mathcal{C})$ and we denote this number by n. We assume that $\mathcal{C}$ admits a $\{\rho\}$ extension for some $\{\rho\}>0$ and that $f:\mathcal{C}^{\{\rho\}}\to X$ is a holomorphic map to a hyperbolic Riemann surface X. We begin by studying the hyperbolic behavior of f on the skeleton $\mathcal{S}(\mathcal{C})$.

Lemma 33. *Let S a component of the skeleton $\mathcal{S}(\mathcal{C})$. Then*

$$\mathrm{diam}(f(S);X) < n\rho. \tag{66}$$

Proof. We proceed by induction and suppose the claim is proved for cells of dimension smaller than n. Let $\mathbf{z},\mathbf{z}''\in S$. We will construct a point $\mathbf{z}'\in S$ such that $\mathrm{dist}(f(\mathbf{z}),f(\mathbf{z}'))<\rho$ and such that $\mathbf{z}'_1=\mathbf{z}''_1$. Then $\mathbf{z}'_{2..n},\mathbf{z}''_{2..n}$ both belong to the $n-1$ dimensional cell $\mathcal{C}_{\mathbf{z}'_1}$ and applying the inductive hypothesis to the restriction of f to this fiber we conclude that $\mathrm{dist}(f(\mathbf{z}'),f(\mathbf{z}''))<(n-1)\rho$ thus finishing the proof.

By definition of the skeleton, we have

$$\mathbf{z}_1=\varepsilon_1 r_1 \qquad \mathbf{z}_2=\varepsilon_2 r_2(\mathbf{z}_1) \qquad \dots \qquad \mathbf{z}_n=\varepsilon_n r_n(\mathbf{z}_{1..n-1}), \quad |\varepsilon_i|=1. \tag{67}$$

where $r_1>0$ and $r_2,\dots,r_n$ are holomorphic functions on $\mathcal{C}^{\{\rho\}}$. Let $\mathcal{F}$ denote the first fiber in $\mathcal{C}$. We define a map $\gamma(\mathbf{z}_1):\mathcal{F}^{\{\rho\}}\to\mathcal{C}^{\{\rho\}}$ by $\gamma(\mathbf{z}_1)=(\mathbf{z}_{1..n})$ where

$$\mathbf{z}_2=\varepsilon_2 r_2(\mathbf{z}_1) \qquad \dots \qquad \mathbf{z}_n=\varepsilon_n r_n(\mathbf{z}_{1..n-1}). \tag{68}$$

Then γ is holomorphic in $\mathcal{F}^{\{\rho\}}$ and satisfies $\gamma(\mathbf{z}_1) = \mathbf{z}$ and $\gamma(\{|\mathbf{z}_1| = r_1\}) \subset S$. We take $\mathbf{z}' := \gamma(\mathbf{z}_1'') \in S$ and note that

$$\operatorname{dist}(f(\mathbf{z}), f(\mathbf{z}'); X) = \operatorname{dist}(f\circ\gamma(\mathbf{z}_1), f\circ\gamma(\mathbf{z}_1''); X) \leqslant \operatorname{dist}(\mathbf{z}_1, \mathbf{z}_1''; \mathcal{F}^{\{\rho\}}) \leqslant \rho \tag{69}$$

where the first inequality follows from the Schwarz-Pick lemma for $f\circ\gamma$ and the second inequality follows since $\mathbf{z}_1, \mathbf{z}_1''$ belong to the boundary of $\mathcal{F}$ in $\mathcal{F}^{\{\rho\}}$. $\square$

Next, we show (essentially by the open mapping theorem) that the boundary of $f(\mathcal{C})$ is controlled by the skeleton.

Lemma 34. *We have the inclusion $\partial f(\mathcal{C}) \subset f(\mathcal{S}(\mathcal{C}))$.*

Proof. We assume for simplicity that the type of $\mathcal{C}$ is $A\cdots A$ (the difference with disc fibers is purely notational). Let $p \in \partial f(\mathcal{C})$ and fix a point $\mathbf{z} \in \mathcal{C}$ such that $f(\mathbf{z}) = p$. If $\mathbf{z} \in S$ then we are done. Otherwise one of the following holds

$$\begin{aligned} r_{1,1} <&|\mathbf{z}_1| < r_{1,2} \\ |r_{2,1}(\mathbf{z}_1)| <&|\mathbf{z}_2| < |r_{2,2}(\mathbf{z}_1)| \\ &\vdots \\ |r_{n,1}(\mathbf{z}_{1..n-1})| <&|\mathbf{z}_n| < |r_{n,2}(\mathbf{z}_{1..n-1})|. \end{aligned} \tag{70}$$

Let k be the largest index for which such inequalities hold. Let

$$A := A(r_{j,1}(\mathbf{z}_{1..j-1}), r_{j,2}(\mathbf{z}_{1..j-1})). \tag{71}$$

We suppose further that $\mathbf{z}$ is chosen such that k is the minimal possible. We define a map $\gamma(z_k) : A^\delta \to \mathcal{C}$ by $\gamma(z_j) = (z_{1..n})$ where $z_{1..k-1} = \mathbf{z}_{1..k-1}$ and

$$z_{j+1} = \frac{\mathbf{z}_{j+1}}{r_{j+1,i(j+1)}(\mathbf{z}_{1..j})} r_{j+1,i(j+1)}(z_{1..j}), \qquad j \geqslant k \tag{72}$$

where $i(j)$ is the index, either 1 or 2, such that equality instead of inequality holds in the jth line of (70). By definition we have $\gamma(\mathbf{z}_j) = \mathbf{z}$.

If $f\circ\gamma$ is non-constant then by the open mapping theorem $p = f\circ\gamma(\mathbf{z}_j)$ is an interior point of $f\circ\gamma(A) \subset f(\mathcal{C})$ contrary to our assumption. Otherwise we have $p = f(\mathbf{z}')$ where $\mathbf{z}' = \gamma(z_j')$ and z_j' is any point on ∂A. The index k obtained for $\mathbf{z}'$ is by definition smaller than that obtained for $\mathbf{z}$, which yields a contradiction to our choice of $\mathbf{z}$. $\square$

Finally, we show that under a suitable topological condition the hyperbolic diameter of $f(\mathcal{C})$ itself can be bounded.

Lemma 35. *Suppose that $f_*(\pi_1(\mathcal{C})) = \{e\} \subset \pi_1(X)$. Then*

$$\operatorname{diam}(f(\mathcal{C}); X) < 2^n n\rho. \tag{73}$$

Proof. Denote by $\pi : \mathbb{D} \to X$ the universal covering map. By our assumption we may lift f to a map $F : \mathcal{C}^{\{\rho\}} \to \mathbb{D}$ satisfying $f = \pi\circ F$. Since π is non-expanding it is enough to prove the claim for the hyperbolic diameter of $F(\mathcal{C})$. By Lemma 33 the hyperbolic diameter of $F(S)$, where S is any component of the skeleton $\mathcal{S}(\mathcal{C})$, is bounded by $n\rho$. By Lemma 34 we also have the inclusion $\partial F(\mathcal{C}) \subset F(\mathcal{S}(\mathcal{C}))$.

Recall that we assume that the type of $\mathcal{C}$ does not contain $D_\circ$ and hence $\bar{\mathcal{C}} \subset \mathcal{C}^\delta$. In particular it follows that $Z := F(\mathcal{C})$ is relatively compact, and hence bounded, in $\mathbb{D}$. Let U denote the unbounded component of $\mathbb{D}\setminus\bar{Z}$ and set $U^c := \mathbb{D}\setminus U$ and $\Gamma :=$

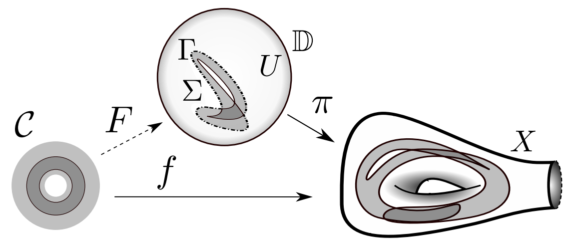

Figure 4. Proof of Lemma 35.

∂U. To avoid pathologies of plane topology we remark that Z is subanalytic, hence Z, U and their boundaries can be triangulated and the Mayer-Vietoris sequence

$$0 = H_1(\mathbb{D}) \to H_0(\Gamma) \to H_0(\bar{U}) \oplus H_0(\overline{U^c}) \to H_0(\mathbb{D}) \to 0 \tag{74}$$

is exact. Since $H_0(\mathbb{D}) = H_0(\bar{U}) = \mathbb{Z}$ we have $H_0(\Gamma) \simeq H_0(\overline{U^c})$. We claim that U^c and hence $\overline{U^c}$ is connected. Assume otherwise and write $\overline{U^c} = V_1 \cup V_2$. Since $\bar{Z}$ is connected we have without loss of generality $\bar{Z}$ is contained in V_1 and therefore disjoint from V_2. In particular ∂V_2 is disjoint from $\bar{Z}$, contradicting

$$\partial V_2 \subset \partial U^c = \partial U \subset \partial Z. \tag{75}$$

In conclusion we have $H_0(\Gamma) \simeq H_0(\overline{U^c}) = \mathbb{Z}$, i.e. Γ is connected.

We claim that the hyperbolic diameter of $F(\mathcal{C})$ is bounded by the hyperbolic diameter of Γ. Indeed, for any two points $p, q \in F(\mathcal{C})$ let ℓ denote the geodesic line connecting them and denote by $p' \in \ell \cap \Gamma$ some point on ℓ before p and by $q' \in \ell \cap \Gamma$ some point on ℓ after q. Then $\operatorname{dist}(p,q) \leqslant \operatorname{dist}(p',q')$ which is bounded by the diameter of Γ. Finally, recall that Γ is connected and contained in the union of $F(S_j)$ where S_j run over the components of $\mathcal{S}(\mathcal{C})$ (whose number is at most 2^n), and the diameter of each $F(S_j)$ is bounded by $n\rho$. From this it follows easily that the diameter of Γ is at most $2^n n\rho$. □

5.3. **Proof of the domination lemma.** We begin by assuming that the type of $\mathcal{C}$ does not contain $D_\circ$. Let

$$U_0 := \{|z| < \tfrac{1}{2}\} \qquad U_1 := \{|z-1| < \tfrac{1}{2}\} \qquad U_\infty = \{|z| > 2\}. \tag{76}$$

We choose $s > 1$ such that the hyperbolic distance between U_q^s and ∂U_q is greater than $n\rho$ for $q = 0, 1, \infty$. From the explicit computations in §5.6 we can take $s = O(\log|\log(n\rho)|)$. If $f(\mathcal{C})$ does not meet U_0^s or U_∞^s then the proof of the domination lemma is completed. Henceforth we assume that $f(\mathcal{C})$ meets both U_0^s and U_∞^s.

Lemma 36. *Suppose $f(\mathcal{C})$ meets both U_0^s and U_∞^s. Then*

$$f_* \pi_1(\mathcal{C}) = \{e\} \subset \pi_1(\mathbb{C} \setminus \{0,1\}). \tag{77}$$

Proof. Recall that we assume that the type of $\mathcal{C}$ does not contain $D_\circ$, and in particular $\bar{\mathcal{C}} \subset \mathcal{C}^\delta$. Thus f extends to a continuous function $\bar{\mathcal{C}}$, so $f(\bar{\mathcal{C}}) \subset \mathbb{C} \setminus \{0,1\}$ is compact and it follows that $\partial f(\mathcal{C})$ meets both U_0^s and U_∞^s. By Lemma 34 we conclude that there exist two components S_0, S_∞ of the skeleton $\mathcal{S}(\mathcal{C})$ such that $f(S_0)$ meets U_0^s and $f(S_\infty)$ meets U_∞^s. From Lemma 33 and the choice of s we conclude that $f(S_0)$ does not meet the boundary of U_0, i.e. $f(S_0) \subset U_0$ and similarly $f(S_\infty) \subset U_\infty$.

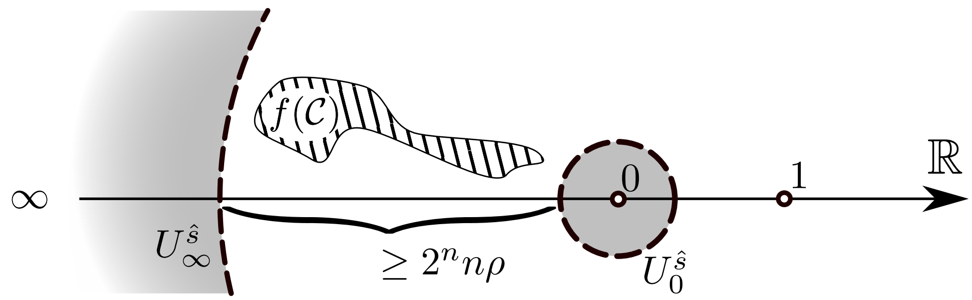

FIGURE 5. Proof of the domination lemma.

It is easy to verify (it is enough to check this for the standard polyannulus) that $\pi_1(S) \to \pi_1(\mathcal{C})$ is epimorphic. In particular, if $\gamma \in \pi_1(\mathcal{C})$ denotes any loop then this loop is free-homotopy equivalent to a loop contained in S_0 and to a loop contained in S_∞. Consequently $f_*(\gamma)$ is free-homotopy equivalent to a loop contained in U_0 and to a loop contained in U_∞. However two such loops cannot be homotopically equivalent in $\mathbb{C} \setminus \{0,1\}$ unless they are both contractible, hence proving the claim. □

Lemma 36 implies the condition of Lemma 35, and we conclude that the hyperbolic diameter of $f(\mathcal{C})$ is bounded by $2^n n\rho$. We now choose $\hat s > 1$ such that the hyperbolic distance between $U_0^{\hat s}$ and $U_\infty^{\hat s}$ is greater than $2^n n\rho$. By §5.6 we may take $\hat s = O(\log(n + \log\rho))$. Then $f(\mathcal{C})$ cannot meet both $U_0^{\hat s}, U_\infty^{\hat s}$ and the domination lemma is proved.

It remains to consider the case that the type of $\mathcal{C}$ contains $D_\circ$. Let $0 < \varepsilon < 1$ and let $\mathcal{C}_\varepsilon$ be the cell obtained from $\mathcal{C}$ by replacing each occurrence of a fiber $D_\circ(r)$ by $A(\varepsilon r, r)$. It is clear that $\mathcal{C}_\varepsilon$ admits a $\{\rho\}$-extension and $\cup_{\varepsilon>0}\mathcal{C}_\varepsilon = \mathcal{C}$. The domination lemma for $\mathcal{C}$ thus follows immediately from the domination lemma for $\mathcal{C}_\varepsilon$, which was already established.

5.4. **Proofs of the fundamental lemmas.** In this section we assume that the type of $\mathcal{C}$ does not contain $D_\circ$. The general case can be reduced to this case as in the end of §5.3. The fundamental lemma for $\mathbb{D}$ is already proved as a consequence of Lemma 35. The proof in the case of $\mathbb{D} \setminus \{0\}$ is based on the following simple geometric lemma.

Lemma 37. *Let $z \in \mathbb{D} \setminus \{0\}$ and let γ_z denote the shortest non-contractible loop passing through z. Then* $\operatorname{length}(\gamma_z) \sim 1/|\log|z||$.

Proof. Let $\zeta = i^{-1}\log z$. Lifting to the universal cover $\exp(i\zeta) : \mathbb{H} \to \mathbb{D} \setminus \{0\}$ we must calculate $\operatorname{dist}(\zeta, \zeta + 2\pi; \mathbb{H})$. By a standard formula for the hyperbolic distance we have

$$\operatorname{dist}(\zeta, \zeta + 2\pi; \mathbb{H}) = 2\ln\left(\frac{\pi + \sqrt{\pi^2 + \operatorname{Im}^2\zeta}}{\operatorname{Im}\zeta}\right) \sim 1/\operatorname{Im}\zeta. \tag{78}$$

□

Let $f : \mathcal{C}^{\{\rho\}} \to \mathbb{D} \setminus \{0\}$ be holomorphic and let z be the maximum value of $|f|$ on $\mathcal{C}$, which by Lemma 34 is obtained on a component S of the skeleton of $\mathcal{C}$. By Lemma 33 we have $\operatorname{diam}(f(S); D \setminus \{0\}) = O_\ell(\rho)$. In particular, if $1/|\log|z|| =$

$\Omega_\ell(\rho)$ then $f(S)$ cannot contain a non-contractible loop in $D\setminus\{0\}$. Then, since $\pi_1(S)\to\pi_1(\mathcal{C})$ is epimorphic we have

$$f_*(\pi_1(\mathcal{C}))=f_*(\pi_1(S))=\{e\}\subset\pi_1(\mathbb{D}\setminus\{0\}). \tag{79}$$

In this case by Lemma 35 we have $\operatorname{diam}(f(\mathcal{C});D\setminus\{0\})=O_\ell(\rho)$. Otherwise we have $\log|z|=-\Omega_\ell(1/\rho)$, which concludes the proof of the first statement. The second statement follows from the simple fact

$$\operatorname{dist}(\zeta_1,\zeta_2;\mathbb{H})\geqslant\operatorname{dist}(\log\operatorname{Im}\zeta_1,\log\operatorname{Im}\zeta_2;\mathbb{R}). \tag{80}$$

We now pass to the proof of the fundamental lemma for $\mathbb{C}\setminus\{0,1\}$. Let S be a component of the skeleton of $\mathcal{C}$. By Lemma 33 we have $\operatorname{diam}(f(S);C\setminus\{0,1\})=O_\ell(\rho)$. In particular, for $\rho=O(1)$ we see that $f(S)$ cannot meet more than one of U_0,U_1,U_∞ in the notation (76). Suppose first the $f(S)$ meets none of these sets. Let ρ_0 denote the length of the shortest non-contractible loop in the compact set $(U_0\cup U_1\cup U_\infty)^c\subset\mathbb{C}\setminus\{0,1\}$. Then for $\rho=O(\rho_0)=O(1)$ we get $\operatorname{diam}(f(\mathcal{C});\mathbb{C}\setminus\{0,1\})=O_\ell(\rho)$ as in the proof of the case $\mathbb{D}\setminus\{0\}$.

Next, suppose that for two different components S,S' the images $f(S),f(S')$ meet two of the sets U_0,U_1,U_∞, say U_0 and U_1 respectively. Then for $\rho=O(1)$ the images $f_*(\pi_1(S))$ (resp. $f_*(\pi_1(S'))$ can only be a power of the fundamental loop around 0 (resp. 1) and as in the proof of the domination lemma we conclude that $f_*(\pi_1(\mathcal{C}))=\{e\}$ and $\operatorname{diam}(f(\mathcal{C});\mathbb{C}\setminus\{0,1\})=O_\ell(\rho)$.

Finally, suppose all skeleton components S meet one of the sets U_0,U_1,U_∞, say U_0. Then for $\rho=O(1)$ we see that $f(S)\subset\mathbb{D}\setminus\{0\}$ for each skeleton component. In this case we can finish the proof as in the case of $\mathbb{D}\setminus\{0\}$. We need only the estimate for the length of the shortest geodesic in $\mathbb{C}\setminus\{0,1\}$ passing through a given point z close to 0: this is asymptotically the same as in the metric of $\mathbb{D}\setminus\{0\}$, for instance by the estimate (87) of [3], given explicitly in §5.6.

5.5. Proof of the monomialization lemma.

The *Voorhoeve index* [28] of a holomorphic function $f:U\to\mathbb{C}$ along a subanalytic curve $\Gamma\subset U$ is defined by

$$V_\Gamma(f):=\frac{1}{2\pi}\int_\Gamma|\,\mathrm{d}\operatorname{Arg}f(z)|. \tag{81}$$

We will need the following basic fact about Voorhoeve indices.

Lemma 38. *Fix $\rho>0$. Let $\mathcal{F}_\lambda$ be a definable family of one-dimensional cells, let $f_\lambda:U^\rho\to\mathbb{C}$ a definable family of holomorphic functions, and let $\Gamma_\lambda\subset\mathcal{F}_\lambda$ be a definable family of curves. Then $V_{\Gamma_\lambda}(f_\lambda)$ is uniformly bounded over λ.*

If f,Γ are algebraic of complexity β then $V_\Gamma(f)=\operatorname{poly}(\beta)$.

Proof. Note that $2\pi V_\Gamma(f)$ is the total length of the curve $(\frac{f}{|f|})(\Gamma)\subset S(1)$, which by standard integral geometry is given by the average number of intersections between the curve and a ray through the origin, i.e. the average number of (isolated) solutions of the equation $\operatorname{Arg}f(z)=\alpha$ for $z\in\Gamma$ $\alpha\in[0,2\pi)$. By o-minimality the *maximal* number (and in particular the average number) of (isolated) solutions for the pair f_λ,Γ_λ is bounded by a constant independent of λ. In the algebraic case the number of solutions is bounded by $O(\beta^2)$ by the Bezout theorem. □

The basic ingredient in the proof of the monomialization lemma is the following one-dimensional version, proved using the Voorhoeve index.

Lemma 39. *Let $\mathcal{F}$ be a one-dimensional cell and $f : \mathcal{F}^{\{\rho\}} \to \mathbb{C} \setminus \{0\}$. Then $f = z^{\alpha(f)} \cdot U(z)$ where*

$$\operatorname{diam}(\operatorname{Re}\log U(\mathcal{F}); \mathbb{R}) < O_f(\rho), \qquad \operatorname{diam}(\operatorname{Im}\log U(\mathcal{F}); \mathbb{R}) < O_f(1).$$

Moreover if $\mathcal{F}, f$ vary in a definable family then $|\alpha(f)|$ and the asymptotic constants in $O_f(\cdot)$ can be taken to be uniform over the family. If f is algebraic of complexity β then $|\boldsymbol{\alpha}(f)| = \operatorname{poly}(\beta)$ and

$$\operatorname{diam}(\operatorname{Re}\log U(\mathcal{F}); \mathbb{R}) < \operatorname{poly}(\beta) \cdot \rho, \qquad \operatorname{diam}(\operatorname{Im}\log U(\mathcal{F}); \mathbb{R}) < \operatorname{poly}(\beta).$$

Proof. We follow the idea of [28, Section 4.5]. By definition $\alpha(f)$ is equal to the total winding number of f along a concentric circle contained in $\mathcal{F}$ for types $D_\circ, A$ and zero in type D. In particular it is bounded by $V_\Gamma(f)$ where Γ is such a circle, and the statements about $\alpha(f)$ then follow from Lemma 38. In the algebraic case we see also that the complexity of U is $\operatorname{poly}(\beta)$.

Any two points in $\mathcal{F}$ can be joined by two consecutive algebraic curve: a radial ray and a circle concentric with $\mathcal{F}$. By Lemma 38 the Voorhoeve index of f along these curves is uniformly bounded (resp. bounded by $\operatorname{poly}(\beta)$ in the algebraic case). Having already established that $|\alpha(f)|$ is uniformly bounded (resp. bounded by $\operatorname{poly}_\ell(\beta)$ in the algebraic case) we conclude that the Voorhoeve index of U along any two such curves is similarly bounded by a uniform constant v (and $v = \operatorname{poly}_\ell(\beta)$ in the algebraic case).

Let $p \in \mathcal{F}$ be an arbitrary point. Since the claim is invariant under scalar multiplication of f we may assume without loss of generality that $U(p) = 1$. Since $U_*\pi_1(\mathcal{F}) = \{e\}$ we have a well-defined root $W : \mathcal{F}^{\{\rho\}} \to \mathbb{C} \setminus \{0\}$ satisfying $W = iU^{1/(4v)}$ and $W(p) = i$. Connecting p to any other point $q \in \mathcal{F}$ by two curves as above, the Voorhoeve index of W along the curves is bounded by $1/4$, hence the total variation of argument is bounded by $\pi/2$. Since $\operatorname{Arg} W(p) = \pi/2$ we conclude that in fact $W : \mathcal{F}^{\{\rho\}} \to \mathbb{H}$. By Lemma 33 we then have $\operatorname{diam}(W(\mathcal{F}); \mathbb{H}) = O(\rho)$. In particular this implies

$$\operatorname{diam}(\operatorname{Re}\log W(\mathcal{F}); \mathbb{R}) = O(\rho) \qquad \operatorname{diam}(\operatorname{Im}\log W(\mathcal{F}); \mathbb{R}) < \pi \tag{82}$$

and the claim for U follows immediately (using the fact that $v = \operatorname{poly}_\ell(\beta)$ in the algebraic case). □

We are now ready to finish the proof of the monomialization lemma by induction on ℓ. Let $\mathcal{C} = \mathcal{C}_{1..\ell} \odot \mathcal{F}$. The case $\mathcal{F} = *$ reduces trivially to the claim for $\mathcal{C}_{1..\ell-1}$ and the case $\ell = 0$ is proved in Lemma 39. Up to renormalization we may assume that the outer radius of $\mathcal{F}$ is 1. Let $\hat{f} : \mathcal{C}_{1..\ell}^{\{\rho\}} \to \mathbb{C} \setminus \{0\}$ be defined by $\hat{f} := f(\mathbf{z}_{1..\ell}, 1)$. If $\mathcal{C}, f$ are algebraic of complexity β then $\hat{f}$ is algebraic of complexity $\operatorname{poly}_\ell(\beta)$.

By definition we have $\boldsymbol{\alpha}(\hat{f}) = \boldsymbol{\alpha}_{1..\ell}(f)$, so $\hat{U} := U(\mathbf{z}_{1..\ell}, 1)$ is equal to $\hat{f}/\mathbf{z}^{\alpha(\hat{f})}$. By the inductive hypothesis we have

$$\operatorname{diam}(\operatorname{Re}\log \hat{U}(\mathcal{C}_{1..\ell}); \mathbb{R}) < O_{\hat{f}}(1) \cdot \rho, \quad \operatorname{diam}(\operatorname{Im}\log \hat{U}(\mathcal{C}_{1..\ell}); \mathbb{R}) < O_{\hat{f}}(1) \tag{83}$$

with uniform constants if $\mathcal{C}, f$ vary in a definable family. In the algebraic case $|\alpha(\hat{f})| = \operatorname{poly}_\ell(\beta)$ and

$$\operatorname{diam}(\operatorname{Re}\log \hat{U}(\mathcal{C}_{1..\ell}); \mathbb{R}) < \operatorname{poly}_\ell(\beta) \cdot \rho, \quad \operatorname{diam}(\operatorname{Im}\log \hat{U}(\mathcal{C}_{1..\ell}); \mathbb{R}) < \operatorname{poly}_\ell(\beta). \tag{84}$$

Also, for each fixed $p \in \mathcal{C}_{1..\ell}$ we have by Lemma 39 for the fiber $\mathcal{C}_p$

$$\operatorname{diam}(\operatorname{Re}\log U(\mathcal{C}_p);\mathbb{R}) < O_f(\rho), \qquad \operatorname{diam}(\operatorname{Im}\log U(\mathcal{C}_p);\mathbb{R}) < O_f(1) \tag{85}$$

with uniform constants if $\mathcal{C}, f$ vary in a definable family. In the algebraic case $|\alpha_{\ell+1}(f)| = \operatorname{poly}(\beta)$ and

$$\operatorname{diam}(\operatorname{Re}\log U(\mathcal{C}_p);\mathbb{R}) < \operatorname{poly}_\ell(\beta)\cdot\rho, \quad \operatorname{diam}(\operatorname{Im}\log U(\mathcal{C}_p);\mathbb{R}) < \operatorname{poly}_\ell(\beta). \tag{86}$$

The triangle inequality now finishes the proof.

5.6. **Hyperbolic lengths.** We fix the hyperbolic metric, $\lambda_D(z)|dz|$ with $\lambda_D(z) = (1-|z|^2)^{-1}$, of constant curvature -4 on the unit disc D. An explicit lower bound for the hyperbolic metric $\lambda_{0,1}(z)|dz|$ on $\mathbb{C}\setminus\{0,1\}$ in U_0 is given in [3],

$$\lambda_{0,1}(z) \geqslant \frac{1}{2|z|\sqrt{2}\left[4+\log(3+2\sqrt{2})-\log|z|\right]}, \quad |z| \leqslant \frac{1}{2}. \tag{87}$$

Integrating (87), we see that the hyperbolic distance from $\{|z|=r\}$ to $\{|z|=1/2\}$ is greater than s for $r < \tilde\rho(s) < \frac{1}{2}$, where

$$\log\tilde\rho(s) = -e^{\Theta(s)} + O(1).$$

As $z \to z^{-1}$ is an isometry of $\mathbb{C}\setminus\{0,1\}$, the hyperbolic distance from $\{|z|=r\}$ to $\{|z|=2\}$ is greater than s for $r > \tilde\rho(s)^{-1}$.

5.6.1. *Proof of Fact 6.* First, consider the case of $\mathcal{F}$ of type D. Up to rescaling, $\mathcal{F}^\delta = D(1)$ and $\partial\mathcal{F} = \{|z|=\delta\}$. Then $\lambda_D(z)\,|dz| \equiv (1-\delta^2)^{-1}\,|dz|$ on $\partial\mathcal{F}$, so the length of $\partial\mathcal{F}$ in D is equal to $\frac{2\pi\delta}{1-\delta^2}$.

Second, consider the cases $\mathcal{F}$ of type $D_\circ, A$. If $S(r)$ is (a component of) $\partial\mathcal{F}$, then $S(r) \subset S^\delta(r) \subset \mathcal{F}^\delta$. Therefore, by Schwartz-Pick lemma, it is enough to prove the following

Lemma 40. *The length of $S(1)$ in the hyperbolic metric of the annulus $S^\delta(1)$ is equal to $\frac{\pi^2}{2|\log\delta|}$.*

Proof. Indeed, the mapping $\phi(z) = \frac{\pi i}{2|\log\delta|}\log z$ maps the universal cover of $S^\delta(1)$ to the strip $\Pi = \{|\operatorname{Im} w| \leqslant \frac{\pi}{2}\}$, with $\phi^{-1}([0,\frac{\pi^2}{|\log\delta|}]) = S^1$. This map preserves hyperbolic distances, so it is enough to find the hyperbolic length of $[0,\frac{\pi^2}{|\log\delta|}]$ in the hyperbolic metric $\lambda_\Pi(w)\,|dw|$ of Π. As Π is invariant under shifts by reals, it is enough to find the $\lambda_\Pi(0)$. The map $\psi(w) = \frac{e^w-1}{e^w+1}$ send Π isometrically to the unit disc, with $\psi(0)=0$, so $\lambda_\Pi(0) = \psi'(0) = 1/2$ and the length of $[0,\frac{\pi^2}{|\log\delta|}]$ in Π is equal to $\frac{\pi^2}{2|\log\delta|}$. $\square$

Remark 41. *The same chain of conformal mappings allows to compute explicitly the distance between $r>0$ and 1 in the hyperbolic metric of $A^\delta = A(\delta r, \delta^{-1})$:*

$$\operatorname{dist}_{A^\delta}(r,1) = \log\frac{1+\tan\phi}{1-\tan\phi}, \quad \textit{where} \quad \phi = \frac{\pi}{4}\left(1 - \frac{\log\delta}{\log(\delta\sqrt{r})}\right). \tag{88}$$

As $r \to 0$, we get

$$\operatorname{dist}_{A^\delta}(r,1) = \log\left(\frac{16}{\pi}\frac{\log(\delta\sqrt{r})}{\log\delta}\right) + O(1) = \log|\log r| + O(1). \tag{89}$$

From the inequalities $\operatorname{dist}_{A^\delta}(r,1) < \operatorname{diam}_{A^\delta} A < \operatorname{dist}_{A^\delta}(r,1) + \frac{\pi^2}{2|\log\delta|}$*, we see that*

$$\operatorname{diam}_{A^\delta} A = \log|\log r| + O(1) \quad as \quad r \to 0. \tag{90}$$

6. Geometric constructions with cells

In this section we develop the two key geometric constructions used in the proofs of our main theorems: cellular refinement and clustering in fibers of proper covers.

6.1. **Refinement of cells.** In this section we show how cells with a $\{\rho\}$-extension can be *refined* into cells with a $\{\sigma\}$-extension for $0<\sigma<\rho$. We remark that while the statement appears innocuous, the proof actually requires the full strength of the fundamental lemma for $\mathbb{D}\setminus\{0\}$. The key difficulty is to construct the refinement of an annulus fiber in a manner that depends holomorphically on the base.

Theorem 9 (Refinement theorem)**.** *Let* $\mathcal{C}^{\{\rho\}}$ *be a (real) cell and* $0<\sigma<\rho$. *Then there exists a (real) cellular cover* $\{f_j : \mathcal{C}_j^{\{\sigma\}} \to \mathcal{C}^{\{\rho\}}\}$ *of size* $\operatorname{poly}_\ell(\rho, 1/\sigma)$ *where each* f_j *is a cellular translate map.*

If $\mathcal{C}$ *varies in a definable family (and* σ,ρ *vary under the condition* $0<\sigma<\rho<\infty$*) then the cells* $\mathcal{C}_j$ *and maps* f_j *can also be chosen from a single definable family. If* $\mathcal{C}$ *is algebraic of complexity* β *then* $\mathcal{C}_j, f_j$ *are algebraic of complexity* $\operatorname{poly}_\ell(\beta)$*.*

Remark 42 (Refinement and monomialization)**.** *The type of any cell* $\mathcal{C}_j$ *obtained by refinement of* $\mathcal{C}^{\{\rho\}}$ *is obtained from the type of* $\mathcal{C}$ *by possibly replacing some* $D_\circ, A$ *fibers by* D *fibers. Moreover* $(f_j)_* : \pi_1(\mathcal{C}_j) \to \pi_1(\mathcal{C})$ *is the natural injection. In particular if* $F : \mathcal{C} \to \mathbb{C}\setminus\{0\}$ *then* $\boldsymbol{\alpha}(f_j^* F)$ *is obtained from* $\boldsymbol{\alpha}(F)$ *by eliminating those indices that correspond to fibers that were replaced by* D *in* $\mathcal{C}_j$*.*

The proof of the refinement theorem will occupy the remainder of this subsection. We begin with the complex case, and indicate the necessary modifications for the real case at the end. We proceed by induction on ℓ (the case $\ell=0$ is vacuous). Consider a cell $\mathcal{C}\odot\mathcal{F}$. By applying the inductive hypothesis to $\mathcal{C}$ and replacing $\mathcal{C}\odot\mathcal{F}$ by each $\mathcal{C}_j \odot f_j^*\mathcal{F}$ we may assume without loss of generality that the base $\mathcal{C}$ already admits E-extension, where E will be chosen later. The case $\mathcal{F} = *$ reduces to the inductive hypothesis directly with $E=\{\sigma\}$.

As a notational convenience we allow ourselves to rescale the fiber $\mathcal{F}$ by a non-vanishing holomorphic function $s \in \mathcal{O}_b(\mathcal{C}^{\{\sigma\}})$, where it is understood that the covering cells $\mathcal{C}_j$ that we construct will eventually be rescaled to cover the original $\mathcal{F}$. When we describe the covering of $\mathcal{F}$ we allow ourselves to use discs centered at a point $p \in \mathcal{O}_b(\mathcal{C}^{\{\sigma\}})$ where it is understood that such discs will be centered at the origin in the covering cells $\mathcal{C}_j$ that we construct, and the maps f_j will translate the origin to p. We note that rescaling has the effect of eventually rescaling the centers of the covering discs that we construct below by a factor of s. In the algebraic case we will always choose s and p to be algebraic of complexity $\operatorname{poly}(\beta)$ so that the maps that we eventually construct will indeed be of complexity $\operatorname{poly}(\beta)$.

We will consider two separate cases: $\rho>1, \sigma=1$ and $\rho=1, \sigma<1$. The general case can be reduced to a composition of these two cases, first refining $\{\rho\}$-extensions into $\{1\}$-extensions and then refining $\{1\}$-extensions into $\{\sigma\}$-extensions

6.1.1. *The case* $\rho>1, \sigma=1$. In this case regardless of the type of $\mathcal{F}$ we have $\mathcal{F}^{\{\rho\}} = \mathcal{F}^\delta$ where $\delta = 1-\Theta(\rho^{-1})$. We will consider the different fiber types separately.

Suppose $\mathcal{F}$ is of type D. Up to rescaling $\mathcal{F} = D(1)$. Then we take $E = \{1\}$. Up to rescaling the fiber, our goal to cover $D(1)$ by discs whose $\Omega(1)$-extensions remain in $D(\delta^{-1}) = D(1 + \Omega(\rho^{-1}))$. One can use for example any disc of radius $O(\rho^{-1})$ centered in $D(1)$; clearly $\operatorname{poly}(\rho)$ such discs suffice to cover $D(1)$.

Suppose $\mathcal{F}$ is of type $D_\circ$. Up to rescaling $\mathcal{F} = D_\circ(1)$. Again we take $E = \{1\}$. We first embed $D_\circ(O(1))$ in $D_\circ(1)$ choosing the radius in such a way that the $\{1\}$-extension remains in $D_\circ(1)$. It remains to cover the annulus $A(\Omega(1), 1)$ using $\operatorname{poly}(\rho)$ discs whose $\Omega(1)$-extensions remain in $D_\circ(1)$. This can be done as in the case $\mathcal{F} = D(r)$.

Suppose $\mathcal{F}$ is of type A. Up to rescaling $\mathcal{F} = A(r, 1)$. We take $E = \{1\}\{\hat\rho\}$ where $\hat\rho$ will be chosen later. By the fundamental lemma for $\mathbb{D} \setminus \{0\}$ applied to r on the cell $\mathcal{C}^{\{1\}}$ one of the following holds

$$\log |r(\mathcal{C}^{\{1\}})| \subset (-\infty, -\Omega_\ell(1/\hat\rho)), \qquad \operatorname{diam}(\log|\log|r(\mathcal{C}^{\{1\}})||; \mathbb{R}) = O_\ell(\hat\rho). \tag{91}$$

We will use the following simple lemma.

Lemma 43. *Let $\alpha \in (0, 1]$ (resp. $\beta \in [1, \infty)$) and suppose $r < \alpha$ (resp. $r\beta < 1$) uniformly in $\mathcal{C}^{\{1\}}$. Then one can cover $\mathcal{C} \odot S^{1-\Omega(1/\rho)}(\alpha)$ (resp. $\mathcal{C} \odot S^{1-\Omega(1/\rho)}(\beta r)$) by $\operatorname{poly}(\rho)$ cells whose $\{1\}$-extensions remain in $(\mathcal{C} \odot \mathcal{F})^{\{\rho\}}$.*

Proof. The $\{1\}$-extension of a disc of radius $O(\alpha/\rho)$ around a point of $S(\alpha)$ remains in $S^{1-\Omega(1/\rho)}(\alpha)$. Moreover for any point in $\mathcal{C}^{\{1\}}$ this remains in $A^{\{\rho\}}(r, 1)$ since $r < \alpha$. We choose a collection of $\operatorname{poly}(\rho)$ such discs to cover $S^{1-\Omega(1/\rho)}(\alpha)$. Then $\mathcal{C} \odot D_i$ gives the required covering. The respective case is similar, with $S(\alpha)$ replaced by $S(\beta r)$. □

Suppose that we are in the first case of (91). Inside $A(r, 1)$ we choose the annulus A such that $A^{\{1\}} = A(r, 1)$: this is possible if we choose $\hat\rho = \Omega(1)$. It remains to cover the two annuli components of $A(r, 1) \setminus A$. Each of these has width $O(1)$ in the logarithmic scale. Lemma 43 allows us to cover subannuli of width $\Omega(1/\rho)$: with α for the outer component and with β for the inner component. Clearly $\operatorname{poly}(\rho)$ applications of the lemma suffice to cover the two components.

Suppose that we are in the second case of (91). If we choose $\hat\rho = \Omega(1)$ we may assume that the ratio of $\log |r|$ between any two points of $\mathcal{C}^{\{1\}}$ is in $(9/10, 10/9)$. We again distinguish two cases: first, suppose that we can uniformly over $\mathcal{C}^{\{1\}}$ choose inside $A(r, 1)$ the annulus A such that $A^{\{1\}} = A(r, 1)$. In this case we can proceed as above. Otherwise, over some point in $\mathcal{C}^{\{1\}}$ we have $r(\mathbf{z}_{1..\ell}) = r_0$ with $\log |r_0| = -O(1)$. We now apply Lemma 43 to cover the annulus given in the logarithmic scale by $(2/3 \log |r_0|, 0)$ (using α) and the annulus given in the logarithmic scale by $(\log |r|, \log |r| - 2/3 \log |r_0|)$ (using β). Crucially, the ratio condition on $\log |r|$ in $\mathcal{C}^{\{1\}}$ ensures that these two annuli remain in $A(r, 1)$ and cover it uniformly over $\mathcal{C}^{\{1\}}$. Since the width of each of these annuli is $O(1)$ we see as above that $\operatorname{poly}(\rho)$ applications of the lemma suffice to cover them.

6.1.2. *The case $\rho = 1, \sigma < 1$.* In this case we have $\mathcal{F}^{\{\sigma\}} = \mathcal{F}^\varepsilon$ where $\varepsilon \sim \sigma$ in type D and $\varepsilon = e^{-\Theta(1/\sigma)}$ in types $D_\circ, A$. We will consider the different fiber types separately.

Suppose $\mathcal{F}$ is of type D. Up to rescaling $\mathcal{F} = D(1)$. Then we take $E = \{\sigma\}$. Up to rescaling the fiber, our goal to cover $D(1)$ by discs whose $\Omega(\sigma)$-extensions remain in $D(1)^{\{1\}}$. One can use for example any disc of radius $O(\sigma)$ centered in $D(1)$; clearly $\operatorname{poly}(1/\sigma)$ such discs suffice to cover $D(1)$.

Suppose now that $\mathcal{F}$ is of type $D_\circ$ or A. Up to rescaling $\mathcal{F} = D_\circ(1)$ or $\mathcal{F} = A(r,1)$. We will use the following analog of Lemma 43. The β-case of the lemma is valid only for type A.

Lemma 44. *Let $\alpha \in (0,1]$ (resp. $\beta \in [1,\infty)$) and suppose $r < \alpha$ (resp. $r\beta < 1$) uniformly in $\mathcal{C}^{\{\sigma\}}$. Then one can cover $\mathcal{C} \odot S^{\Omega(1)}(\alpha)$ (resp. $\mathcal{C} \odot S^{\Omega(1)}(\beta r)$) by* $\operatorname{poly}(1/\sigma)$ *cells whose $\{\sigma\}$-extensions remain in $(\mathcal{C} \odot \mathcal{F})^{\{1\}}$.*

Proof. The $\{\Omega(1)\}$-extension of a disc of radius $\Omega(\alpha)$ around a point of $S(\alpha)$ remain in $S^{\Omega(1)}(\alpha)$. We choose a collection of $O(1)$ such discs to cover $S^{\Omega(1)}(\alpha)$. By what was already proved for discs, we can further cover each of the discs by $\operatorname{poly}(1/\sigma)$ discs whose $\{\sigma\}$-extensions remain in $S^{\Omega(1)}(\alpha)$. Moreover for any point in $\mathcal{C}^{\{\sigma\}}$ this remains in $\mathcal{F}^{\{1\}}$ (in type A we use that $r < \alpha$). If $\{D_i\}$ denotes the collection of all discs obtained in this process then $\mathcal{C} \odot D_i$ gives the required covering. The respective case is similar, with $S(\alpha)$ replaced by $S(\beta r)$. □

Suppose $\mathcal{F} = D_\circ(1)$. We take $E = \{\sigma\}$. We first embed $D_\circ(e^{-\Theta(1/\sigma)})$ in $D_\circ(1)$ so that the $\{\sigma\}$-extensions remains in $D_\circ(1)$. It remains to cover the annulus $A(e^{-\Theta(1/\sigma)}, 1)$, or in the logarithmic scale $(-\Theta(1/\sigma), 0)$. Lemma 44 allows us to cover subannuli of logarithmic width $\Omega(1)$, and indeed $\operatorname{poly}(1/\sigma)$ applications suffice to cover the annulus.

Finally suppose $\mathcal{F} = A(r,1)$. We take $E = \{\sigma\}\{\hat\rho\}$ where $\hat\rho$ will be chosen later. By the fundamental lemma for $\mathbb{D} \setminus \{0\}$ applied to r on the cell $\mathcal{C}^{\{\sigma\}}$ one of the following holds

$$\log|r(\mathcal{C}^{\{\sigma\}})| \subset (-\infty, -\Omega_\ell(1/\hat\rho)), \qquad \operatorname{diam}(\log|\log|r(\mathcal{C}^{\{\sigma\}})||;\mathbb{R}) = O_\ell(\hat\rho). \tag{92}$$

Suppose that we are in the first case of (92). Inside $A(r,1)$ we choose the annulus A (uniformly over $\mathcal{C}^{\{\sigma\}}$) such that $A^{\{\sigma\}} = A(r,1)$: this is possible if $\log|r| = -\Omega(1/\sigma)$, i.e. if we choose $\hat\rho = \Omega(\sigma)$. It remains to cover the two annuli components of $A(r,1) \setminus A$. Each of these has width $O(1/\sigma)$ in the logarithmic scale. Lemma 44 allows us to cover subannuli of width $\Omega(1)$: with α for the outer component and with β for the inner component. Clearly $\operatorname{poly}(1/\sigma)$ applications of the lemma suffice to cover the two components.

Suppose that we are in the second case of (92). If we choose $\hat\rho = \Omega(1)$ we may assume that the ratio of $\log|r|$ between any two points of $\mathcal{C}^{\{1\}}$ is in $(9/10, 10/9)$. We again distinguish two cases: first, suppose that we can uniformly over $\mathcal{C}^{\{\sigma\}}$ choose inside $A(r,1)$ the annulus A such that $A^{\{\sigma\}} = A(r,1)$. In this case we can proceed as above. Otherwise, over some point in $\mathcal{C}^{\{\sigma\}}$ we have $r(\mathbf{z}_{1..\ell}) = r_0$ with $\log|r_0| = -O(1/\sigma)$. We now apply Lemma 44 to cover the annulus given in the logarithmic scale by $(2/3\log|r_0|, 0)$ (using α) and the annulus given in the logarithmic scale by $(\log|r|, \log|r| - 2/3\log|r_0|)$ (using β). Crucially, the ratio condition on $\log|r|$ in $\mathcal{C}^{\{\sigma\}}$ ensures that these two annuli remain in $A(r,1)$ and cover it uniformly over $\mathcal{C}^{\{\sigma\}}$. Since the logarithmic width of each of these annuli is $O(1/\sigma)$ we see as above that $\operatorname{poly}(1/\sigma)$ applications of the lemma suffice to cover them.

6.1.3. *The real case.* Only a very minor modification is needed in order to treat the real case. If $\mathcal{C}$ is real then all the rescalings performed during the proof are also real. Wherever we construct a collection of discs to cover a domain $\mathcal{F}$ by discs in the complex case, we now choose a collection of discs with real centers to cover $\mathbb{R}_+\mathcal{F}$. This ensures that the cells that we construct are real, and the rest of the proof remains unchanged.

6.1.4. *Uniformity over families and Remark 42.* The statement regarding uniformity over families follows by inspection of the proof. By induction the refinement maps chosen for the base can all be chosen from a single definable family. The covering constructed for the fiber, after a uniform rescaling, consists of discs or annuli with a constant center and radius. The family of all such discs or annuli is certainly definable.

Remark 42 follows immediately from the proof: it suffices to note that D fibers are always covered by D fibers, and $D_\circ, A$ fibers are covered by a collection of D fibers and possibly one additional fibers of type $D_\circ, A$; and this additional fiber is always centered at zero, and hence homotopy equivalent to the original fiber by the injection map.

6.2. **Monomial cells.** Let $\mathcal{C}$ be a cell of length ℓ. An *admissible monomial* on $\mathcal{C}$ is a function of the form $c\cdot\mathbf{z}^{\boldsymbol{\alpha}}$ where $c\in\mathbb{C}$ and $\boldsymbol{\alpha}=\boldsymbol{\alpha}(f)$ is the associated monomial of some function $f\in\mathcal{O}_b(\mathcal{C})$.

Definition 45 (Monomial cell)**.** *We will say that a cell $\mathcal{C}$ is* monomial *if $\mathcal{C}=*$; or if $\mathcal{C}=\mathcal{C}_{1..\ell}\odot\mathcal{F}$ and the radii involved in $\mathcal{F}$ are admissible monomials on $\mathcal{C}_{1..\ell}$.*

Using the refinement theorem and the monomialization lemma we prove the following proposition.

Proposition 46. *In the conclusion of the refinement theorem one can further require that each $\mathcal{C}_j$ is a monomial cell.*

Proposition 46 implies that we could use monomial cells, rather than general complex cells, as our standard models for cellular covers. In some cases this is more convenient, as the monomial cells have a more transparent combinatorial and algebraic structure. However for the most part we have chosen in this paper to state our constructions for general complex cells.

The proof of Proposition 46 will occupy the remainder of this section. We start by applying the refinement theorem to $\mathcal{C}$ with some $\{\hat\sigma\}=\{\hat\sigma_1\}\{\hat\sigma_2\}$ to be chosen later. Recall that all of the cells constructed in the refinement theorem can be chosen from one definable family independent of $\hat\sigma$. By the monomialization lemma, if $r(\mathbf{z})$ is any of the radii involved in the definition of one of these cells $\mathcal{C}_j$ then we have $r(\mathbf{z})=\mathbf{z}^{\boldsymbol{\alpha}(r)}U(\mathbf{z})$ where

$$\begin{aligned}&\operatorname{diam}(\operatorname{Re}\log U(\mathcal{C}_j^{\{\hat\sigma_2\}});\mathbb{R})<O_{\mathcal{C}}(\hat\sigma_1)\\&\operatorname{diam}(\operatorname{Re}\log U(\mathcal{C}_j^{\{\hat\sigma_2\}});\mathbb{R})<\operatorname{poly}_\ell(\beta)\cdot\hat\sigma_1\end{aligned}\tag{93}$$

in the subanalytic and algebraic cases respectively. We will now construct a monomial cell $\tilde{\mathcal{C}}_j^{\{\sigma\}}$ such that $\mathcal{C}_j\subset\tilde{\mathcal{C}}_j\subset\tilde{\mathcal{C}}_j^{\{\sigma\}}\subset\mathcal{C}^{\{\hat\sigma\}}$, and the identity map then gives the covering of $\mathcal{C}_j$ by a monomial cell as required.

We construct $\tilde{\mathcal{C}}_j$ by induction. If $\mathcal{C}_j$ is of length zero or one then it is already monomial so we may take $\tilde{\mathcal{C}}_j:=\mathcal{C}_j$. If $\mathcal{C}_j:=(\mathcal{C}_j)_{1..\ell}\odot\mathcal{F}$ then we set $\tilde{\mathcal{C}}_j:=\widetilde{(\mathcal{C}_j)_{1..\ell}}\odot\tilde{\mathcal{F}}$

where $\tilde{\mathcal{F}}$ is defined as follows. Suppose $\mathcal{F} = A(r_1, r_2)$ and write $r_1 = \mathbf{z}^{\boldsymbol{\alpha}_1} U_1$ and $r_2 = \mathbf{z}^{\boldsymbol{\alpha}_2} U_2$. Then $\tilde{\mathcal{F}} = A(\tilde{r}_1, \tilde{r}_2)$ where

$$\tilde{r}_1 = \mathbf{z}^{\boldsymbol{\alpha}_1} \min_{\widetilde{\mathbf{z}_{1..\ell} \in (\mathcal{C}_j)^{\{\sigma\}}_{1..\ell}}} |r_1(z)| \qquad \tilde{r}_2 = \mathbf{z}^{\boldsymbol{\alpha}_1} \max_{\widetilde{\mathbf{z}_{1..\ell} \in (\mathcal{C}_j)^{\{\sigma\}}_{1..\ell}}} |r_2(z)|. \tag{94}$$

It is clear that $\mathcal{C}_j \subset \tilde{\mathcal{C}}_j$, and what remains to be verified is that $\tilde{\mathcal{C}}_j^{\{\sigma\}} \subset \mathcal{C}_j^{\{\hat{\sigma}\}}$. By (93), for an appropriate choice of $\hat{\sigma}_1$ we can make $\operatorname{diam}(\operatorname{Re}\log U(\mathcal{C}_j^{\{\hat{\sigma}_2\}}); \mathbb{R}) < 1$, which implies that $\tilde{r}_1/r_1 \geqslant 1/e$ and $\tilde{r}_2/r_1 < e$ on $\mathcal{C}_j^{\{\hat{\sigma}_2\}}$. Now choosing $\hat{\sigma}_2 < \sigma$ and also $\{\hat{\sigma}\} < \{\sigma\}/e$ we have indeed $\tilde{\mathcal{C}}_j^{\{\sigma\}} \subset \mathcal{C}_j^{\{\hat{\sigma}\}}$.

6.3. **Clustering in fibers of proper covering maps.**

6.3.1. *The general (complex) setting.* Let $\mathcal{C}^{\{\rho\}}$ be a cell and let $Z \subset \mathcal{C}^{\{\rho\}} \times \mathbb{C}$ be an analytic set such that the natural projection $\pi : Z \to \mathcal{C}^{\{\rho\}}$ is a proper covering map. Let ν denote the degree of π and set $\hat{\mathcal{C}} := \mathcal{C}_{\times \nu!}$. Let $\hat{Z} \subset \hat{\mathcal{C}} \times \mathbb{C}$ be the pullback $(R_{\nu!}, \mathrm{id})^* Z$. Then the sections $y_j : \hat{\mathcal{C}} \to \mathbb{C}$ of $\hat{Z}$ over $\hat{\mathcal{C}}$ are univalued, and we denote their collection by Σ. Each section y_j in fact extends holomorphically to the cell $\hat{\mathcal{C}}^{\{\nu! \cdot \rho\}}$ by Proposition 22, but this exponential (in ν) loss in the size of the extension is too coarse for our purposes. Instead, we denote by $\nu_j \leqslant \nu$ the size of the $\pi_1(\mathcal{C})$-orbit of y_j thought of as a multivalued section over $\mathcal{C}$.

Lemma 47. *Let $y_j \in \Sigma$. Then y_j is already univalued as a section over the cover $\hat{\mathcal{C}}_j := \mathcal{C}_{\times \nu_j}$, and extends holomorphically to $\hat{\mathcal{C}}_j^{\{\nu_j \cdot \rho\}}$.*

Proof. The group $\pi_1(\mathcal{C})$ is abelian. It is enough to check that the for any $g \in \pi_1(\mathcal{C})$ we have $g^{\nu_j}(y_j) = y_j$. This is elementary: the $\langle g \rangle$-action induces a partition of the orbit $\pi_1(\mathcal{C}) \cdot y_j$ of size ν_j into $\langle g \rangle$-orbits, and $\pi_1(\mathcal{C})$ acts transitively on these orbits. Thus they are all of the same size, which in particular divides ν_j, and this is true in particular for the orbit $\langle g \rangle \cdot y_j$. □

Let $y_i, y_j, y_k \in \Sigma$ be three distinct sections. Since π is unramified, the sections are pairwise distinct over any point of $\hat{\mathcal{C}}$. We let $\nu_{i,j,k} := \operatorname{lcm}(\nu_i, \nu_j, \nu_k)$ and $\hat{\mathcal{C}}_{i,j,k} := \mathcal{C}_{\times \nu_{i,j,k}}$. We define a map $s_{i,j,k}$ as follows,

$$s_{i,j,k} : \hat{\mathcal{C}}_{i,j,k}^{\{\nu_{i,j,k} \cdot \rho\}} \to \mathbb{C} \setminus \{0,1\}, \qquad s_{i,j,k} = \frac{y_i - y_j}{y_i - y_k}. \tag{95}$$

By the fundamental lemma for maps into $\mathbb{C} \setminus \{0,1\}$ one of the following holds:

$$\begin{gathered} s_{i,j,k}(\hat{\mathcal{C}}_{i,j,k}) \subset B(\{0,1,\infty\}, e^{-\Omega_\ell(1/(\nu^3 \rho))}; \mathbb{C}P^1), \\ \operatorname{diam}(s_{i,j,k}(\hat{\mathcal{C}}_{i,j,k}); \mathbb{C} \setminus \{0,1\}) = O_\ell(\nu^3 \rho). \end{gathered} \tag{96}$$

We remark that equivalently (96) holds if we replace $\hat{\mathcal{C}}_{i,j,k}$ by $\hat{\mathcal{C}}$, since $s_{i,j,k}$ on $\hat{\mathcal{C}}$ factors through $\hat{\mathcal{C}}_{i,j,k}$.

Fix $y_i \in \Sigma$. We will cluster the remaining sections into annuli according to their relative distances from y_i, which are expressed by the quantities $s_{i,j,k}$. Since these quantities are invariant under affine transformations of $\mathbb{C}$, we may assume for simplicity of the notation that $y_i = 0$. We record a useful corollary of (96) in this normalization.

Lemma 48. *Suppose $\rho = O_\ell(1/\nu^3)$. Let $j,k \neq i$ and write $R := \log|y_j/y_k|$. One of the following holds:*

$$
\begin{aligned}
R|_{\hat{\mathcal{C}}} < -\Omega_\ell(1/\nu^3\rho) \quad &or \quad R|_{\hat{\mathcal{C}}} > \Omega_\ell(1/\nu^3\rho),\\
\operatorname{diam}(R(\hat{\mathcal{C}}),\mathbb{R}) &= O_\ell(\nu^3\rho)\\
\frac{\max_{\mathbf{z}\in\hat{\mathcal{C}}} R(z)}{\min_{\mathbf{z}\in\hat{\mathcal{C}}} R(z)} &< 1 + O_\ell(\nu^3\rho).
\end{aligned}
\tag{97}
$$

Proof. If at some point in $\hat{\mathcal{C}}$ we have $y_j/y_k \in A(1/2,2)$ then for $\rho = O(1)$ we have by (96) that $y_j/y_k(\hat{\mathcal{C}}) \subset A(1/4,4)$. In this domain the $\log|\cdot|$-distance is bounded up to constant by the $\mathbb{C}\setminus\{0,1\}$ distance so $\operatorname{diam}(R(\hat{\mathcal{C}});\mathbb{R}) = O_\ell(\nu^3\rho)$. It remains to consider the case $y_j/y_k(\hat{\mathcal{C}}) \subset D_\circ(1/2)$ (or $A(2,\infty)$ which is the same up to inversion). In the first case of (96) we have $R = -\Omega_\ell(1/(\nu^3\rho))$. In the second case of (96), since the $\mathbb{C}\setminus\{0,1\}$ metric is equivalent to the $D_\circ(1)$ metric in $D_\circ(1/2)$ we have (as in the fundamental lemma for $D_\circ$) the estimate

$$
\operatorname{diam}(\log|R(\hat{\mathcal{C}})|;\mathbb{R}) = O_\ell(\nu^3\rho). \tag{98}
$$

With our assumption on ρ this means that $R(\hat{\mathcal{C}})$ varies multiplicatively by a factor of size at most $1 + O_\ell(\nu^3\rho)$. □

We fix a quantity $0 < \gamma < 1$ which we call the *gap*. Pick an arbitrary point $p \in \hat{\mathcal{C}}$ and let

$$
S_i := \{\log|y_j(p)| : y_j \in \Sigma, y_j \neq y_i\} \subset \mathbb{R}. \tag{99}
$$

We say that two points s, s' in S_i belong to the same cluster if they are connected in the transitive closure of the relation $|s-s'| < 5|\log\gamma|$. We order the clusters with respect to $<$ on $\mathbb{R}$. For cluster number $q = 1,\dots,m_i$ we let $I_{i,q}$ denote the minimal closed interval containing the $2|\log\gamma|$-neighborhood of the cluster. For each q we arbitrarily choose $\hat{y}_{i,q} \in \Sigma$ to be one of the sections with $\log|\hat{y}_{i,q}(p)| \in I_{i,q}$, and call it the *center* of the cluster. We define

$$
\gamma^{5\nu} < l_{i,q} < \gamma^2, \qquad\qquad \gamma^{-2} < r_{i,q} < \gamma^{-5\nu} \tag{100}
$$

by the condition

$$
e^{I_{i,q}} = [l_{i,q}|\hat{y}_{i,q}(p)|, r_{i,q}|\hat{y}_{i,q}(p)|]. \tag{101}
$$

We also fix some $\delta < 1$ arbitrarily close to 1 (merely to ensure that the boundary circles below are covered).

We define three types of fibers over $\hat{\mathcal{C}}$ as follows:

$$
\mathcal{F}_{i,q} = A^\delta(l_{i,q}\hat{y}_{i,q}, r_{i,q}\hat{y}_{i,q}) \qquad q = 1,\dots,m_i \tag{102}
$$

$$
\mathcal{F}_{i,q+} = A^\delta(r_{i,q}\hat{y}_{i,q}, l_{i,q+1}\hat{y}_{i,q+1}), \qquad q = 1,\dots,m_i - 1 \tag{103}
$$

$$
\mathcal{F}_{i,0+} = D_\circ^\delta(l_{i,1}\hat{y}_{i,1}), \qquad \mathcal{F}_{i,m+} = A^\delta(r_{i,q}\hat{y}_{i,q}, \infty). \tag{104}
$$

Note that each of these fibers actually depends only on $y_i, \hat{y}_{i,q}$ for $\mathcal{F}_{i,q}$ and on $y_i, \hat{y}_{i,q}, \hat{y}_{i,q+1}$ for $\mathcal{F}_{i,q+}$. Thus they actually factor as domains defined over a cover $\hat{\mathcal{C}}_{i,j,k}$ for an appropriate choice of the indices. We denote this cover by $\hat{\mathcal{C}}_{i,q}$ or $\hat{\mathcal{C}}_{i,q+}$.

The key properties of these domains are summarized in the following proposition.

Proposition 49. *Suppose $1/\rho > \operatorname{poly}_\ell(\nu, |\log\gamma|)$. Then the following hold uniformly over $\hat{\mathcal{C}}$:*

(1) The fibers $\mathcal{F}_{i,q}, \mathcal{F}_{i,q+}$ are well-defined and cover $\mathbb{C}\setminus\{0\}$.

(2) *The domains* $\mathcal{F}^\gamma_{i,q} \setminus \mathcal{F}_{i,q}$ *do not contain any of the points* y_j *for* $q = 1, \dots, m_i$.
(3) *The domains* $\mathcal{F}^\gamma_{i,q+}$ *do not contain any of the points* y_j *for* $q = 0, \dots, m_i$.

Proof. The only non-trivial assertion in the first statement is that $\mathcal{F}_{i,q+}$ is well-defined, i.e. that $r_{i,q}\hat{y}_{i,q} < l_{i,q+1}\hat{y}_{i,q+1}$ uniformly over $\hat{\mathcal{C}}$. At p we have by construction

$$\log|\hat{y}_{i,q+1}(p)/\hat{y}_{i,q}(p)| > \log(r_{i,q}/l_{i,q+1}) + |\log\gamma|. \tag{105}$$

We need to prove that

$$\log|\hat{y}_{i,q+1}/\hat{y}_{i,q}| > \log(r_{i,q}/l_{i,q+1}) \tag{106}$$

uniformly over $\hat{\mathcal{C}}$. This follows easily from Lemma 48 for an appropriate choice of ρ. The only non-trivial case is the last one, which follows if one recalls that $\log(r_{i,q}/l_{i,q+1}) < 10\nu|\log\gamma|$.

The third statement follows from the second: over p all the point y_j except $y_i = 0$ lie in the domains $\mathcal{F}_{i,q}$, and assuming that they never cross into the boundaries $\mathcal{F}^\gamma_{i,q} \setminus \mathcal{F}_{i,q}$ this remains true uniformly over $\hat{\mathcal{C}}$. We proceed to the proof of the second statement. We will show that y_j does not belong to the outer boundary of $\mathcal{F}^\gamma_{i,q} \setminus \mathcal{F}_{i,q}$ (the case of the inner boundary is similar). By construction over p one of the following holds:

$$\log|y_j(p)/\hat{y}_q(p)| < \log r_{i,q} - 2|\log\gamma|, \qquad \log|y_j(p)/\hat{y}_q(p)| > \log r_{i,q} + 3|\log\gamma|. \tag{107}$$

We must prove that one of

$$\log|y_j/\hat{y}_q| < \log r_{i,q}, \qquad \log|y_j/\hat{y}_q| > \log r_{i,q} + |\log\gamma|, \tag{108}$$

holds uniformly over $\hat{\mathcal{C}}$. This follows in the same manner as the first statement. □

Since the sections y_j do not meet the $\mathcal{F}_{i,q+}$ and $\partial\mathcal{F}_{i,q}$, it follows that each section $y_j \neq y_i$ uniformly lies in a single $\mathcal{F}_{i,q}$. We say that such sections belong to the cluster $\mathcal{F}_{i,q}$.

Remark 50. *If we focus our attention on a single cluster* $\mathcal{F}_{i,q}$ *then it is convenient, up to an affine transformation over* $\hat{\mathcal{C}}_{i,q}$, *to assume that both* $y_i = 0$ *and* $\hat{y}_{i,q} = 1$. *With* γ, ρ *as in Proposition 49 we then have* $\mathcal{F}_{i,q} = A^\delta(l_{i,q}, r_{i,q})$. *For any* y_j *in the* $\mathcal{F}_{i,q}$ *cluster our choice of* ρ *ensures that the second option of* (96) *holds, and we have* $\operatorname{diam}(y_j(\hat{\mathcal{C}}_{i,q}), \mathbb{C} \setminus \{0,1\}) = O(\nu^3\rho)$.

6.4. **The real setting.** Suppose that $\mathcal{C}^{\{\rho\}}$ is a real cell and $Z \subset \mathcal{C}^{\{\rho\}} \times \mathbb{C}$ is real, i.e. invariant under $\mathbf{z} \to \bar{\mathbf{z}}$. In this case we would like to construct the fibers $\mathcal{F}_{i,q}, \mathcal{F}_{i,q+}$ to be real as well. However, the construction above produces fibers whose centers and radii are given in terms of the sections y_j, which are generally not real. We now modify this construction to produce real fibers. We fix $p \in \mathbb{R}\hat{\mathcal{C}}$.

The pullbacks $\hat{\mathcal{C}}, \hat{Z}$ of $\mathcal{C}, Z$ by $R_{\nu!}$ are also real. Since $\hat{Z}$ is a cover, each section y_j is either always real or always non-real on $\mathbb{R}\hat{\mathcal{C}}$. We denote the former sections by $\Sigma_\mathbb{R}$ and the latter by $\Sigma_\mathbb{C}$. If $y_j \in \Sigma$ then by the symmetry of $\hat{Z}$ there exists a symmetric section $y_{\bar{j}} \in \Sigma$ defined by $y_{\bar{j}}(\mathbf{z}) := \overline{y_j(\bar{\mathbf{z}})}$.

Definition 51. *Let* $y_j \in \Sigma_\mathbb{C}$. *Making a real affine transformation we pass to a chart where* $y_j(p) = i, y_{\bar{j}}(p) = -i$. *Then the pair* $y_j, y_{\bar{j}}$ *is called* admissible *if* $D(1/10)$ *contains none of the points* $y_k(p)$ *for* $y_k \in \Sigma$.

An *admissible center* is a map of the form $(y_j+y_{\bar{j}})/2$ where $y_j \in \Sigma_{\mathbb{R}}$ or $y_j, y_{\bar{j}} \in \Sigma_{\mathbb{C}}$ is an admissible pair. Every admissible center is real on $\mathbb{R}\hat{\mathcal{C}}$. We denote the set of admissible centers by $\{c_1, \ldots, c_t\}$.

Lemma 52. *Let $y_j \in \Sigma_{\mathbb{C}}$ and by a real affine transformation pass to a chart where $y_j(p) = i, y_{\bar{j}}(p) = -i$. Then $D(1/5)$ contains $c_l(p)$ for an admissible center c_l.*

Proof. If $y_j, y_{\bar{j}}$ is admissible then the claim is obvious. Otherwise there exists a section $y_k(p) \in D(1/10)$. Rescaling to make $y_k(p) = i$ we see that it will be enough to prove the claim for y_k. Clearly this process must terminate after no more than ν steps. □

The motivation for the notion of admissibility comes from the following lemma.

Lemma 53. *Suppose $1/\rho = \mathrm{poly}_\ell(\nu)$. For every section y_j and admissible center c_l we have $c_l \neq y_j$ uniformly over $\hat{\mathcal{C}}$, unless $c_l = y_j \in \Sigma_{\mathbb{R}}$.*

Proof. If $c_l = y_k \in \Sigma_{\mathbb{R}}$ for some k then the claim follows from the fact that $\hat{Z}$ is a cover whose sections are pairwise distinct over any point of $\hat{\mathcal{C}}$. Assume therefore that $c_l = (y_k+y_{\bar{k}})/2$ for $y_k \in \Sigma_{\mathbb{C}}$. We make a real affine transform to pass to a chart where $y_k(p) = i$. Suppose toward contradiction that at some point $\tilde{p} \in \hat{\mathcal{C}}$ we have $y_j = (y_k+y_{\bar{k}})/2$, i.e. $s_{j,k,\bar{k}}(\tilde{p}) = -1$. Then by (96) we have $|s_{j,k,\bar{k}}(p)+1| = O_\ell(\nu^3\rho)$. For a suitable choice of ρ this readily implies that $|y_j(p)| < 1/10$, contradicting the admissibility of c_l. □

Lemma 53 implies that we can define maps $\tilde{s}_{l,j,k}$ as follows

$$\tilde{s}_{l,j,k} : \hat{\mathcal{C}}_{l,j,k}^{\{\nu_{l,j,k}\cdot\rho\}} \to \mathbb{C} \setminus \{0,1\}, \qquad s_{i,j,k} = \frac{c_l - y_j}{c_l - y_k}. \tag{109}$$

Fix an admissible center c_l. We will cluster the sections y_j into annuli according to their relative distances from c_l in analogy with the complex construction. As before, we make a real affine transformation and assume that $c_l = 0$.

We define the intervals $I_{l,q}$ in the same way as in the complex setting. However, a small variation is needed in the choice of $\hat{y}_{l,q}$, which must be real on $\mathbb{R}\hat{\mathcal{C}}$ to maintain the real structure of our construction. Let y_j be one of the sections with $y_j(p) \in I_{l,q}$. We define $\hat{y}_{l,q} := \sqrt{y_j y_{\bar{j}}}$. Clearly $\hat{y}_{l,q}$ is real on $\mathbb{R}\hat{\mathcal{C}}$. To see that it is univalued on $\hat{\mathcal{C}}$ note that y_j and $y_{\bar{j}}$ have the same associated monomial by symmetry, hence $y_j y_{\bar{j}}$ has an even associated monomial and admits a univalued square root.

With $\hat{y}_{l,q}$ chosen as above we continue the construction as in the complex case. To verify that all the arguments remain valid we need an analog of Lemma 48 where either y_j or y_k (or both) are replaced by a cluster center $\hat{y}_{l,q} := \sqrt{y_h y_{\bar{h}}}$. This follows essentially from the same lemma applied to y_h and to $y_{\bar{h}}$, and we leave the details for the reader. As a consequence we see that Proposition 49 and Remark 50 continue to hold, with the fibers $\mathcal{F}_{i,q}$ and $\mathcal{F}_{i,q+}$ now real on $\mathbb{R}\hat{\mathcal{C}}$.

7. Cellular Weierstrass Preparation Theorem

In this section we state and prove a cellular analog of the Weierstrass preparation theorem. We stress that unlike the classical theorem, the cellular version does not involve a change in the order of coordinates.

Definition 54. *Let $\gamma > 0$ and $\mathcal{C} \odot \mathcal{F}^\gamma$ be a cell and let $F \in \mathcal{O}_b(\mathcal{C} \odot \mathcal{F}^\gamma)$. We say that $\mathcal{C} \odot \mathcal{F}^\gamma$ is a* Weierstrass cell *with* gap γ *for F if:*

- *F vanishes identically on $\mathcal{C}\odot\mathcal{F}^\gamma$, or*
- *F is non-vanishing on $\mathcal{C}\odot *$ if $\mathcal{F}=*$, or*
- *F is non-vanishing on $\mathcal{C}\odot(\mathcal{F}^\gamma\setminus\mathcal{F})$ if $\mathcal{F}=D,D_\circ,A$.*

If $\hat{\mathcal{C}}$ is a cell and $F\in\mathcal{O}_b(\hat{\mathcal{C}})$ we say that a cellular map $f:\mathcal{C}\odot\mathcal{F}^\gamma\to\hat{\mathcal{C}}$ is Weierstrass with gap γ for F if $\mathcal{C}\odot\mathcal{F}^\gamma$ is a Weierstrass cell with gap γ for f^*F.

Theorem 10 (Cellular Weierstrass Preparation Theorem (WPT))**.** *Let $\rho,\sigma>0$. Let $\mathcal{C}^{\{\rho\}}$ be a (real) cell and $F\in\mathcal{O}_b(\mathcal{C}^{\{\rho\}})$ a (real) function. Then there exist $N=\operatorname{poly}_{\mathcal{F},F}(\rho,1/\sigma)$ (real) Weierstrass maps $f_j:\mathcal{C}_j^{\{\sigma\}}\odot\mathcal{F}_j^\gamma\to\mathcal{C}^{\{\rho\}}$ for F with gap $\gamma<1$ such that $\mathcal{C}\subset\cup_j f_j(\mathcal{C}_j\odot\mathcal{F}_j)$.*

If $\mathcal{C},F$ vary in a definable family then N,γ can be taken uniform over the family and the maps f_j can be chosen from a single definable family. If $\mathcal{C},F$ are algebraic of complexity β then one may take $N=\operatorname{poly}_\ell(\beta,\rho,1/\sigma)$ and $\gamma=1-1/\operatorname{poly}_\ell(\beta)$.

We will prove the WPT and CPT by induction on $\ell:=\ell(\mathcal{C})$ and $\dim\mathcal{C}$. The cases $\ell=0$ and $\dim\mathcal{C}=0$ are vacuous. We now prove the WPT for a cell $\mathcal{C}\odot\mathcal{F}$ of length $\ell+1$ assuming that the CPT and WPT are true for cells of smaller length or equal length and smaller dimension. The case $\mathcal{F}=*$ reduces to the CPT for $\mathcal{C}$ so we assume $\mathcal{F}$ is $D,D_\circ,A$.

We will give two separate proofs: one in the algebraic case, and one in the analytic case. This is the only part of the proof of the main theorems where our arguments significantly diverge for these two cases.

7.1. **Proof in the algebraic case.**

7.1.1. *Algebraic discriminants.* We will require the following simple lemma on discriminants.

Lemma 55. *Let $\mathcal{C}$ be a cell of length ℓ and $F\in\mathcal{O}_b(\mathcal{C})$, both algebraic of complexity β. Suppose that F does not vanish identically on $\mathcal{C}$. Then there exist:*

- *A polynomial $P\in\mathbb{C}[\mathbf{z}_{1..\ell}]$ of complexity* $\operatorname{poly}(\beta)$, *not identically vanishing on $\mathcal{C}$, and satisfying $\{F=0\}\subset\{P=0\}$.*
- *A polynomial $D\in\mathbb{C}[\mathbf{z}_{1..\ell-1}]$ of complexity* $\operatorname{poly}(\beta)$, *not identically vanishing on $\mathcal{C}_{1..\ell-1}$, such that the projection*
$$\pi:(\mathcal{C}_{1..\ell}\times\mathbb{C})\cap\{P=0\}\to\mathcal{C}_{1..\ell-1},\qquad \pi(\mathbf{z}_{1..\ell})=\mathbf{z}_{1..\ell-1}\tag{110}$$
is proper covering map outside $\{D=0\}$.

If F is real then P,D can be chosen real as well.

Proof. We may assume without loss of generality that the type of $\mathcal{C}$ does not contain $*$, since any such coordinate can be ignored for the statement of the lemma. Then $\mathcal{C}\subset\mathbb{C}^\ell$ is open.

By definition the graph of F is contained in some irreducible algebraic variety $G_F\subset\mathbb{C}^\ell\times\mathbb{C}_w$. Since F is not identically vanishing on $\mathcal{C}$, the equation $w=0$ cuts G_F properly in a subvariety of dimension $\ell-1$, and the projection of this variety to $\mathbb{C}^\ell$ contains $\{F=0\}$ and is contained in a proper algebraic hypersurface of degree $\operatorname{poly}(\beta)$, i.e. in a set $\{P=0\}$ where $P\in\mathbb{C}[\mathbf{z}_1..\ell]$ is not identically vanishing (on $\mathcal{C}$, since $\mathcal{C}$ is open). We may also assume that P is square-free as a polynomial in $\mathbb{C}(\mathbf{z}_{1..\ell-1})[\mathbf{z}_\ell]$, and in particular has no multiple roots for a generic value of $\mathbf{z}_{1..\ell-1}$. Then the classical discriminant D of P satisfies the conditions of the lemma.

For the final statement, if F is real then its graph is invariant under conjugation, and the same is then also true for G_F. The polynomial P is then also real by construction. $\square$

7.1.2. *Proof of the algebraic WPT.* We now proceed to the proof of the WPT. Suppose $\mathcal{F} = A(r_1, r_2)$ (the cases $D(r), D_\circ(r)$ are similar). Applying the refinement theorem, we may assume that ρ is already as small as we wish as long as $1/\rho = \operatorname{poly}_\ell(1/\sigma, \beta)$.

There is no harm in replacing F by the polynomial P obtained from Lemma 55 applied to the function $F \cdot (\mathbf{z}_{\ell+1} - r_1) \cdot (\mathbf{z}_{\ell+1} - r_2)$. In other words we may assume without loss of generality that F is a polynomial in the $\mathbf{z}_\ell$ variable and that it vanishes when $\mathbf{z}_\ell$ is either 0, $r_1(\mathbf{z}_{1..\ell})$ or $r_2(\mathbf{z}_{1..\ell})$. We let D denote the corresponding discriminant.

We apply the CPT to $\mathcal{C}$ with D and let $f_j : \mathcal{C}_j^{\{\rho\}} \to \mathcal{C}^{\{\rho\}}$ denote the resulting cellular cover. If $f_j(\mathcal{C}_j^{\{\rho\}})$ is contained in $\{D = 0\}$ then since cellular maps preserve dimension $\dim \mathcal{C}_j \leqslant \dim\{D = 0\} \leqslant \dim \mathcal{C} - 2$. In this case we set

$$\hat{\mathcal{C}}_j := \mathcal{C}_j \odot f_j^* \mathcal{F} \qquad\qquad \hat{f}_j := (f_j, \mathrm{id}) : \hat{\mathcal{C}}_j^{\{\rho\}} \to \mathcal{C}^{\{\rho\}} \tag{111}$$

and inductively apply the WPT to $\hat{\mathcal{C}}_j$ and $\hat{f}_j^* F$. We obtain Weierstrass maps $f_{j,k} : \mathcal{C}_{j,k}^{\{\sigma\}} \odot \mathcal{F}^\gamma \to \hat{\mathcal{C}}_j^{\{\rho\}}$ for F, and the compositions $f_{j,k} \circ \hat{f}_j : \mathcal{C}_{j,k}^{\{\sigma\}} \odot \mathcal{F}^\gamma \to \mathcal{C}^{\{\rho\}}$ are Weierstrass maps for F which cover $f(\mathcal{C}_j) \odot \mathcal{F}$.

It remains to consider the case that $f_j(\mathcal{C}_j^{\{\rho\}})$ is disjoint from $\{D = 0\}$. In the same way as before, it will suffice to prove the WPT for $\hat{\mathcal{C}}_j$ and $\hat{f}_j^* F$. We return now to the original notation replacing this pair by $\mathcal{C}, F$. We note that F is still polynomial in $\mathbf{z}_\ell$ (though perhaps not in the other coordinates) and the projection

$$\pi : (\mathcal{C}^{\{\rho\}} \times \mathbb{C}) \cap \{P = 0\} \to \mathcal{C}^{\{\rho\}}, \qquad \pi(\mathbf{z}_{1..\ell+1}) = \mathbf{z}_{1..\ell} \tag{112}$$

is a proper covering map. It will suffice to prove the WPT under these conditions. We are now in a position to use the constructions of §6.3.1. Note that $\nu = \operatorname{poly}_\ell(\beta)$. We will choose $\gamma = 1 - 1/\operatorname{poly}_\ell(\beta)$ (the precise choice will be determined later).

Recall that we may choose ρ small as long as $1/\rho = \operatorname{poly}_\ell(1/\sigma, \beta)$. We take $\{\rho\} < \{\sigma\} \cdot \{\hat{\rho}\}$ where $\hat{\rho}$ is chosen in such a way that Proposition 49 holds over $\mathcal{C}^{\{\sigma\}}$. The zero map is a section of π which we denote by y_0. Since r_1, r_2 are sections of π they belong to certain clusters around $y_0 = 0$, say with indices $q_1 \leqslant q_2$. We may also assume that $r_1 = \hat{y}_{q_1}$ and if $q_1 \neq q_2$ then $r_2 = \hat{y}_{q_2}$.

If $q_1 = q_2 = q$ then we define $\tilde{\mathcal{F}} = \mathcal{F}_{0,q}$. If $q_1 < q_2$ then we define

$$\tilde{\mathcal{F}} := A^\delta(l_{q_1} r_1, r_{q_2} r_2), \tag{113}$$

i.e. we take the left endpoint of the $\mathcal{F}_{0,q_1}$ cluster and the right endpoint of the $\mathcal{F}_{0,q_2}$ cluster. Since r_1, r_2 are univalued over $\mathcal{C}$ this is actually a fiber over $\mathcal{C}$ and the Weierstrass condition is provided by Proposition 49. The inclusion $\tilde{\mathcal{F}}^\gamma \subset \mathcal{F}^{\{\rho\}}$ follows from (100) for an appropriate choice of γ.

7.1.3. *The real setting.* If $\mathcal{C}$ and F are real then P is also real, and consequently the coverings constructed by inductive applications of the CPT can be taken to be real. After these reductions, the fiber constructed in (113) is clearly real as well.

7.2. **Proof in the analytic case.** Before giving the proof of the WPT in the analytic case we develop some general results concerning the Laurent coefficients of definable families of holomorphic functions.

7.3. **Laurent domination in definable families.** We will study the following property of the Taylor/Laurent coefficients of a holomorphic function.

Definition 56 (Taylor domination). *A holomorphic function $f : D(r) \to \mathbb{C}$ is said to posses the (p, M) Taylor domination property*[2] [2] *if its Taylor expansion $f(z) = \sum a_k(z - z_0)^k$ satisfies the estimate*

$$|a_k|r^k < M \max_{j=0,\ldots,p} |a_j|r^j, \qquad k = p+1, p+2, \ldots \tag{114}$$

Similarly, for $r_2 > r_1 > 0$ a holomorphic function $f : A(r_1, r_2) \to \mathbb{C}$ is said to posses the (p, M) Laurent domination property if its Laurent expansion $f(z) = \sum a_k(z - z_0)^k$ satisfies the estimates

$$\begin{aligned} |a_k|r_2^k &< M \max_{j=-p,\ldots,p} |a_j|r_2^j, \qquad k = p+1, p+2, \ldots \\ |a_k|r_1^k &< M \max_{j=-p,\ldots,p} |a_j|r_1^j, \qquad k = -p-1, -p-2, \ldots \end{aligned} \tag{115}$$

We will need the following lemma on Laurent expansions.

Lemma 57. *Let $S := S(1)$ and $f_\lambda : S^{\delta^2} \to \mathbb{C}$ a definable family of holomorphic functions for some $0 < \delta < 1$. Then there exists B such that for every $\lambda \in \Lambda$ we have*

$$\|f_\lambda\|_{S^\delta} \leqslant B \|f_\lambda\|_S . \tag{116}$$

If we write $f_\lambda(z) = \sum a_k(\lambda) z^k$ then there exists $p \in \mathbb{N}$ and $m > 0$ such that for all λ we have

$$|a_j(\lambda)| > m \|f_\lambda\|_S \qquad \text{for some } j = j(\lambda) \in \{-p, \ldots, p\}. \tag{117}$$

Proof. As we have seen in the proof of Lemma 39, the Voorhoeve index of f_λ along any circle in S^{δ^2} is uniformly bounded over λ (and over the circle). Fix λ and assume without loss of generality (up to rotation) that the maximum of f_λ on S^δ is attained at δi or $\delta^{-1} i$. Let C be the circle intersecting $i\mathbb{R}$ at $\delta^{\pm 1} i$ and fix some disc D with $C \subset D$ and $\bar{D} \subset S^{\delta^2}$. The Voorhoeve index of f_λ over ∂D is uniformly bounded over λ, and it follows from [28, Theorem 3][3] that $B_{S \cap D, D}(f_\lambda)$ is uniformly bounded by some constant B_1, where

$$B_{K,U}(f) = \log \max_{z \in \bar{U}} |f(z)| - \log \max_{z \in K} |f(z)|. \tag{118}$$

Thus

$$\|f_\lambda\|_{S^\delta} = \max_{z \in \bar{D}} |f(z)| \leqslant e^{B_1} \|f_\lambda\|_S \tag{119}$$

proving the first part.

From the first part and the Cauchy estimate it follows that

$$|a_k(\lambda)| \leqslant B \delta^{|k|} \|f_\lambda\|_S . \tag{120}$$

[2] note that we use a slightly simplified form of the definition given in [2].

[3] The result there is stated for K with nonempty interior, but actually holds for K of the form $S \cap D$, see [28, Section 4.1].

Then

$$\|f_\lambda\|_S \leqslant \sum_{k\in\mathbb{Z}} |a_k(\lambda)| = \sum_{k\in\{-p,\dots,p\}} |a_k(\lambda)| + \sum_{k\notin\{-p,\dots,p\}} |a_k(\lambda)| \leqslant \tag{121}$$

$$\sum_{k\in\{-p,\dots,p\}} |a_k(\lambda)| + \frac{2B\delta^{p+1}}{1-\delta}\,\|f_\lambda\|_S \tag{122}$$

and the result now follows with p such that $\frac{2B\delta^{p+1}}{1-\delta} < 1/2$ and $m = 1/(4p+2)$. □

As a direct corollary of Lemma 57 we obtain the following result. We remark that a similar statement for families of discs appeared (in a slightly different form and with a different proof) in [17]. The disc case also follows fairly directly from a classical theorem of Biernacki [4]. However the annulus case does not seem to follow in a similar manner.

Corollary 58. *Let $\mathcal{F}_\lambda$ be a definable family of discs or annuli and let $f_\lambda : \mathcal{F}^\varepsilon_\lambda \to \mathbb{C}$ be a definable family of holomorphic functions, for some $0 < \varepsilon < 1$. Then the functions f_λ have the (p, M) Laurent domination property in $\mathcal{F}_\lambda$ for some uniformly bounded p, M.*

Proof. We prove the annuli case, the disc case being simpler. Since the claim is invariant under rescaling, we may assume without loss of generality that $\mathcal{F}_\lambda$ is of the form $A(r_1, 1)$. We need to prove

$$|a_k| < M \max_{j=-p,\dots,p} |a_j|, \qquad k = p+1, p+2, \dots \tag{123}$$

And indeed by the Cauchy estimates and Lemma 57 with $\delta^2 = \varepsilon$ we have

$$|a_k(\lambda)| \leqslant B\,\|f_\lambda\|_s\,\delta^k \leqslant \frac{B}{m}|a_j(\lambda)| \qquad \text{for some } j \in \{-p,\dots,p\}. \tag{124}$$

For $k = -p-1, \dots$ we proceed similarly, assuming now that $\mathcal{F}_\lambda$ is of the form $A(1, r_2)$. □

We record a simple consequence of the Taylor domination property.

Proposition 59. *Let $A := A(r_1, r_2)$ and $0 < \delta < 1$. Let $f : A^\delta \to \mathbb{C}$ be a holomorphic function with (p, M) Taylor domination. Write*

$$f(z) = \sum_{j=-p}^{p} a_j z^j + R_p(z). \tag{125}$$

Then for $z \in A$ we have

$$|R_p(z)| < \frac{2\delta}{1-\delta} M \max_{j=-p,\dots,p} |a_j z^j|. \tag{126}$$

Proof. The Taylor domination property in A^δ gives

$$\begin{aligned} |a_k| r_2^k &< \delta^{k-p} M \max_{j=-p,\dots,p} |a_j| r_2^j, \qquad k = p+1, p+2, \dots \\ |a_k| r_1^k &< \delta^{-k-p} M \max_{j=-p,\dots,p} |a_j| r_1^j, \qquad k = -p-1, -p-2, \dots \end{aligned} \tag{127}$$

and the same clearly holds with r_1, r_2 replaced by $z \in A$. Summing over $k = p+1, \dots, \infty$ and $k = -p-1, \dots, -\infty$ we obtain (126). □

7.3.1. *Proof of the analytic WPT.* We may assume without loss of generality that $\mathcal{C}\odot\mathcal{F}$ admits an extension slightly larger than $\{\rho\}$. Then $(\mathcal{C}\odot\mathcal{F})^{\{\rho\}}$ is subanalytic, and in particular the family of all discs or annuli of the form $\{\mathbf{z}_{1..\ell}\}\times\tilde{\mathcal{F}}$ with $\{\mathbf{z}_{1..\ell}\}\times\tilde{\mathcal{F}}^{1/2}\subset(\mathcal{C}\odot\mathcal{F})^{\{\rho\}}$ is definable. Applying Lemma 58 we find p,M such that F satisfies the (p,M) Laurent domination property on every fiber $\{\mathbf{z}_{1..\ell}\}\times\tilde{\mathcal{F}}$ as above.

Applying the refinement theorem, we may assume that ρ is already as small as we wish as long as $1/\rho=\operatorname{poly}_\ell(1/\sigma)$. Since the maps constructed in the refinement theorem are cellular translate maps, every disc/annulus $\{\mathbf{z}_{1..\ell}\}\times\tilde{\mathcal{F}}^{1/2}\subset(\mathcal{C}\odot\mathcal{F})^{\{\rho\}}$ in a refined cell maps to a disc/annulus satisfying the same requirement in our original cell $\mathcal{C}$. Thus we may assume that after refinement our function F still satisfies the (p,M) Laurent domination property on every fiber $\{\mathbf{z}_{1..\ell}\}\times\tilde{\mathcal{F}}$ as above. Note that crucially (p,M) does not depend on our choice of $\{\rho\}$.

Suppose $\mathcal{F}=A(r_1,r_2)$ (the cases $D(r),D_\circ(r)$ are similar). If F vanishes identically on $\mathcal{C}\odot\mathcal{F}$ then there is nothing to prove, so suppose otherwise. Rescaling $\mathcal{F}$ by r_2 we may assume that $r_2=1$ and $|r_1|<1$. We write a Laurent expansion

$$F(\mathbf{z}_{1..\ell+1})=\sum_{k=-\infty}^{\infty}a_k(\mathbf{z}_{1..\ell})\mathbf{z}_{\ell+1}^k \tag{128}$$

where $a_k\in\mathcal{O}_b(\mathcal{C}^{\{\rho\}})$. We apply the CPT to the collection $a_{-p},\dots,a_p$ on $\mathcal{C}^{\{\rho\}}$. In the same way as in §7.1 we may reduce to the case where every a_k is either identically or non-vanishing on $\mathcal{C}^{\{\rho\}}$. Note also that since this step only reparametrizes the base without affecting the fiber, the (p,M) Laurent domination property still holds uniformly for every $\mathbf{z}_{1..\ell}\in\mathcal{C}^{\{\rho\}}$.

Let $\Pi\subset\{-p,\dots,p\}$ denote the indices k such that $a_k\neq0$. For $j\neq k\in\Pi$ we define

$$r_{jk}(\mathbf{z}_{1..\ell})=\sqrt[k-j]{\frac{a_j(\mathbf{z}_{1..\ell})}{a_k(\mathbf{z}_{1..\ell})}}, \tag{129}$$

that is, $S(|r_{jk}|)$ is the circle in $\mathbf{z}_{\ell+1}$ where the j-th and k-th terms of (128) are of the same modulus. Note that r_{jk} is multivalued. We let Σ denote the set consisting of $1,2$ and the pairs $(j,k)\in\Pi^2$, $j\neq k$, and set $\mu:=2+(2p+1)p\geqslant\#\Sigma$.

We show that the zeros of F can only occur in concentric annuli of bounded width around the radii r_{jk}. It is more convenient to state this in the logarithmic scale. We set $s_\alpha:=\log|r_\alpha|$ for every $\alpha\in\Sigma$ and note that s_α is single valued on $\mathcal{C}_{1..\ell}$.

Lemma 60. *There exists a constant $B=B(M,p)$ such that for any $\mathbf{z}_{1..\ell+1}\in\mathcal{C}$ satisfying*

$$|\log|\mathbf{z}_{\ell+1}|-s_\alpha(\mathbf{z}_{1..\ell})|>B\qquad\text{for }\alpha\in\Sigma. \tag{130}$$

we have $F(\mathbf{z}_{1..\ell+1})\neq0$.

Proof. Assume that (130) holds for some unspecified constant B which will be chosen later. Set $r:=|\mathbf{z}_{\ell+1}|$. For $k,j\in\Pi^2$ with $k\neq j$ we have

$$\left|\log\frac{|a_j\mathbf{z}_{\ell+1}^j|}{|a_k\mathbf{z}_{\ell+1}^k|}\right|=|(j-k)[-\log(|r_{jk}|+\log r]|>B. \tag{131}$$

In particular there exist one term $j_0 \in \Pi$ such that $|a_{j_0}\mathbf{z}_{\ell+1}^{j_0}|$ is maximal and we have

$$|a_k\mathbf{z}_{\ell+1}^k| < e^{-B}|a_{j_0}\mathbf{z}_{\ell+1}^{j_0}|, \qquad k \in \{-p,\dots,p\} \setminus \{j_0\}. \tag{132}$$

Set

$$R_p(\mathbf{z}_{1..\ell+1}) := F(\mathbf{z}_{1..\ell+1}) - \sum_{k=-p}^{p} a_j(\mathbf{z}_{1..\ell})\mathbf{z}_{\ell+1}^k. \tag{133}$$

Since $1,2 \in \Sigma$ we have $e^B r_1(\mathbf{z}_{1..\ell}) < r < e^{-B} r_2(\mathbf{z}_{1..\ell})$. Then Proposition 59 implies (with $\delta = e^{-B}$) that

$$|R_p(\mathbf{z}_{1..\ell})| < \frac{2M}{e^B - 1}|a_{j_0}\mathbf{z}_{\ell+1}^{j_0}|. \tag{134}$$

In particular choosing B such that $2pe^{-B} + \frac{2M}{e^B-1} < 1$ we see that

$$|R_p(\mathbf{z}_{1..\ell})| + \sum_{k\in\Pi\setminus\{j_0\}} |a_k\mathbf{z}_{\ell+1}^k| < |a_{j_0}\mathbf{z}_{\ell+1}^{j_0}| \tag{135}$$

so $F(\mathbf{z}_{1..\ell+1}) \neq 0$ as claimed. $\square$

In light of Lemma 60 our goal will be to construct an annulus $\mathcal{A} \subset \mathcal{F}^{\{\rho\}}$ which contains $\mathcal{F}$ and such that $\mathcal{A}^\gamma \setminus \mathcal{A}$ remains at logarithmic distance at least B from each of the $|r_\alpha|$, uniformly over $\mathcal{C}^{\{\sigma\}}$.

Apply the CPT to the cell $\mathcal{C}^{\{\rho\}}$ and the hypersurfaces $\{r_\alpha = r_\beta\}$ for $\alpha \neq \beta \in \Sigma$ (formally we raise both sides to a power and clear denominators to obtain a bounded analytic equation). Once again we may reduce to the case that each of these equations is satisfied either identically or nowhere in $\mathcal{C}^{\{\rho\}}$.

We let Z denote the union of the graphs of the zero function and of r_α over $\mathcal{C}^{\{\rho\}}$ for $\alpha \in \Sigma$. By the condition above $\pi : Z \to \mathcal{C}^{\{\rho\}}$ is a proper covering map. We are now in a position to use the constructions of §6.3.1. We choose $\gamma = e^B$.

Recall that we may choose ρ small as long as $1/\rho = \mathrm{poly}_\ell(1/\sigma)$. We take $\{\rho\} < \{\sigma\}\cdot\{\hat\rho\}$ where $\hat\rho$ is chosen in such a way that Proposition 49 holds over $\mathcal{C}^{\{\sigma\}}$. The zero map is a section of π which we denote by y_0. Since r_1, r_2 are sections of π they belong to certain clusters around $y_0 = 0$, say with indices $q_1 \leqslant q_2$. We may also assume that $r_1 = \hat y_{q_1}$ and if $q_1 \neq q_2$ then $r_2 = \hat y_{q_2}$.

If $q_1 = q_2$ then we define $\tilde{\mathcal{F}} = \mathcal{F}_{0,q}$. If $q_1 < q_2$ then we define

$$\tilde{\mathcal{F}} := A^\delta(r_{q_1} r_1, l_{q_2} r_2), \tag{136}$$

i.e. we take the left endpoint of the $\mathcal{F}_{0,q_1}$ cluster and the right endpoint of the $\mathcal{F}_{0,q_2}$ cluster. Since r_1, r_2 are univalued over $\mathcal{C}$ this is actually a fiber over $\mathcal{C}$ and the Weierstrass condition is provided by Proposition 49. The inclusion $\tilde{\mathcal{F}}^\gamma \subset \mathcal{F}^{\{\rho\}}$ follows from (100) provided that we choose $\{\rho\}$ large enough (note that B and hence γ did not depend on our choice of $\{\rho\}$).

7.3.2. *The real setting.* If $\mathcal{C}$ and F are real then the Laurent coefficients a_j are also real, and consequently the coverings constructed by inductive applications of the CPT can be taken to be real. After these reductions, the fiber constructed in (136) is clearly real as well.

7.3.3. *Uniformity over families.* Uniformity of the number of cells over definable families follows readily from the proof. Indeed the indices p, M have already been shown to be uniformly bounded over families. The fact that all Weierstrass cells can be chosen from a single definable family follows exactly as in the case of the refinement theorem.

7.3.4. *Analytic discriminants.* Later we will also require the following lemma on analytic discriminants.

Lemma 61. *Let $\mathcal{C}^{\{\rho\}}$ be a cell and let $Z \subset \mathcal{C}^{\{\rho\}} \times \mathbb{C}$ such that the projection $\pi : Z \to \mathcal{C}^{\{\rho\}}$ is proper. Then there exists $0 \neq D \in \mathcal{O}_b(\mathcal{C})$ such that π is a covering map outside $\{D = 0\}$. If $\mathcal{C}, Z$ are real then D can be chosen real as well. If $\mathcal{C}, Z$ vary in a definable family then D can also be chosen to vary from a definable family.*

Proof. According to [22, Theorem III.21] the map π is an *analytic cover* and in particular it is a λ-sheeted covering map, for some $\lambda \in \mathbb{N}$, outside a negligible set $A \subset \mathcal{C}^{\{\rho\}}$. Then letting

$$P_Z(\mathbf{z}, w) := \prod_{\eta \in \pi^{-1}(\mathbf{z})} (w - \eta) \tag{137}$$

we obtain a monic polynomial of degree λ with holomorphic locally bounded coefficients in $\mathcal{C}^{\{\rho\}} \setminus A$, which by the removable singularity theorem extend to holomorphic locally bounded coefficients in $\mathcal{C}^{\{\rho\}}$. Then D can be taken to be the classical discriminant of P_Z with respect to w. If $\mathcal{C}, Z$ are real then P_Z and hence D are also real. It is clear that this construction can be made uniform over definable families. □

8. Proofs of the CPT and CPrT

In this section we finish the proof of the CPT and CPrT for a cell $\mathcal{C} \odot \mathcal{F}$. We proceed by induction on the length and dimension. By the note following Theorem 10, we may already assume that the WPT holds for $\mathcal{C} \odot \mathcal{F}$.

8.1. **Proof of the CPT.** We will prove the CPT in a slightly weakened form, replacing the prepared maps f_j by arbitrary cellular maps. We later prove that this weaker form implies the CPrT, which in turn directly implies the stronger form of the CPT. We describe the proof for the complex version of the CPT, and at the end indicate the changes required for the real version. To avoid cluttering the text, we state our proof for the algebraic version of the CPT with polynomial estimates in the complexity β. The subanalytic version where these polynomial estimates are replaced by uniformity over definable families is obtained in a completely analogous manner.

8.1.1. *Reduction to the case of a single function.* We claim that it is enough to prove the CPT for a single function F. We suppose that this is already proved and prove the result for an arbitrary collection $F_1, \ldots, F_M$. We may suppose that none of the F_j vanish identically on $\mathcal{C}$. Let $F := F_1 \cdots F_M$ and apply the CPT to $\mathcal{C}, F$ to obtain a cellular cover $f_j : \mathcal{C}_j^{\{\sigma\}} \to (\mathcal{C} \odot \mathcal{F})^{\{\rho\}}$ compatible with F. Fix some f_j. If $f_j(\mathcal{C}_j^{\{\sigma\}})$ lies outside the zeros of F then it is already compatible with $F_1, \ldots, F_M$. Otherwise it lies in the zeros of F, and since cellular maps preserve dimension we have $\dim \mathcal{C}_j < \dim \mathcal{C} \odot \mathcal{F}$. By induction we obtain cellular maps f_{jk} :

$\mathcal{C}_{jk}^{\{\sigma\}} \to \mathcal{C}_j^{\{\sigma\}}$ which are compatible with $f_j^* F_1, \ldots, f_j^* F_M$. Then the compositions $f_j \circ f_{jk}$ are compatible with $F_1, \ldots, F_M$ and cover $f_j(\mathcal{C}_j)$, and taken together this gives a cellular cover of $\mathcal{C} \odot \mathcal{F}$ with $\operatorname{poly}_\ell(\beta, 1/\sigma, \rho)$ maps as required.

8.1.2. *Reduction to large σ and small ρ.* Applying the refinement theorem to $\mathcal{C}$ we may suppose that it already admits a $\{\hat\rho\}$-extension for $\hat\rho^{-1} = \operatorname{poly}_\ell(\rho, \beta)$. Below we will assume that ρ is already as small as we wish subject to this asymptotic. Similarly, it is enough to prove the CPT with any $\hat\sigma = \operatorname{poly}_\ell(\beta)$ since we may afterwards apply the refinement theorem to the resulting cells to obtain cells with σ-extensions. Below we will assume that σ is already as large as we wish subject to this asymptotic.

8.1.3. *Reduction to a Weierstrass cell for F.* We apply the WPT to cover $\mathcal{C} \odot \mathcal{F}$ by cells of the form $f_j : \mathcal{C}_j^{\{\rho\}} \odot \mathcal{F}_j^\gamma \to (\mathcal{C} \odot \mathcal{F})^{\{\rho\}}$ which are Weierstrass for F with gap $\gamma = 1 - 1/\operatorname{poly}_\ell(\beta)$. It is enough to prove the CPT for each of these cells separately, i.e. we may assume that $F \in \mathcal{O}_b(\mathcal{C}^{\{\rho\}} \odot \mathcal{F}^\gamma)$ does not vanish in $\mathcal{C}^{\{\rho\}} \odot (\mathcal{F}^\gamma \setminus \mathcal{F})$. If $\mathcal{F}$ is of type $*$ the CPT reduces to the CPT for $\mathcal{C}_{1..\ell-1}$. Suppose $\mathcal{F} = A(r_1, r_2)$ (the cases $D(r), D_\circ(r)$ are similar).

8.1.4. *Reduction to a proper covering map.* Since $\mathcal{C}^{\{\rho\}} \odot \mathcal{F}$ is a Weierstrass cell for F the zero locus of F in this cell is proper cover of $\mathcal{C}$ under the projection $\pi : \mathcal{C}^{\{\rho\}} \times \mathbb{C} \to \mathcal{C}^{\{\rho\}}$. We define Z to be the union of this zero locus with the graphs of the functions $0, r_1, r_2 : \mathcal{C}^{\{\rho\}} \to \mathbb{C}$. Then by Lemma 61 there exists $D \in \mathcal{O}_b(\mathcal{C}^{\{\rho\}})$ not identically vanishing on $\mathcal{C}$ such that the projection $\pi|_Z$ is a proper covering map outside $\{D = 0\}$. In the algebraic case, applying Lemma 55 to the function $F \cdot \mathbf{z}_{\ell+1}(\mathbf{z}_{\ell+1} - r_1)(\mathbf{z}_{\ell+1} - r_2)$ we see that D can be taken to be a polynomial of complexity $\operatorname{poly}(\beta)$.

We apply the CPT to $\mathcal{C}$ with D and let $f_j : \mathcal{C}_j^{\{\rho\}} \to \mathcal{C}^{\{\rho\}}$ denote the resulting cellular cover. If $f_j(\mathcal{C}_j^{\{\rho\}})$ is contained in $\{D = 0\}$ then since cellular maps preserve dimension $\dim \mathcal{C}_j \leqslant \dim\{D = 0\} \leqslant \dim \mathcal{C} - 2$. In this case we set

$$\hat{\mathcal{C}}_j := \mathcal{C}_j \odot f_j^* \mathcal{F} \qquad\qquad \hat f_j := (f_j, \mathrm{id}) : \hat{\mathcal{C}}_j^{\{\rho\}} \to \mathcal{C}^{\{\rho\}} \tag{138}$$

and note that $f_j^* F \in \mathcal{O}_b(\hat{\mathcal{C}}_j^{\{\hat\rho\}})$ for $\hat\rho = \operatorname{poly}(\rho, \beta)$. We inductively apply the CPT to $\hat{\mathcal{C}}_j$ with its $\hat\rho$-extension and $\hat f_j^* F$. We obtain a cellular cover $f_{j,k} : \mathcal{C}_{j,k}^{\{\sigma\}} \to \hat{\mathcal{C}}_j^{\{\hat\rho\}}$ compatible with F, and the compositions $f_{j,k} \circ \hat f_j : \mathcal{C}_{j,k}^{\{\sigma\}} \to \mathcal{C}^{\{\rho\}}$ are compatible with F and cover $f(\mathcal{C}_j) \odot \mathcal{F}$.

It remains to consider the case that $f_j(\mathcal{C}_j^{\{\rho\}})$ is disjoint from $\{D = 0\}$. In the same way as before, it will suffice to prove the CPT for $\hat{\mathcal{C}}_j$ and $\hat f_j^* F$. We return now to the original notation replacing this pair by $\mathcal{C}, F$ so that

$$\pi : (\mathcal{C}^{\{\rho\}} \odot \mathcal{F}^\gamma) \cap \{F = 0\} \to \mathcal{C}, \qquad \pi(\mathbf{z}_{1..\ell+1}) = \mathbf{z}_{1..\ell} \tag{139}$$

is a proper covering map. We are now in a position to use the constructions of §6.3.1. We denote the degree of π by ν, and note that $\nu = \operatorname{poly}_\ell(\beta)$.

8.1.5. *Covering the zeros.* By Lemma 47, each section y_j of π lifts to a univalued map $y_j : \hat{\mathcal{C}}_j^{\nu_j \cdot \rho} \to \mathbb{C}$. By definition the image of this maps lies in $(\mathcal{C}^{\{\rho\}} \odot \mathcal{F}^\gamma) \cap \{F = 0\}$, i.e. it gives a cell compatible with F. The collections of all of these cells cover the zeros of F in $\mathcal{C} \odot \mathcal{F}$. The remaining (and far more non-trivial) task is to cover the complement of this set of zeros.

8.1.6. *The Voronoi cells associated to $y_0 = 0$.* For the remainder of the proof we fix a point $p \in \hat{\mathcal{C}}$. We may assume that ρ is small enough that Proposition 49 holds, not only over $\mathcal{C}$ but over $\mathcal{C}^{\{\sigma\}}$. The zero map is a section of π which we denote by y_0. We will construct a collection of cells which we call the *Voronoi cells* of y_0.

Since r_1, r_2 are sections of π they belong to certain clusters around $y_0 = 0$, say with indices $q_1 \leqslant q_2$. For $q = q_1, \dots, q_2 - 1$ the fibers $\mathcal{F}^\gamma_{0,q+}$ over $\hat{\mathcal{C}}_{0,q}$ contain none of the zeros y_j and are contained in $\mathcal{F}^\gamma$. Therefore the maps $\hat{\mathcal{C}}_{0,q+} \odot \mathcal{F}_{0,q+} \to \mathcal{C} \odot \mathcal{F}$ admit γ-extensions compatible with F. These give our first Voronoi cells.

Now fix $q = q_1, \dots, q_2$ and consider the cluster $\mathcal{F}_{0,q}$. Up to an affine transformation as in Remark 50 we may assume that $\hat{y}_{0,q} = 1$ and $\mathcal{F}_{0,q} = A^\delta(l_{0,q}, r_{0,q})$. Recall that $\gamma = 1 - 1/\operatorname{poly}_\ell(\beta)$, and therefore we have $\mathcal{F}_{0,q} \subset A(1/2, 2)$. Recall also that

$$\operatorname{diam}(y_j(\hat{\mathcal{C}}), \mathbb{C} \setminus \{0, 1\}) = O(\nu^3 \rho) \tag{140}$$

for any of the y_j belonging to $\mathcal{F}_{0,q}$.

Let $\alpha > 0$ be such that the balls of radius α around $y_j(p)$ belongs to $\mathcal{F}^\gamma_{0,q}(p)$. Clearly we can choose $\alpha^{-1} = \operatorname{poly}_\ell(\beta)$. Let $U_p(\alpha)$ be the set obtained from $\mathcal{F}_{0,q}(p)$ by removing all of these balls. We can cover $U_p(\alpha)$ be $\operatorname{poly}_\ell(\beta)$ discs $\mathcal{D}_k$ centered at $U_p(\alpha)$ such that $\mathcal{D}_k^{1/2}$ does not meet the balls of radius $\alpha/2$ around $y_j(p)$. These give our remaining Voronoi cells. We claim that $\hat{\mathcal{C}}_{0,q} \odot \mathcal{D}_k^{1/2}$ is compatible with F. Indeed, over p the discs $\mathcal{D}_k$ remain at distance $\alpha/2$ from the zeros, and by (140) the points y_j do not move enough to meet $\mathcal{D}_i$ for ρ sufficiently small. By the same reasoning we obtain the following fundamental property of the Voronoi cells.

Lemma 62. *The Voronoi cells associated to y_0 cover every point $\mathbf{z}_{1..\ell+1} \in \mathcal{C} \odot \mathcal{F}$ such that*

$$\operatorname{dist}(\mathbf{z}_{\ell+1}, y_0) < (2\alpha)^{-1} \operatorname{dist}(\mathbf{z}_{\ell+1}, y_j) \tag{141}$$

for every y_j.

8.1.7. *The Voronoi cells associated to y_i.* Let y_i be one of the non-zero sections of π. We will construct a collection of *Voronoi cells* for y_i by analogy with y_0: the only additional difficulty is making sure that all cells remain in $\mathcal{F}^\gamma$. Our goal will be to construct cells with the following property.

Lemma 63. *The Voronoi cells associated to y_i cover every point $\mathbf{z}_{1..\ell+1} \in \mathcal{C} \odot \mathcal{F}$ such that*

$$\operatorname{dist}(\mathbf{z}_{\ell+1}, y_i) \leqslant 2\alpha \operatorname{dist}(\mathbf{z}_{\ell+1}, y_0) \tag{142}$$

and

$$\operatorname{dist}(\mathbf{z}_{\ell+1}, y_i) \leqslant (2\alpha)^{-1} \operatorname{dist}(\mathbf{z}_{\ell+1}, y_j) \tag{143}$$

for every $y_j \neq y_0, y_i$.

We make an affine transformation such that $y_i = 0$. Since $y_i \in \mathcal{F}$, we know that in these coordinates $D(ry_0) \subset \mathcal{F}^\gamma$ for $r \sim 1 - \gamma = 1/\operatorname{poly}(\beta)$. In the notations of §6.3.1, let $\mathcal{F}_{i,q_0}$ be the last cluster whose outer boundary at p is smaller than $ry_0(p)$. With our choice of γ the logarithmic width of each cluster can be assumed to be bounded by $\log 2$. Therefore the inner boundary of $\mathcal{F}_{i,q_0+1}$ at p is at least $ry_0(p)/2$. If the outer boundary of $\mathcal{F}_{i,q_0}$ is smaller than $\gamma^2 ry_0(p)/2$ then we also set

$$\tilde{\mathcal{F}}_{i,q_0+} = A^\delta(r_{i,q_0}\hat{y}_{i,q_0}, \gamma r y_0/2). \tag{144}$$

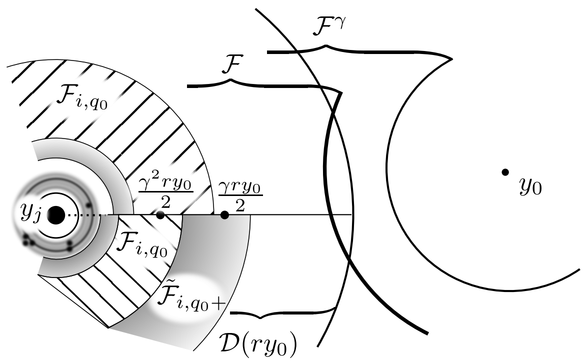

FIGURE 6. Construction of the $\tilde{\mathcal{F}}_{i,q_0+}$ fiber near the boundary.

One can check in the same way as in Proposition 49 that when defined, $\tilde{\mathcal{F}}_{i,q_0+}$ is a well defined annulus over $\hat{\mathcal{C}}_{i,q_0+}$ and $\tilde{\mathcal{F}}^\gamma_{i,q_0+} \subset \mathcal{F}^\gamma$ contains no zeros.

The fibers $\mathcal{F}_{i,q}, \mathcal{F}_{i,q+}$ for $q = 1, \ldots, q_0 - 1$, the fiber $\mathcal{F}_{i,q_0}$, and $\tilde{\mathcal{F}}_{i,q_0+}$ if it is defined cover $D(\gamma^2 r y_0/2)$ uniformly over $\hat{\mathcal{C}}$. We can choose our $\alpha = 1/\operatorname{poly}_\ell(\beta)$ in such a way that this disc contains all points satisfying (142). It will be enough to construct the Voronoi cells covering the points in these fibers which also satisfy (143). This is done in a manner completely analogous to the Voronoi cells of y_0, except that in place of $\mathcal{F}_{0,q_1+}, \ldots, \mathcal{F}_{0,(q_2-1)+}$ we use $\mathcal{F}_{i,0+}, \ldots, \mathcal{F}_{i,(q_0-1)+}$ and $\tilde{\mathcal{F}}_{i,q_0+}$ if it is defined; and instead of $\mathcal{F}_{0,q_1}, \ldots, \mathcal{F}_{0,q_2}$ we use $\mathcal{F}_{i,1}, \ldots, \mathcal{F}_{i,q_0-1}$.

8.1.8. *Conclusion.* We claim that the Voronoi cells for y_0 and the roots y_j cover $\mathcal{C} \odot \mathcal{F} \setminus \{F = 0\}$. Indeed, let $\mathbf{z} \in \mathcal{C} \odot \mathcal{F}$. If $\operatorname{dist}(\mathbf{z}_{\ell+1}, y_0) < (2\alpha)^{-1} \operatorname{dist}(\mathbf{z}_{\ell+1}, y_j)$ for every y_j then $\mathbf{z}$ is covered by the Voronoi cells of y_0. Otherwise it is covered by the Voronoi cell of y_j for the root y_j closest to $\mathbf{z}_{\ell+1}$. Since the roots of $\{F = 0\}$ were already covered in §8.1.5, this concludes the proof of the CPT in the complex case.

8.1.9. *The real case of the CPT.* The proof of the CPT in the real case proceeds in the same manner up to §8.1.5. At this point one should cover only the sections y_j that are real, ensuring that we indeed obtain real cells. Since the sections y_j are locally unramified, each of them is either purely real or purely non-real on $\mathbb{R}_+\mathcal{C}$ so we can indeed choose those y_j that are real to obtain a covering of the zeros of F in $\mathbb{R}_+(\mathcal{C} \odot \mathcal{F})$.

The next step is the construction of the Voronoi cells. To ensure that the constructed cells are real, one should replace the construction of §6.3.1 by that of §6.4 as we explain below.

The section $y_0 = 0$ is real and therefore forms an admissible center. The Voronoi cells for y_0 are constructed as in the complex case, except that the discs $\mathcal{D}_k$ are now chosen with real centers such that their positive parts cover $\mathbb{R}_+ U_p(\alpha)$. The points $\mathbf{z}_{\ell+1}$ that are left uncovered over p are those that have distance at most α to some root y_k. If $\mathbf{z}_{\ell+1} \in \mathbb{R}$ then by Lemma 52 such points also have distance at most α to some admissible center $c_l = (y_j + y_{\bar{j}})/2$. By analogy with Lemma 62 we obtain the following lemma.

Lemma 64. *The Voronoi cells associated to y_0 cover every point $\mathbf{z}_{1..\ell+1} \in \mathbb{R}_+(\mathcal{C} \odot \mathcal{F})$ such that*

$$\operatorname{dist}(\mathbf{z}_{\ell+1}, y_0) < (2\alpha)^{-1} \operatorname{dist}(\mathbf{z}_{\ell+1}, c_l) \tag{145}$$

for every admissible center c_l.

For the remaining Voronoi cells, we construct them centered around the admissible centers c_l rather than the sections y_j. The construction is analogous, replacing again the covering of $U_p(\alpha)$ by a real covering of $\mathbb{R}_+U_p(\alpha)$. We similarly obtain the following lemma.

Lemma 65. *The Voronoi cells associated to c_l cover every point $\mathbf{z}_{1..\ell+1} \in \mathbb{R}_+(\mathcal{C}\odot\mathcal{F})$ such that*

$$\operatorname{dist}(\mathbf{z}_{\ell+1}, c_l) \leqslant 2\alpha \operatorname{dist}(\mathbf{z}_{\ell+1}, y_0) \tag{146}$$

and

$$\operatorname{dist}(\mathbf{z}_{\ell+1}, c_l) \leqslant (2\alpha)^{-1} \operatorname{dist}(\mathbf{z}_{\ell+1}, c_m) \tag{147}$$

for every $c_m \neq y_0, y_l$.

The proof is then concluded in the same manner.

8.2. **Proof of the CPrT.** In this section we will prove the CPrT using the WPT and CPT. We will need a simple remark on the structure of the maps constructed in these two theorems.

Remark 66. *The maps constructed in the CPT can be assumed to be translates in the final variable, as the reader may easily verify by examining the inductive proof. Similarly, the maps constructed in the WPT can be assumed to be translates in the final two variables: the CPT and refinement theorems are applied in the base to give a translate in the next-to-last variable, whereas in the last variable the map is the identity.*

The main inductive step for the proof of the CPrT is contained in the following lemma.

Lemma 67. *Let $f : \mathcal{C}^{\{\rho\}} \to \hat{\mathcal{C}}$ be a (real) cellular map. Then there exists a (real) cellular cover $\{g_j : \mathcal{C}_j^{\{\rho\}} \to \mathcal{C}^{\{\rho\}}\}$ of size $\operatorname{poly}_f(\rho)$ such that each $f \circ g_j$ is prepared* in the last variable.

If $\mathcal{C}, \hat{\mathcal{C}}, f$ vary in a definable family then the size of the cover is $\operatorname{poly}(\rho)$ uniformly over the family, and the maps g_j can be chosen to vary definably over the family. If $\mathcal{C}, \hat{\mathcal{C}}, f$ are algebraic of complexity β then the cover has size $\operatorname{poly}(\beta, \rho)$ and complexity $\operatorname{poly}(\beta)$.

Proof. We describe the proof for the complex case, and at the end indicate the changes required for the real version. We treat the case of one cell and briefly indicate the minor points related to uniformity over families.

Let $\mathcal{C} := \mathcal{C}_{1..\ell} \odot \mathcal{F}$. Recall that the last coordinate of f has the form $P(\mathbf{z}_{1..\ell}; \mathbf{z}_{\ell+1})$ where P is a monic polynomial in $\mathbf{z}_{\ell+1}$ (say of degree μ) with coefficients holomorphic in $\mathcal{C}^{\delta}_{1..\ell}$. If $\mathcal{F} = *$ then we can just replace $\mathbf{z}_{\ell+1}$ by zero in the expression above, so f itself is already prepared. So assume $\mathcal{F}$ is of type $D, D_\circ, A$.

Let $\Sigma \subset \mathcal{C}$ denote the set critical points of P with respect to $\mathbf{z}_{\ell+1}$, i.e.

$$\Sigma := \left\{ \frac{\partial P}{\partial \mathbf{z}_{\ell+1}}(\mathbf{z}_{1..\ell+1}) = 0 \right\}. \tag{148}$$

Note that since P is a monic polynomial of positive degree in $\mathbf{z}_{\ell+1}$ the hypersurface Σ has zero dimensional fibers over $\mathcal{C}_{1..\ell}$.

We apply the CPT to $\mathcal{C}$ and Σ to obtain a cellular cover $\{g_j : \mathcal{C}_j^{\{\rho\}} \to \mathcal{C}^{\{\rho\}}\}$ compatible with Σ. If the image of g_j is contained in Σ then, since the fibers of Σ over $\mathcal{C}_{1..\ell}$ are zero dimensional, $\mathcal{C}_j$ has type ending with $*$ and in this case $f \circ g_j$ is already prepared as noted above. It remains to consider the case that the image of g_j is disjoint from Σ. Recall from Remark 66 that g_j is a translate in its last variable $\mathbf{w}_{\ell+1}$. Then we have

$$\frac{\partial}{\partial \mathbf{w}_{\ell+1}}(P \circ g_j) = \frac{\partial P}{\partial \mathbf{z}_{\ell+1}}(g_j(\mathbf{w}_{1..\ell+1})) \neq 0 \tag{149}$$

since the image of g_j is disjoint from Σ. It will now be enough to prove the lemma with f replaced by each $f \circ g_j$ as above. In other words we may assume without loss of generality that $\Sigma = \emptyset$.

Recall that P is bounded in absolute value by some constant N on $\mathcal{C}^{\{\rho\}}$. Write $D := D(N)$ and consider the cell

$$\tilde{\mathcal{C}} := \mathcal{C}_{1..\ell} \odot D \odot \mathcal{F} \tag{150}$$

with coordinates $\mathbf{z}_{1..\ell}, w, \mathbf{z}_{\ell+1}$, which also admits a $\{\rho\}$-extension. Let $\Gamma \subset \tilde{\mathcal{C}}^{\{\rho\}}$ be the hypersurface given by

$$\Gamma := \{w = P(\mathbf{z}_{1..\ell}; \mathbf{z}_{\ell+1})\}. \tag{151}$$

We apply the WPT to $\tilde{\mathcal{C}}^{\{\rho\}}, \Gamma$ to obtain a cellular covering $\{g_\alpha : \mathcal{C}_\alpha^{\{\rho/\mu\}} \odot \mathcal{F}_\alpha^\gamma \to \mathcal{C}^{\{\rho\}}\}$ of $\tilde{\mathcal{C}}$ by Weierstrass cells for Γ.

Suppose first that $g_\alpha(\mathcal{C}_\alpha) \subset \Gamma$. Then since Γ has zero-dimensional fibers over $\mathcal{C}_{1..\ell} \odot D$ (because P is monic) the type of $\mathcal{C}_\alpha$ ends with $*$, and we have

$$\psi_\alpha(\mathbf{z}_{1..\ell}, w) := g_\alpha(\mathbf{z}_{1..\ell}, w, *) = (\cdots, w + \phi_\alpha(\mathbf{z}_{1..\ell}), \zeta_\alpha(\mathbf{z}_{1..\ell}, w)) \tag{152}$$

where $\psi_\alpha : (\mathcal{C}_\alpha)_{1..\ell} \to \Gamma$ is a cellular map admitting a $\{\rho/\mu\}$-extension. Here we used the fact that g_α is a translate in its next to last variable. For later purposes we set $\nu(\alpha) = 1$.

Suppose now that g_α is not compatible with Γ. Then by the definition of a Weierstrass cell it follows that $g_\alpha^*\Gamma$ forms a cover of $(\mathcal{C}_\alpha)_{1..\ell+1}$. Since we are assuming that $\Sigma = 0$, this is a covering map of degree $\nu(\alpha) \leqslant \mu$. Then $g_\alpha^*\Gamma$ admits $\nu(\alpha)$ multivalued sections. Each section $s_{\alpha,j} : (\mathcal{C}_\alpha)_{1..\ell+1} \to g_\alpha^*\Gamma$ takes the form

$$s_{\alpha,j}(\mathbf{z}_{1..\ell}, w) = (\mathbf{z}_{1..\ell}, w, \hat{\zeta}_{\alpha,j}(\mathbf{z}_{1..\ell}, w)) \tag{153}$$

where $\hat{\zeta}$ is ramified of order at most $\nu(\alpha, j) \leqslant \nu(\alpha)$ (see Lemma 47). We note that in the algebraic case, $s_{\alpha,j}$ is algebraic of degree $\operatorname{poly}_\ell(\beta)$ since it is the section of a hypersurface of complexity $\operatorname{poly}_\ell(\beta)$. In the family case $\nu(\alpha)$ and the sections $s_{\alpha,j}$ vary definably.

We set

$$\mathcal{C}_{\alpha,j} := ((\mathcal{C}_\alpha)_{1..\ell+1})_{\times\nu(\alpha,j)}. \tag{154}$$

Precomposing with $R_{\nu(\alpha,j)} : \mathcal{C}_{\alpha,j} \to (\mathcal{C}_\alpha)_{1..\ell+1}$ we obtain a univalued map $s_{\alpha,j} \circ R_{\nu(\alpha,j)} : \mathcal{C}_{\alpha,j} \to g_\alpha^*\Gamma$ and finally composing with g_α we obtain a map $\psi_{\alpha,j} := g_\alpha \circ s_{\alpha,j} \circ R_{\nu(\alpha,j)} : \mathcal{C}_{\alpha,j} \to \Gamma$ of the form

$$\psi_{\alpha,j}(\mathbf{z}_{1..\ell}, w) = (\cdots, w^{\nu(\alpha,j)} + \phi_{\alpha,j}(\mathbf{z}_{1..\ell}), \zeta_{\alpha,j}(\mathbf{z}_{1..\ell}, w)). \tag{155}$$

Here again we used the fact that g is a translate in its next to last variable. By construction ψ admits a $\{\rho\}$-extension.

Let β be one of the indices α or (α, j) for the maps ψ constructed in (152),(155). Set $f_\beta := ((\psi_\beta)_{1..\ell}, \zeta_\beta) : \mathcal{C}_\beta^{\{\rho\}} \to \mathcal{C}$. Then $f \circ f_\beta$ is prepared in its final variable: indeed, since ψ maps into Γ we have

$$w^{\nu(\alpha)} + \phi_\beta(\mathbf{z}_{1..\ell}) = P((\psi_\beta)_{1..\ell}(\mathbf{z}_{1..\ell}), \zeta_\beta(z_{1..\ell}, w)) = (f \circ f_\beta)_{\ell+1} \tag{156}$$

for every $(\mathbf{z}_{1..\ell}, w) \in \mathcal{C}_\beta^{\{\rho\}}$. It remains to show that $f_\beta(\mathcal{C}_\beta)$ cover $\mathcal{C}$. But this is clear, since $\psi_\beta(\mathcal{C}_\beta)$ covers $\tilde{\mathcal{C}} \cap \Gamma$ by construction and the projection of $\tilde{\mathcal{C}} \cap \Gamma$ to $\mathbf{z}_{1..\ell+1}$ equals $\mathcal{C}$ by our choice of D.

We now consider the real case. We proceed in the same manner up to the construction of the sections $\psi_{\alpha,j}$. Note that some of these sections may not be real. However, since the critical locus Σ is empty it follows that on $\mathbb{R}_+\mathcal{C}_{\alpha,j}$ the section $\psi_{\alpha,j}$ is either always real or always non-real (real roots of a real holomorphic function can only leave the real line if they collide). Since we are only interested in covering $\mathbb{R}_+\Gamma$ we may replace the full set of sections $\psi_{\alpha,j}$ by those sections that are real on $\mathbb{R}_+\mathcal{C}_{\alpha,j}$. With this modification we obtain a real cover of $\mathcal{C}$ as required. $\square$

We are now ready to finish the proof of the CPrT by induction: suppose that the CPrT is already proved for cells of length ℓ, and we will prove it for a cell $\mathcal{C} := \mathcal{C}_{1..\ell} \odot \mathcal{F}$. By Lemma 67 we may assume that the map f is already prepared in its final variable. By the inductive hypothesis we can find a cellular cover $g_j : \mathcal{C}_j^{\{\rho\}} \to \mathcal{C}_{1..\ell}^{\{\rho\}}$ such that $f_{1..\ell} \circ g_j$ is prepared. Then $\hat{g}_j := g_j \odot \mathrm{id} : (\mathcal{C}_j \odot (g_j^*\mathcal{F}))^{\{\rho\}} \to \mathcal{C}^{\{\rho\}}$ is a cellular cover for $\mathcal{C}$ with $f \circ \hat{g}_j$ is prepared as required.

9. Analysis on complex cells

In this section we prove some basic estimates on holomorphic functions in complex cells. We also introduce the notion of a *quadric* cell which allows for some finer estimates and is used in the construction of smooth parametrizations in the sequel.

We fix the notation for the remainder of this section. We let $\mathcal{C} = \mathcal{F}_1 \odot \cdots \odot \mathcal{F}_\ell$ denote a complex cell. As a matter of normalization, we assume that $\mathcal{F}_j$ is is one of the forms $*, D(1), D_\circ(1)$ or $A(r_j, 1)$. Every cell is isomorphic to a cell of this form by an appropriate rescaling map. We say that such a cell $\mathcal{C}$ is *normalized*. We fix $0 < \delta < 1$ and assume that $\mathcal{C}$ admits a δ-extension.

9.1. Logarithmic derivatives.

The following proposition is a cellular analog of the classical fact that a logarithmic derivative of a holomorphic function admits at most first order poles.

Proposition 68. *Let $0 < \delta < 1/8$ and let $f \in \mathcal{O}_b(\mathcal{C}^\delta)$ be non-vanishing. Then for $\mathbf{z} \in \mathcal{C}$ we have*

$$\left| \frac{1}{f} \frac{\partial f}{\partial \mathbf{z}_i} \right| \leqslant O_\ell(|\mathbf{z}_i|^{-1}) \qquad \text{for } i = 1, \ldots, \ell. \tag{157}$$

If $\mathcal{C}, f$ are algebraic of complexity β then

$$\left| \frac{1}{f} \frac{\partial f}{\partial \mathbf{z}_i} \right| \leqslant \mathrm{poly}_\ell(\beta) \cdot |\mathbf{z}_i|^{-1} \qquad \text{for } i = 1, \ldots, \ell. \tag{158}$$

Proof. By the monomialization Lemma 17 we have $\log f = \sum_k m_k \log \mathbf{z}_k + \log U$ where $g = \log U$ is holomorphic and bounded in $\mathcal{C}^{1/2}$. The derivative of the first term is bounded by $|m_i \mathbf{z}_i|^{-1}$, and in the algebraic case $|m_i| = \mathrm{poly}_\ell(\beta)$. It remains to give a similar estimate for $\frac{\partial g}{\partial \mathbf{z}_i}$. Since we only care about the derivatives we may, after a translate, assume that the image of g contains 0. Then by the monomialization lemma in the algebraic case $\|g\|_{C^{1/4}} \leqslant \mathrm{poly}_\ell(\beta)$.

Write a decomposition $g = \sum_{\boldsymbol{\sigma}} g_{\boldsymbol{\sigma}}(\mathbf{z}^{[\boldsymbol{\sigma}]})$ as in Corollary 26,

$$g_{\boldsymbol{\sigma}} \in \mathcal{O}_b(\mathcal{P}^\delta), \qquad \|g_{\boldsymbol{\sigma}}\|_{\mathcal{P}^{1/2}} = O_\ell(\|g\|_{C^{1/4}}). \tag{159}$$

We estimate the derivative by

$$\left|\frac{\partial g_{\boldsymbol{\sigma}}}{\partial \mathbf{z}_i}\right| \leqslant \sum_{j=1}^{\ell} \left| (g_{\boldsymbol{\sigma}})'_j(\mathbf{z}^{[\boldsymbol{\sigma}]}) \frac{\partial(\mathbf{z}_j^{[\boldsymbol{\sigma}_j]})}{\partial \mathbf{z}_i} \right|. \tag{160}$$

By the Cauchy formula for $g_{\boldsymbol{\sigma}}$ in $\mathcal{P}^{1/2}$ we see that $\left\|(g_{\boldsymbol{\sigma}})'_j\right\|_{\mathcal{P}} = O_\ell(\|g_{\boldsymbol{\sigma}}\|_{\mathcal{P}^{1/2}})$, which is $\mathrm{poly}_\ell(\beta)$ in the algebraic case.

The derivatives $\frac{\partial(\mathbf{z}_j^{[\boldsymbol{\sigma}_j]})}{\partial \mathbf{z}_i}$ are either 0 or 1 for $\boldsymbol{\sigma}_j = 1$; and either 0 or $-r_i/\mathbf{z}_i^2$ or $\mathbf{z}_j^{-1}\frac{\partial r_j}{\partial \mathbf{z}_i}$ for $\boldsymbol{\sigma} = -1$. But

$$\left|-\frac{r_i}{\mathbf{z}_i^2}\right| = \left|\frac{r_i}{\mathbf{z}_i}\cdot\frac{1}{\mathbf{z}_i}\right| \leqslant |\mathbf{z}_i|^{-1}, \qquad \left|\mathbf{z}_j^{-1}\frac{\partial r_j}{\partial \mathbf{z}_i}\right| = \left|\frac{r_j}{\mathbf{z}_j}\cdot\frac{1}{r_j}\frac{\partial r_j}{\partial \mathbf{z}_i}\right| \leqslant K|\mathbf{z}_i|^{-1} \tag{161}$$

where and the final inequality follows by induction on ℓ and $K = \mathrm{poly}_\ell(\beta)$ in the algebraic case. From the above we conclude that $\frac{\partial g}{\partial \mathbf{z}_i} < K'|\mathbf{z}_i|^{-1}$ for an appropriate constant K' as claimed. □

9.2. **Quadric cells.** We say that the $\mathcal{C}$ cell is *quadric* if the radii r_j (for $\mathcal{F}_j$ of annulus type) have a univalued square root, $r_j = \rho_j^2$ on $\mathcal{C}_{1..j-1}$. This can also be stated by saying that the associated monomial of r_j has even degrees. We note if $\mathcal{C}$ is quadric then $\mathcal{C}^\delta$ is quadric as well. We denote by $\mathcal{QC}$ the *positive quadrant* defined by replacing each fiber $\mathcal{F}_j = A(r_j, 1)$ in the definition of $\mathcal{C}$ by $A(\rho_j, 1)$. We denote by $\mathcal{Q}^\delta\mathcal{C}$ the cell obtained by taking $\tilde{\mathcal{F}}_j = A(r_j, \delta^{-1})$. Note that this is the same as $\mathcal{Q}(\mathcal{C}^\delta)$ up to renormalizing all outer radii to 1.

Proposition 69. *Let $\mathcal{C}$ be a cell admitting a δ-extension. Set $\boldsymbol{\nu} := (2^{\ell-1}, \ldots, 1)$. Then $\mathcal{C}_{\times\boldsymbol{\nu}}$ is quadric and admits a $\delta^{1/2^{\ell-1}}$-extension.*

Proof. If $(\alpha_1, \ldots, \alpha_{j-1})$ are the degrees of the associated monomial of r_j then the associated monomial of the j-th fiber of $\mathcal{C}_{\times\boldsymbol{\nu}}$ has degrees $2^{j-\ell}(2^{\ell-1}\alpha_1, \ldots, 2^{\ell-j+1}\alpha_{j-1})$, all of which are even as claimed. The existence of a $\delta^{1/2^{\ell-1}}$-extension is a simple exercise. □

Suppose $\mathcal{C}$ is a quadric cell and $\boldsymbol{\sigma} \in \{-1, 1\}^\ell$, with negative entries allowed only for indices j such that $\mathcal{F}^j$ is an annulus. We define the inversion map $I_{\boldsymbol{\sigma}} : \mathcal{C} \to \mathcal{C}$ by the identity on coordinates $\mathbf{z}_j$ with $\boldsymbol{\sigma} = 1$ and by $r_j/\mathbf{z}_j$ on coordinates $\mathbf{z}_j$ with $\boldsymbol{\sigma} = -1$. Using inversions we can prove the following.

Proposition 70. *Let $\mathcal{C}$ be a real cell admitting a δ-extension. There exists a collection of quadric normalized cells $\mathcal{C}_j$ and real cellular maps $f_j : \mathcal{C}_j \to \mathcal{C}^\delta$ admitting δ-extensions such that $f_j(\mathcal{QC}_j)$ covers $\mathcal{C}$ and $f_j(\mathbb{R}_+\mathcal{QC}_j)$ covers $\mathbb{R}_+\mathcal{C}$. If $\mathcal{C}$ is algebraic of complexity β then $\mathcal{C}_j, f_j$ are algebraic of complexity* $\mathrm{poly}_\ell(\beta)$.

Proof. First, by subdividing $\mathcal{C}$ using the real CPT we may assume that it admits a δ^{2^ℓ}-extension. Applying Proposition 69 we see that $\mathcal{C}_{\times\boldsymbol{\nu}}$ admits a δ^2-extension, is quadric, and after rescaling may also be assumed to be normalized. It will suffice to prove the claim for $\mathcal{C}_{\times\boldsymbol{\nu}}$, so henceforth we replace $\mathcal{C}$ by $\mathcal{C}_{\times\boldsymbol{\nu}}$.

We essentially want to use the collection of all the inversion maps $I_{\boldsymbol{\sigma}}$ to cover $\mathcal{C}$ by $I_{\boldsymbol{\sigma}}(\mathcal{Q}\mathcal{C})$ and $\mathbb{R}_+\mathcal{C}$ by $I_{\boldsymbol{\sigma}}(\mathbb{R}_+\mathcal{Q}\mathcal{C})$. The minor technical issue is that this does not cover the equators $\{\mathbf{z}_j=\rho_j\}$. To avoid this problem, in the case $\sigma=(1,\dots,1)$ we use a slightly larger cell $\tilde{\mathcal{C}}$. Namely, we replace each fiber $\mathcal{F}_j=A(r_j,1)$ by $\tilde{\mathcal{F}}_j=A(\delta r_j,1)$. Since $\mathcal{C}$ admits a δ^2-extension $\tilde{\mathcal{C}}$ admits a δ-extension, and now the equator of $\mathcal{C}$ is covered by $\mathcal{Q}\tilde{\mathcal{C}}$. □

The following proposition is our principal motivation for introducing the notion of a quadric cell.

Lemma 71. *Let $\delta<1/4$ and let $\mathcal{C}^\delta$ be a quadric cell and $f\in\mathcal{O}_b(\mathcal{C}^\delta)$. Then*

$$\left\|\frac{\partial f}{\partial\mathbf{z}_j}\right\|_{\mathcal{Q}\mathcal{C}}\leqslant O_\ell(\|f\|_{\mathcal{C}^\delta}\cdot\delta)\qquad\text{for } j=1,\dots,\ell. \tag{162}$$

Proof. We assume without loss of generality that $\|f\|_{\mathcal{C}^\delta}=1$. By Corollary 26, $f=\sum_{\boldsymbol{\sigma}}f_{\boldsymbol{\sigma}}(\mathbf{z}^{[\boldsymbol{\sigma}]})$ with $\|f_\sigma\|_{\mathcal{P}^{2\delta}}=O_\ell(1)$. It will suffice to prove the claim for each of these summands, so fix $\boldsymbol{\sigma}\in\{-1,1\}^\ell$.

If $\boldsymbol{\sigma}_j=1$ then

$$\frac{\partial}{\partial\mathbf{z}_j}f_{\boldsymbol{\sigma}}(\mathbf{z}^{[\boldsymbol{\sigma}]})=(f_\sigma)'_j(\mathbf{z}^{[\boldsymbol{\sigma}]})+\sum_{k>j,\boldsymbol{\sigma}_k=-1}(f_\sigma)'_k(\mathbf{z}^{[\boldsymbol{\sigma}]})\cdot\frac{1}{\mathbf{z}_k}\cdot\frac{\partial r_k}{\partial\mathbf{z}_j}. \tag{163}$$

By the Cauchy formula

$$\left\|(f_{\boldsymbol{\sigma}})'_j\right\|_{\mathcal{P}}=O_\ell(\delta\,\|f_\sigma\|_{\mathcal{P}^{2\delta}})=O_\ell(\delta). \tag{164}$$

Also, since $\rho_k=\sqrt{r_k}$ is holomorphic of norm at most 1 in $\mathcal{C}^\delta_{1..k-1}$ we have by induction

$$\left\|\frac{1}{z_k}\frac{\partial r_k}{\partial\mathbf{z}_j}\right\|_{\mathcal{Q}\mathcal{C}}=\left\|2\frac{\partial\sqrt{r_k}}{\partial\mathbf{z}_j}\frac{\sqrt{r_k}}{\mathbf{z}_k}\right\|_{\mathcal{Q}\mathcal{C}}\leqslant O_\ell(\delta) \tag{165}$$

where we used the fact that $\sqrt{r_k}/\mathbf{z}_k<1$ in $\mathcal{Q}\mathcal{C}$. Combining these estimates we see that $\left\|\frac{\partial}{\partial\mathbf{z}_j}f_{\boldsymbol{\sigma}}(\mathbf{z}^{[\boldsymbol{\sigma}]})\right\|_{\mathcal{Q}\mathcal{C}}=O_\ell(\delta)$.

The case $\boldsymbol{\sigma}_j=-1$ is similar. We have

$$\frac{\partial}{\partial\mathbf{z}_j}f_{\boldsymbol{\sigma}}(\mathbf{z}^{[\boldsymbol{\sigma}]})=-(f_\sigma)'_j(\mathbf{z}^{[\boldsymbol{\sigma}]})\cdot\frac{r_j}{\mathbf{z}_j^2}+\sum_{k>j,\boldsymbol{\sigma}_k=-1}(f_\sigma)'_k(\mathbf{z}^{[\boldsymbol{\sigma}]})\cdot\frac{1}{\mathbf{z}_k}\cdot\frac{\partial r_k}{\partial\mathbf{z}_j}, \tag{166}$$

which can be estimated in the same manner noting that $\left\|\frac{r_j}{\mathbf{z}_j^2}\right\|_{\mathcal{Q}\mathcal{C}}\leqslant1$ by definition. □

9.3. **Straightening the positive quadrant.** Let $\mathcal{C}$ be a normalized quadric cell admitting a $\delta=1/100$ extension. To simplify the notations we assume that $\mathcal{C}$ contains no fibers of type D (we can replace each fiber of this type by fibers of type $D_\circ$ and $*$). To further simplify the notation we further assume that $\mathcal{C}$ contains no fibers of type $*$ (if it does then one should add additional single coordinates to B defined below in order to preserve the cellular structure of the map).

Denote $n := \dim \mathcal{C}$ and let $B := (0,1)^{\times n}$. We define a map $h : B \to \mathbb{R}_+\mathcal{C}^\delta$ inductively by

$$h_i(\mathbf{u}) = \mathbf{u}_i + R_i(\mathbf{u}_{1..i-1}), \qquad R_i := \sqrt{r_i \circ h_{1..i-1}}. \tag{167}$$

Here r_i denotes the inner radius of $\mathcal{F}_i$ if it is of type A, and 0 if it is of type $D, D_\circ$. Since $\mathcal{C}$ is normalized it follows that $R_+\mathcal{Q}\mathcal{C} \subset h(B) \subset \mathcal{C}^\delta$. We think of h as a straightening of the positive quadrant. Our goal will be to show that h can be analytically continued to a sector

$$S_\ell(\varepsilon) := \{|\operatorname{Arg} u_i| < \varepsilon, |u_i| < 2 : i = 1, \ldots, \ell\} \tag{168}$$

for some positive $\varepsilon > 0$. This will later be used to construct parametrizations of $\mathbb{R}_+\mathcal{C}$ with control on derivatives.

Proposition 72. *There exists $\varepsilon > 0$ such that h can be analytically extended to $S_\ell(\varepsilon)$ and we have*

$$h(S_\ell(\varepsilon)) \subset \mathcal{Q}^{1/4}\mathcal{C} \cap \{\operatorname{Arg} \mathbf{z}_j \leqslant \pi/4 : j = 1, \ldots, \ell\}. \tag{169}$$

Moreover for any j such that $\mathcal{F}_j$ is an annulus we have

$$\left\| \frac{\mathbf{u}_i}{R_j} \frac{\partial R_j}{\partial \mathbf{u}_i} \right\|_{S_\ell(\varepsilon)} \leqslant M_\ell, \quad i = 1, \ldots, \ell \tag{170}$$

for some $M_\ell > 0$, i.e. $\log R_j$ is a C^1-bounded function of $\log \mathbf{u}$. If $\mathcal{C}$ is algebraic of complexity β then $\varepsilon^{-1}, M_\ell = \operatorname{poly}_\ell(\beta)$.

Proof. By induction on ℓ we may assume that $h_{1..\ell-1}$ extends to $S_{\ell-1}(\varepsilon)$ for a sufficiently small ε, that

$$h_{1..\ell-1}(S_{\ell-1}(\varepsilon)) \subset \mathcal{Q}^{1/4}\mathcal{C}_{1..\ell-1} \cap \{\operatorname{Arg} \mathbf{z}_j \leqslant \pi/4 : j = 1, \ldots, \ell\}, \tag{171}$$

and that (170) already holds for $R_{1..\ell-1}$ with an appropriate constant $M_{\ell-1}$ (as a shorthand notation we write $R_{1..\ell-1}$ to mean only those R_j which are defined, i.e. such that $\mathcal{F}_j$ is an annulus). Our first goal is to prove (170) for R_ℓ assuming $\mathcal{F}_\ell$ is an annulus.

Since $\log R_{1..\ell-1}$ is a C^1-bounded function of $\log \mathbf{u}_{1..\ell-2}$ we may, taking $\varepsilon < \pi/(4M_{\ell-1})$, suppose that on $S_{\ell-1}(\varepsilon)$ we have

$$|\operatorname{Arg} \mathbf{u}_i|, |\operatorname{Arg} R_i| \leqslant \pi/4, \qquad i = 1, \ldots, \ell-1. \tag{172}$$

Indeed R_i is real on $\mathbb{R}S_{i-1}(\varepsilon)$, and since every point of $S_{i-1}(\varepsilon)$ lies at logarithmic distance at most ε from this real part, it follows that $\log R_i$ and in particular $\operatorname{Arg} R_i$ is at distance at most $M_{\ell-1} \cdot \varepsilon$ from zero. We note that this implies

$$|\mathbf{u}_i|, |R_i| \leqslant |\mathbf{u}_i + R_i| = |\mathbf{z}_i|, \qquad i = 1, \ldots, \ell-1. \tag{173}$$

Now computing the left hand side of (170) with $j = \ell$ we have

$$\frac{\mathbf{u}_i}{R_\ell} \frac{\partial R_\ell}{\partial \mathbf{u}_i} = \mathbf{u}_i \left(\frac{1}{\sqrt{r_\ell}} \frac{\partial \sqrt{r_\ell}}{\partial \mathbf{z}_i} \circ h_{1..\ell-1} \right) + \mathbf{u}_i \sum_{i<j<\ell} \frac{\partial R_j}{\partial \mathbf{u}_i} \left(\frac{1}{\sqrt{r_\ell}} \frac{\partial \sqrt{r_\ell}}{\partial \mathbf{z}_j} \circ h_{1..\ell-1} \right). \tag{174}$$

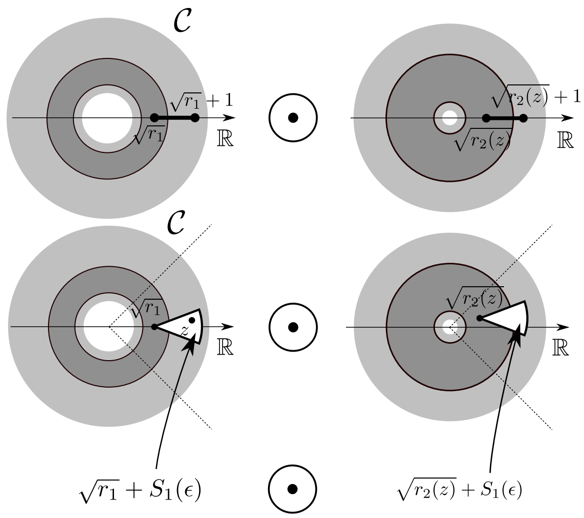

FIGURE 7. Analytic continuation of h to $S_\ell(\varepsilon)$.

By (171) the image $h_{1..\ell-1}(S_{\ell-1}(\varepsilon))$ is contained in $\mathcal{C}^{1/4}_{1..\ell-1}$. Applying Proposition 68 and using (173) we have for some $K>0$

$$\left|\mathbf{u}_i\Big(\frac{1}{\sqrt{r_\ell}}\frac{\partial\sqrt{r_\ell}}{\partial\mathbf{z}_i}\circ h_{1..\ell-1}\Big)\right| \leqslant \left|\frac{K\mathbf{u}_i}{\mathbf{z}_i}\right| \leqslant K \tag{175}$$

$$\left|\frac{1}{\sqrt{r_\ell}}\frac{\partial\sqrt{r_\ell}}{\partial\mathbf{z}_j}\circ h_{1..\ell-1}\right| \leqslant \frac{K}{|\mathbf{z}_j|} \leqslant \frac{K}{|R_j|}. \tag{176}$$

Plugging into (174) and using the inductive (170) we get

$$\left\|\frac{\mathbf{u}_i}{R_\ell}\frac{\partial R_\ell}{\partial\mathbf{u}_i}\right\|_{S_{\ell-2}(\varepsilon)} \leqslant K\left(1+\sum_{i<j<\ell}\left|\frac{\mathbf{u}_i}{R_j}\frac{\partial R_j}{\partial\mathbf{u}_i}\right|\right) \leqslant K\big(1+(\ell-i-1)M_{\ell-1}\big) \tag{177}$$

as claimed. In the algebraic case $K=\operatorname{poly}_\ell(\beta)$ and by induction we have indeed $M_\ell=\operatorname{poly}_\ell(\beta)$.

We now pass to proving that h can be analytically extended to $S_\ell(\varepsilon)$, satisfying (169). By assumption $h_{1..\ell-1}$ is already extended and satisfies (171). If $\mathcal{F}_\ell$ is an annulus (the other cases are similar but easier) what remains to be verified is that

$$R_l(\mathbf{u}_{1..\ell-1})+\{|\operatorname{Arg}\mathbf{u}_\ell|<\varepsilon, |\mathbf{u}_\ell|<2\} \subset \{|R_\ell|<|\mathbf{z}_\ell|<4\} \tag{178}$$

for $\mathbf{u}_{1..\ell-1}\in S_{\ell-1}(\varepsilon)$. Having established (170) for R_ℓ, we can now assume that (172) holds for R_ℓ as well (for a sufficiently small ε). Then (178) follows by an elementary geometric argument illustrated in Figure 7. □

Corollary 73. *Let $S_\ell(\varepsilon)$ be as in Proposition 72. Then the Jacobian of h satisfies*

$$\left\|\left[\frac{\partial h_i}{\partial\mathbf{u}_j}\right]-I_{\ell\times\ell}\right\|_{S_\ell(\varepsilon)} \leqslant O_\ell(\delta). \tag{179}$$

Proof. By induction on ℓ we may assume that the Jacobian of $h_{1..\ell-1}$ already satisfies the condition. It remains to produce a bound for each $\frac{\partial h_\ell}{\partial\mathbf{u}_j}$ with $j<\ell$. We have

$$\frac{\partial h_\ell}{\partial \mathbf{u}_j} = \frac{\partial\big(\sqrt{R_\ell} \circ h_{1..\ell-1}\big)}{\partial \mathbf{u}_j}, \tag{180}$$

and since the norm of the Jacobian of $h_{1..\ell-1}$ is $O_\ell(1)$ it remains to show that the derivatives $\frac{\partial \sqrt{R_\ell}}{\partial \mathbf{z}_k} \circ h$ are bounded by $O_\ell(\delta)$ for $k = 1, \ldots, \ell-1$. Since $h(S_\ell(\varepsilon)) \subset \mathcal{Q}^{1/4}\mathcal{C}$ and $\sqrt{R_\ell}$ is bounded by 1 in $\mathcal{C}^\delta$, this follows from Lemma 71. □

The following is a direct corollary of Lemma 71 and Corollary 73.

Corollary 74. *Let $F \in \mathcal{O}_b(\mathcal{C}^\delta)$ and let $\Phi := F \circ h$. Then*

$$\left\| \frac{\partial \Phi}{\partial \mathbf{u}_i} \right\|_{S_\ell(\varepsilon)} < O_\ell(\|F\| \cdot \delta) \qquad i = 1, \ldots, \ell. \tag{181}$$

10. Smooth parametrization results

In this section we use complex cells to prove the various refinements of the Yomdin-Gromov algebraic lemma described in §1.1. All of these results follow fairly directly from a parametrization result involving functions with a holomorphic continuation to a complex sector, which we describe first.

10.1. **Sectorial parametrizations of subanalytic sets.**

We say that $B \subset \mathbb{R}^\ell$ is a cube if it is a direct product of intervals $(0,1)$ and singletons $\{0\}$. We will say that a map $h : B \to \mathbb{R}^\ell$ is *cellular* if h_j depends only on $\mathbf{x}_{1..j}$. The *sectorial cube* $B(\varepsilon) \subset \mathbb{C}^\ell$ corresponding to B is the direct product where we replace each interval $(0,1)$ by the sector $S(\varepsilon) := \{\operatorname{Arg} z < \varepsilon, |z| < 2\}$. We call ε the *angle* of $B(\varepsilon)$.

Theorem 11. *Let $S \subset [0,1]^\ell$ be subanalytic. There exists a collection of cubes $B_1, \ldots, B_N$ and injective cellular maps $\phi_j : B_j \to S$ such that $S = \cup_j \phi_j(B_j)$. Moreover, there exists $\varepsilon > 0$ such that each ϕ_j extends to a holomorphic map $\phi_j : B_j(\varepsilon) \to \mathcal{P}_\ell^{1/2}$ with unit C^1-norm.*

If S is semialgebraic of complexity β then we have $N, 1/\varepsilon = \operatorname{poly}_\ell(\beta)$.

Proof. Set $\delta = 1/100$ as in §9.3. We begin by applying Corollary 29 in the semialgebraic case or Corollary 30 in the subanalytic case with $\{\rho\} < 1/2$ and $\{\sigma\} < \delta$, followed by Proposition 70 to construct a collection quadric normalized cells $\mathcal{C}_j$ and real cellular maps $f_j : \mathcal{C}_j^\delta \to \mathcal{P}_\ell^{1/2}$ such that $f_j(\mathbb{R}_+\mathcal{C}^\delta) \subset A$ and $f_j(\mathbb{R}_+\mathcal{Q}\mathcal{C}_j)$ covers $\mathbb{R}_+\mathcal{C}$. By construction each f_j is a composition of a prepared cellular map, a covering map and an inversion maps. Since each of these maps is injective on the positive real part we see that f_j is injective on $\mathbb{R}_+\mathcal{C}_j$.

Let $h_j : B_j(\varepsilon) \to \mathbb{R}_+\mathcal{C}^\delta$ be the straightening map constructed for $\mathcal{C}_j$ in §9.3 and set $\phi_j := f_j \circ h_j$. The map h_j is cellular and injective by construction, so ϕ_j is cellular and injective on B_j. We have

$$A \subset \bigcup_j f_j(\mathbb{R}_+\mathcal{Q}\mathcal{C}_j) \subset \bigcup_j \phi_j(B_j) \subset A \tag{182}$$

Finally, by Corollary 74 the C^1-norm of ϕ_j is bounded by $O_\ell(1)$. Making an additional subdivision of B_j into $O_\ell(1)$ parts and rescaling one can make this norm bounded by 1, thus proving the claim. □

Later we will use the Cauchy estimate to pass from the sectorial parametrizations of Theorem 11 to parametrizations with bounded higher-order derivatives. Toward this end the following simple combinatorial lemma will be useful.

Proposition 75. *Let $B(\varepsilon)$ be a sectorial cube and let $\Phi : B(\varepsilon) \to \mathbb{C}$ have C^1-norm 1. There is a decomposition*

$$\Phi(\mathbf{u}) = \sum_{I\subset\{1,\dots,\ell\}} \Phi_I(\mathbf{u}) \tag{183}$$

where $\Phi_I : B(\varepsilon) \to \mathbb{C}$ depends on $\mathbf{u}_i : i \in I$ only, vanishes on $\cup_{i\in I}\{\mathbf{u}_i = 0\}$ and has C^1-norm bounded by $O_\ell(1)$. In particular

$$|\Phi_I(\mathbf{u})| \leqslant O_\ell(\min_{i\in I} |\mathbf{u}_i|), \qquad \text{for } \mathbf{u} \in B(\varepsilon). \tag{184}$$

Proof. We proceed by reverse induction on j, defined as the minimal $i \in \{1, \dots, \ell\}$ such that Φ depends on $\mathbf{u}_j$ and does not vanish identically on $\{\mathbf{u}_j = 0\}$. If the condition defining j is empty then we may take $\Phi = \Phi_I$ where I denotes the set of indices that Φ depends on. Otherwise write

$$\Phi = (\Phi - R) + R, \qquad R(\mathbf{u}) = \Phi(\mathbf{u}_1, \dots, \mathbf{u}_{j-1}, 0, \mathbf{u}_{j+1}, \dots, \mathbf{u}_\ell). \tag{185}$$

Since the first summand vanishes on $\{\mathbf{u}_j = 0\}$ and the second summand does not depend on $\mathbf{u}_j$ we may finish for each of them by induction (the C^1-norm of the summands are majorated by $2, 1$ respectively). ☐

10.2. **The algebraic reparametrization lemmas.** We give more detailed statements of the algebraic lemmas in the C^r-smooth and mild contexts, and show that these statements imply the statements appearing in §1.1. Our result in the C^r case is as follows.

Theorem 12. *Let $X \subset [0, 1]^\ell$ be subanalytic. There exists $A > 0$, cubes $B_1, \dots, B_C$ and subanalytic injective cellular maps $\phi_j : B_j \to X$ whose images cover X, with*

$$\|D^{\boldsymbol{\alpha}}\phi_j\| \leqslant \boldsymbol{\alpha}!(Ar)^{|\boldsymbol{\alpha}|}, \quad \text{for } |\boldsymbol{\alpha}| \leqslant r. \tag{186}$$

If X is semialgebraic of complexity β then $A, C = \operatorname{poly}_\ell(\beta)$ and the maps ϕ_j are semialgebraic of complexity $\operatorname{poly}_\ell(\beta, r)$.

The cellular structure and injectivity of the parametrizing maps implies that the result of Theorem 12 is automatically uniform over families. Indeed, suppose $X \subset \mathbb{R}^{n+m}$ is a bounded subanalytic set viewed as a family X_p of subsets of $\mathbb{R}^m$. Construct $\phi_1, \dots, \phi_N$ as above. For any $p \in \mathbb{R}^n$ and any $j = 1, \dots, N$, the fiber of $(\phi_j)^{-1}_{1..n}(p)$ is either empty or a cube $B_{p,j} \subset \mathbb{R}^m$. The ϕ_j of the latter type restrict to parameterizing maps for X_p with the same C^r-norm bound (186) (now in m variables). One can subdivide the domains of the ϕ_j into smaller cubes to obtain a parametrization with unit C^r norms. Let $\mu := \dim X_p$. Subdividing B_j into N^μ subcubes of side length $1/N$ and rescaling back to the unit cube gives maps $\phi_{j,k} : B_{j,k} \to \mathbb{R}^\ell$ with

$$\frac{\|D^{\boldsymbol{\alpha}}\phi_{j,k}\|}{\boldsymbol{\alpha}!} \leqslant \left(\frac{Ar}{N}\right)^{|\boldsymbol{\alpha}|}, \quad \text{for } |\boldsymbol{\alpha}| \leqslant r. \tag{187}$$

Taking $N = Ar$ one obtains unit norms above. This gives the C^r statements of §1.1.

Our result in the mild case is as follows.

Theorem 13. *Let $X \subset [0,1]^\ell$ be subanalytic. There exists $A < \infty$, cubes $B_1, \dots, B_C$ and $(A,2)$-mild injective cellular maps $\phi_j : B_j \to X$ whose images cover X.*

If X is semialgebraic of complexity β then one may take $A, C = \operatorname{poly}_\ell(\beta)$.

As in the case of Theorem 12, the result of Theorem 13 is automatically uniform over families due to the cellular structure and the injectivity of the maps.

10.3. **Proofs of the smooth parametrization results.** Let X be as in Theorem 12 or Theorem 13. By Theorem 11 there exists a collection of cubes $B_1, \dots, B_N$ and injective cellular maps $\phi_j : B_j \to X$ such that $X = \cup_j \phi_j(B_j)$, and each ϕ_j extends to a holomorphic map $\phi_j : B_j(\varepsilon) \to \mathbb{C}^\ell$ with unit C^1-norm. Moreover in the algebraic case $N, 1/\varepsilon = \operatorname{poly}_\ell(\beta)$. It will be enough to prove the parametrization results for each $\phi_j(B_j)$ separately, so we fix a pair $(B,\phi) := (B_j, \phi_j)$ and proceed to consider the problem of parametrizing $\phi(B)$.

Let Φ denote one of the coordinates of ϕ and recall that Φ has unit C^1-norm. To control higher derivatives we will consider the map $\tilde\phi := \phi \circ S$ where S is a "flattening" bijection $S : B \to B$. We will see that an appropriate choice of S (one for the C^r version and another for the mild version) allows one to control the higher derivatives of $\Phi \circ S$ using the Cauchy estimate.

To simplify the notation, we will suppose below that B is of the form $(0,1)^\ell$ (in general the direct product may also contain copies of $*$, but these obviously have no impact on parametrization questions). We will take Φ to be any function of unit C^1-norm in $B(\varepsilon)$.

10.3.1. *Proof of the C^r-parametrization result.* Consider the bijection

$$P_r : B \to B, \qquad P_r(\mathbf{w}_{1..\ell}) = (\mathbf{w}_1^r, \dots, \mathbf{w}_\ell^r). \tag{188}$$

In the notation of §10.3, Theorem 12 follows from the following lemma.

Lemma 76. *The function $\Phi \circ P_r$ is has bounded C^r-norm,*

$$\left|(\Phi \circ P_r)^{(\boldsymbol{\alpha})}(\mathbf{w})\right| \leqslant \boldsymbol{\alpha}!\,(O_\ell(r/\varepsilon))^{|\boldsymbol{\alpha}|} \qquad \textit{for } |\boldsymbol{\alpha}| \leqslant r. \tag{189}$$

Proof. Since Φ extends holomorphically to $S_\ell(\varepsilon)$, the composition $\Phi \circ P_r$ extends holomorphically to

$$\Omega := S_\ell(\varepsilon/r) \cap \{|\zeta_i| < 2^{1/r}, i = 1, \dots, \ell\}. \tag{190}$$

In particular for every $\mathbf{w} \in B$ it contains the polydisc $D_{\mathbf{w}} = \{|\xi_i - \mathbf{w}_i| \leqslant (\sin\frac{\varepsilon}{r})\mathbf{w}_i\}$. Denote by $\mathcal{SC}(D_{\mathbf{w}})$ its skeleton.

Using Proposition 75 we see that

$$(\Phi \circ P_r)^{(\boldsymbol{\alpha})}(\mathbf{w}) = \sum_I (\Phi_I \circ P_r)^{(\boldsymbol{\alpha})}(\mathbf{w}) = \sum_{\operatorname{supp}\boldsymbol{\alpha} \subset I} (\Phi_I \circ P_r)^{(\boldsymbol{\alpha})}(\mathbf{w}), \tag{191}$$

so it is enough to consider $(\Phi_I \circ P_r)^{(\boldsymbol{\alpha})}(\mathbf{w})$ only for those summands with $\operatorname{supp}\boldsymbol{\alpha} \subset I$. Applying the Cauchy formula we have

$$\begin{aligned}\left|(\Phi_I \circ P_r)^{(\boldsymbol{\alpha})}(\mathbf{w})\right| &= \left|(2\pi i)^{-\ell}\boldsymbol{\alpha}! \int_{\mathcal{SC}(D_{\mathbf{w}})} \frac{\Phi_I \circ P_r(\xi)}{\prod(\xi_i - \mathbf{w}_i)^{\boldsymbol{\alpha}_i+1}} d\xi_1 \dots d\xi_\ell\right| \\ &\leqslant \boldsymbol{\alpha}! \prod \left(\mathbf{w}_i \sin\frac{\varepsilon}{r}\right)^{-\boldsymbol{\alpha}_i} \max_{\xi \in \mathcal{SC}(D_{\mathbf{w}})} |\Phi_I \circ P_r(\xi)|. \end{aligned} \tag{192}$$

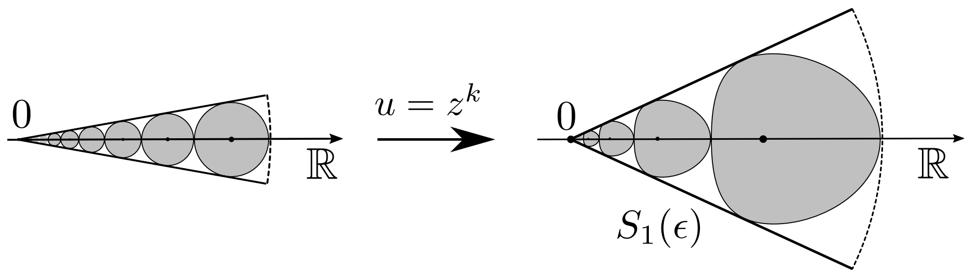

FIGURE 8. Reparametrization of $S_\ell(\varepsilon)$ in the C^r case.

Denote $\mathbf{w}_{\min} := \min_{i\in I}\mathbf{w}_i$. By Proposition 75 we have

$$|\Phi_I \circ P_r(\xi)| \leqslant O_\ell(1)\cdot \min_{i\in I}|\xi_i^r| \leqslant O_\ell(1)\cdot |\mathbf{w}_{\min}^r|, \tag{193}$$

where in the last inequality we used $\left(1+\sin\frac{\epsilon}{r}\right)^r < e$. Then (192) implies

$$\left|(\Phi_I\circ P_r)^{(\boldsymbol{\alpha})}(\mathbf{w})\right| \leqslant O_\ell(1)\left(\sin\frac{\epsilon}{r}\right)^{-|\boldsymbol{\alpha}|}\boldsymbol{\alpha}!\mathbf{w}_{\min}^{r-|\boldsymbol{\alpha}|} \leqslant \boldsymbol{\alpha}!\left(O_\ell(r/\varepsilon)\right)^{|\boldsymbol{\alpha}|} \tag{194}$$

as long as $|\boldsymbol{\alpha}|\leqslant r$. □

10.3.2. *Proof of the mild parametrization result.* Consider the bijection

$$\mathrm{Exp}: B\to B, \qquad \mathrm{Exp}(\mathbf{w}_{1..\ell}) = \left(\exp\left(1-\frac{1}{\mathbf{w}_1}\right),\dots,\exp\left(1-\frac{1}{\mathbf{w}_\ell}\right)\right). \tag{195}$$

In the notation of §10.3, Theorem 13 follows from the following lemma.

Lemma 77. *The function $\Phi\circ\mathrm{Exp}$ is $(A,2)$-mild,*

$$\left|(\Phi\circ\mathrm{Exp})^{(\boldsymbol{\alpha})}(\mathbf{w})\right| \leqslant \boldsymbol{\alpha}!(A|\boldsymbol{\alpha}|^2)^{|\boldsymbol{\alpha}|} \qquad \textit{for } \boldsymbol{\alpha}\in\mathbb{N}^\ell \tag{196}$$

with $A = O_\ell(1/\varepsilon)$.

Proof. Since Φ extends holomorphically to $S_\ell(\varepsilon)$, the composition $\Phi\circ\mathrm{Exp}$ extends holomorphically to the domain $\Omega := \mathrm{Exp}^{-1}S_\ell(\varepsilon)$ and it is easy to check that for any $\mathbf{w}\in B$,

$$D_{\mathbf{w}} := \{|\xi_i - \mathbf{w}_i| \leqslant \frac{1}{2}\varepsilon\mathbf{w}_i^2\}\subset\Omega. \tag{197}$$

Denote by $\mathcal{SC}(D_{\mathbf{w}}) := \{|\xi_i-\mathbf{w}_i| = \frac12\varepsilon\mathbf{w}_i^2\}$ the skeleton of $D_{\mathbf{w}}$.

Using Proposition 75 in the same way as in §10.3.1 we see that it is enough to consider $(\Phi_I\circ\mathrm{Exp})^{(\boldsymbol{\alpha})}(\mathbf{w})$ only for those summands with $\operatorname{supp}\boldsymbol{\alpha}\subset I$. Applying the Cauchy formula we have

$$\begin{aligned}\left|(\Phi_I\circ\mathrm{Exp})^{(\boldsymbol{\alpha})}(\mathbf{w})\right| &= \left|(2\pi i)^{-\ell}\boldsymbol{\alpha}!\int_{\mathcal{SC}(D_{\mathbf{w}})}\frac{\Phi_I\circ\mathrm{Exp}(\xi)}{\prod(\xi_i-\mathbf{w}_i)^{\boldsymbol{\alpha}_i+1}}d\xi_1\dots d\xi_\ell\right| \\ &\leqslant \boldsymbol{\alpha}!\prod\left(\mathbf{w}_i^2\frac{\varepsilon}{2}\right)^{-\boldsymbol{\alpha}_i}\max_{\xi\in\mathcal{SC}(D_{\mathbf{w}})}|\Phi_I\circ\mathrm{Exp}(\xi)|.\end{aligned} \tag{198}$$

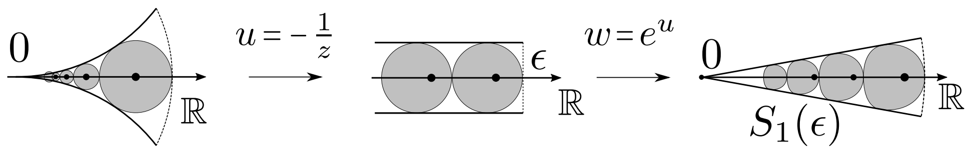

FIGURE 9. Reparametrization of $S_\ell(\varepsilon)$ in the mild case.

By Proposition 75 we have

$$\begin{aligned}|\Phi_I\circ \operatorname{Exp}(\xi)| &\leqslant O_\ell(1)\min_{i\in I}\left|\exp\left(1-\xi_i^{-1}\right)\right|\\ &\leqslant O_\ell(1)/\exp(\max_{i\in I}\operatorname{Re}\xi_i^{-1})\leqslant O_\ell(1)/\exp(\max_{i\in I}\operatorname{Re}\mathbf{w}_i^{-1}),\end{aligned}\tag{199}$$

where in the last inequality we used the fact that for $w<\varepsilon^{-1}$, the mapping $\xi\mapsto 1/\xi$ maps the circle $\{|\xi-w|\leqslant\frac12\varepsilon w^2\}$ inside the circle $\{\left|\xi-w^{-1}\right|<2\}$.

Denote $\mathbf{w}_{\min}=\min_{i\in I}\mathbf{w}_i$. Then (198) implies

$$\begin{aligned}\left|(\Phi_I\circ\operatorname{Exp})^{(\boldsymbol\alpha)}(\mathbf{w})\right| &\leqslant \boldsymbol\alpha! O_\ell(1)(2/\varepsilon)^{|\boldsymbol\alpha|}\mathbf{w}_{\min}^{-2|\boldsymbol\alpha|}e^{-\frac{1}{\mathbf{w}_{\min}}}\\ &\leqslant \boldsymbol\alpha! O_\ell\left((2/\varepsilon)^{|\boldsymbol\alpha|}\right)|\boldsymbol\alpha|^{2|\boldsymbol\alpha|}e^{-2|\boldsymbol\alpha|}\end{aligned}\tag{200}$$

where the final estimate is a simple maximization exercise (with the maximum attained at $\mathbf{w}_{\min}=-\frac{1}{2|\boldsymbol\alpha|}$). □

10.4. **Uniform parametrizations in $\mathbb{R}_{\mathbf{an}}$.** In this section we give a proof of Proposition 2. We begin by proving Lemma 3.

Proof of Lemma 3. It is enough to prove that $\log\xi$ has derivative bounded by $4\pi p(p+1)$ in $[-1,1]$. Note that $\log\xi$ is well defined and p-valent in $D(2)$. Let $r:=|(\log\xi)'(z_0)|$ for some $z_0\in D(1)$. By the p-valent analog of the Koebe 1/4-theorem $\log\xi(z_0+D(1))$ contains a disc of radius at least $1/4p\cdot r$, see [24, Theorem 5.1]. On the other hand, it cannot contain a disc of radius larger than $\pi(p+1)$ since then $\xi=e^{\log\xi}$ would not be p-valent. Thus $r\leqslant 4\pi p(p+1)$ as claimed. □

We now proceed to the proof of Proposition 2. As a first step we will reduce to the case of a single complex cell. Let $G_F\subset[0,1]^2\times[0,1]^2$ denote the graph of $F(e,t)$. According to Theorem 30 and Remark 31 there exists a collection of prepared cellular maps $f_j:\mathcal{C}_j^{1/4}\to\mathcal{P}_4^{1/2}$ such that $f_j(\mathbb{R}_+\mathcal{C}_j)$ covers G_F and $f_j(\mathbb{R}_+\mathcal{C}_j^{1/2})\subset G_F$. By Remark 31 we may also assume that f_j is compatible with the function e. Let $\mathcal{C}$ denote one of the cells $\mathcal{C}_j$ and $f:=f_j$. It will be enough to prove that

$$\log[f(\mathbb{R}_+\mathcal{C})\cap\{e=e_0\}]\tag{201}$$

has bounded Euclidean length independent of $e_0\neq 0$.

We first note that since $\mathcal{C}$ maps into a graph, its type must end with $**$. We proceed to consider the first two fiber types. If the type of $\mathcal{C}$ begins with $*$ then $f(\mathbb{R}_+\mathcal{C})$ meets only one line $e=e_0$ and the length of (201) is bounded by the (finite) log-length of the hyperbola $xy=e_0$ in $[0,1]^2$. Similarly if the type of $\mathcal{C}$ begins with D or A then since f_j is compatible with e we conclude that $f_j(\mathbb{R}_+\mathcal{C})$ meets only lines $e=e_0$ with e_0 bounded away from 0, and the same argument holds. The only non-trivial case is therefore when the type of $\mathcal{C}$ begins with $D_\circ$ and f maps this punctured disc to a disc centered around $e=0$.

Let $\mathcal{C}$ be of the form $\mathcal{C}=D_\circ(R)\odot\mathcal{F}\odot*\odot*$. If $\mathcal{F}=*$, then $f(\mathbb{R}_+\mathcal{C})$ meets, for every e_0, only one point in $xy=e_0$. Otherwise we may write f in the form

$$(e,t,x,y)=f(\varepsilon,\tau,*,*)=(\varepsilon^m,\tau^n+\phi(\varepsilon),\xi(\varepsilon,\tau),\eta(\epsilon,\tau)). \tag{202}$$

If $\mathcal{F}=D(r)$, then the functions $\xi(\varepsilon,\tau),\eta(\epsilon,\tau)$ are p-valent in $\tau\in D^{1/2}(r)$, for some p uniform over $\varepsilon\in D_\circ(R)$ by subanalyticity of f restricted to $\mathcal{C}^{1/2}$ (see Proposition 27). The claim then follows from Lemma 3. If $\mathcal{F}=D_\circ(r)$, then by removable singularity theorem both $\xi(\varepsilon,\tau)$ and $\eta(\epsilon,\tau)$ can be extended to $D_\circ(R)\odot D(r)$, with extensions satisfying the same relation $\xi\cdot\eta=\varepsilon^m$ for $\varepsilon\in D_\circ(R)$. In particular, both $\xi(\varepsilon,\tau)$ and $\eta(\epsilon,\tau)$ do not vanish, and we can proceed as above.

The remaining case is $\mathcal{F}=A(r_1,r_2)$. We now make two reductions. First, we replace f with

$$(e,t,x,y)=g(\varepsilon,\tau,*,*)=(\varepsilon^m,\tau^n/r_2^n(\varepsilon),\xi(\varepsilon,\tau),\eta(\varepsilon,\tau)) \tag{203}$$

defined on the same cell. This amounts to an affine transformation of the time variable. Evidently, the image of ξ,τ is unchanged, and the constant M_r in (3) is replaced by $|r_2^{nr}(\varepsilon)|\,M_r$, which is again uniformly bounded in ε. Second, we replace $\mathcal{C}$ with $\tilde{\mathcal{C}}=D_\circ(R)\odot A(r,1)\odot*\odot*$ for $r:=r_1(\varepsilon)/r_2(\varepsilon)$, and replace g with the composition

$$(e,t,x,y)=\tilde g(\varepsilon,\tau,*,*)=g\circ\pi=(\varepsilon^m,\tau^n,\xi(\varepsilon,\tau r_2(\varepsilon)),\eta(\varepsilon,\tau r_2(\varepsilon)))\,, \tag{204}$$

where $\pi:\tilde{\mathcal{C}}\to\mathcal{C}$ is the mapping $(\varepsilon,\tau,*,*)\to(\varepsilon,\tau r_2(\varepsilon),*,*)$. This only amounts to a reparametrization of the set $g(\mathcal{C})$ leaving both the image of ξ,η and the bounds (3) unchanged. Finally by a similar rescaling of ε we may assume that $R=1$. Without loss of generality we return to our original notation, assuming now that $\mathcal{C}=D_\circ(1)\odot A(r,1)\odot*\odot*$ and $f(\varepsilon,\tau,*,*)=(\varepsilon^m,\tau^n,\xi(\varepsilon,\tau),\eta(\varepsilon,\tau))$.

Decompose η as

$$\eta(\varepsilon,\tau)=a(\varepsilon,\tau)+b(\varepsilon,\tfrac{r}{\tau}), \tag{205}$$

where both a,b are holomorphic and bounded in a neighborhood of $\mathcal{P}_2^{1/2}$. We have $\partial_t=\frac{\tau^{1-n}}{n}\partial_\tau$ and a straightforward induction shows that if $\partial_t^r\eta$ is bounded then $\partial_\tau^r\eta$ is bounded as well. Since $\partial_\tau^r a$ is bounded uniformly in ε by the Cauchy estimates for a, we conclude that $\partial_\tau^r b(\varepsilon,\frac{r}{\tau})$ is also bounded uniformly in ε. We claim that this is impossible unless $b\equiv b(\varepsilon)$ or $r\sim 1$. In the former case η itself extends as a holomorphic function to $\mathcal{P}_2^{1/2}$ and we return to the case of functions on discs treated above using Lemma 3. In the latter case, we may (for instance following the proof of the refinement theorem) cover $D_\circ(1)\odot A(r,1)$ by finitely many cells of the type $D_\circ(1)\odot D(1)$ return again to the disc case.

It remains to prove our claim above. Let $v=\frac{r}{\tau}$, so $b(\varepsilon,v)$ is holomorphic and bounded in a neighborhood of $\mathcal{P}^{1/2}$. Suppose

$$r(\varepsilon)=\varepsilon^m(c+\dots), \qquad m>0,c\neq 0 \tag{206}$$

$$b(\varepsilon,v)=\sum_{\ell\geqslant 0}\varepsilon^\ell b_\ell(v), \qquad b_\ell\in\mathcal{O}_b(\mathbb{D}^{1/2}), \tag{207}$$

and let ℓ_0 be the first index such that b_{ℓ_0} is non-constant. A simple computation using $\partial_\tau = -\frac{v^2}{r}\partial_v$ gives

$$\partial_\tau^k b(\varepsilon, v) = \sum_{\ell \geqslant \ell_0 - mk} \varepsilon^\ell \tilde{b}_{\ell,k}(v), \tag{208}$$

where

$$\tilde{b}_{\ell_0 - mk,k} = \left(-\frac{v^2}{c}\partial_v\right)^k b_{\ell_0}(v) \not\equiv 0 \tag{209}$$

and the final inequality follows since b_{ℓ_0} is non-constant, and the operator $(v^2\partial_v)^k$ does not kill any non-constant holomorphic functions as can be seen by examining its action on Taylor expansions. For k such that $\ell_0 - mk < 0$ we see that $\partial_\tau^k b(\varepsilon, v)$ has a pole in $\varepsilon = 0$ for any v such that $\tilde{b}_{\ell_0-mk,k}(v) \neq 0$ and is therefore unbounded as $\varepsilon \to 0$, contradicting our assumption.

Appendix A. Applications in dynamics

YOSEF YOMDIN

A.1. **Non-dynamical parametrization results.** In this section we present a non-dynamical result, based on the refined algebraic lemma. This result reduces the problem of bounding dynamical local entropy (considered in §A.2 below) to a "combinatorial" analysis of the complexity of long compositions of smooth functions. Set $Q := [0, 1]$. Our basic unit of complexity is defined as follows.

Definition 78 (k-complexity unit (k-cu))**.** *A k-complexity unit is a map $\phi : Q^l \to \mathbb{R}^m$ with $\|\phi\|_k \leqslant 1$. We say that $\{\phi_q : Q^l \to \mathbb{R}^m\}$ is a k-cu cover of a set $A \subset \mathbb{R}^m$ if A is contained in the union of the images of ϕ_q.*

Theorem 14 below can be considered as a kind of "Taylor formula" for C^k-parametrizations: it measures the complexity of the graph of a C^k-map in terms of a covering by k-cu's; exactly as in the usual Taylor formula, the estimates depend only on the k-th derivative. A result of this type appears (in a weaker form) in [50, Theorem 2.1]. In [21] it is improved and split into "Main lemma" and "Main corollary". Following [21], we split below Theorem 2.1 of [50] into Theorem 14 and Corollary 79, and incorporate other improvements due to Gromov. Below we denote by G_g the graph of a map g.

Theorem 14. *Let $g : Q^l \to \mathbb{R}^m$ be a C^k-mapping such that $\max_{x \in Q^l} \left\|d^k g\right\| \leqslant M$. Then $G_g \cap Q^{l+m}$ admits a k-cu cover of the form $\{(\phi_q, g \circ \phi_q)\}$ of size $\operatorname{poly}_{l,m}(k) M^{l/k}$.*

Proof. Let $S := \{x \in Q^l : \|g(x)\| \leqslant 1\}$. We subdivide Q^l into sub-cubes Q^l_j of size $\gamma = (kM^{1/k})^{-1}$ and parametrize each Q^l_j by an affine mapping $\eta_j : Q^l \to Q^l_j$. The number N_1 of the sub-cubes is $k^l M^{l/k}$. Since the k-th derivative under this parametrization is multiplied by γ^k, we get for $\theta_j := g \circ \eta_j$ the bound $\left\|d^k \theta_j\right\| \leqslant k^{-k}$.

Let P_j be the Taylor polynomial of θ_j of degree $k-1$ at the center of Q^l. By the Taylor remainder formula we have

$$\left\|d^l \theta_j - d^l P_j\right\| \leq \frac{1}{(k-l)!k^k}, \qquad l = 0, \ldots, k. \tag{210}$$

Put $S_j = \{x \in Q^l : \ ||P_j(x)|| \leqslant \frac{3}{2}\}$. By (210) we have $|\theta_j - P_j| < \frac{1}{2}$, and hence the sets $\eta_j(S_j)$ for $j = 1, \ldots, N_1$ cover S.

In the next step, which is central for all the construction, we apply the refined algebraic lemma to the graph of P_j over S_j with smoothness order k. We get k-cu's $\psi_{ij} : Q^l \to S_j$ for $i = 1, \dots, N_2$ with the images of ψ_{ij} covering S_j, such that $P_j \circ \psi_{ij}$ are also k-cu's. Moreover $N_2 \leqslant \operatorname{poly}_{l,m}(k)$. Now we estimate the derivatives $(\theta_j \circ \psi_{ij})^{(s)}$ for $s = 1, \dots, k$, comparing them with the derivatives of $P_j \circ \psi_{ij}$ via the Faà di Bruno formula:

$$(F \circ G)^{(s)} = \sum_{l=1}^{s} B_s^l(G', G'', \dots, G^{(s-l+1)}) \cdot F^{(l)} \circ G, \tag{211}$$

with B_s^l the so-called Bell polynomials, satisfying in particular

$$B_s := \sum_{l=1}^{s} B_s^l(1, \dots, 1) < s^s. \tag{212}$$

This formula is linear with respect to F, and applying it to $F = \theta_j - P_j$ and $G = \psi_{ij}$ we get for $s = 1, \dots, k$,

$$\begin{aligned} |(\theta_j \circ \psi_{ij})^{(s)} - (P_j \circ \psi_{ij})^{(s)}| = |((\theta_j - P_j) \circ \psi_{ij})^{(s)}| \leq \\ \sum_{l=1}^{s} B_s^l(\psi'_{ij}, \psi''_{ij}, \dots, \psi_{ij}^{(k-l+1)}) \cdot |(\theta_j - P_j)^{(l)} \circ \psi_{ij}| \leqslant \frac{B_s}{k^k} \leqslant 1, \end{aligned} \tag{213}$$

by the bounds (210), and because ψ_{ij} are k-cu's. Since $P_j \circ \psi_{ij}$ is also a k-cu we conclude that $|(\theta_j \circ \psi_{ij})^{(s)}| \leqslant 2$. Therefore, another subdivision of Q^l into sub-cubes of size $\frac{1}{2}$ and the corresponding affine parametrizations of these sub-cubes reduces the derivative bounds to 1. We denote by $\{\psi_{i,j,p}\}$ the collection of $N_1 N_2 2^l = \operatorname{poly}_{l,m}(k) M^{l/k}$ maps obtained in this way. Set $\phi_{i,j,p} := \eta_j \circ \psi_{i,j,p}$. Then $\phi_{i,j,p}$ is a k-cu since $\psi_{i,j,p}$ is, and $g \circ \phi_{i,j,p} = \theta_j \circ \psi_{i,j,k}$ is a k-cu as shown above. $\square$

Next we modify Theorem 14 to make it suitable to apply to n-fold iterations $f^{\circ n} = f \circ f \circ \dots \circ f$. The main reason to separate Theorem 14 from Corollary 79 below is that in the latter it is not enough to assume that $||d^k g|| \leqslant M$ only for the highest derivative: we need $||d^s g|| \leqslant M$ for all $s = 1, \dots, k$.

Corollary 79. *Let $g : Q^l \to \mathbb{R}^m$ be a C^k-mapping such that $\max_{x \in B'} \|d^s g\| \leqslant M$ for $s = 1, \dots, k$. Let $\sigma : Q^l \to Q^l$ be a k-cu and put $h = g \circ \sigma$. Then $G_h \cap Q^{l+m}$ admits a k-cu cover of the form $\{(\phi_q, h \circ \phi_q)\}$ of size $\operatorname{poly}_{l,m}(k) M^{l/k}$.*

Proof. By the Faà di Bruno formula (211) we have for the k-th derivative of h

$$|h^{(k)}| = |(g \circ \sigma)^{(k)}| = |\sum_{l=1}^{k} B_k^l(\sigma', \sigma'', \dots, \sigma^{(k-l+1)}) \cdot g^{(l)} \circ \sigma| < k^k M. \tag{214}$$

Now we apply Theorem 14 to h, and get the required covering by k-cu's. $\square$

A.2. **Applications to volume growth and entropy.** Let Y be a compact m-dimensional C^∞-smooth Riemannian manifold, and let $f : Y \to Y$ be a smooth (at least $C^{1+\mu}$, with $\mu > 0$) mapping. The topological entropy $h(f)$ was defined in §1.3. Below we consider another "entropy-type" dynamical invariant called the *volume growth.*

A.2.1. *Volume growth.* For $\sigma^l : Q^l \to Y$ a C^k-mapping, and for $\mathcal{S} \subset Q^l$ a measurable subset, let $v(\sigma^l, \mathcal{S})$ denote the l-dimensional volume of the image of $\sigma^l(\mathcal{S})$ in Y. Put $v(f, \sigma^l, \mathcal{S}, n) = v(f^{\circ n} \circ \sigma^l, \mathcal{S})$, and put

$$v(f, \sigma^l, \mathcal{S}) = \limsup_{n\to\infty} \frac{1}{n} \log v(f, \sigma^l, \mathcal{S}, n), \qquad v(f, \sigma^l) = v(f, \sigma^l, Q^l). \tag{215}$$

Finally, we put

$$v_{l,k}(f) = \sup_{\sigma^l} v(f, \sigma^l), \qquad v_k(f) = \max_l v_{l,k}(f), \qquad v(f) = v_\infty \tag{216}$$

where the supremum is taken over all C^k maps $\sigma^l : Q^l \to Y$. An important inequality obtained by Newhouse in [34] connects the volume growth to the topological entropy,

$$h(f) \leqslant v(f). \tag{217}$$

The inverse inequality is not always true, but the question of its validity is important in many dynamical applications. In particular, as it was explained in §1.3.3, this inverse inequality implies Shub's entropy conjecture. In order to analyze its validity we define a δ-local volume growth at $x \in Y$ as

$$v_{l,k}(f, \delta, x) = \sup_{\sigma^l} \ v(f, \sigma^l, B^n_\delta(x)) = \sup_{\sigma^l} \limsup_{n\to\infty} \ \frac{1}{n} \log v(f, \sigma^l, B^n_\delta(x), n) \tag{218}$$

where $B^n_\delta(x)$ is the δ-ball around x in the n-th iterated metric d_n. The quantity $v_{l,k}(f, \delta, x)$ measures the exponential rate of volume growth of the part of the images of $f^{\circ n} \circ \sigma^l$ which always stay at a distance at most δ from the orbit of $f^{\circ n}(x)$ for $n \to \infty$. Finally, we put

$$v_{l,k}(f, \delta) = \sup_{x\in Y} v_{l,k}(f, \delta, x), \qquad v^0_{l,k}(f) = \lim_{\delta\to 0} v_{l,k}(f, \delta). \tag{219}$$

The δ-tail entropy can be also be defined in an analogous manner setting $h^*(f, \delta, x) = h(f, B^n_\delta(x))$ and $h^*(f, \delta) := \sup_{x\in Y} h^*(f, \delta, x)$. There is a version of Newhouse's inequality (217) for these invariants.

A.2.2. *Local volume growth and entropy.* The following partial inverse to (217) is almost immediate:

$$v_{l,k}(f) \leqslant h(f, \delta) + v_{l,k}(f, \delta). \tag{220}$$

Indeed, $h(f, \delta)$ counts the minimal number of the balls $B^n_\delta(x)$ covering Y, while $v_{l,k}(f, \delta)$ bounds the volume growth on each of these balls. As $\delta \to 0$ we obtain the bound

$$v_{l,k}(f) \leqslant h(f) + v^0_{l,k}(f). \tag{221}$$

Thus the inverse to the Newhouse inequality is satisfied if $v^0_{l,k}(f) = 0$. Our first goal is to establish the following result.

Theorem 15 ([50])**.** *If $f : Y \to Y$ is C^∞-smooth then $v^0_{l,k}(f) = 0$.*

A.2.3. *Local C^k-complexity growth.* From now on we concentrate on bounding from above the local volume growth $v_{k,l}(f, \delta, x)$ for a given $x \in Y$. For this purpose we define yet another entropy-like invariant $h_{l,k}(f, \delta, x)$, which measures the "local C^k-complexity growth" in a δ-neighborhood of the orbit of x (cf. [21, 51]):

$$h_{l,k}(f, \delta, x) = \sup_{\sigma^l} \limsup_{n\to\infty} \frac{1}{n} \log N_k(f, \sigma^l, \delta, x, n), \tag{222}$$

where $\sigma^l : Q^l \to Y$ varies over all C^k-smooth mappings and $N_k(f, \sigma^l, \delta, x, n)$ is the minimal number of k-cu's $\psi_j : Q^l \to Q^l$, whose images cover $(\sigma^l)^{-1}(B^n_\delta(x))$, and for which $f^{\circ n} \circ \sigma^l \circ \psi_j$ are k-cu's.

An almost immediate fact is that

$$v_{l,k}(f, \delta, x) \leqslant h_{l,k}(f, \delta, x). \tag{223}$$

Indeed, since $f^{\circ n} \circ \sigma^l \circ \psi_j$ are k-cu's, the l-volume of their image does not exceed a certain constant depending only on l, m and hence

$$v_{l,k}(f, \sigma^l, B^n_\delta(x), n) \leqslant O_m(N_k(f, \sigma^l, \delta, x, n)) \tag{224}$$

As $n \to \infty$ in (215) and (222) the asymptotic constant disappears. Thus it remains to bound from above $h_{l,k}(f, \delta, x)$.

Below the derivatives of $f : Y \to Y$, and their norms, are defined via the local coordinate charts on Y.

Proposition 80. *Assume that for certain positive constants L, M the inequalities*

$$\|df\| \leqslant L, \qquad \max_{s=2,\dots,k} \|d^s f\|^{\frac{1}{s-1}} \leqslant M \tag{225}$$

are satisfied. Then for each $x \in Y$ and for each $\delta \leqslant M^{-1}$ we have

$$h_{l,k}(f, \delta, x) \leqslant \frac{l \log L}{k} + \log \operatorname{poly}_m(k). \tag{226}$$

Proof. Using the local coordinate charts at the points $x, f(x), f^{\circ 2}(x), \dots$ of the orbit of x under f, we replace iterations of f with compositions of a non-autonomous sequence F of mappings $f_i : B^m_1 \to \mathbb{R}^m$ with $f_i(0) = 0$, where B^m_1 is the unit ball in $\mathbb{R}^m$, centered at the origin.

Fix $\delta \leqslant M^{-1}$, and consider the concentric ball $B^m_\delta \subset B^m_1$. Let $\eta : B^m_1 \to B^m_\delta$ be a linear contraction $\eta(x) = \delta x$. We consider instead of the sequence F of mappings f_i a sequence $\bar F$ of δ-rescaled mappings $\bar f_i = \eta^{-1} \circ f_i \circ \eta : B^m_1 \to \mathbb{R}^m$. For the derivatives we have $d^s \bar f_i = \delta^{s-1} d^s f_i$, and by the assumptions we get

$$||d\bar f_i|| \leqslant L, \ \max_{s=2,\dots,k} ||d^s \bar f_i|| \leqslant 1. \tag{227}$$

Thus δ-rescaling "kills" all the derivatives of f_i, starting with the second one. The first derivative (having a dynamical meaning) does not change.

We have to estimate the number $\bar N_{k,n} := N_k(\bar F, \sigma^l, 1, 0, n)$, which, by an evident change of notations, is the minimal number of k-cu's $\psi_j : Q^l \to Q^l$, whose images cover $(\sigma^l)^{-1}(\bar B^n_1(0))$, and for which $\bar f_n \circ \bar f_{n-1} \circ \dots \circ \bar f_1 \circ \sigma^l \circ \psi_j$ are k-cu's.

We will prove via induction by n that

$$\bar N_{k,n} \leq L^{\frac{nl}{k}} C^n \tag{228}$$

where $C = \operatorname{poly}_m(k)$ is the constant from Corollary 79. Assume that (228) is valid for n. To prove it for $n+1$ we use Corollary 79. We apply it to $g = \bar f_{n+1}$ and to each k-cu ν of the form

$$\nu = \bar f_n \circ \bar f_{n-1} \circ \dots \circ \bar f_1 \circ \sigma^l \circ \psi_j, \tag{229}$$

obtained in the n-th step. For each ν we get at most $N = CL^{\frac{l}{k}}$ new k-cu's $\phi_q : Q^l \to Q^l$ whose images cover $(\sigma^l)^{-1}(\bar B^{n+1}_1(0))$, and for which the compositions

$\bar{f}_{n+1} \circ \nu \circ \phi_q$ are also k-cu's. By the induction assumption, the number $\bar{N}_{k,n}$ of the k-cu's ν is at most $L^{\frac{nl}{k}} C^n$. Therefore

$$\bar{N}_{k,n+1} \leqslant \bar{N}_{k,n} N \leqslant \bar{N}_{k,n} C L^{\frac{l}{k}} \leqslant L^{\frac{(n+1)l}{k}} C^{n+1}. \tag{230}$$

This completes the induction, proving (228) for $n \in \mathbb{N}$. Taking log and dividing by n finishes the proof. □

To finish the proof of Theorem 15 we should eliminate the extra term $\log \operatorname{poly}_m(k)$ in (226). We do this by replacing f by its iterate $f^{\circ q}$ for some sufficiently large q.

Let f be as in Proposition 80. For each $q \in \mathbb{N}$ the first derivative of $f^{\circ q}$ is bounded by L^q. Denote by M_q a constant satisfying

$$\max_{s=2,\dots,k} ||d^s f^q||^{\frac{1}{s-1}} \leqslant M_q. \tag{231}$$

The constants M_q can be estimated from L, M using Faà di Bruno formula (see, e.g. [14]). We do not provide here an explicit expression, since below, for analytic f, we get it in a much easier way. Using the equality $h_{l,k}(f^{\circ q}, \delta, x) = q h_{l,k}(f, \delta, x)$ and applying Proposition 80 to $f^{\circ q}$ we obtain the following result, which contains Theorem 15.

Corollary 81. *For each $\delta \leqslant 1/M_q$ we have*

$$h_{l,k}(f, \delta, x) \leqslant \frac{l \log L}{k} + \frac{\log \operatorname{poly}_m(k)}{q}. \tag{232}$$

In particular, for $q \to \infty$ and $\delta \to 0$,

$$v^0_{l,k}(x) \leqslant h^0_{l,k}(x) \leqslant \frac{l \log L}{k}, \qquad v^0_{l,\infty}(x) = h^0_{l,\infty}(x) = 0. \tag{233}$$

A.2.4. *Volume growth for analytic maps.* Now we show, following [14] how the Corollary 81 can be used to provide an explicit bound for the local volume growth and local entropy. Setting $q(k) = O_m(1) \cdot \frac{k \log k}{l \log L}$ we immediately obtain the following.

Corollary 82. *For each $\delta \leq \delta(k) = \frac{1}{M_{q(k)}}$ we have*

$$h_{l,k}(f, \delta, x) \leqslant \frac{2l \log L}{k}. \tag{234}$$

For C^∞ functions f Corollary 82 reduces the problem of estimating the asymptotic behavior of $h_{l,k}(f, \delta)$ as $\delta \to 0$ to the problem of estimating the high-order derivatives of $f^{\circ q(k)}$. We find it explicitly for analytic f, referring the reader to [14], where a much more general setting is presented.

Let $Y \subset \mathbb{R}^p$ be a compact m-dimensional real analytic submanifold, and let $f : Y \to Y$ be a real analytic mapping. We assume that f is extendible to a complex analytic mapping $\tilde{f} : U \to \mathbb{C}^p$ in a neighborhood U of Y in $\mathbb{C}^p$ of size ρ. More accurately, for each point $x \in Y$ the complex polydisc at x of radius ρ is contained in U. We assume that the norm $||\tilde{f}||$ is bounded by D in U, and the norm of its first derivative $||d\tilde{f}||$ is bounded there by L. The following theorem immediately implies the statement about volume growth in Theorem 5 since the volume of the image of a k-cu is universally bounded; the statement about tail entropy follows similarly but we omit the details.

Theorem 16. *For f as above, for each $x \in Y$, and for all k sufficiently big we have*

$$h_{l,k}(f,\delta,x) \leqslant O_m(\log L) \cdot \frac{\log|\log\delta|}{|\log\delta|}. \tag{235}$$

Proof. Since, by the assumptions, $||d\tilde{f}|| \leqslant L$, this complex mapping can expand distances at most L times. Therefore, the q-th iterate $f^{\circ q}$ is extendible to at least a complex neighborhood U_q of Y of the size $\frac{\rho}{L^q}$, and it is bounded there by D. Applying Cauchy formula, we conclude that for each $x \in Y$ we have

$$\left\| d^s \tilde{f}^{\circ q} \right\| \leqslant D s! \left(\frac{L^q}{\rho} \right)^s, \qquad s = 0, 1, 2, \dots \tag{236}$$

Therefore

$$\left\| d^s \tilde{f}^{\circ q} \right\|^{1/s} \leqslant D^{1/s} \cdot s \cdot \frac{L^q}{\rho}, \qquad s = 0, 1, 2, \dots \tag{237}$$

In particular, for $s = 0, 1, 2, \dots, k$ we have

$$\left\| d^s \tilde{f}^{\circ q} \right\|^{1/s} \leqslant \frac{DkL^q}{\rho} = 2^{O_m(k \log k)} \tag{238}$$

for k sufficiently big. Therefore $M_{q(k)} = 2^{O_m(k \log k)}$ (note that the difference between the $1/s$ and $1/(s-1)$ power is absorbed into the asymptotic constant), and Corollary 82 applies with $\delta = 2^{-O_m(k \log k)}$. Solving for fixed δ, we see that Corollary 82 applies with

$$k = \Omega_m \left(\frac{|\log\delta|}{\log|\log\delta|} \right). \tag{239}$$

With this k, Corollary 82 gives

$$h_{l,k}(f,\delta,x) \leqslant \frac{2l \log L}{k} = O_m(\log L) \cdot \frac{\log|\log\delta|}{|\log\delta|}. \tag{240}$$

as claimed. □

Appendix B. Applications in diophantine geometry

For $p \in \mathbb{P}^\ell(\mathbb{Q})$ we define $H(p)$ to be $\max_i |\mathbf{p}_i|$ where $\mathbf{p} \in \mathbb{Z}^{\ell+1}$ is a projective representative of p with $\gcd(\mathbf{p}_0, \dots, \mathbf{p}_\ell) = 1$. For $\mathbf{x} \in \mathbb{A}^\ell(\mathbb{Q})$ we define its height to be the height of $\iota(\mathbf{x})$ for the standard embedding

$$\iota : \mathbb{A}^\ell \to \mathbb{P}^\ell, \qquad \iota(\mathbf{x}_{1..\ell}) = (1 : \mathbf{x}_1 : \cdots : \mathbf{x}_\ell). \tag{241}$$

For a set $X \subset \mathbb{P}^\ell(\mathbb{R})$ we denote

$$X(\mathbb{Q}, H) := \{\mathbf{x} \in X \cap \mathbb{P}(\mathbb{Q})^\ell : H(x) \leqslant H\}, \tag{242}$$

and similarly for $X \subset \mathbb{R}^\ell$.

In [9] Bombieri and Pila introduced a method for studying the quantity $\#X(\mathbb{Q}, H)$ as a function of H when X is the graph of a C^r (or C^∞) smooth function $f : [0,1] \to \mathbb{R}$. It turns out that two very different asymptotic behaviors are obtained depending on whether the graph of f belongs to an algebraic plane curve. The Yomdin-Gromov algebraic lemma has been used in both of these directions, to generalize from graphs of functions to more general sets. In §B.1 we give a complex-cellular analog of the Bombieri-Pila determinant method. We then present an application in the algebraic context in §B.2 and in the transcendental context in §B.3.

B.1. **The Bombieri-Pila method for complex cells.** The Bombieri-Pila method is based on a clever estimate for certain *interpolation determinants* of a collection of functions, which is obtained using Taylor expansions. In this section we develop a parallel theory over complex cells, replacing Taylor expansions by Laurent expansions.

B.1.1. *Interpolation determinants.* Let $\mu \in \mathbb{N}$. Let $\mathbf{f} := (f_1, \dots, f_\mu)$ be a collection of functions with a common domain U and $\mathbf{p} := (p_1, \dots, p_\mu) \in U^\mu$ a collection of points. For minor technical simplifications we will assume that $f_1 \equiv 1$.

We define the *interpolation determinant*

$$\Delta(\mathbf{f}; \mathbf{p}) := \det(f_i(p_j))_{1 \leqslant i,j \leqslant \mu}. \tag{243}$$

The key to the Bombieri-Pila [9] is an estimate for the interpolation determinant assuming that $\mathbf{f}$ are sufficiently smooth and $\mathbf{p}$ are contained in a sufficiently small ball. In our complex analytic analog the small ball will be replaced by a cell $\mathcal{C}$ with a $\boldsymbol{\delta}$-extension and the smoothness assumption will be replaced by boundedness in $\mathcal{C}^{\boldsymbol{\delta}}$.

Let $\mathcal{C}$ be a complex cell and let $m \leqslant \dim \mathcal{C}$ denote the number of D-fibers. Set

$$E = E(m) := \frac{m}{m+1}(m!)^{1/m}. \tag{244}$$

Fix $0 < \delta < 1/2$, and let $\boldsymbol{\delta}$ be given by δ in the D-coordinates and $\delta^{E\mu^{1+1/m}}$ in the $A, D_\circ$ coordinates.

Lemma 83. *With $\mathcal{C}^{\boldsymbol{\delta}}$ as above, suppose $f_1, \dots, f_\mu \in \mathcal{O}_b(\mathcal{C}^{\boldsymbol{\delta}})$ with $\operatorname{diam}(f_i(\mathcal{C}^{\boldsymbol{\delta}}), \mathbb{C}) \leqslant M$ and $p_1, \dots, p_\mu \in \mathcal{C}$. Then*

$$|\Delta(\mathbf{f}; \mathbf{p})| \leqslant M^\mu \mu^{O_\ell(\mu)} \delta^{E\mu^{1+1/m} + O_\ell(\mu)}. \tag{245}$$

When $m = 0$ the bound above holds[4] *if we replace $m = 0$ by $m = 1$.*

Proof. Since $f_1 \equiv 1$ the interpolation determinant is unchanged if we replace each f_j by $f_j - \text{const}$ for $j \geqslant 2$. In this way we may assume without loss of generality that $\|f_i\|_{\mathcal{C}^{\boldsymbol{\delta}}} \leqslant M$. We may also assume by rescaling that $M = 1$.

Write a Laurent expansion

$$f_i(\mathbf{z}) = \left(\sum_{\boldsymbol{\alpha} \in \mathbb{Z}^\ell, |\boldsymbol{\alpha}| < E\mu^{1+1/m}} c_{i,\boldsymbol{\alpha}} \mathbf{z}^{[\boldsymbol{\alpha}]} \right) + R_i(\mathbf{z}) \tag{246}$$

and note that by Proposition 25 we have $|c_{i,\boldsymbol{\alpha}}| < \boldsymbol{\delta}^{\operatorname{pos} \boldsymbol{\alpha}}$ and

$$\|R_i\|_{\mathcal{C}} \leqslant \sum_{|\boldsymbol{\alpha}| \geqslant E\mu^{1+1/m}} \boldsymbol{\delta}^{\operatorname{pos} \boldsymbol{\alpha}} \leqslant \sum_{k \geqslant E\mu^{1+1/m}} (2k+1)^\ell \delta^k = O_\ell(\mu^{2\ell}) \delta^{E\mu^{1+1/m}} \tag{247}$$

where the final two estimates are elementary and left to the reader. We expand the determinant $\Delta(\mathbf{f}, \mathbf{p})$ multilinearly into $\mu^{O_\ell(\mu)}$ determinants Δ_I where each f_i is replaced by one of the summands in (246). We may consider only those Δ_I where each normalized monomial appears at most once: otherwise the determinant has two identical columns. We claim that each such Δ_I is majorated in $\mathcal{C}$ by $\mu^{O_\ell(\mu)} \delta^{E\mu^{1+1/m}}$.

Consider first the case that Δ_I contains a residue R_i or one of the coefficients $c_{i,\boldsymbol{\alpha}}$ where $\boldsymbol{\alpha}$ contains a non-vanishing $A, D_\circ$ coordinate. Expand Δ_I into $\mu!$ summands.

[4] In fact one can easily derive better estimates in this case but this is not needed in this paper.

Each summand contains either a term $R_i(p_j)$ or $c_{i,|\boldsymbol{\alpha}|}$ as above, and the estimate then follows from (247) or from our assumption on $\boldsymbol{\delta}$.

The remaining determinants Δ_I involve only the m variables of type D, which we assume for simplicity of the notation are given by $\mathbf{z}_{1..m}$. We expand Δ_I into a $\mu!$ summands. Each summand is a product

$$c_{1,\boldsymbol{\alpha}^1}\mathbf{z}^{[\boldsymbol{\alpha}^1]}\cdots c_{\mu,\boldsymbol{\alpha}^\mu}\mathbf{z}^{[\boldsymbol{\alpha}^\mu]}, \qquad \boldsymbol{\alpha}^1,\dots,\boldsymbol{\alpha}^\mu\in\mathbb{N}^m \tag{248}$$

with distinct $\boldsymbol{\alpha}^j$. By the estimate on the Laurent coefficients this summand is bounded by $\delta^{\sum_{i,j}\boldsymbol{\alpha}^j_i}$. Let $L_m(k)$ denote the number of monomials of degree k in m variables. The minimal term δ^q will clearly be obtained if we choose $L_m(0)$ of the $\boldsymbol{\alpha}^j$s of degree 0, $L_m(1)$ of degree 1 and so on. Let ν be the largest integer satisfying $\sum_{k=0}^{\nu}L_m(k)\leqslant\mu$. Then $q\geqslant\sum_{k=0}^{\nu}L_m(k)\cdot k$. Simple computations give

$$\begin{aligned} L_m(k)&=\frac{k^{m-1}}{(m-1)!}+O_m(k^{m-2}) \qquad \sum_{j=0}^{k}L_m(j)=\frac{k^m}{m!}+O_m(k^{m-1})\\ \mu&=\frac{\nu^m}{m!}+O(\nu^{m-1}) \qquad \nu=(m!\mu)^{1/m}+O(1)\end{aligned} \tag{249}$$

and finally $q=E\mu^{1+1/m}+O(\mu)$ as claimed. $\square$

Lemma 83 gives essentially the same estimate as the classical Bombieri-Pila method, *but with dimension m instead of* $\dim\mathcal{C}$. The key point is that for $\boldsymbol{\delta}$ corresponding to $\{\rho\}$-extensions with $\rho\ll 1$ the $D_\circ, A$-coordinates have order $2^{-1/\rho}$ (compared with order ρ for the D coordinates), which allows us to produce cellular covers satisfying the conditions of Lemma 83. Roughly one may say that the punctured discs and annuli that we produce are so thin that they may be ignored for the purposes of the Bombieri-Pila method. More accurately we have the following version of the refinement theorem

Lemma 84. *Let $\mathcal{C}^{1/2}$ be a (real) complex cell with m coordinates of type D. Let $\mu\in\mathbb{N}$ and $0<\delta<1$, and define $\boldsymbol{\delta}$ as in Lemma 83. Then there exists a (real) cellular cover $\{f_j:\mathcal{C}_j^{\boldsymbol{\delta}}\to\mathcal{C}^{1/2}\}$ of size*

$$N=\begin{cases}\operatorname{poly}_\ell(\mu\log(1/\delta))\cdot\delta^{-2m} & \textit{for complex covers}\\ \operatorname{poly}_\ell(\mu\log(1/\delta))\cdot\delta^{-m} & \textit{for real covers}\end{cases} \tag{250}$$

where each f_j is a cellular translate map.

If $\mathcal{C}$ varies in a definable family (and σ,ρ vary under the condition $0<\sigma<\rho<\infty$) then the cells $\mathcal{C}_j$ and maps f_j can also be chosen from a single definable family. If $\mathcal{C}$ is algebraic of complexity β then $\mathcal{C}_j, f_j$ are algebraic of complexity $\operatorname{poly}_\ell(\beta)$.

Proof. In the $D_\circ, A$ coordinates $\boldsymbol{\delta}$ has value $\delta^{E\mu^{1+1/m}}$. We begin by constructing, using the refinement theorem, a cellular cover $\{g_j:\mathcal{C}_j^{\{\rho\}}\to\mathcal{C}^{1/2}\}$ with $2^{-1/\rho}=O_\ell(\delta^{E\mu^{1+1/m}})$. The size of this cover is $\operatorname{poly}_\ell(\mu\log\delta)$. We now fix some $g=g_j$. It will be enough to construct a cellular cover $\{f_j:\mathcal{C}_j^{\boldsymbol{\delta}}\to\mathcal{C}_j^{\{\rho\}}\}$ of size $O_\ell(\delta^{-m})$ where each f_j is as in the conclusion of the lemma. Since each $D_\circ, A$ coordinate already has the desired extension, we need only refine the D fibers to achieve a δ-extension. This is elementary: after rescaling each D-fiber to $D(1)$ we simply cover $D(1)$ (resp. $(-1,1)$) by δ^2 (resp. δ) discs (resp. real discs) of radius δ. $\square$

B.1.2. *Polynomial interpolation determinants.* Let $\Lambda \subset \mathbb{N}^\ell$ be a finite set and set $d := \max_{\boldsymbol{\alpha}\in\Lambda} |\boldsymbol{\alpha}|$ and $\mu := \#\Lambda$. For minor technical simplifications we assume $\mathbf{0} \in \Delta$. Let $\mathbf{f} := (f_1, \ldots, f_\ell)$ be a collection of functions with a common domain U and $\mathbf{p} := (p_1, \ldots, p_\mu) \in U^\mu$ a collection of points. We define the *polynomial interpolation determinant*

$$\Delta^\Lambda(\mathbf{f}; \mathbf{p}) := \Delta(\mathbf{g}; \mathbf{p}), \qquad \mathbf{g} := (\mathbf{f}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \Lambda). \tag{251}$$

Basic linear algebra shows that for $S \subset U$ the set $\mathbf{f}(S) \subset \mathbb{C}^\ell$ belongs to a hypersurface $\{P = 0\}$ with $\operatorname{supp} P \subset \Lambda$ if and only if $\Delta^\Lambda(\mathbf{f}; \mathbf{p})$ vanishes for any $\mathbf{p} \subset S$ of size μ.

The following is due to Bombieri and Pila [9].

Lemma 85. *Let $H \in \mathbb{N}$ and suppose $H(\mathbf{f}(\mathbf{p}_j)) \leqslant H$ for any $i = 1, \ldots, \ell$ and $j = 1, \ldots, \mu$. Then $\Delta^\Lambda(\mathbf{f}; \mathbf{p})$ either vanishes or satisfies*

$$|\Delta^{\mathbf{d}}(\mathbf{f}; \mathbf{p})| \geqslant H^{-\mu d}. \tag{252}$$

Proof. Let $Q_j \leqslant H$ denote the common denominator of $\mathbf{f}(\mathbf{p}_j)$. The row corresponding to $\mathbf{p}_j$ in $\Delta^\Lambda(\mathbf{f}; \mathbf{p})$ consists of rational numbers with common denominator dividing Q_j^d. Factoring this common denominator from each row we obtain a matrix with integer entries whose determinant is either vanishing or an integer. In the latter case

$$|\Delta^\Lambda(\mathbf{f}; \mathbf{p})| \geqslant \prod_{j=1}^{\mu} Q_j^{-d} \geqslant H^{-\mu d}. \tag{253}$$

□

B.2. **Rational points on algebraic hypersurfaces.**

B.2.1. *Previous work.* Let $X \subset \mathbb{P}^2(\mathbb{C})$ be an irreducible curve of degree d. Let $f : I \to X$ be a C^r-smooth function with $\|f\|_r \leqslant 1$ and write $X_f := f(I)$. Pila [36] proves, using a generalization of the method of [9], that for a sufficiently large $r = r(d)$ one has $\#X_f(\mathbb{Q}, H) = O_{\varepsilon,d}(H^{2/d+\varepsilon})$ for any $\varepsilon > 0$.

If one allows $r = r(d, H)$, for instance if f is C^∞ smooth, then the Bombieri-Pila method can be pushed further to replace $H^{2/d+\varepsilon}$ by $H^{2/d} \log^k H$ for some $k > 0$. The study of $\#X(\mathbb{Q}, H)$ is therefore directly related to the C^r parameterization complexity of X in the sense of the Yomdin-Gromov algebraic lemma. Using an explicit parameterization construction Pila [40] proved that $\#X(\mathbb{Q}, H) \leqslant (6d)^{10} 4^d H^{2/d} (\log H)^5$.

In [25] Heath-Brown has developed a non-Archimedean version of the Bombieri-Pila method and used it to derive the estimate $\#X(\mathbb{Q}, H) = O_{\varepsilon,d}(H^{2/d+\varepsilon})$. This was later improved (in arbitrary dimension ℓ) by Salberger [45] to $O_{\ell,d}(H^{2/d} \log H)$ and very recently by Walsh [48] to $O_{\ell,d}(H^{2/d})$.

For a general irreducible variety $X \subset \mathbb{P}(\mathbb{C})^\ell$ of dimension m, Broberg [12, Theorem 1] (generalizing a result of Heath-Brown [25]) has proved that $X(\mathbb{Q}, H)$ is contained in $O_{\ell,d,\varepsilon}(H^{(m+1)d^{-1/m}+\varepsilon})$ hypersurfaces $\{P_i = 0\}$, which have degrees $O_{\ell,d,\varepsilon}(1)$ and which do not contain X. Intersecting X with each of the hypersurfaces one obtains varieties of dimension $m - 1$, and could potentially proceed by induction to eventually obtain estimates on $\#X(\mathbb{Q}, H)$. Marmon [30] has recovered this result using the Bombieri-Pila method by appealing to the Yomdin-Gromov algebraic lemma.

B.2.2. *Proof of Theorem 6.* In this section we will prove Theorem 6 using the complex-cellular version of the Bombieri-Pila method. We remark that a result similar to Theorem 6 could probably also be proved by following the argument of [30] and replacing the Yomdin-Gromov algebraic lemma by our refined version.

We will work in the affine setting. Let ι_j denote the j-th standard chart $\mathbf{x}_{1..\ell} \to (\cdots : \mathbf{x}_j : 1 : \mathbf{x}_{j+1} : \cdots)$. If $\mathbf{z} = (\mathbf{z}_0 : \cdots : \mathbf{z}_\ell) \in \mathbb{P}^\ell(\mathbb{C})$ with $|\mathbf{z}_j|$ maximal then in the ι_j-chart $\mathbf{z}$ corresponds to a point $\mathbf{x} \in \bar{\mathcal{P}}_\ell$, and moreover $H(\mathbf{z}) = H(\mathbf{x})$. We assume without loss of generality that $j = 0$. Below $\mathbf{x}$ denotes the affine coordinates in this chart. It will suffice to count the points of height H in $\bar{\mathcal{P}}_\ell \cap \iota_0^{-1}(X)$.

We fix some degree-lexicographic monomial ordering on $\mathbb{C}[\mathbf{x}_{1..\ell}]$ and let $\Lambda := \mathbb{N}^\ell \setminus \mathrm{LT}(I)$ where $I \subset \mathbb{C}[\mathbf{x}_{1..\ell}]$ denotes the ideal of X and $\mathrm{LT}(I)$ the set of leading terms of I with respect to the fixed ordering. Set $\Lambda(k) := \Lambda \cap \{|\boldsymbol{\alpha}| \leqslant k\}$. Then $\mu(k) := \#\Lambda(k)$ is the Hilbert function of X. This function eventually equals the Hilbert polynomial, and by Nesterenko's estimate [33] we have

$$\mu(k) \geqslant \binom{k+m+1}{m+1} - \binom{k+m+1-d}{m+1} = dk^m/m! + \mathrm{poly}_m(d)k^{m-1}. \tag{254}$$

We will apply the Bombieri-Pila method with $\Lambda(k)$ interpolation determinants.

Let $\mathcal{C} \subset \mathcal{P}_\ell^{1/2}$ be a complex cell as in Lemma 83 and let $\mathbf{p} \subset \mathcal{C}(\mathbb{Q}, H)$. Then by Lemmas 83 and 85 we have

$$H^{-\mu k} \leqslant |\Delta^{\Lambda(k)}(\mathbf{x}; \mathbf{p})| \leqslant 2^\mu \mu^{O_\ell(\mu)} \delta^{E\mu^{1+1/m} + O_\ell(\mu)} \tag{255}$$

unless $\Delta^{\Lambda(k)}(\mathbf{x}; \mathbf{p}) = 0$. Note that in the right hand side of (255), Lemma 83 gives the estimate with m equal to the number of disc fibers in $\mathcal{C}$ (which is at most $\dim \mathcal{C}$), and this implies the same estimate with $m = \dim X$. Thus unless $\Delta^{\Lambda(k)}(\mathbf{x}; \mathbf{p}) = 0$ we have, for $k > \mathrm{poly}_m(d)$,

$$\begin{gathered}\log \delta > \frac{-O_\ell(\log \mu) - k \log H}{E\mu^{1/m} + O_\ell(1)} = \frac{-k \log H - O_\ell(1)}{E\mu^{1/m} + O_\ell(1)} = \\ \frac{m+1}{m} \cdot \frac{-k \log H - O_\ell(1)}{kd^{1/m}(1 + \mathrm{poly}_m(d)/k)^{1/m}} = -\frac{m+1}{m} \cdot \frac{\log H}{d^{1/m}}(1 + \mathrm{poly}_\ell(d)/k).\end{gathered} \tag{256}$$

Therefore, for

$$\delta = H^{-\frac{m+1}{m} d^{-1/m}(1+\mathrm{poly}_\ell(d)/k)} \tag{257}$$

we have $\Delta^{\Lambda(k)}(\mathbf{x}; \mathbf{p}) = 0$ for any $\mathbf{p}$ as above, and $\mathcal{C}(\mathbb{Q}, H)$ is therefore contained in a hypersurface $\{P = 0\}$ satisfying $\operatorname{supp} P \subset \Lambda(k)$. In particular P is of degree at most k and does not vanish identically on X. It remains to show that $\mathbb{R}\mathcal{P}_\ell^{1/2} \cap \iota_0^{-1}(X)$ can be covered by $\delta^{-m} \operatorname{poly}(\mu \log(1/\delta))$ real cells satisfying the conditions of Lemma 83. For this we first use the CPT to construct a real cover of $\mathbb{R}(\mathcal{P}_\ell^{1/2} \cap \iota_0^{-1}(X))$ of size $\mathrm{poly}_\ell(\beta)$ admitting 1/2-extensions, and then apply Lemma 84 to further refine each of the cells to satisfy the conditions of Lemma 83.

B.3. **Rational points on transcendental sets.**

B.3.1. *Log-sets in diophantine applications of the Pila-Wilkie theorem.* In some of the most remarkable applications of the Pila-Wilkie theorem, particularly those related to modular curves and more general Shimura varieties, it is necessary to count rational points on sets that are definable in $\mathbb{R}_{\mathrm{an,exp}}$ but *not* in $\mathbb{R}_{\mathrm{an}}$. We briefly recall the most standard example coming from the universal covers of modular curves.

Recall that the universal covering map $j : \mathbb{H} \to \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \simeq \mathbb{C}$ can be factored as a composition $j(\tau) = J(q)$ where $q = e^{2\pi i\tau}$ and $J(q) : D_\circ(1) \to \mathbb{C}$ is meromorphic in a neighborhood of $q = 0$. We extend j, J as a function of several variables coordinate-wise, and denote by $\Omega \subset \mathbb{H}^n$ the (coordinate-wise) standard fundamental domain for j. In [43] a key step is the application of the Pila-Wilkie theorem to sets of the form $Y := \Omega \cap j^{-1}(X)$ where $X \subset \mathbb{C}^n$ is an algebraic variety. This set is not (in general) subanalytic: it is of the form $Y = \log A$ where $\log(\cdot) = (2\pi i)^{-1}\log_e(\cdot)$ and $A := J^{-1}(X)$. Note that since J is meromorphic the set A is subanalytic, so that Y is a log-set.

B.3.2. *An interpolation result for logarithms of complex cells.* Let $\mathcal{C}^{\boldsymbol{\delta}/2}$ be a complex cell as in Lemma 83, and let $n := \dim \mathcal{C}$ and m denote the number of fibers of type D. Let $\mathbf{f} := (f_0, \ldots, f_n)$ be a tuple of non-vanishing holomorphic functions on $\mathcal{C}^{\boldsymbol{\delta}/2}$. We write $\mathbf{x}_{0..n} = \log \mathbf{f}_{0..n}$ and denote $X := \mathbf{x}(\mathcal{C})$.

Recall that $\pi_1(\mathcal{C}) \simeq \mathbb{Z}^{n-m}$, so we may take $\boldsymbol{\alpha}(\mathbf{f}_j) \in \mathbb{Z}^{n-m}$ for $j = 0, \ldots, n$. Then there exist $m+1$ vectors $\boldsymbol{\gamma}^0, \ldots, \boldsymbol{\gamma}^m \in \mathbb{Z}^{n+1}$ linearly independent over $\mathbb{Z}$ and satisfying $\sum \boldsymbol{\gamma}_j^k \boldsymbol{\alpha}(\mathbf{f}_j) = 0$. We write $\mathbf{g}_k = \prod_j \mathbf{f}_j^{\boldsymbol{\gamma}_j^k}$. Then $\mathbf{g}_{0..m}$ are non-vanishing holomorphic functions on $\mathcal{C}^{\boldsymbol{\delta}/2}$ with trivial associated monomials. Then the logarithms $\mathbf{y}_{0..m} := \log \mathbf{g}_{0..m}$, which are $\mathbb{Z}$-linear combinations of the $\mathbf{x}$-variables, are holomorphic univalued functions in $\mathcal{C}^{\boldsymbol{\delta}/2}$.

Proposition 86. *Let $H \in \mathbb{N}$. If*

$$\log \delta < -O_{\mathbf{f}}(k^{-1/m} \log H) \tag{258}$$

then $X(\mathbb{Q}, H)$ is contained in an algebraic hypersurface $\{P(\mathbf{y}) = 0\}$ of degree at most k in $\mathbb{C}^{n+1}$.

If $\mathcal{C}, \mathbf{f}$ vary over a definable family such that the type of $\mathcal{C}$ and the associated monomials of $\mathbf{f}_{0..n}$ are fixed then the asymptotic constants can be taken uniformly over the family.

Proof. Since $\mathbf{y}$ are fixed $\mathbb{Z}$-linear combinations of $\mathbf{x}$ we have $H(\mathbf{y}) = O_{\mathbf{f}}(H(\mathbf{x}))$. Thus it will be enough to prove the statement for $Y(\mathbb{Q}, H)$ instead of $X(\mathbb{Q}, H)$ where $Y := \mathbf{y}(\mathcal{C})$.

By the monomialization lemma the diameter of $\mathbf{y}_j(\mathcal{C}^{\boldsymbol{\delta}})$ is bounded in $\mathbb{C}$ for $j = 0, \ldots, m$ by some quantity $M = O_{\mathbf{f}}(1)$ which can be taken uniform over definable families. We are thus in position to apply Lemma 83 with

$$\Lambda(k) = \{\boldsymbol{\alpha} \in \mathbb{N}^{m+1} : |\boldsymbol{\alpha}| \leqslant k\}, \qquad \mu(k) \sim k^{m+1}. \tag{259}$$

Let $\mathbf{p} \subset Y(\mathbb{Q}, H)$. Then by Lemmas 83 and 85 we have

$$H^{-\mu k} \leqslant |\Delta^{\Lambda(k)}(\mathbf{y}; \mathbf{p})| \leqslant M^\mu \mu^{O_\ell(\mu)} \delta^{E\mu^{1+1/m} + O_\ell(\mu)} \tag{260}$$

unless $\Delta^{\Lambda(k)}(\mathbf{x}; \mathbf{p}) = 0$. In the former case we have

$$\log \delta > \frac{-O_{\mathbf{g}}(k \log H)}{k^{1+1/m} + O_\ell(1)} = -O_{\mathbf{g}}(k^{-1/m} \log H). \tag{261}$$

Therefore, for δ satisfying (258) we have $\Delta^{\Lambda(k)}(\mathbf{y}; \mathbf{p}) = 0$ for any $\mathbf{p}$ as above, and $Y(\mathbb{Q}, H)$ is indeed contained in a hypersurface of degree at most k in the $\mathbf{y}$-variables. $\square$

Remark 87 (Logarithms in families)**.** *We remark that if* $g_\lambda : \mathcal{C}_\lambda^{1/2} \to \mathbb{C} \setminus \{0\}$ *is a definable family of functions with trivial associated monomials then* $\log g_\lambda : \mathcal{C}_\lambda \to \mathbb{C}$ *is not necessarily definable (in* $\mathbb{R}_{an}$*) as illustrated by the example* $g_\lambda \equiv \lambda$ *for* $\lambda \in (0,1]$*. However, if we define* $\tilde{g}_\lambda = g_\lambda / \|g_\lambda\|_{\mathcal{C}_\lambda}$ *then* $\log \tilde{g}_\lambda$ *is definable. Indeed by the monomialization lemma we know that* $w = \log \tilde{g}_\lambda(\mathcal{C}) \subset D(M+2\pi)$ *for some uniformly bounded* M*. Then the graph of a (univalued) branch of* $\log \tilde{g}_\lambda$ *is definable, being one of the components of the definable set* $e^w = \tilde{g}_\lambda(\mathbf{z})$ *with* $w \in D(M)$ *and* $\mathbf{z} \in \mathcal{C}_\lambda$ *(note that the exponential here is* restricted *to a compact set).*

B.3.3. *Proof of Proposition 5.* We apply Corollary 30 to the subanalytic set given by the total space $X \subset \mathbb{R}^K$ of the family $\{X_\lambda\}$, where we order the parameter variables before the fiber variables. We obtain real cellular maps $f_j : \mathcal{C}_j^{1/2} \to \mathcal{P}_K^{1/2}$ such that $f_j(\mathbb{R}_+\mathcal{C}_j^{1/2}) \subset X$ and $\cup_j f_j(\mathbb{R}_+\mathcal{C}_j) = X$. We may also require by Remark 31 that each f_j is compatible with the fiber variables $\mathbf{x}_{1..\ell}$. It will be enough to prove the statement for each $f_j(\mathbb{R}_+\mathcal{C}_j)$ separately, so fix $\mathcal{C}, f = \mathcal{C}_j, f_j$ and assume $X = f(\mathbb{R}_+\mathcal{C})$.

We view $\mathcal{C}$ as a subanalytic family of cells $\mathcal{C}_\lambda$ of length ℓ. If the type of $\mathcal{C}_\lambda$ has dimension strictly less then n then we can take $Y = X$, so assume it has dimension n. Below we let $\mathbf{f}_{0..n}$ denote the f-pullback of some fixed $n+1$-tuple of the variables $\mathbf{x}_{1..\ell}$. Note that the associated monomials $\boldsymbol{\alpha}(\mathbf{f}_j)$ of $\mathbf{f}_j$ on $\mathcal{C}_\lambda$ are independent of λ (it is obtained from the associated monomial of $\mathbf{f}_j$ on $\mathcal{C}$ by eliminating the λ variables).

Let $\mathcal{R} = \{f_{\lambda,\theta} : \mathcal{C}_{\lambda,\theta} \to \mathcal{C}_\lambda\}$ denote the refinement family consisting of the refinement cells of $\mathcal{C}_\lambda$ as in Lemma 84. We split $\mathcal{R}$ into subfamilies $\mathcal{R}(\mathcal{T})$ consisting of the cells $\mathcal{C}_{\lambda,\theta}$ of type $\mathcal{T}$. According to Remark 42 the types $\mathcal{T}$ are those obtained from the type of $\mathcal{C}_\lambda$ by possibly replacing some $D_\circ, A$ fibers by D. Fix some $\mathcal{T}$ and let m denote the number of fibers of type D. The associated monomial of $f_{\lambda,\theta}^*\mathbf{f}_j$ is constant over $\mathcal{R}(\mathcal{T})$: it is obtained from $\boldsymbol{\alpha}(\mathbf{f}_j)$ by eliminating those indices that correspond to fibers that were replaced by D in $\mathcal{T}$.

We can thus apply Proposition 86 to $f_{\lambda,\theta}^*\mathbf{f}_{0..n}$ to deduce that if the cell $\mathcal{C}_{\lambda,\theta}$ admits $\boldsymbol{\delta}/2$ extension and if $\delta = H^{-O_X(k^{-1/m})}$ then

$$[\mathbf{x}_{\lambda,\theta}(\mathcal{C}_{\lambda,\theta})](\mathbb{Q}, H) \subset \{P(\mathbf{y}_{\lambda,\theta}) = 0\} \tag{262}$$

for some polynomial $P(\mathbf{y}_{\lambda,\theta})$ of degree k, where $\mathbf{y}_{\lambda,\theta}$ are some fixed $\mathbb{Z}$-linear combinations of $\mathbf{x}_{\lambda,\theta} := \log(f_{\lambda,\theta}^*\mathbf{f}_{0..n})$ which are holomorphic in $\mathcal{C}_{\lambda,\theta}^{\boldsymbol{\delta}/2}$. Note that $\mathbf{y}_{\lambda,\theta}$ does *not* necessarily depend definably on the parameters. However, the normalized $\tilde{\mathbf{y}}_{\lambda,\theta}$ defined by replacing $\mathbf{x}_{\lambda,\theta}$ with their normalized versions $\tilde{\mathbf{x}}_{\lambda,\theta}$ is definable according to Remark 87. For each fixed value of the parameters we have $\tilde{\mathbf{y}}_{\lambda,\theta} = \mathbf{y}_{\lambda,\theta} + \mathrm{const}$, and in particular the hypersurface $\{P(\mathbf{y}_{\lambda,\theta}) = 0\}$ can be rewritten as a hypersurface $\{\tilde{P}(\tilde{\mathbf{y}}_{\lambda,\theta}) = 0\}$ in the $\tilde{\mathbf{y}}_{\lambda,\theta}$-variables.

We now define a family $\tilde{Y}_{\lambda,\theta,c}$ with the parameter θ of $\mathcal{R}_{\lambda,\theta}$ and the parameter c encoding the coefficients of an arbitrary non-zero polynomial P_c of degree k,

$$\tilde{Y}_{\lambda,\theta,c} := \mathbb{R}(f_{\lambda,\theta}(\mathcal{C}_{\lambda,\theta}) \cap \{P_c(\tilde{\mathbf{y}}_{\lambda,\theta}) = 0\}). \tag{263}$$

For any λ we can choose, according to Lemma 84 a collection of

$$N = \mathrm{poly}_\ell(\delta) = H^{-O_X(k^{-1/n})} \tag{264}$$

cells $\mathcal{C}_{\lambda,\theta_j}$ which admit δ-extensions, satisfy the conditions of Lemma 83, and cover $\mathcal{C}_\lambda$. For each of them there exists a polynomial $P_{c_j}(\tilde{\mathbf{y}}_{\lambda,\theta_j})$ whose zeros contain $[\mathbf{x}_{\lambda,\theta}(\mathcal{C}_{\lambda,\theta})](\mathbb{Q},H)$. The union over $j=1,\dots,N$ of $\log\tilde{Y}_{\lambda,\theta_j,c_j}$ thus contains $(\log X_\lambda)(\mathbb{Q},H)$.

Recall that $\tilde{Y}$ was constructed for some choice of $n+1$ of the coordinates $\mathbf{x}_{1..\ell}$. We now repeat this construction for each such choice S. We now define a family $\hat{Y}_{\lambda,\mu}$ with the parameters μ encoding the pair θ_S, c_S for each of the choices S,

$$\hat{Y}_{\lambda,\mu} = \bigcap_S \tilde{Y}_{\lambda,\theta_S,c_S}. \tag{265}$$

As before, for every λ there is a choice of $N = H^{-O_X(k^{-1/n})}$ parameters μ_j such that the union $(\log\hat{Y}_{\lambda,\mu_j})(\mathbb{Q},H)$ contains $(\log X_\lambda)(\mathbb{Q},H)$.

If $\hat{Y}_{\lambda,\mu}$ has dimension n then X_λ satisfies a polynomial equation of degree k in $\log\mathbf{f}_{0..n}$ for every $n+1$ tuple of coordinates $\mathbf{f}_{0..n}$ among $\mathbf{x}_{1..\ell}$ and in this case $\log X_\lambda$ is contained in an algebraic variety of dimension n. Since X_λ has pure dimension n this implies that $(\log X_\lambda)^{\text{trans}} = \emptyset$. Thus if we define Y by removing from $\hat{Y}$ any fibers of dimension n, the conditions of the proposition are satisfied.

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