

**SIMPLE EXPONENTIAL ESTIMATE FOR THE NUMBER
OF REAL ZEROS OF COMPLETE ABELIAN INTEGRALS**

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ABSTRACT. We show that for a generic real polynomial $H = H(x, y)$ and an arbitrary real differential 1-form $\omega = P(x, y) dx + Q(x, y) dy$ with polynomial coefficients of degree $\leq d$, the number of ovals of the foliation $H = \text{const}$, which yield the zero value of the complete Abelian integral $I(t) = \oint_{H=t} \omega$, grows at most as $\exp O_H(d)$ as $d \rightarrow \infty$, where $O_H(d)$ depends only on H .

This result essentially improves the previous double exponential estimate obtained in [1] for the number of zero ovals on a bounded distance from singular level curves of H .

**La borne simplement exponentielle pour le nombre
de zéros réels isolés des intégrales complètes abéliennes**

Résumé. Soit $H = H(x, y)$ un polynôme réel de deux variables et $\omega = P(x, y) dx + Q(x, y) dy$ une forme différentielle quelconque avec des coefficients polynomiaux réels de degré d . Nous montrons que le nombre des ovals (c'est-à-dire les composantes compactes connexes) des courbes de niveau $H = \text{const}$, telles que l'intégrale de la forme s'annule, est au plus $\exp O_H(d)$ quand $d \rightarrow \infty$, où $O_H(d)$ ne dépend que du polynôme H .

Ce théorème améliore de manière essentielle la borne doublement exponentielle obtenue dans [1]. Nous établissons aussi les bornes supérieures exponentielles similaires pour le nombre de zéros isolés pour les enveloppes polynomiales des équations différentielles ordinaires linéaires irréductibles ou presque irréductibles.

Version française abrégée. — Soit $H = H(x, y)$ un polynôme réel de deux variables et $\omega = P(x, y) dx + Q(x, y) dy$ une forme différentielle quelconque avec des coefficients polynomiaux réels. Alors pour chaque valeur régulière $t \in \mathbf{R}$ de H l'ensemble de niveau $H = t$ comporte des composantes connexes compactes (*ovales*) et des composantes noncompactes. La forme ω peut être intégrée le long de chaque ovale, il en résulte une fonction multivoque analytique $I_{H,\omega}(t)$ du paramètre t , dite *intégral abélienne complète*.

Le problème suivant a été posé plusieurs fois pendant les vingt-cinq dernières années : *combien de zéros réels isolés peut avoir la fonction $I_{H,\omega}(t)$?* Dans l'article précédent [1], on a établi la borne supérieure doublement exponentielle : si H est un polynôme générique, alors pour chaque segment fermé $K \subseteq \mathbf{R}$ sans valeurs critiques de H , le nombre de zéros isolés de $I_{H,\omega}(t)$ dans K est au plus $\exp(\exp O_{H,K}(d))$, où $d = \max(\deg P, \deg Q) = \deg \omega$, et $O_{H,K}(d)$ est un terme de croissance linéaire en d ne dépendant que de H et K .

Théorème 1. *Si la compléxification de H est une fonction de Morse $\mathbf{C}^2 \rightarrow \mathbf{C}$ et la partie principale homogène de H est un produit de facteurs linéaires tous différents, alors le nombre total des ovals sur lesquelles l'intégrale abélienne complète s'annule est au plus $\exp O_H(d)$, quand $d = \deg \omega$ tend vers l'infini.*

En fait, on obtient ce résultat comme corollaire d'un théorème plus général sur les zéros de fonctions obtenues comme enveloppes polynomiales. Soit L un opérateur

différentiel linéaire d'ordre n avec des coefficients rationnels. Pour n'importe quel système fondamental de solutions f_1, \dots, f_n de l'équation $Ly = 0$, on introduit l'*enveloppe polynomiale d'ordre d de L* comme la classe des fonctions représentables sous la forme $\sum_{i,j=1}^n p_{ij}(t) f_i^{(j-1)}(t)$ avec $p_{ij} \in \mathbf{R}[t]$ et $\deg p_{ij} \leq d$. Il est facile de montrer que cette condition ne dépend ni du choix des fonctions f_i ni du choix de leurs branches. On peut aussi montrer que chaque intégrale abélienne, à facteur polynomial près, appartient à l'enveloppe polynomiale d'ordre $O(\deg \omega)$ d'un certain opérateur L déterminé seulement par H [7].

Théorème 2. *Si l'opérateur L est irréductible (ne peut être factorisé comme une composition $L = L_1 L_2$ avec des opérateurs L_i du même type), et si $K \in \mathbf{R}$ est un segment compact sans points singuliers de L , alors chaque fonction appartenant à l'enveloppe polynomiale d'ordre d a au plus $\exp O_{K,L}(d)$ zéros réels isolés sur K , quand $d \rightarrow \infty$.*

La même borne supérieure exponentielle reste valable pour un segment dont les extrémités sont des points singuliers de L , pourvu qu'ils soient du type de Fuchs [10], et que les opérateurs de monodromie en chaque point n'aient que des valeurs propres unitaires (théorème 3).

La condition d'irréductibilité peut être affaiblie. L'opérateur différentiel ordinaire L est dit *presque irréductible*, s'il peut être factorisé comme $L_0 L_1 \cdots L_k$ avec tous les facteurs irréductibles à coefficients rationnels et chaque opérateur L_i d'ordre 1, sauf le premier L_0 qui peut être arbitraire irréductible. Si L est presque irréductible, alors la même borne exponentielle est valable pour L sur chaque segment K sans points singuliers de L (théorème 4).

Les deux dernières généralisations sont indépendantes : pour un opérateur du type de Fuchs et presque irréductible, on peut trouver une borne supérieure du nombre de zéros isolés pour les enveloppes polynomiales de façon indépendante du choix du segment K (théorème 5).

1. ZEROS OF COMPLETE ABELIAN INTEGRALS. — Let $H = H(x, y)$ be a real polynomial in two variables, and $\omega = P(x, y) dx + Q(x, y) dy$ a differential 1-form with real polynomial coefficients. Then for any regular value $t \in \mathbf{R}$ of H the form ω may be integrated over any oval (=connected compact component) of the level curve $\{H(x, y) = t\}$, yielding thus a (multivalued, if there is more than one such oval) function

$$I_{H,\omega}(\cdot): t \longmapsto I_{H,\omega}(t) = \oint_{\delta(t)} \omega, \quad \delta(t) \text{ is an oval } \subseteq \{H = t\},$$

called the *complete Abelian integral*. An oval is called an *isolated zero* of the Abelian integral, if the above integral vanishes for this oval, but is nonzero for all sufficiently close ovals.

The *tangential* (sometimes also referred to as *weakened*) *Hilbert 16th problem* is formulated as follows: *find an upper bound for the number of isolated zeros of the integral $I_{H,\omega}(\cdot)$, in terms of $\deg H$ and $\deg \omega = \max(\deg P, \deg Q)$.* This problem is closely related to the Hilbert 16th problem (the second part), concerning the number of *limit cycles* of polynomial differential equations on the plane. The zeros of Abelian integrals correspond to the limit position of cycles generated in a polynomial perturbation $dH + \varepsilon\omega = 0$ from the Hamiltonian system $dH = 0$ (written in the Pfaffian form): a limit cycle can be born from an oval $H = t$ only if the corresponding Abelian integral is zero (H. Poincaré–L. Pontryagin).

The mere existence of such an estimate was established by A. Varchenko and A. Khovanskiĭ in 1984, see [2,3]. Since then, in numerous papers many particular cases were

analyzed; G. Petrov obtained a complete solution in the *elliptic case* for $H = y^2 + 3x - x^3$ [4]; many other results are surveyed in [5]. P. Mardešić found a (linear in $\deg \omega$) upper bound for the *multiplicity* of an isolated zero [12]. In [1,4] the *asymptotic problem* was considered, with H fixed (and generic) and ω variable. If $H(x, y) = y^2 + p(x)$, where p is a Morse polynomial in one variable (the *hyperelliptic case*), then the complete Abelian integral may have at most $O_H(\deg \omega)$ real isolated zeros, provided that all roots of p are real [4]. The main result of [1] is the following theorem: *for almost any Hamiltonian H and for any small positive ρ the number of isolated zeros of $I_{H,\omega}(t)$ in the interval $|t| < 1/\rho$ and at least ρ -distant from the critical values of H , can be at most $\exp(\exp O_{H,\rho}(\deg \omega))$ as $\deg \omega \rightarrow +\infty$ (the double exponential upper estimate), the term $O_{H,\rho}$ being dependent on H and ρ .*

In this note we show that the double exponential upper estimate can be improved to a simple exponential one. Moreover, we formulate explicitly the genericity conditions on Hamiltonian and show that this upper bound is valid for real zeros arbitrarily close to the critical locus of H .

Theorem 1. *If H is a Morse function on \mathbf{C}^2 , real on \mathbf{R}^2 , and the principal homogeneous part of H expands as a product of pairwise different linear factors, then the number of ovals yielding the zero value of the integral of ω , grows at most as $\exp O_H(\deg \omega)$ as $\deg \omega \rightarrow +\infty$.*

The same result is valid for the hyperelliptic case without additional assumption on the realness of the roots of the polynomial p . In fact, these results are corollaries to a more general theorem describing occurrence of zeros in *polynomial envelopes* of linear differential operators with specific restrictions on their reducibility.

2. POLYNOMIAL ENVELOPES OF IRREDUCIBLE OPERATORS. — Let $L = a_0(t)\partial^n + a_1(t)\partial^{n-1} + \dots + a_1(t)\partial + a_0(t)$, $\partial = \frac{d}{dt}$, $a_i \in \mathbf{R}(t)$, be a linear differential operator of degree n with meromorphic coefficients on the Riemann sphere \mathbf{CP}^1 . Denote by $\Sigma_L = \{t_1, \dots, t_q\}$ the singular locus of L , i.e. the union of singular points. We consider only *real operators and equations*, whose coefficients are real on the real axis $\mathbf{R} \subset \mathbf{C}$. The space of such operators is a (noncommutative) ring $\mathbf{R}(t)[\partial]$ with respect to the operation of composition. An operator L is *irreducible*, if it cannot be represented as a composition $L = L_1 L_2$ with nontrivial $L_i \in \mathbf{R}(t)[\partial]$.

Let $f_1(t), \dots, f_n(t)$ be any fundamental system of solutions to the differential equation $Ly = 0$. Then for any finite $d \in \mathbf{N}$ we may consider the *polynomial d -envelope* of the functions f_i as the collection of all linear combinations of the form $F(t) = \sum_{i,j=1}^n \sum_{k=0}^d c_{ijk} t^k \partial^{j-1} f_i(t) = \sum_{i,j=1}^n p_{ij}(t) \partial^{j-1} f_i(t)$, $p_{ij}(\cdot) \in \mathbf{R}[t]$, $\deg p_{ij} \leq d$. Obviously, this is a linear space determined only by the operator L : choosing another fundamental system or taking different branches of the same solutions f_i will not change the polynomial envelope, which will be referred to as the polynomial d -envelope of L . In a similar way we can introduce the *rational d -envelope* as the set of functions representable in the form $\sum_{i,j=1}^n r_{ij}(t) \partial^{j-1} f_i(t)$, where $r_{ij}(\cdot) \in \mathbf{R}(t)$ are arbitrary rational functions of degrees $\leq d$ (this class, however, will not be a linear space).

Choose any compact segment $K \Subset \mathbf{R} \setminus \Sigma_L$ free from singular points of L , and introduce the index $\mathcal{N}(d; K, L)$ as the maximal number of isolated zeros which a real function from the polynomial d -envelope of L may have on K .

Theorem 2. *If the operator L is irreducible, then on any compact segment $K \Subset \mathbf{R} \setminus \Sigma_L$*

$$\mathcal{N}(d; K, L) \leq \exp O_{K,L}(d) \quad \text{as } d \rightarrow +\infty. \quad (1)$$

Corollary. *If $f(t)$ is a real solution to an irreducible linear equation $Ly = 0$ on a compact segment $K \Subset \mathbf{R} \setminus \Sigma_L$, then the ν th derivative $f^{(\nu)}(t)$ may have at most $\exp O_{K,L}(\nu)$ isolated zeros on K , as $\nu \rightarrow \infty$.*

Remark. There are no restrictions on the type of singularities of the operator L . For example, they can be irregular (non-Fuchsian).

Theorem 2 admits an independent extension in two directions: under certain assumptions a closed segment K on a positive distance from the singular locus, can be replaced by an open interval with endpoints in Σ_L , and the irreducibility condition can be also relaxed to a certain extent.

3. FUCHSIAN SINGULARITIES AT THE ENDPOINTS. — Assume now that the real operator L has a singularity at one endpoint a of the segment $K = [a, b] \Subset \mathbf{R}$, while all interior points of K and the other endpoint b are nonsingular. Denote by \mathbf{M} the monodromy operator for that singularity, describing the result of analytic continuation of solutions along a small loop γ around a : $\Delta_\gamma f = \mathbf{M}f$. Since L and f_j are real and $t \in \mathbf{R}$, we have $\bar{\mathbf{M}} = \mathbf{M}^{-1}$, hence the spectrum of \mathbf{M} must be symmetric with respect to the unit circle $\{|\lambda| = 1\}$. If all eigenvalues are *unitary* (on the unit circle), then it is known that zeros of solutions of $Ly = 0$ and their polynomial envelopes cannot accumulate to the corresponding singularity $t = a$ [3]. All eigenvalues of the monodromy \mathbf{M} will be unitary, if the singularity is *regular*, or *Fuchsian*, and corresponding *indicial equation* has only real roots [10].

Theorem 3. *Let be an irreducible operator L , $K = (a, b) \subset \mathbf{R} \setminus \Sigma_L$ a semiinterval without singular points of L inside and $a \in \Sigma_L$ a Fuchsian singular point. If the monodromy operator \mathbf{M} has only unitary eigenvalues, then the simple exponential estimate (1) holds for that pair (K, L) .*

In particular, if all real singularities from $\Sigma_L \cap (\mathbf{R} \cup \infty)$ are Fuchsian and have only unitary eigenvalues, then the total number of real isolated zeros in a polynomial d -envelope of L is at most $\exp O_L(d)$ as $d \rightarrow \infty$.

4. ESSENTIALLY IRREDUCIBLE OPERATORS. — Any reducible operator $L \in \mathbf{R}(t)[\partial]$ admits a factorization in the ring $\mathbf{R}(t)[\partial]$ with irreducible factors.

Definition. An operator L is *essentially irreducible*, if it admits factorization $L = L_0 L_1 \dots L_s$ with all L_i , except for the first one L_0 , of order 1: $L_i = p_i(t)\partial + q_i(t)$, $p_i, q_i \in \mathbf{R}(t)$, $i = 1, \dots, s$, and $L_0 \in \mathbf{R}(t)[\partial]$ irreducible.

Theorem 4. *If the linear operator L is essentially irreducible, and K is a compact segment on a positive distance from the singular locus, then the exponential estimate (1) for the number of real isolated zeros in polynomial envelopes holds for the pair (L, K) .*

Theorems 3 and 4 can be combined into a single theorem.

Theorem 5. *Let $L \in \mathbf{R}(t)[\partial]$ be an essentially irreducible differential operator. If the all singularities of L on the real axis are Fuchsian, and the corresponding monodromy operators have only unitary eigenvalues, then the exponential estimate (1) holds for the pair (L, K) with an arbitrary segment K . Moreover, in this case the exponential estimate depends only on the operator L .*

5. ABELIAN INTEGRALS AS RATIONAL ENVELOPES. THE MONODROMY GROUP AND ITS ESSENTIAL IRREDUCIBILITY. — Below follows a brief description of the proofs.

The function $I(t) = \oint_{H=t} \omega$ is originally defined as a real multivalued function of the real parameter t ranging over the set of regular values of H . This function can

be analytically extended as a multivalued function on the set $\mathbf{CP}^1 \setminus \Sigma_L$, see [6]. The monodromy of this function (the result of an analytic continuation along any loop avoiding singular points t_s) depends only on the Hamiltonian H and does not depend on the form. The following result can be proved.

Lemma 1, see [7]. *If H is a polynomial satisfying the assumptions of Theorem 1, then there exists a real Fuchsian equation $Ly = 0$ with the following properties:*

1. *The operator $L \in \mathbf{R}(t)[\partial]$ with $n = \text{ord } L = (\text{deg } H - 1)^2$, is essentially irreducible;*
2. *All singular points of L have monodromy operators with all eigenvalues equal to 1;*
3. *For any choice of the form ω , the Abelian integral after the analytic extension can be represented as a function from the rational d -envelope of any fundamental system of solutions of the equation $Ly = 0$, where $d \leq O_{H,L}(\text{deg } \omega)$.*

Since H will be treated as fixed, we fix also once and forever the operator L . If we are interested in the number of zeros, then by getting rid of the denominators of the rational coefficients we may reduce that problem to the problem of estimating the number of zeros in polynomial d -envelopes of L with $d = O_L(\text{deg } \omega)$.

6. FROBENIUS–SCHLESINGER–POLYA DECOMPOSITION OF A DIFFERENTIAL OPERATOR. — If D is a linear ordinary differential operator of order n , and $F_1(t), \dots, F_n(t)$ is its fundamental system of solutions, then one can construct the sequence of Wronskians $W_i(t)$, $i = 0, 1, \dots, n$, $W_0(t) \equiv 1$, $W_i(t)$ being the Wronski determinant of the first i functions F_1, \dots, F_i . Since F_i are real analytic (multivalued) functions on $\mathbf{C} \setminus \Sigma_L$, so are the Wronskians W_i .

Lemma 2 [8,9,10].

$$D = \frac{W_n}{W_{n-1}} \partial \frac{W_{n-1}^2}{W_n W_{n-2}} \partial \frac{W_{n-2}^2}{W_{n-1} W_{n-3}} \partial \dots \partial \frac{W_1^2}{W_2 W_0} \partial \frac{W_0}{W_1}. \tag{2}$$

Denote by $N_K(f)$ the number of isolated zeros of a function f , real meromorphic on K , with their multiplicities. In [9] G. Polya proved that if for a differential operator D all Wronski determinants W_i are nonvanishing on a real segment $K = [a, b] \subset \mathbf{R}$, then $N_K(Df) \geq N_K(f) - n$.

Lemma 3. *If each Wronskian W_i has at most ν real isolated zeros on K , not vanishing identically, then for any function f analytic on $K \Subset \mathbf{C} \setminus \Sigma_D$, one has $N_K(Df) \geq N_K(f) - (2n + 1)(\nu + 1)$. In particular, if $Du = 0$, then $N_K(u) \leq (2n + 1)(\nu + 1)$.*

Now the problem is to construct a nontrivial differential operator vanishing on all functions from the polynomial envelope, and then estimate the number of isolated zeros for the corresponding Wronskians.

7. GENERALIZED JENSEN INEQUALITY. — Let $K \subset U \subset \mathbf{C}$ be a pair of sets, say, connected, simply connected, with smooth disjoint boundaries. Let $f(t)$ be a function analytic in a neighborhood of the closure of the larger set U . Denote by $N_X(f)$ the number of (complex) isolated zeros of f on a set $X \subset U$, and by $\mathcal{M}(f, X)$ the maximum modulus of f on the closure \overline{X} .

Lemma 4 [11]. *There exists a constant $\alpha = \alpha(K, U)$ depending only on the sets $K \subset U$, such that for any function analytic on \overline{U} , $N_K(f) \leq \alpha(K, U) \cdot (\ln \mathcal{M}(f, U) - \ln \mathcal{M}(f, K))$.*

8. TWO-SIDED ESTIMATES FOR WRONSKIANs RELATED TO POLYNOMIAL ENVELOPES OF IRREDUCIBLE DIFFERENTIAL EQUATIONS. — If f_1, \dots, f_n is a fundamental system of solutions for an *irreducible* differential equation $Ly = 0$, then the functions

$F_{ijk}(t) = t^i f_j^{(k-1)}$, $i = 0, 1, \dots, d$, $j, k = 1, \dots, n$, constitute a basis in the polynomial envelope. Denote by $W_s^d(t)$, $s = 1, 2, \dots, n^2(d+1)$, the Wronski determinants of the first s functions F_{ijk} ordered lexicographically as $f_1, \dots, f_d, tf_1, \dots, tf_n, \dots, t^d f_1, \dots, t^d f_n, f'_1, \dots, f'_n, tf'_1, \dots, tf'_n, \dots, f_1^{(n-1)}, \dots, f_n^{(n-1)}, \dots, tf_1^{(n-1)}, \dots, tf_n^{(n-1)}, t^d f_1^{(n-1)}, \dots, t^d f_n^{(n-1)}$.

Let $U \in \mathbf{C} \setminus \Sigma_L$. Then one can easily establish the following upper estimate for the Wronskians: for all $s = 1, 2, \dots, n^2(d+1)$ one has $\mathcal{M}(W_s^d, U) \leq \exp O_{L,U}(d^3)$. The lower estimate is much more subtle.

Main Lemma [1]. *If the operator L irreducible, and $K \in \mathbf{C} \setminus \Sigma_L$ is an arbitrary disk, then for all $s = 1, 2, \dots, n^2(d+1) - 1$ one has $\mathcal{M}(W_{s+1}^d, K) \geq \exp(-O_L(d^3)) \cdot \mathcal{M}(W_s^d, K)^\beta$ with some $\beta = \beta(K, L) < +\infty$.*

These inequalities imply the double exponential lower estimate for the Wronskians: $\mathcal{M}(W_s^d, K) \geq \exp(-\exp O_{K,L}(d))$ for all $s = 1, 2, \dots, n^2(d+1)$. Substituting this and the previous estimate into the generalized Jensen inequality, we conclude with a simple exponential estimate for the number of zeros for each W_s^d , which is sufficient to prove Theorem 2.

9. OTHER THEOREMS. — Theorem 3 can be proved by similar arguments. The monodromy operator \mathbf{M} with a unitary spectrum can be put into the upper triangular form by a *real* linear transformation. Thus without loss of generality one may assume that the linear subspace spanned by the first k functions f_1, \dots, f_k is invariant by \mathbf{M} , therefore the functions $\widetilde{W}_s^d(t) = (t-a)^{\lambda_{sd}} W_s^d(t)$ become analytic on the interval $[a, b]$ after an appropriate choice of the *real* numbers λ_{sd} . Now the Jensen inequality can be applied to the functions \widetilde{W}_s^d , yielding a simple exponential estimate for the number of their zeros on $(a, b]$; the rest of the proof remains unchanged.

The proof of Theorem 4 goes in the same way as that in §3.2.1 from [1] and amounts to suppressing the order of a differential operator when knowing some of its solutions.

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