

# L-CONVEX-CONCAVE SETS IN REAL PROJECTIVE SPACE AND L-DUALITY\*

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We define a class of  $L$ -convex-concave subsets of  $\mathbb{R}P^n$ , where  $L$  is a projective subspace of dimension  $l$  in  $\mathbb{R}P^n$ . These are sets whose sections by any  $(l+1)$ -dimensional space  $L'$  containing  $L$  are convex and concavely depend on  $L'$ . We introduce an  $L$ -duality for these sets, and prove that the  $L$ -dual to an  $L$ -convex-concave set is an  $L^*$ -convex-concave subset of  $(\mathbb{R}P^n)^*$ . We discuss a version of Arnold hypothesis for these sets and prove that it is true (or wrong) for an  $L$ -convex-concave set and its  $L$ -dual simultaneously.

## Introduction

**Convex-concave sets and Arnold hypothesis.** The notion of convexity is usually defined for subsets of affine spaces, but it can be generalized for subsets of projective spaces. Namely, a subset of a projective space  $\mathbb{R}P^n$  is called *convex* if it doesn't intersect some hyperplane  $L \subset \mathbb{R}P^n$  and is convex in the affine space  $\mathbb{R}P^n \setminus L$ . In the very definition of the convex subset of a projective space appears a hyperplane  $L$ . In projective space there are subspaces  $L$  of different dimensions, not only hyperplanes. For any subspace  $L$  one can define a class of  $L$ -convex-concave sets. These sets are the main object of investigation in this paper. If  $L$  is a hyperplane then this class coincides with the class of closed convex sets lying in the affine chart  $\mathbb{R}P^n \setminus L$ . Here is the definition of  $L$ -convex-concave sets.

A closed set  $A \subset \mathbb{R}P^n$  is *L-convex-concave* if: 1) the set  $A$  doesn't intersect the projective subspace  $L$ , 2) for any  $(\dim L + 1)$ -dimensional subspace  $N \subset \mathbb{R}P^n$  containing  $L$  the section  $A \cap N$  of the set  $A$  by  $N$  is convex, 3) for any  $(\dim L - 1)$ -dimensional subspace  $T \subset L$  the complement to the projection of the set  $A$  from the center  $T$  on the factor-space  $\mathbb{R}P^n/T$  is an open convex set.

**Example.** In a projective space  $\mathbb{R}P^n$  with homogeneous coordinates  $x_0 : \dots : x_n$  one can consider a set  $A \subset \mathbb{R}P^n$  defined by the inequality  $\{K(x) \leq 0\}$ , where  $K$  is a non-degenerate quadratic form on  $\mathbb{R}^{n+1}$ . Suppose that  $K$  is positively defined on some  $(k+1)$ -dimensional subspace, and is negatively defined on some  $(n-k)$ -dimensional subspace. In other words, suppose that (up to a linear change of coordinates) the form  $K$  is of the form  $K(x) = x_0^2 + \dots + x_k^2 - x_{k+1}^2 - \dots - x_n^2$ . In this case the set  $A$  is  $L$ -convex-concave with respect to projectivization  $L$  of any  $(k+1)$ -dimensional subspace of  $\mathbb{R}^{n+1}$  on which  $K$  is positively defined.

We are mainly interested in the following hypothesis.

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**The Main Hypothesis.** *Any  $L$ -convex-concave subset  $A$  of an  $n$ -dimensional projective space contains a projective subspace  $M$  of dimension equal to  $(n - 1 - \dim L)$ .*

Note that any projective subspace of dimension bigger than  $(n - 1 - \dim L)$  necessarily intersects  $L$ , so it cannot be contained in  $A$ . For the quadratic set  $A$  from the previous example the Main Hypothesis is evidently true: as  $M$  one can take projectivization of any  $(n - k)$ -dimensional subspaces of  $\mathbb{R}^{n+1}$  on which  $K$  is negatively defined.

For an  $L$ -convex-concave set  $A$  with a smooth non-degenerate boundary  $B$  the Main Hypothesis is a particular case of the following hypothesis due to Arnold, see [Ar1, Ar2].

**Arnold hypothesis.** *Let  $B \subset \mathbb{R}P^n$  be a connected smooth hypersurface bounding some domain  $U \subset \mathbb{R}P^n$ . Suppose that at any point of  $B$  the second fundamental form of  $B$  with respect to the outward normal vector is nondegenerate. Suppose that this form has a (necessarily constant) signature  $(n - k - 1, k)$ , i.e. at each point  $b \in B$  the restriction of the second quadratic form to some  $k$ -dimensional subspace of  $T_b B$  is negatively defined and its restriction to some  $(n - k - 1)$ -dimensional subspace of  $T_b B$  is positively defined.*

*Then one can find a projective subspace of dimension  $(n - k - 1)$  contained in the domain  $U$  and a projective subspace of dimension  $k$  in the complement  $\mathbb{R}P^n \setminus \bar{U}$ .*

Our Main Hypothesis and the very notion of  $L$ -convex-concavity were invented during an attempt to prove or disprove the Arnold hypothesis. We didn't succeed to prove it in full generality. However, we obtained several results in this direction.

We proved Arnold hypothesis for hypersurfaces satisfying the following additional assumption: there exists a non-degenerate quadratic cone  $K$  and a hyperplane  $\pi \subset \mathbb{R}P^n$  not passing through the vertex of the cone, such that, first, the hypersurface and the cone  $K$  have the same intersection with the hyperplane  $\pi$ , and, second, at each point of this intersection the tangent planes to the hypersurface and to the cone coincide (paper in preparation).

There is an *affine* version of the Arnold hypothesis: one should change  $\mathbb{R}P^n$  to  $\mathbb{R}^n$  in its formulation (and ask if there exist affine subspaces of dimensions  $k$  and  $(n - k - 1)$  in  $U$  and  $\mathbb{R}^n \setminus \bar{U}$  respectively). Our second result is an explicit construction of a counterexample to this affine version of Arnold conjecture (paper in preparation). The main role in this construction is played by *affine* convex-concave sets.

Here is the definition of the class of  $(L)$ -convex-concave subsets of  $\mathbb{R}^n$ . Fix a class  $(L)$  of  $(k + 1)$ -dimensional affine subspaces of  $\mathbb{R}^n$  parallel to  $L$ . Its elements are parameterized by points of the quotient space  $\mathbb{R}^n/N$ , where  $N$  is the (only) linear subspace of this class. A set  $A$  is called *affine  $(L)$ -convex-concave* if

- 1) any section  $A \cap N$  of  $A$  by a subspace  $N \in (L)$  is convex and
- 2) the section  $A \cap N_a$  depends concavely on the parameter  $a \in \mathbb{R}^n/N$ .

The last condition means that for any segment  $a_t = ta + (1 - t)b$ ,  $0 \leq t \leq 1$ , in the parameter space  $\mathbb{R}^n/N$  the section  $A \cap N_{a_t}$  is contained inside the linear combination (in the Minkowski sense)  $t(A \cap N_a) + (1 - t)(A \cap N_b)$  of the sections  $A \cap N_a$  and  $A \cap N_b$ . Any projective  $L$ -convex-concave set is affine  $(L)$ -convex-concave in any affine chart not containing  $L$  with respect to the class  $(L)$  of  $(\dim L + 1)$ -dimensional affine subspaces whose closures in  $\mathbb{R}P^n$  contain  $L$ .

For a class  $(L)$  of parallel planes in  $\mathbb{R}^3$  we constructed a  $(L)$ -convex-concave set  $A \subset \mathbb{R}^3$  not containing lines with smooth and everywhere non-degenerate boundary.

However, all our attempts to modify the example in such a way that its closure  $\overline{A} \subset \mathbb{R}P^3$  will be  $L$ -convex-concave failed. Finally we proved that this is impossible: the Main Hypothesis is true for  $\mathbb{R}^3$  and any  $L$ -convex-concave set with  $\dim L = 1$ . This is the only case of the Main Hypothesis we were able to prove (except trivially true cases of  $\dim L = 0$  and  $\dim L = n - 1$  in projective space  $\mathbb{R}P^n$  of any dimension  $n$ ).

**The Main Hypothesis in the three-dimensional case.** Our proof of the Main hypothesis in three-dimensional case is quite lengthy. In this paper we construct an  $L$ -duality needed for the fourth step of the proof (see below). The third step of the proof requires a cumbersome combinatorics and will be given in a separate paper.

We will give a sketch of this proof and will clarify the role of  $L$ -duality.

**Sketch of the proof.** Any line lying inside a  $L$ -convex-concave set  $A \subset \mathbb{R}P^3$  intersects all convex sections  $A \cap N$  of  $A$  by planes  $N$  containing the line  $L$ . Vice versa, any line intersecting all these sections lies in  $A$ . **The first step** of the proof is an application of a Helly theorem [He1, He2]. Consider a four-dimensional affine space of all lines in  $\mathbb{R}P^3$  not intersecting  $L$ , and convex subsets  $U_N$  of this space consisting of all line intersecting the section  $A \cap N$ . Applying the Helly theorem to the family  $U_N$ , we conclude that if for any five sections  $A \cap N_i$ ,  $i = 1, \dots, 5$ , one can find a line intersecting all of them, then there is a line intersecting all sections.

For any four section one can prove existence of a line intersecting all of them. **The second step** of the proof consists of the proof of this claim (in any dimension).

**Proposition 1** (about four sections). *Let  $A$  be a  $L$ -convex-concave subset of  $\mathbb{R}P^n$ , and let  $\dim L = n - 2$ . Then for any four sections  $A \cap N_i$  of the set  $A$  by hyperplanes  $N_i$ ,  $i = 1, \dots, 4$ ,  $N_i \supset L$ , one can find a line intersecting all of them.*

The proof uses a theorem due to Browder [Br]. This theorem is a version of a Brouwer fixed point theorem claiming existence of a fixed point of a continuous map of a closed  $n$ -dimensional ball into itself. The Browder theorem deals with set-valued upper semi-continuous maps of a convex set  $B^n$  into the set of all its closed convex subsets of  $B^n$ . The Browder theorem claims that there is a point  $a \in B^n$  such that  $a \in f(a)$ .

Here is how we use it. From the  $L$ -convex-concavity property of the set  $A \subset \mathbb{R}P^n$  with  $\text{codim } L = 2$ , one can easily deduce that for any three sections  $A_i = A \cap N_i$ ,  $i = 1, 2, 3$ , and any point  $a_1 \in A_1$  there is a line passing through  $a_1$  and intersecting both  $A_2$  and  $A_3$ . For four sections  $A_i = A \cap N_i$ ,  $i = 1, \dots, 4$ , and a point  $a_1 \in A_1$  consider all pairs of lines  $l_1$  and  $l_2$  such that

- 1) the line  $l_1$  passes through  $a_1$  and intersects  $A_2$  and  $A_3$ ,
- 2) the line  $l_2$  passes through the point of intersection of  $l_1$  and  $A_3$ , intersects  $A_4$  and intersects  $A_1$  at point  $a'_1$ .

Consider a set-valued mapping  $f$  of the section  $A_1$  to the set of all its subsets mapping the point  $a_1$  to the set of all points  $a'_1$  obtainable in this way. We prove that  $f$  satisfies conditions of the Browder theorem. Therefore there exists a point  $a_1 \in A_1$  such that  $a_1 \in f(a_1)$ . It means that there is a line  $l_1$  passing through this point and coinciding with the corresponding line  $l_2$ . Therefore this line intersects the sections  $A_2$ ,  $A_3$ ,  $A_4$  and the second step of the proof ends here.

Proof of the existence of a line intersecting (fixed from now on) sections  $A \cap N_i$ ,  $i = 1, \dots, 5$ , is quite complicated and goes as follows. Choose an affine chart containing all five sections and not containing the line  $L$ . Fix a Euclidean metric in this chart.

Define a *distance* from a line  $l$  to the collection of sections  $A \cap N_i$ ,  $i = 1, \dots, 5$ , as the maximum of distances from the point  $a_i = l \cap N_i$  to the section  $A \cap N_i$ ,  $i = 1, \dots, 5$ . A line  $l$  is a *Chebyshev line* if the distance from  $l$  to the sections  $A \cap N_i$ ,  $i = 1, \dots, 5$ , is the minimal one. We prove that for the Chebyshev line these distances are all equal. With the Chebyshev line  $l$  one can associate five half-planes  $p_i^+ \subset N_i$ . These half-planes are supporting to the sections  $A \cap N_i$  at the points  $b_i \in A \cap N_i$ , the closest to  $a_i$  points of the section  $A \cap N_i$ . We have to prove that the distance from  $L$  to the sections is equal to zero, i.e. that  $a_i = b_i$ .

To prove it is enough to find a line  $l'$  intersecting all half-planes  $p_i^+$ ,  $i = 1, \dots, 5$ . Indeed, if  $a_i \neq b_i$  then, moving slightly the line  $l$  into the direction of the line  $l'$ , one can decrease the distance from the line  $l$  to the sections  $A \cap N_i$ ,  $i = 1, \dots, 5$ , which is impossible. So, it is enough to prove that there exists a line  $l$  intersecting the five support half-planes  $p_i^+ \subset N_i$ ,  $i = 1, \dots, 5$ .

We will call the configuration of the five half-planes  $p_i^+ \subset N_i$ ,  $i = 1, \dots, 5$ , *non-degenerate* if their boundaries intersect the line  $L$  in five different points. Otherwise, i.e. if they intersect  $L$  in less than five points, we will call the configuration *degenerate*. We prove the existence of the line  $l'$  separately for non-degenerate (Step 3) and degenerate (Step 4) cases.

Detailed proof of the third step is given in our paper “A convex-concave domain in  $\mathbb{R}P^3$  contains a line” (in preparation).

Here is a brief sketch of this third step. The proof of an existence of a line intersecting all five half-planes  $p_i^+ \subset N_i$  of a non-degenerate configuration is based on a detailed analysis of combinatorial properties of each possible configuration. It turns out that there are essentially only six possible combinatorial types. For different combinatorial types of configurations the proofs differ, though share the same spirit.

Here is a rough description of the most common scheme. Instead of half-planes  $p_i^+ \subset N_i$ ,  $i = 1, \dots, 5$ , consider extended half-planes  $p_i$  such that

- 1)  $p_i^+ \subset p_i \subset N_i$ ;
- 2) boundaries of the half-planes  $p_i$  intersect the Chebyshev line and
- 3) intersections of the boundaries of  $p_i$  and  $p_i^+$  with the line  $L$  coincide.

It is enough to prove that there exists a line intersecting all extended half-planes  $p_1 \subset N_1$ ,  $i = 1, \dots, 5$ , and at least one of them at an interior point. Take planes  $\pi_i$  containing the Chebyshev line  $l$  and boundaries of half-planes  $p_i$ ,  $i = 1, \dots, 5$ . Each half-plane  $p_j$  is divided by planes  $\pi_i$  into five sectors. The minimizing property of the Chebyshev line  $l$  implies that some particular sectors necessarily intersect the convex-concave set  $A$ .

Using combinatorial properties of the configuration, we choose four half-planes and a particular sector on one of them intersecting the set  $A$ . Applying the Browder theorem (as on the step 2), we prove existence of a line intersecting the four sections *in some prescribed sectors* of the corresponding half-planes. From the combinatorial properties of the configuration follows that the constructed line intersects the fifth half-plane, q.e.d.

In the present paper we prove, among other results, the claim of the **fourth step**, i.e. existence of a line intersecting all five half-planes  $p_i^+ \subset N_i$  of a degenerate configuration. The proof goes as follows. All hyperplanes  $N \subset \mathbb{R}P^n$  containing a fixed subspace  $L$  of codimension 2, can be parameterized by points of a projective line  $\mathbb{R}P^n/L$ , so have a natural cyclic order. We say that a  $L$ -convex-concave set  $A$  with  $\dim L = n - 2$  is *linear between cyclically ordered sections*  $A_i = A \cap N_i$  if the

intersection  $A_{ij}$  of the set  $A$  with a half-space of the projective space bounded by two adjacent hyperplanes  $N_i$  and  $N_{i+1}$  coincides with a convex hull of the sections  $A_i = A \cap N_i$  and  $A_{i+1} = A \cap N_{i+1}$  (the convex hull is taken in any affine chart  $\mathbb{R}P^n \setminus N_j$ ,  $j \neq i, j \neq i+1$ , and doesn't depend on the choice of the chart.)

**Proposition 2.** *Let  $A$  be a  $L$ -convex-concave subset of  $\mathbb{R}P^n$ , and  $\dim L = n - 2$ . Suppose that there exist four sections of the set  $A$  such that  $A$  is linear between these sections. Then the set  $A$  contains a line.*

This is a reformulation of the Proposition 1.

We prove the following, dual to the Proposition 2, claim.

**Proposition 3** (about sets with octagonal sections). *Let  $D \subset \mathbb{R}P^n$  be a  $L$ -convex-concave set, and  $\dim L = 1$ . Suppose that any section  $D \cap N$  of  $D$  by any two-dimensional plane  $N$  containing the line  $L$ , is an octagon whose sides lie on lines intersecting the line  $L$  in four fixed (i.e. not depending on  $N$ ) points. In other words, each octagon has four pairs of "parallel" sides intersecting  $L$  in a fixed point. Then there exists an  $(n - 2)$ -dimensional projective subspace intersecting all planar sections  $D \cap N$ ,  $L \subset N$ , of the set  $D$ .*

In fact, the main goal of this paper is to give a definition of an  $L$ -duality with respect to which the two propositions above are dual, and to establish general properties of this duality required for reduction of the Proposition 3 to the Proposition 2.

Let's return to the Step 4 of the proof. In degenerate cases the boundaries of the five half-planes  $p_i^+$ ,  $i = 1, \dots, 5$  intersect the line  $L$  in at most four points. Assume that their number is exactly four and denote them by  $Q_1, Q_2, Q_3, Q_4$ . Perform now the following surgery of the set  $A$ . Replace each convex section  $A \cap N$  of the set  $A$ ,  $L \subset N$ , by a circumscribed octagon whose four pairs of parallel sides intersect the line  $L$  at the points  $Q_1, \dots, Q_4$ . In §6 we prove that application of this surgery to a  $L$ -convex-concave set  $A$  results in a  $L$ -convex-concave set  $D$ . The set  $D$  satisfies conditions of the Proposition 3, so there exists a line intersecting all octagonal sections of the set  $D$ . This line intersects all half-planes  $p_i^+$ ,  $i = 1, \dots, 5$ , and the proof of the main hypothesis in three-dimensional case is finished.

**$L$ -duality and plan of the paper.** There are several well-known types of duality, e.g. a usual projective duality or a duality between convex subsets of  $\mathbb{R}^n$  containing the origin and convex subsets of the dual space. Different types of duality are useful for different purposes. Here we will construct a  $L$ -duality mapping a  $L$ -convex-concave subset  $A$  of a projective space  $\mathbb{R}P^n$  to a set  $A_L^\perp$  in the dual projective space  $(\mathbb{R}P^n)^*$ . The set  $A_L^\perp$  turns out to be  $L^*$ -convex-concave, where  $L^* \subset (\mathbb{R}P^n)^*$  is a subspace dual to  $L$ . The main duality property holds for  $L$ -duality:  $A = (A_L^\perp)_{L^*}^\perp$ . The Main Hypotheses for a set  $A \subset \mathbb{R}P^n$  and for its dual  $A_L^\perp \subset (\mathbb{R}P^n)^*$  turn out to be equivalent: if the set  $A_L^\perp$  contains a projective subspace  $M^*$  such that  $\dim M^* + \dim L^* = n - 1$ , then the set  $A$  contains the dual subspace  $M$  such that  $\dim M + \dim L = n - 1$ . This is why  $L$ -duality is useful for us: the problem for the  $L$ -dual set may be easier than for the initial set. This is how the  $L$ -duality is used in the Step 4 of the proof of the Main Hypothesis in three-dimensional case.

In this paper we give a detailed description of the  $L$ -duality. Its meaning is easy to understand if the  $L$ -convex-concave set  $A$  is a domain with a smooth boundary. Assume that the boundary  $B$  of  $A$  is strictly convex-concave, i.e. that its second

quadratic form is nondegenerate at each point. Consider a hypersurface  $B^*$  in the dual projective space  $(\mathbb{R}P^n)^*$  projectively dual (in the classical sense) to  $B$ . The smooth hypersurface  $B^*$  divides  $(\mathbb{R}P^n)^*$  into two parts. The subspace  $L^*$  dual to  $L$  doesn't intersect hypersurface  $B^*$ , so exactly one of the connected component of  $(\mathbb{R}P^n)^* \setminus B^*$  does not contain  $L^*$ . The  $L$ -dual of the set  $A$  coincides with the closure of this component.

This definition does not work for sets whose boundary is not smooth and strictly convex-concave. However, we are forced to deal with such sets (in particular with sets whose sections are closed convex polygons and whose complements to projections are open convex polygons). Therefore we have to give a different, more suitable to our settings definition. An example of how one can define such a thing is the classical definition of dual convex sets. We follow closely this example.

Here is the plan of the paper. First, in §1, we give a definition of projective separability, mimicking the standard definition of separability for affine spaces. All statements formulated in this paragraph are immediate, so we omit the proofs. In §2 we discuss the notion of projective duality, the notion mimicking the classical definition of duality for containing the origin convex subsets of linear spaces. Here all statements are also very simple, but for the sake of completeness we give their proofs and explain why all of them are parallel to the classical ones.

After that, in §3, we define  $L$ -duality and prove its basic properties (using already defined projective separability and projective duality). At the end of §3 we discuss semi-algebraic  $L$ -convex-concave sets and a relation between the  $L$ -duality and integration by Euler characteristics. The results of §5 and §6 will be used in the Step 4 of the proof of the Main Hypothesis in the three-dimensional case. From the results of §4 follows, in particular, the proposition about convex-concave sets with octagonal sections (the Proposition 3 above). In §6 we describe, in particular, the surgery allowing to circumscribe convex octagons around planar convex sections.

## §1. Projective and affine separability

We recall the terminology related to the notion of separability in projective and affine spaces.

**Projective case.** We say that a subset  $A \subset \mathbb{R}P^n$  is *projectively separable* if any point of its complement lies on a hyperplane not intersecting the set  $A$ .

**Proposition.** *Complement to a projectively separable set  $A$  coincides with a union of all hyperplanes not intersecting the set  $A$ . Vice versa, complement to any union of hyperplanes has property of projective separability.*

This proposition can be reformulated:

**Proposition.** *Any subset of projective space defined by a system of linear homogeneous inequalities  $L_\alpha \neq 0$ , where  $\alpha$  belongs to some set of indexes and  $L_\alpha$  is a homogeneous polynomial of degree one, is projectively separable. Vice versa, any projectively separable set can be defined in this way.*

We define a *projective separability hull* of the set  $A$  as the smallest projectively separable set containing the set  $A$ .

**Proposition.** *The projective separability hull of a set  $A$  is exactly the complement to a union of all hyperplanes in  $\mathbb{R}P^n$  not intersecting the set  $A$ . In other words, a*

point lies in the projective separability hull of the set  $A$  if and only if any hyperplane containing this point intersects the set  $A$ .

**Affine case.** Recall the well known notion of separability in the affine case. Namely, a subset  $A$  of an affine space is *affinely separable* if any point of the complement to the set  $A$  belongs to a closed half-space not intersecting the set  $A$ . Evidently, any affinely separable set is convex and connected.

**Proposition.** *The complement to an affinely separable set  $A$  coincides with a union of closed half-spaces not intersecting the set  $A$ . Vice versa, a complement to any union of closed half-spaces is affinely separable.*

This property can be reformulated.

**Proposition.** *Any subset of an affine space defined by a system of linear inequalities  $\{L_\alpha(x) < 0\}$ , where  $\alpha$  belongs to some set of indices and  $L_\alpha$  is a polynomial of degree at most one, is affinely separable. Vice versa, any affinely separable set can be defined in this way.*

We define an *affine separability hull* of a set  $A$  as the smallest set containing the set  $A$  and having the property of affine separability.

**Proposition.** *Affine separability hull of a set  $A$  is equal to a complement to a union of all closed not intersecting the set  $A$  half-spaces of the affine space. In other words, a point lies in the affine separability hull of the set  $A$  if and only if any closed half-space containing this point also intersects the set  $A$ .*

**Convex subsets of projective spaces and separability.** Projective and affine separability are closely connected.

**Proposition.** *Let  $L$  be a hyperplane in a projective space  $\mathbb{R}P^n$  and  $U = \mathbb{R}P^n \setminus L$  be a corresponding affine chart.*

1. *Any affinely separable subset of the affine chart  $U$  (so, in particular, connected and convex in  $U$ ), is also projectively separable as a subset of a projective space.*
2. *Any connected projectively separable subset of the affine chart  $U$  is also affinely separable as a subset of an affine space  $U$ .*

A connected projectively separable subset of a projective space not intersecting at least one hyperplane will be called a *separable convex* subset of the projective space. (There is exactly one projective separable subset of projective space intersecting all hyperplanes, namely the projective space itself.)

**Remark.** We defined above a notion of a (not necessarily projectively separable) convex subset of a projective space: a nonempty subset  $A$  of a projective space  $\mathbb{R}P^n$  is called *convex* if, first, there is a hyperplane  $L \subset \mathbb{R}P^n$  not intersecting the set  $A$  and, second, any two points of the set  $A$  can be joined by a segment lying in  $A$ . We will not need convex non-separable sets.

## §2. Projective and linear duality

We construct here a variant of a projective duality. To a subset  $A$  of a projective space  $\mathbb{R}P^n$  corresponds in virtue of this duality a subset  $A_p^*$  of the dual projective space  $(\mathbb{R}P^n)^*$ . This duality is completely different from the usual projective duality and is similar to a linear duality used in convex analysis. For the sake of completeness we describe here this parallelism as well.

**Projective duality.** Projective space  $\mathbb{R}P^n$  is obtained as a factor of a linear space  $\mathbb{R}^{n+1} \setminus 0$  by a proportionality relation. The dual projective space, by definition, is a factor of the set of all nonzero covectors  $\alpha \in (\mathbb{R}^{n+1})^* \setminus 0$  by a proportionality relation.

There is a one-to-one correspondence between hyperplanes in the space and points of the dual space. More general, to any subspace  $L \subset \mathbb{R}P^n$  corresponds a dual subspace  $L^* \subset (\mathbb{R}P^n)^*$  of all hyperplanes containing  $L$ , and the duality property  $(L^*)^* = L$  holds.

For any set  $A \subset \mathbb{R}P^n$  we define its dual set  $A_p^* \subset (\mathbb{R}P^n)^*$  to be a set of all hyperplanes in  $\mathbb{R}P^n$  not intersecting the set  $A$ . (The symbol  $A^*$  denotes the dual space, so we introduce the new notation  $A_p^*$ .)

**Proposition.** 1. *If  $A$  is non-empty, then the set  $A_p^*$  is contained in some affine chart of the dual space.*

2. *The set  $A_p^*$  is projectively separable.*

**Proof.** 1. The set  $A$  is nonempty, so contains some point  $b$ . A hyperplane  $b^* \in (\mathbb{R}P^n)^*$  corresponding to the point  $b$ , doesn't intersect the set  $A_p^*$ . Therefore the set  $A_p^*$  is contained in the affine chart  $(\mathbb{R}P^n)^* \setminus b^*$ .

2. If a hyperplane  $L \subset \mathbb{R}P^n$ , considered as a point in the space  $(\mathbb{R}P^n)^*$ , is not contained in the set  $A_p^*$ , then, by definition, the hyperplane  $L$  intersects the set  $A$ . Let  $b \in A \cap L$ . The hyperplane  $b^*$  dual to the point  $b$  doesn't intersect the set  $A_p^*$ . So this hyperplane separates the point corresponding to the hyperplane  $L$  from the set  $A_p^*$ .

The following theorem gives a full description of the set  $(A_p^*)_p^*$ .

**Theorem.** *For any set  $A \subset \mathbb{R}P^n$  the corresponding set  $(A_p^*)_p^*$  consists of all points  $a$  such that any hyperplane containing  $a$  intersects the set  $A$ . In other words, the set  $(A_p^*)_p^*$  coincides with the projective separability hull of the set  $A$ .*

**Proof.** The point  $a$  belongs to  $(A_p^*)_p^*$  if and only if the corresponding hyperplane  $a^* \subset (\mathbb{R}P^n)^*$  doesn't intersect the set  $A_p^*$ . To any point  $p$  in  $(\mathbb{R}P^n)^*$  of this hyperplane corresponds a hyperplane  $p^* \subset \mathbb{R}P^n$  containing the point  $a$ . The point  $p \in (\mathbb{R}P^n)^*$  doesn't belong to  $A_p^*$  if and only if the hyperplane  $p^* \subset \mathbb{R}P^n$  intersects the set  $A$ . So the condition that all points of the hyperplane  $a^* \subset (\mathbb{R}P^n)^*$  does not belong to  $A_p^*$ , means that all hyperplanes in  $\mathbb{R}P^n$  containing the point  $a$ , intersect the set  $A$ .

**Corollary.** *The duality property  $(A_p^*)_p^* = A$  holds for all projectively separable subsets of a projective space, and only for them.*

**Linear duality.** The property of affine separability differs from the property of projective separability: we use closed half-spaces in the affine definition and hyperplanes in the projective definition. One can do the same with the duality theory developed above and define the set  $A_a^*$  corresponding to a subset of an affine space as a set of all closed half-spaces not intersecting the set  $A$ . This definition is not very convenient because the set of all closed half-spaces doesn't have a structure of an affine space. Moreover, this set is topologically different from affine space: it is homeomorphic to the sphere  $S^n$  with two removed points (one point corresponding to an empty set and another to the whole space). One can avoid this difficulty by considering instead a set of all closed half-spaces not containing some fixed point



with one added element (this element corresponds to an empty set regarded as a half-space on an infinite distance from the fixed point). This set has a natural structure of an affine space. Namely, taking the fixed point as the origin and denoting the resulting linear space by  $\mathbb{R}^n$ , one can parameterize the set described above by  $(\mathbb{R}^n)^*$ : to any nonzero  $\alpha \in (\mathbb{R}^n)^*$  corresponds a closed half-space defined by inequality  $\langle \alpha, x \rangle \geq 1$ . To  $\alpha = 0$  corresponds an empty set (defined by the same inequality  $\langle \alpha, x \rangle \geq 1$ ).

It is more convenient to consider only sets containing some fixed point when talking about affine duality. Taking this point as the origin, we get the well-known theory of affine duality, which is parallel to the theory of projective duality. Here are its main points.

To any subset  $A$  of a linear space  $\mathbb{R}^n$  corresponds a subset  $A_l^*$  of a dual space  $(\mathbb{R}^n)^*$  consisting of all  $\alpha \in (\mathbb{R}^n)^*$  such that the inequality  $\langle \alpha, x \rangle < 1$  holds for all  $x \in A$ .

**Proposition.** *For any set  $A \subset \mathbb{R}^n$  containing the origin the corresponding dual set  $A_l^*$  in the dual space has the property of affine separability. In particular, it is convex.*

**Proposition.** *For any set  $A \subset \mathbb{R}^n$  containing the origin the set  $(A_l^*)_l^*$  consists of all points  $a \in \mathbb{R}^n$  with the following property: any closed half-space containing  $a$  intersects the set  $A$ . In other words, the set  $(A_l^*)_l^*$  is equal to the affine separability hull of the set  $A$ .*

**Corollary.** *The duality property  $(A_p^*)_p^* = A$  holds for all containing the origin convex sets with the property of affine separability, and only for them.*

### §3. L-duality

Here we construct a  $L$ -duality. A subset  $A$  of a projective space  $\mathbb{R}P^n$  disjoint from some subspace  $L$ , will be  $L$ -dual to a subset  $A_L^\perp$  of a dual projective space  $(\mathbb{R}P^n)^*$  disjoint from the subspace  $L^*$ .

Any subset  $C$  in the projective space  $(\mathbb{R}P^n)^*$  can be considered as a subset of a set of all hyperplanes in the projective space  $\mathbb{R}P^n$ . We will also denote it by  $C$ .

Let  $L$  be some projective subspace of  $\mathbb{R}P^n$ , and  $A$  be any set not intersecting  $L$ . For a hyperplane  $\pi$  not containing the subspace  $L$ , denote by  $L_\pi$  the subspace  $L \cap \pi$ . Consider a factor-space  $(\mathbb{R}P^n)/L_\pi$ . The image  $\pi_L$  of a hyperplane  $\pi$  is a hyperplane in the factor-space  $(\mathbb{R}P^n)/L_\pi$ .

**Definition.** We say that the hyperplane  $\pi$  belongs to the  $L$ -dual set  $A_L^\perp$  if  $\pi$  doesn't contain  $L$  and the hyperplane  $\pi_L$  is contained in the projection of the set  $A$  on the factor-space  $(\mathbb{R}P^n)/L_\pi$ .

In other words, a hyperplane  $\pi$  belongs to the set  $A_L^\perp$  if projection of  $\pi$  from the center  $L_\pi$  belongs to  $B_p^*$ , where  $B$  is the complement to the projection of the set  $A$  on the space  $\mathbb{R}P^n/L_\pi$ .

Here is another description of the set  $A_L^\perp$ . The complement  $\mathbb{R}P^n \setminus L$  to the subspace  $L$  is fibered by spaces  $N \supset L$  of dimension  $\dim N = \dim L + 1$ . A hyperplane  $\pi$  belongs to  $A_L^\perp$ , if and only if for any fiber  $N$  its intersection with the set  $A \cap \pi$  is non-empty,  $N \cap A \cap \pi \neq \emptyset$ . In other words,  $\pi \in A_L^\perp$  if and only if  $\pi$  intersects any section of  $A$  by any  $(\dim L + 1)$ -dimensional space containing  $L$ .

**Example.** Let  $L$  be a hyperplane, and  $A$  be a set disjoint from  $L$ ,  $A \cap L = \emptyset$ . Then  $A_L^\perp$  is a union of all hyperplanes intersecting the set  $A$ . In other words, the set  $A_L^\perp$  is a complement to the set  $A_p^*$ . Indeed, in this case the only space  $N$  containing  $L$  is the projective space  $\mathbb{R}P^n$  itself. Note that in this case the set  $L$ -dual to  $A$  doesn't depend on the choice of a hyperplane  $L$  (as long as  $L$  doesn't intersect the set  $A$ ).

**Proposition.** *If  $A \subset B$  and  $B \cap L = \emptyset$ , then  $(A_L^\perp) \subset (B_L^\perp)$ .*

**Proof.** If a hyperplane intersects all sections  $A \cap N$ , then it intersects all sections  $B \cap N$ .

**Proposition.** *Let  $M$  be a projective subspace in  $\mathbb{R}P^n$  not intersecting  $L$  and of a maximal possible dimension, i.e.  $\dim M = \dim L^* = n - \dim L - 1$ . Then  $M_L^\perp = M^*$ .*

**Proof.** Any section of  $M$  by  $(\dim L + 1)$ -dimensional space containing  $L$  is just a point, and any point of  $M$  is a section of  $M$  by such a space. By definition of  $M_L^\perp$ , a hyperplane  $\pi$  belongs to  $M_L^\perp$  if and only if it intersects any such section, i.e. contains any point of  $M$ . This is exactly the definition of  $M^*$ .

Let  $L^* \subset (\mathbb{R}P^n)^*$  be the space dual to  $L$ . What can be said about: a) sections of the set  $A_L^\perp$  by  $(\dim L^* + 1)$ -dimensional spaces  $N \supset L^*$ ; b) projections of the set  $A_L^\perp$  from a  $(\dim L^* - 1)$ -dimensional subspace  $T$  of a space  $L^*$ ? We give below answers to these questions.

**Sections of the  $L$ -dual set.** Recall first a duality between sections and projections. Let  $N$  be a projective subspace in the space  $(\mathbb{R}P^n)^*$ . Consider a dual to  $N$  subspace  $N^* \subset \mathbb{R}P^n$ . We will need later an isomorphism and a projection described below.

There is a natural isomorphism between a space dual to the quotient space  $\mathbb{R}P^n/N^*$  and the space  $N$ . This isomorphism is a projectivization of a natural isomorphism between a space dual to a factor-space and a subspace of a dual space dual to the kernel of the factorization. Each hyperplane containing the space  $N^*$ , projects to a hyperplane in  $\mathbb{R}P^n/N^*$ . (If a hyperplane doesn't contain the space  $N^*$ , then its projection is the whole space  $\mathbb{R}P^n/N^*$ .)

Using this isomorphism one can describe a section of the set  $C \subset (\mathbb{R}P^n)^*$  by the space  $N$  in terms of the space  $\mathbb{R}P^n$ . Consider a subset  $C_{N^*} \subset (\mathbb{R}P^n)^*$  of the set of hyperplanes  $C$  consisting of all hyperplanes containing  $N^*$  (this is equivalent to  $C_{N^*} = C \cap N$ ). Each hyperplane from  $C_{N^*}$  projects to a hyperplane in the factor-space  $\mathbb{R}P^n/N^*$ . But the space  $(\mathbb{R}P^n/N^*)^*$  is identified with the space  $N$ . After projection and identifying we get the required section  $C \cap N$  from the set  $C_{N^*}$ .

**Theorem 1.** *Let  $A$  be a subset of  $\mathbb{R}P^n$  not intersecting  $L$ , and  $N$  be any subspace of  $(\mathbb{R}P^n)^*$ , containing  $L^*$  as a hyperplane (i.e.  $\dim N = \dim L + 1$  and  $N \supset L^*$ ). Then the section  $A_L^\perp \cap N$  is equal to  $B_p^*$ , where  $B \subset (\mathbb{R}P^n/N^*)$  is a complement to the projection of the set  $A$  on the space  $(\mathbb{R}P^n)/N^*$ .*

**Proof.** This Theorem follows from the description above of sections of subsets of  $(\mathbb{R}P^n)^*$ . Consider the set of hyperplanes  $C = A_L^\perp$ . By the definition of the set  $A_L^\perp$ , the set  $C_{N^*}$  consists of all hyperplanes containing the projective space  $N^*$ , such that their projections on  $\mathbb{R}P^n/N^*$  after projection from  $N^*$  are contained

in projection of the set  $A$ . In other words, their projections are hyperplanes in  $\mathbb{R}P^N/N^*$  not intersecting the complement to the projection of the set  $A$ . Vice versa, any hyperplane not intersecting this complement  $B$  is, by definition of the set  $A_L^\perp$ , a projection of some hyperplane belonging to the set  $C_{N^*}$ . Therefore  $A_L^\perp \cap N = B_p^*$ .

**Projections of  $L$ -dual sets.** Recall a duality between projections and sections.

Denote by  $Q$  a subspace in  $\mathbb{R}P^n$  dual to the center of projection  $T \subset (\mathbb{R}P^n)^*$ . There is a natural isomorphism between the space  $Q^*$ , consisting of all hyperplanes of the space  $Q$ , and the factor-space  $(\mathbb{R}P^n)^*/T$ . Namely, one should consider points of  $(\mathbb{R}P^n)^*/T$  as equivalency classes in the set of all hyperplanes in the space  $\mathbb{R}P^n$  not containing the space  $Q$ , of the following equivalency relation: two hyperplanes are equivalent if and only if their intersections with  $Q$  coincide. This intersection is the hyperplane in the space  $Q$  corresponding to this equivalency class.

Projection of a subset  $C$  of  $(\mathbb{R}P^n)^*$  from a center  $T$  can be described in the following way. A set of hyperplanes  $C$  in  $\mathbb{R}P^n$  defines some set of hyperplanes  $C(Q)$  in the subspace  $Q = T^\perp$ : a hyperplane  $Q_1 \subset Q$  belongs to the set  $C(Q)$  if and only if there exists a hyperplane belonging to the set  $C$  intersecting  $Q$  exactly by  $Q_1$ . Projection of the set  $C$  from the center  $T$  is exactly the set  $C(Q)$  of hyperplanes in  $Q$  after identifying  $Q^*$  and  $(\mathbb{R}P^n)^*/T$ .

**Theorem 2.** *Let  $A$  be a set in  $\mathbb{R}P^n$  not intersecting  $L$ , and  $T$  be a hyperplane in the dual space  $L^* \subset (\mathbb{R}P^n)^*$ . Then the projection of the set  $A_L^\perp$  from the center  $T$  can be described as a set of all hyperplanes  $p$  in space  $Q = T^\perp \supset L$  with the following property: there exists a hyperplane  $\pi \subset A_L^\perp$  whose intersection with  $Q$  is equal to  $p$ ,  $p = \pi \cap Q$ .*

**Proof.** This Theorem follows from the description of projections of subsets  $C \subset (\mathbb{R}P^n)^*$  given above.

**Definition.** We say that a set  $A$  is *coseparable* relative to  $L$  if  $A \cap L = \emptyset$  and for any hyperplane  $L_1 \subset L$  a complement to projection of the set  $A$  from the center  $L_1$  has the property of affine separability in space  $(\mathbb{R}P^n)/L_1$ .

**Corollary.** *If, in addition to all conditions of the Theorem 2, the set  $A$  is coseparable relative to  $L$ , then the complement to the projection of the set  $A_L^\perp$  from the center  $T$  is dual to the section  $A \cap T^*$  (i.e. equal to  $(A \cap T^*)_p^*$ ).*

**Description of the set  $(A_L^\perp)_{L^*}^\perp$ .** Let  $A$  be a subset of  $\mathbb{R}P^n$  not intersecting a subspace  $L$ , and  $L^*$  be a dual to  $L$  subspace of  $(\mathbb{R}P^n)^*$ . What can be said about a subset of  $\mathbb{R}P^n$   $L^*$ -dual to the subset  $A_L^\perp$  of the space  $(\mathbb{R}P^n)^*$ ? From the theorems 1 and 2 we easily obtain the description of this set  $(A_L^\perp)_{L^*}^\perp$ .

**Theorem 3.** *The set  $(A_L^\perp)_{L^*}^\perp$  doesn't intersect  $L$  and consists of all points  $a \in \mathbb{R}P^n$  satisfying the following condition: in the space  $L_a$  spanned by  $L$  and  $a$ , for any hyperplane  $p$  in  $L_a$  containing the point  $a$ ,  $a \in p \subset L_a$  there is a hyperplane  $\pi \subset \mathbb{R}P^n$ ,  $\pi \in A_L^\perp$ , such that  $p = \pi \cap L_a$ .*

**Proof.** A section of the set  $(A_L^\perp)_{L^*}^\perp$  by the subspace  $L_a$  can be described, according to the theorem 1 (applied to the subset  $A_L^\perp$  of the space  $(\mathbb{R}P^n)^*$  and the subspace  $L^*$  of this space), as the set of hyperplanes in the factor-space  $(\mathbb{R}P^n)^*/L_a^*$  not intersecting a complement to the projection of the set  $A_L^\perp$  on the space  $(\mathbb{R}P^n)^*/L_a^*$ .

So the point  $a \in \mathbb{R}P^n$  lies in  $(A_L^\perp)_{L^*}^\perp$  if and only if a hyperplane in  $(\mathbb{R}P^n)^*/L_a^*$ , corresponding to this point  $a \in \mathbb{R}P^n$ ,  $a \in L_a$ , is contained in the projection of the set  $A_L^\perp$ . This means that any hyperplane  $p$  of  $L_a$ ,  $P \subset L_a$ , containing the point  $a$ , lies in the projection of the set  $A_L^\perp$ , if considered as a point of the space  $(\mathbb{R}P^n)^*/L_a^*$ . This means, according to the theorem 2, that for the hyperplane  $p$  there exists a hyperplane  $\pi \in A_L^\perp$  such that  $\pi \cap L_a = p$ , q.e.d.

Let's reformulate the Theorem 3. The point  $a$  belongs to the set  $(A_L^\perp)_{L^*}^\perp$  if the following two conditions hold:

**Condition 1.** The point  $a$  in the space  $L_a$ , spanned by  $L$  and  $a$ , has the following property: any hyperplane  $p \subset L_a$ , containing the point  $a$ , intersects the set  $L_a \cap A$ . In other words, the point  $a$  lies in the set  $((L_a \cap A)_p^*)^*$ .

**Condition 2.** Projection of the point  $a$  from any center  $L_1 \subset L$ , where  $L_1$  is a hyperplane in  $L$ , is contained in some hyperplane in the space  $(\mathbb{R}P^n)/L_1$  contained in the projection of the set  $A$  on the space  $(\mathbb{R}P^n)/L_1$ .

**Theorem 4.** *The conditions 1 and 2 are equivalent to the condition that the point  $a$  belongs to the set  $(A_L^\perp)_{L^*}^\perp$ .*

**Proof.** Indeed, according to the Theorem 3, if  $a \in (A_L^\perp)_{L^*}^\perp$ , then any hyperplane  $p$  in the space  $L_a$  containing the point  $a$ , is an intersection of  $L_a$  and a hyperplane  $\pi \in A_L^\perp$ . This means that, first, the hyperplane  $p$  intersects  $A$  and, second, that the projection of the point  $a$  from  $L_1 = L \cap \pi$  is containing in a hyperplane in the factor-space  $\mathbb{R}P^n/L_1$ , which, in turn, is contained in the projection of the set  $A$ . The first property is equivalent to the Condition 1, and the second is equivalent to the Condition 2.

**Corollary.** *Suppose that a set  $A$  doesn't intersect the space  $L$ , and intersection of  $A$  with any subspace  $N$  containing  $L$  as a hyperplane, is projectively separable in projective space  $N$ . Then  $(A_L^\perp)_{L^*}^\perp \subset A$ .*

**Proof.** Indeed, the Condition 1 guarantees that for any space  $N$ , containing  $L$  as a hyperplane, the inclusion  $(A_L^\perp)_{L^*}^\perp \cap N \subset ((N \cap A)_p^*)^*$  holds. But  $((N \cap A)_p^*)^* = N \cap A$ , since  $N \cap A$  is projectively separable. Therefore  $(A_L^\perp)_{L^*}^\perp \subset A$ .

**Corollary.** *Suppose that the set  $A$  is coseparable relative to  $L$ . Then the intersection of the set  $(A_L^\perp)_{L^*}^\perp$  with any space  $N$ , containing  $L$  as a hyperplane, depends on the subset  $A \cap N$  of the projective space  $N$  only and coincides with the set  $((A \cap N)_p^*)^*$ . In particular, in this case  $A \subseteq (A_L^\perp)_{L^*}^\perp$ .*

**Proof.** If the set  $A$  is coseparable relative to  $L$ , then the Condition 2 holds for points satisfying to the Condition 1. This is exactly what the Corollary claims.

**Properties of  $L$ -coseparable and  $L$ -separable sets.** Let's sum up the facts about  $L$ -coseparable and  $L$ -separable subsets of a projective space proved above.

Let a subset  $A$  of a projective space  $\mathbb{R}P^n$  be coseparable relative to a space  $L$ , and suppose that any section of  $A$  by a space containing  $L$  as a hyperplane, is projectively separable.

Then the set  $A_L^\perp$  in the dual projective space  $(\mathbb{R}P^n)^*$  has the same properties relative to the dual space  $L^*$ . Moreover, any section of  $A_L^\perp$  by a subspace  $N$  containing  $L^*$  as a hyperplane, is dual to the set  $B$  (i.e. is equal to  $B_p^*$ ), where  $B$  is a complement to the projection of the set  $A$  on  $(\mathbb{R}P^n)/N^*$  from the center  $N^*$ .

Projection of the set  $A_L^\perp$  from the center  $T$ , where  $T$  is any hyperplane in space  $L^*$ , is dual to the section of  $A$  by  $T^*$  (i.e. is equal to  $(A \cap T^*)^*$ ). Also, the duality relation  $(A_L^\perp)_L^\perp = A$  holds.

If the set  $A_L^\perp$  contains a projective space  $M^*$  of dimension equal to the dimension of the space  $L$ , then the set  $A$  contains its dual space  $M$  of dimension equal to the dimension of the space  $L^*$ .

$L$ -convex-concave sets are  $L$ -separable and  $L$ -coseparable, because closed sets and open sets are both separable. Therefore for  $L$ -convex-concave all the aforementioned properties hold.

**Semialgebraic  $L$ -convex-concave sets.** Here we will use the integration by Euler characteristics, introduced by O. Viro (see [Vi]). We will denote Euler characteristics of a set  $X$  by  $\chi(X)$ .

**Theorem.** *Let  $A$  be a  $L$ -convex-concave closed semialgebraic set in  $\mathbb{R}P^n$ , and let  $\dim L = k$ . Then for any hyperplane  $\pi \subset \mathbb{R}P^n$  the  $\chi(A \cap \pi)$  is equal to  $\chi(\mathbb{R}P^{n-k-1})$  or to  $\chi(\mathbb{R}P^{n-k-2})$ . In the first case the hyperplane  $\pi$ , considered as a point of  $(\mathbb{R}P^n)^*$ , belongs to the  $L$ -dual to  $A$  set  $A_L^\perp$ . In the second case the hyperplane  $\pi$  doesn't belong to the set  $A_L^\perp$ .*

**Proof.** The complement to  $L$  in  $\mathbb{R}P^n$  is a union of nonintersecting fibers, each fiber being a  $(k+1)$ -dimensional space  $N$  containing  $L$ . The set  $A$  is  $L$ -convex-concave, so its intersection with each fiber  $N$  is convex and closed. Therefore for each space  $N$  the intersection  $A \cap N \cap \pi$  of the set  $A \cap N$  with a hyperplane  $\pi$  either is empty or is a closed convex set.

Suppose that the hyperplane  $\pi$  doesn't contain the space  $L$ , and denote by  $L_\pi$  the space  $L \cap \pi$ . In the factor-space  $\mathbb{R}P^n/L_\pi$  we have a fixed point  $\pi(L)$  (projection of the space  $L$ ), a set  $B$  (the complement to the projection of the set  $A$  from  $L_\pi$ ), and a hyperplane  $\pi_L$  (projection of the hyperplane  $\pi$ ). To each point  $a$  of the hyperplane  $\pi_L$  in the factor-space corresponds a space  $N(a)$  in  $\mathbb{R}P^n$ ,  $N(a) \supset L$ , whose projection is equal to the line passing through  $a$  and  $\pi(L)$ . The intersection  $N(a) \cap A \cap \pi$  is empty if  $a$  belongs to the set  $B$ . Otherwise, the intersection  $N(a) \cap A \cap \pi$  is a closed convex set. The Euler characteristics of the set  $N(a) \cap A \cap \pi$  is equal to zero in the first case, and is equal to one in the second case. Using Fubini theorem for an integral by Euler characteristics for the projection of the set  $A \cap \pi$  on the factor-space  $\mathbb{R}P^n/L_\pi$ , we get

$$\chi(A \cap \pi) = \chi(\pi_L \setminus (\pi_L \cap B)).$$

So  $\chi(A \cap \pi) = \chi(\pi_L) = \chi(\mathbb{R}P^{n-k-1})$ , if  $\pi_L \cap B = \emptyset$ . Otherwise, i.e. if  $\pi_L \cap B \neq \emptyset$ , the  $\chi(A \cap \pi) = \chi(\mathbb{R}P^{n-k-2})$ . In the first case  $\pi_L \in A_L^\perp$  by definition, and in the second case  $\pi_L \notin A_L^\perp$ . Therefore the theorem is proved for hyperplanes not containing the space  $L$ . If  $L \subset \pi$ , then from similar considerations one can see that  $\chi(\pi \cap A) = \chi(\mathbb{R}P^{n-k-2})$ , q.e.d.

**Corollary.** *For semi-algebraic  $L$ -convex-concave sets  $A$  the  $L$ -dual set is defined canonically (i.e.  $A_L^\perp$  doesn't depend on the choice of the space  $L$ , relative to which the set  $A$  is  $L$ -convex-concave).*

**Remark.** For the semialgebraic  $L$ -convex-concave sets one can prove the duality relation

$$(A_L^\perp)_{L^*}^\perp = A,$$

using only this theorem and a Radon transform for the integral by Euler characteristics, see [Vi], and also [PKh].

#### §4. Duality between pointed convex sections of convex-concave sets and affine dependence of convex sections on parameter

In this section we define properties of pointedness (with respect to a cone) and of affine dependence on parameter (for parameters belonging to some convex domain) of sections.

We begin with affine versions of these notions and then give corresponding projective definitions. We prove that the property of pointedness and the property of affine dependence on parameters are dual.

**Pointedness of sections.** We start with affine settings. Let  $K$  be a pointed (i.e. not containing linear subspaces) closed convex cone in a linear space  $N$  with vertex at the origin.

We say that a set  $A$  is *pointed* with respect to  $K$ , if there is a point  $a \in A$  such that the set  $A$  lies entirely in a translated cone  $(K + a)$  with the vertex at the point  $a$ . This point  $a$  will be called a *vertex* of the set  $A$  relative to the cone  $K$ . The vertex of the set  $A$  relative to  $K$  is evidently uniquely defined.



Fig. 1. a) Cone  $K$ , b) Pointed with respect to the cone  $K$  set  $A$ .

In affine space we deal with pointed cones  $K$ , which are unions of rays beginning at the vertex of the cone not containing lines.

In the projective setting it is more natural to consider cones  $\tilde{K}$  which are unions of lines. Such a cone  $\tilde{K}$  will be called *projectively pointed*, if the set of lines lying in the cone forms a convex set in  $\mathbb{R}P^{n-1}$ . Evidently, a cone  $\tilde{K}$  is projectively pointed if and only if it is a union of an affine pointed cone  $K$  with its opposite cone  $(-K)$ ,  $\tilde{K} = K \cup (-K)$ .

We say that a set  $A$  in affine space is *pointed with respect to a cone*  $\tilde{K} = K \cup (-K)$ , if the set  $A$  is pointed with respect to both the cone  $K$  and the cone  $(-K)$ .

A set  $A$  pointed with respect to a cone  $\tilde{K}$  has two vertices  $a$  and  $b$ , relative to the cones  $K$  and  $(-K)$  correspondingly.

The following statement is evident.

**Proposition.** Suppose a connected set  $A$  is pointed with respect to a cone  $\tilde{K} = K \cup (-K)$ , and let  $a$  and  $b$  be vertices of  $A$  relative to  $\tilde{K}$ . Let  $\tilde{Q}$  be a hyperplane intersecting  $\tilde{K}$  at one point (the origin) only. Then an affine hyperplane  $Q$ , parallel to a hyperplane  $\tilde{Q}$ , intersects the set  $A$  if and only if  $Q$  intersects the segment joining the points  $a$  and  $b$ . Vice versa, if a connected set  $A$  with fixed points  $a$  and  $b$  has

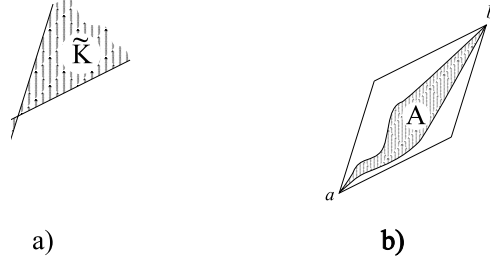


Fig. 2. a) Projectively pointed cone  $\tilde{K}$ , b) A set  $A$  pointed with respect to the cone  $\tilde{K}$ .

this property, then the set  $A$  is pointed with respect to the cone  $\tilde{K} = K \cup (-K)$  and  $a$  and  $b$  are the vertices of  $A$ .

Let's turn now to a projective setting. Let  $N$  be a projective space,  $L \subset N$  be a fixed hyperplane and  $\Delta \subset L$  be a closed convex set in  $L$ .

We say that a connected set  $A \subset N$ , not intersecting the hyperplane  $L$ , is *pointed with respect to the convex set  $\Delta$* , if there exist two points  $a$  and  $b$  in the set  $A$  (so-called *vertices* of the set  $A$  with respect to  $\Delta$ ) such that any hyperplane  $p$  in projective space  $N$ , not intersecting the convex set  $\Delta \subset L$ , intersects  $A$  if and only if  $p$  intersects the segment joining the points  $a$  and  $b$  and lying in the affine space  $N \subset L$ .

This projective definition is a projective reformulation of the affine definition. Indeed, the projective space is a linear space with an added hyperplane at infinity. To the convex set  $\Delta$ , lying in the hyperplane at infinity, corresponds a pointed cone  $\tilde{K}$  equal to the union of all lines passing through the origin and points of the set  $\Delta$ .

According to the Proposition, the set  $A$  in the affine space  $N \setminus L$  is pointed with respect to the cone  $\tilde{K}$  if and only if the set  $A$ , considered as a subset of projective space, is pointed with respect to the convex set  $\Delta = \tilde{K} \cap L$ .

**Families of convex sets affinely dependent on parameters.** We begin with an affine setting. Fix a linear subspace  $N$  of a linear space  $\mathbb{R}^n$ . The linear space  $\mathbb{R}^n$  is fibered by affine subspaces  $N_m$  parallel to  $N$  and parameterized by points  $m$  of a factor-space  $\mathbb{R}^n/N$ . Fix a convex domain  $\Delta$  in the space of parameters  $\mathbb{R}^n/N$ . Suppose that for each point  $m \in \Delta$  in the affine space  $N_m$  a closed convex set  $A_m \subset N_m$  is given.

We say that a *family of convex sets  $\{A_m\}$  depends affinely on parameter  $m \in \Delta$* , if for any two points  $m_1, m_2 \in \Delta$  and any  $0 \leq t \leq 1$ , the set  $A_{m_t}$  corresponding to the parameter  $m_t = tm_1 + (1-t)m_2$  is a linear combination  $tA_{m_1} + (1-t)A_{m_2}$  of sets  $A_{m_1}$  and  $A_{m_2}$  in Minkowski sense.

**Proposition.** *A family of convex sets  $A_m$ ,  $m \in \Delta$  depends affinely on parameters if and only if for any simplex  $\Delta(a_1, \dots, a_k) \subset \Delta$  with linearly independent vertices  $a_1, \dots, a_k \in \Delta$  a convex hull of a union of the sets  $A_{a_1}, \dots, A_{a_k}$  coincides with a union of the sets  $A_m$  for all parameters  $m \in \Delta(a_1, \dots, a_k)$ .*

A particular case of one-dimensional space  $N = (l)$  is especially simple. In this case the convex sets  $A_m$  are simply segments, and the Proposition reads as follows.

**Proposition.** *A family of parallel segments  $A_m$  in  $\mathbb{R}^n$  depends affinely on parameter  $m$  belonging to a convex domain  $\Delta \subset \mathbb{R}^n/(l)$ , if and only if there exist two hyperplanes  $\Gamma_1$  and  $\Gamma_2$  in the space  $\mathbb{R}^n$  such that, first, for any  $m \in \Delta$  ends of the segment  $A(m)$  coincide with points of intersections of the line  $N_m$  with the hyperplanes  $\Gamma_1$  and  $\Gamma_2$ , and, second, projection along  $N$  of an intersection of  $\Gamma_1$  and  $\Gamma_2$  doesn't intersect the interior of  $\Delta$ .*

The general definition of affine dependence on parameter can be reduced, using projections, to the case of one-dimensional space. Let  $Q$  be a subspace of the space  $N$ . A quotient space  $\mathbb{R}^n/Q$  contains a subspace  $\pi(N) = N/Q$ . Spaces  $(\mathbb{R}^n/Q)/\pi(N)$  and  $\mathbb{R}^n/N$  are naturally isomorphic and we will use this isomorphism.

We say that a family of convex sets  $A_m \subset N_m$  depends affinely on parameter  $m \in \Delta \subset \mathbb{R}^n/N$  in direction of the hyperplane  $Q$  in space  $N$ , if after the projection  $\pi: \mathbb{R}^n \rightarrow \mathbb{R}^n/Q$  the segments  $\pi(A_m)$  on lines  $N_m/Q$  depend affinely on parameter  $m \in \Delta$ . (Using the isomorphism of  $(\mathbb{R}^n/Q)/\pi(N)$  and  $\mathbb{R}^n/N$ , we consider  $\Delta \subset \mathbb{R}^n/N$  as a set in  $(\mathbb{R}^n/G)/\pi(N)$ .)

**Theorem.** *A family of convex sets  $A_m \subset N_m$  depends affinely on parameter  $m \in \Delta$  if and only if the family  $A_m \subset N_m$  depends affinely on parameter  $m \in \Delta$  in direction of  $Q$  for any hyperplane  $Q$ .*

**Proof.** Taking a subspace  $M$  transversal to  $N$ , we can identify all parallel spaces  $N_m$  (two points of different sections are identified if they lie in the same translate of  $M$ ). Then all dual spaces  $N_m^*$  are identified with the space  $N^*$  and all support functions  $H_m(\xi) = \max_{x \in A_m} \langle \xi, x \rangle$  of convex sets  $A_m$  can be considered as functions on the same space  $N^*$ .

To a linear combination (in Minkowski sense) of convex sets corresponds a linear combination of their support functions. So the dependence of the family of convex sets  $A_m$  on parameter  $m \in \Delta$  is affine if and only if for any fixed covector  $\xi \in N^*$  the support function  $H_m(\xi)$  is a linear polynomial on parameter  $m$ .

Let's rewrite this condition for  $\xi$  and  $-\xi$  simultaneously. Denote by  $Q$  a hyperplane in  $N$  defined by an equation  $\langle \xi, x \rangle = \langle -\xi, x \rangle = 0$ . Project the set  $A = \bigcup_{m \in \Delta} A_m$  along the space  $Q$ . The projection  $\pi(A)$  lies in the space  $\mathbb{R}^n/Q$  with a marked one-dimensional subspace  $l = N/Q$ . On each line  $l_m$ ,  $m \in \mathbb{R}^n/N = (\mathbb{R}^n/Q)/l$  lies a segment  $\pi(A_m)$  equal to the projection of the convex set  $A_m$ .

By assumption, the segments  $\pi(A_m)$  lie between two hyperplanes  $\Gamma_1$  and  $\Gamma_2$ . Also, the ends  $x(m)$  and  $y(m)$  of these segments lie on the line  $l_m$ , and are defined by equations  $H_m(\xi) = \langle \xi, x(m) \rangle$ ,  $H_m(-\xi) = \langle -\xi, y(m) \rangle$ . Therefore the affine dependence of convex sets  $A_m$ ,  $m \in Q$  in direction  $Q$  means that the support functions  $H_\xi(m)$  and  $H_{-\xi}(m)$ , where  $\xi$  are covectors orthogonal to  $Q$ , are polynomials of first degree in  $m \in \Delta$ . Since this is true for any hyperplane  $Q \subset N$ , the function  $H_\xi(m)$  depends linearly on  $m$  for any fixed  $\xi$ .

Consider now projective settings. Instead of a linear space  $\mathbb{R}^n$  fibered by affine subspaces  $N_m$  parallel to a space  $N$  and parameterized by points of the factor-space  $\mathbb{R}^n/N$ , we will have the following objects: a projective space  $\mathbb{R}P^n$  with a projective subspace  $L$ , fibered by subspaces  $N_m$  of dimension  $\dim N_m = \dim L + 1$  and containing the space  $L$ . The subspaces  $N_m$  are parameterized by points of a factor-space  $M = (\mathbb{R}P^n)/N$ . Consider parameters  $m$  belonging to a convex set  $\Delta \subset M$ .



Let  $T \subset L$  be a hyperplane in  $L$ . Denote a projection of the projective space from the center  $T$  by  $\pi$ . Projection of the space  $L$  is just a point  $\pi(L)$ . Projection of the space  $N$  is a line  $l$  belonging to a bundle of all lines  $l_m = \pi(N_m)$  containing the marked point  $\pi(L)$ . After a natural identification of factor-spaces  $(\mathbb{R}P^n)/L$  and  $(\mathbb{R}P^n/T)/\pi(L)$ , the space  $N_m \subset \mathbb{R}P^n$  and the line  $l_m = \pi(N_m) \subset \mathbb{R}P^n/T$  correspond to the same parameter  $m \in \mathbb{R}P^n/L = (\mathbb{R}P^n/T)/\pi(L)$ . The domain  $\Delta \subset \mathbb{R}P^n/L$  can be considered as a domain in the space  $(\mathbb{R}P^n/T)/\pi(L)$ .

Introduce the following notation. Let  $\Gamma_1$  and  $\Gamma_2$  be two hyperplanes in projective space, not containing the point  $\pi(L)$ , and  $l$  be a line containing this point. Points of intersection of  $\Gamma_1$  and  $\Gamma_2$  with the line  $l$  divide it into two segments. The segment not containing the point  $\pi(L)$  will be called exterior relative to the point  $\pi(L)$  segment between hyperplanes  $\Gamma_1$  and  $\Gamma_2$  on the line  $l$ .

Let  $A$  be a set not intersecting space  $L$ , whose sections  $A_m$  by the spaces  $N_m \supset L$  are convex. We say that *sections  $A_m$  depend affinely on parameter  $m$*  belonging to a convex domain  $\Delta \subset \mathbb{R}P^n/L$  in direction of the hyperplane  $T \subset L$ , if the sections of the set  $\pi(A)$  by lines  $l_m$  containing the point  $\pi(L)$ , depend affinely on  $m \in \Delta \subset \mathbb{R}P^n/L (= (\mathbb{R}P^n/T)/(\pi(L)))$ . In other words, there exist two hyperplanes  $\Gamma_1$  and  $\Gamma_2$  in  $\mathbb{R}P^n/T$ , not containing  $\pi(L)$ , such that, first, the intersection of  $\pi(A)$  with any line  $l_m$ ,  $m \in \Delta$ , is equal to the exterior relative to  $\pi(L)$  segment of the line  $l_m$  lying between  $\Gamma_1$  and  $\Gamma_2$ , and, second, the projection of  $\Gamma_1 \cap \Gamma_2$  on  $\mathbb{R}P^n/L$  doesn't intersect  $\Delta$ .

Now we can give a definition of affine dependence of sections on parameter belonging to a convex domain of the space of parameters.

We say that *sections  $A_m$  of the set  $A$  by projective spaces  $N_m \supset L$  depend affinely on parameter  $m$*  in domain  $\Delta$ , if  $A_m$  depend affinely on parameter  $m$  in domain  $\Delta$  with respect to any hyperplane  $T \subset L$ . The following statement can be easily checked.

**Proposition.** *Let  $\Gamma$  be a projective hyperplane containing the space  $L$ , such that its projection to the space  $(\mathbb{R}P^n)/L$  doesn't intersect a convex set  $\Delta \subset \mathbb{R}P^n/L$ . Consider an affine chart  $U$  of the projective space,  $U = \mathbb{R}P^n \setminus \Gamma$ . Sections of a set  $A \subset \mathbb{R}P^n$ ,  $A \cap L = \emptyset$ , by spaces  $N_m$  depend affinely on parameter  $m$  in domain  $\Delta$ , if and only if the sections of the set  $A \cap U$  in the affine space  $U$  by parallel spaces  $N_m \setminus \Gamma$  depend affinely on parameter  $m$  in domain  $\Delta \subset ((\mathbb{R}P^n)/L) \setminus (\Gamma/L)$ .*

**Duality.** Let  $\Delta$  be a convex domain in the space  $L$ , and let  $\Delta_p^*$  be a dual convex domain in the space  $(\mathbb{R}P^n)^*/L^*$ . The space  $(\mathbb{R}P^n)^*/L^*$  parameterizes  $(\dim L^* + 1)$ -dimensional subspaces of  $(\mathbb{R}P^n)^*$  containing  $L^*$ . The domain  $\Delta^*$  corresponds to subspaces  $Q^* \subset (\mathbb{R}P^n)^*$  of this type dual to subspaces  $Q \subset L$  not intersecting the domain  $\Delta$ .

**Theorem.** *Let  $A$  be a  $L$ -convex-concave subset of a projective space  $\mathbb{R}P^n$ . A section  $A \cap N$  of the set  $A$  by a  $(\dim L + 1)$ -dimensional subspace  $N$  containing  $L$ , is pointed relative to a convex domain  $\Delta \subset L$ , if and only if the following dual condition holds:  $L$ -dual to the  $A$  subset  $A_L^\perp$  of the dual space  $(\mathbb{R}P^n)^*$  depends affinely on parameters belonging to the domain  $\Delta_p^*$  in direction of the hyperplane  $N^* \subset L^*$ .*

**Proof.** The set  $A$  is  $L$ -convex-concave, so the section  $A \cap N$  is dual to the complement to the projection from the center  $N^* \subset L^*$  of the set  $A_L^\perp$ .

Let  $a$  and  $b$  be vertices of the pointed set  $A \cap N$  relative to the convex set  $\Delta \subset L$ . Fix a hyperplane  $q_L$  in the space  $L$ , not intersecting the convex set  $\Delta \subset L$ .

Consider a one-dimensional bundle  $\{p^t\}$  of hyperplanes containing the space  $q_L$  in space  $N$ . This bundle contains the following three hyperplanes: the hyperplane  $L$ , a hyperplane  $p_a$  containing the vertex  $a$  of the set  $A$ , and hyperplane  $p_b$ , containing the vertex  $b$  of the set  $A$ .

Take a segment  $[p_a, p_b]$  with ends corresponding to  $p_a$  and  $p_b$  and not containing the point  $L$  on a projective line corresponding to the one-dimensional bundle of hyperplanes  $\{p^t\}$ . Any hyperplane  $p^t$  (except the hyperplane  $L$  itself) intersects  $L$  by a subspace  $q_L$ , and  $q_L$  doesn't intersect  $\Delta$ . The set  $A$  is pointed with respect to  $\Delta$ , so a hyperplane  $p^{t_0}$  intersects  $A \cap N$  if and only if the point  $p^{t_0}$  belongs to the segment  $[p_a, p_b]$ .

Consider the dual space  $(\mathbb{R}P^n)^*$ . To the section  $A \cap N$  of the set  $A$  corresponds a projection of the set  $A_L^\perp$  from the center  $N^*$ . To hyperplanes in  $N$  correspond points in a factor-space  $(\mathbb{R}P^n)^*/N^*$ . In particular, to the hyperplane  $L$  in  $N$  corresponds the marked point  $\pi(L^*)$  in the factor-space  $(\mathbb{R}P^n)^*/N^*$ , namely the projection of the space  $L^*$  from the center  $N^*$ . To the bundle of hyperplanes  $\{p^t\}$  corresponds a line passing through  $\pi(L^*)$ . This line intersects projection of the set  $A_L^\perp$  exactly by a segment  $[p^a, p^b]$  not containing the point  $\pi(L^*)$ .

To different hyperplanes  $q_L$  in space  $L$  correspond different one-dimensional bundles of hyperplanes  $\{p^t\}$  in  $N$ , i.e. different lines in  $(\mathbb{R}P^n)^*/N^*$ , containing the marked point  $\pi(L^*)$ . The hyperplane  $q_L$  in space  $L$  doesn't intersect set  $\Delta$ , so the dual space  $q_L^* \supset L^*$  is parameterized by a point of  $\Delta_p^* \subset (\mathbb{R}P^n)^*/L^*$ . Projection of the space  $q_L^*$  from the center  $N^*$  is a line in the space  $(\mathbb{R}P^n)^*/N$ , parameterized by the same point of the domain  $\Delta^*$ . Each such line intersects projection of the set  $A_L^\perp$  by segment  $[p^a, p^b]$ . The point  $p^a$  lies in the hyperplane  $\Gamma^a$  in the space  $(\mathbb{R}P^n)^*/N^*$  dual to the point  $a \in N$ . The point  $p^b$  lies in the hyperplane  $\Gamma^b$  in the space  $(\mathbb{R}P^n)^*/N^*$  dual to the point  $b \in N$ .

Two hyperplanes  $\Gamma^a$  and  $\Gamma^b$  divide the space  $(\mathbb{R}P^n)^*/N^*$  into two parts. Denote by  $\Gamma(a, b)$  the closure of the part not containing the point  $\pi(L^*)$ . We just proved that the set  $\Gamma(a, b)$  and projection of the set  $A_L^\perp$  to the space  $(\mathbb{R}P^n)^*/N^*$  have the same intersections with lines passing through the point  $\pi(L^*)$ , and parameterized by point of the domain  $\Delta_p^*$ . The theorem proved.

### §5. $L$ -convex-concave sets with planar sections being octagons with four pairs of parallel sides

Consider a subset  $A$  of  $\mathbb{R}P^n$  convex-concave with respect to a one-dimensional space  $L$ . Fix four points  $a_1, \dots, a_4$  lying on the line  $L$  in this order. These points divide  $L$  into four pairwise non-intersecting intervals  $\langle a_1, a_2 \rangle$ ,  $\langle a_2, a_3 \rangle$ ,  $\langle a_3, a_4 \rangle$ ,  $\langle a_4, a_1 \rangle$ . Denote their complements to  $L$  by  $I_1 = [a_1, a_2] = L \setminus \langle a_1, a_2 \rangle, \dots, I_4 = [a_4, a_1] = L \setminus \langle a_4, a_1 \rangle$  (these segments are intersecting). In this paragraph we prove the Main Hypothesis for  $L$ -convex-concave sets  $A$  whose sections  $N \cap A$  by two-dimensional planes  $N$  containing the line  $L$ , are pointed relative to the segments  $I_1, \dots, I_4$ .

**Theorem.** *Suppose that all planar sections  $A \cap N$  of a  $L$ -convex-concave set  $A$ ,  $A \subset \mathbb{R}P^n$ ,  $\dim L = 1$ , by two-dimensional planes  $N$  containing  $L$ , are pointed with respect to four segments  $I_1, \dots, I_4$  on the line  $L$ . Suppose that the union of  $I_i$  coincides with  $L$  and that the complements  $L \setminus I_i$  are pairwise non-intersecting. Then the set  $A$  contains a projective space  $M$  of dimension  $(n - 2)$ .*

Before the proof we will make two remarks.

First, the assumptions of the theorem about the convex-concave set  $A$ , are easier to understand in an affine chart  $\mathbb{R}^n$  not containing the line  $L$ . In this chart the family of two-dimensional planes containing  $L$  becomes a family of parallel two-dimensional planes. In the space  $\mathbb{R}^n$  four classes of parallel lines are fixed, each passing through one of the points  $a_1, \dots, a_4$  of the line  $L$  at infinity. The assumptions of the theorem mean that each section of the set  $A$  by a plane  $N$  is an octagon with sides belonging to these four fixed classes of parallel lines. (Some sides of this octagon can degenerate to a point, and number of sides of the octagon  $(A \cap N)$  will then be smaller than 8.)

Also, there is a natural isomorphism between  $(\mathbb{R}P^1)^*$  and  $\mathbb{R}P^1$ . Indeed, each point  $c \in \mathbb{R}P^1$  of a projective line is also a hyperplane in  $\mathbb{R}P^1$ . However, a segment  $[a, b]$  on the projective line  $\mathbb{R}P^1$  will be dual to its complement  $\langle a, b \rangle = (\mathbb{R}P^1) \setminus [a, b]$ , and not to itself. Indeed, by definition, a dual to a convex set  $\Delta$  set  $\Delta_p^*$ , consists of all hyperplanes not intersecting  $\Delta$ .

**Proof of the Theorem.** Consider the dual projective space  $(\mathbb{R}P^n)^*$  and its subspace  $L^*$ ,  $\dim L^* = (n - 2)$ , dual to the line  $L$ . Projective line  $(\mathbb{R}P^n)^*/L^*$ , isomorphic to a line dual to  $L$ , is divided by points  $a_1^*, \dots, a_4^*$  into four intervals  $\langle a_1^*, a_2^* \rangle$ ,  $\langle a_2^*, a_3^* \rangle$ ,  $\langle a_3^*, a_4^* \rangle$ ,  $\langle a_4^*, a_1^* \rangle$ , dual to segments  $I_1, \dots, I_4$ . The set  $A_L^\perp$ ,  $L$ -dual to the set  $A$ , will depend affinely on parameter on these intervals, since the set  $A$  is pointed relative to the segments  $I_1, \dots, I_4$ . Therefore the set  $A_L^\perp$  is a linear interpolation of its four sections. In other words, this set has four sections by planes corresponding to  $a_1^*, \dots, a_4^*$ , and all other sections of  $A_L^\perp$  are affine combinations (in Minkowski sense) of sections corresponding to the ends of intervals.  $L^*$ -convex-concave sets of this type contain a line (see Introduction and our paper “A convex-concave domain in  $\mathbb{R}P^3$  contains a line”, in preparation). Denote this line by  $l$ . The set  $A$  will contain an  $(n - 2)$ -dimensional space  $l^* \subset \mathbb{R}P^n$  dual to the line  $l$ , q.e.d.

## §6. Surgeries on convex-concave sets

In this section we describe two special surgeries on  $L$ -convex-concave subsets of  $\mathbb{R}P^n$ , one applicable when  $\dim L = n - 2$  and another when  $\dim L = 1$ . These two surgeries are dual.

**The first surgery:**  $\dim L = n - 2$ . To a  $(n - 2)$ -dimensional subspace  $L$  of  $\mathbb{R}P^n$  corresponds a one-dimensional bundle of hyperplanes containing  $L$ . These hyperplanes are parameterized by points of the projective line  $\mathbb{R}P^n/L$ . Fix two points  $a$  and  $b$  and a segment  $[a, b]$  on this line, one of two segments into which the points  $a$  and  $b$  divide the projective line  $\mathbb{R}P^n/L$ .

For any  $L$ -convex-concave set  $A$  and the segment  $[a, b] \subset \mathbb{R}P^n/L$  we define a set  $S_{[a, b]}(A)$ , which is also  $L$ -convex-concave. Here is the definition of the set  $S_{[a, b]}(A)$ . The hyperplanes  $\Gamma_a$  and  $\Gamma_b$  corresponding to parameters  $a$  and  $b$ ,  $L = \Gamma_a \cap \Gamma_b$ , divide the set  $\mathbb{R}P^n \setminus L$  into two half-spaces: the first half-space  $\Gamma^1[a, b]$  is projected to the segment  $[a, b]$ , and the second one  $\Gamma^2[a, b]$  is projected to its complement.

Let  $c$  be some point on the line  $\mathbb{R}P^n/L$  not belonging to the segment  $[a, b]$ , and let  $\Gamma_c$  be the corresponding hyperplane in  $\mathbb{R}P^n$ .

**Definition.** The set  $S_{[a, b]}(A)$  is defined by the following requirements:

- 1) the set  $S_{[a, b]}(A)$  doesn't intersect the space  $L$ , i.e.  $S_{[a, b]}(A) \cap L = \emptyset$ ;
- 2) the set  $S_{[a, b]}(A) \cap \Gamma_{[a, b]}^1$  coincides with a convex hull of the union of sections  $A \cap \Gamma(a)$  and  $A \cap \Gamma(b)$  in an affine chart  $\mathbb{R}P^n \setminus \Gamma_c$ ;

3) the set  $S_{[a,b]}(A) \cap \Gamma_{[a,b]}^2$  coincides with  $A \cap \Gamma_{[a,b]}^2$ .

It is easy to see that the set  $S_{[a,b]}(A)$  is correctly defined, i.e. it doesn't depend on the choice of the hyperplane  $\Gamma_c$ .

**Theorem.** *For any  $L$ -convex-concave set  $A$  the set  $S_{[a,b]}(A)$  is also  $L$ -convex-concave.*

**Proof.** Any section of the set  $S_{[a,b]}(A)$  by a hyperplane  $\Gamma_d$  containing  $L$ , is convex. Indeed, if  $d \notin [a, b]$ , then  $\Gamma_d \cap S_{[a,b]}(A) = \Gamma_d \cap A$ , and the set  $\Gamma_d \cap A$  is convex by definition. Otherwise, i.e. if  $d \in [a, b]$ , the  $\Gamma_d \cap S_{[a,b]}(A)$  is a linear combination (in Minkowski sense) of convex sections  $A \cap \Gamma_a$  and  $A \cap \Gamma_b$  (in any affine chart  $\mathbb{R}P^n \setminus \Gamma_c$ ,  $c \notin [a, d]$ ), so is convex.

Let's prove that a complement to a projection of the set  $S_{[a,b]}(A)$  from any center  $L_1 \subset L$ , where  $L_1$  is a hyperplane in  $L$ , is a convex open set. Consider a projection  $\pi(A)$  of the set  $A$  on the projective plane  $\mathbb{R}P^n/L_1$ . The set  $A$  is  $L$ -convex-concave, so the complement  $B$  to the projection  $\pi(A)$  is an open convex set, containing the marked point  $\pi(L)$ . The plane  $\mathbb{R}P^n/L_1$  contains two lines,  $l_a = \pi(\Gamma_a)$  and  $l_b = \pi(\Gamma_b)$ , passing through the point  $\pi(L)$ , a half-plane  $l_{[a,b]}^1 = \pi(\Gamma_{[a,b]}^1)$  and a complementary half-plane  $l_{[a,b]}^2 = \pi(\Gamma_{[a,b]}^2)$ .

From the definition of the set  $S_{[a,b]}(A)$  we see that the complement  $B_{[a,b]}$  to its projection  $\pi(S_{[a,b]}(A))$  has the following structure.

- 1) set  $B_{[a,b]}$  contains the point  $\pi(L)$ ;
- 2) Consider two closed triangles with vertices at the point  $\pi(L)$ , lying in  $l^1[a, b]$ , with one side being the segment lying inside  $l^1[a, b]$  and joining the points of intersection of lines  $l_a$  and  $l_b$  with the boundary of the domain  $B$  (see Fig. 3). The set  $B_{[a,b]} \cap l^1[a, b]$  is a union of these triangles with the sides described above removed;
- 3) set  $B_{[a,b]} \cap l^2[a, b]$  coincides with the set  $B \cap l^2[a, b]$ .

From this description we see that the set  $B_{[a,b]}$  is convex and open, q.e.d.

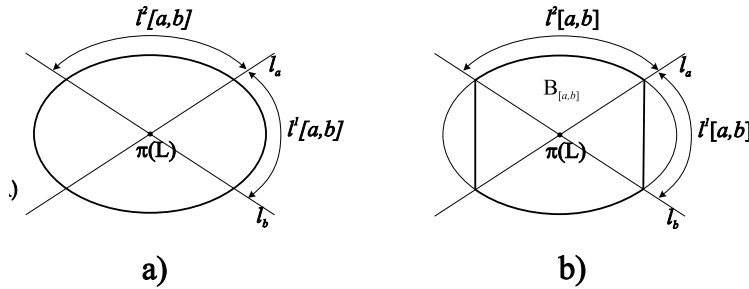


Fig. 3. a) The complement  $B$  to the projection  $\pi(A)$  of the set  $A$ , b) The complement  $B_{[a,b]}$  to the projection  $\pi(S_{[a,b]}(A))$  of the set  $S_{[a,b]}(A)$ .

If two segment  $[a, b]$  and  $[c, d]$  on the line  $\mathbb{R}P^n/L$  do not have common interior points, then the surgeries  $S_{[a,b]}$  and  $S_{[c,d]}$  commute. We can divide the line  $\mathbb{R}P^n/L$  into a finite set of segments  $[a_1, a_2], \dots, [a_{k-1}, a_k], [a_k, a_1]$  and apply to a  $L$ -convex-concave set  $A$  the surgeries corresponding to these segments. As a result we will get a  $L$ -convex-concave set  $D$ , such that the sections of  $D$  by hyperplanes  $\Gamma_{a_1}, \dots, \Gamma_{a_n}$  coincide with sections  $A \cap \Gamma_{a_i}$  of the set  $A$  by the same hyperplanes. For an

intermediate value of parameter  $a_i < a < a_{i+1}$  the section  $D \cap \Gamma_a$  coincides with the section by the same hyperplane of the convex hull of the union of sections  $A \cap \Gamma_{a_i}$  and  $A \cap \Gamma_{a_{i+1}}$  in affine chart  $\mathbb{R}P^n \setminus \Gamma_c$  (where  $c$  is any point of the line  $\mathbb{R}P^n/L$ , not belonging to the segment  $[a_i, a_{i+1}]$ ).

**The second surgery:**  $\dim L = 1$ . To a one-dimensional space  $L$  corresponds a  $(n - 2)$ -dimensional bundle of two-dimensional planes containing the line  $L$ . Fix two points  $a$  and  $b$  and a segment  $[a, b]$  on the line  $L$  — one of two segments into which the points  $a$  and  $b$  divide the line  $L$ . For any  $L$ -convex-concave set  $A$  and the segment  $[a, b] \subset L$  we construct a new  $L$ -convex-concave set  $P_{[a,b]}(A)$ . A section of  $P_{[a,b]}(A)$  by any two-dimensional plane  $N$ ,  $N \supset L$ , will depend on the section of the set  $A$  by this plane  $N$  only.

We define first an operation  $F_{[a,b]}$  applicable to two-dimensional convex sets. This operation  $F_{[a,b]}$  transforms planar sections  $A \cap N$  of the set  $A$  to planar sections  $P_{[a,b]}(A) \cap N$  of the set  $P_{[a,b]}(A)$ .

Consider a two-dimensional projective plane  $N$  with a distinguished projective line  $L$  and a segment  $[a, b] \subset L$ . Let  $\Delta \subset N$  be any closed convex subset of the plane  $N$ , not intersecting the line  $L$ .

By definition, the operation  $F_{[a,b]}$  transforms a set  $\Delta \subset N$  to the smallest convex set  $F_{[a,b]}(\Delta)$  containing the set  $\Delta$  and pointed relative to the segment  $[a, b]$ .

Here is a more explicit description of the set  $F_{(a,b)}(\Delta)$ .

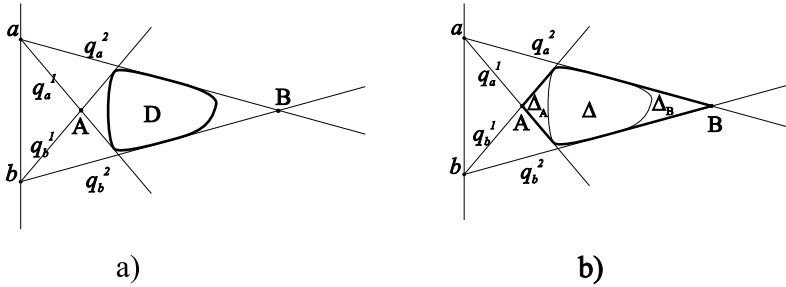


Fig. 4. a) The set  $\Delta$  and tangents to it passing through the points  $a$  and  $b$ , b) The set  $F_{(a,b)}(\Delta)$ .

Draw four tangents,  $q_a^1, q_a^2$  and  $q_b^1, q_b^2$ , to the set  $\Delta$  passing through the points  $a$  and  $b$  correspondingly (see Fig. 4). In the convex quadrangle  $\Delta_1$  in the affine plane  $N \setminus L$  with sides on the lines  $q_a^1, q_a^2$  and  $q_b^1, q_b^2$  there are exactly two vertices  $A$  and  $B$  satisfying the following condition: the support lines to the quadrangle  $\Delta_1$  at the vertex do not intersect the segment  $[a, b]$ . To the vertex  $A$  corresponds a curvilinear triangle  $\Delta_A$  with two sides lying on two sides of the quadrangle  $\Delta_1$  joint to the vertex  $A$ . The third side of  $\Delta_A$  coincides with the part of the boundary of the set  $\Delta$  visible from the point  $A$ .

A similar curvilinear triangle  $\Delta_b$  corresponds to the vertex  $B$ . Evidently the set  $F_{[a,b]}(\Delta)$  coincides with the set  $\Delta_A \cup \Delta \cup \Delta_B$ .

Now we can define the set  $P_{[a,b]}(A)$ .

**Definition.** For any  $L$ -convex-concave subset  $A$  of  $\mathbb{R}P^n$ ,  $\dim L = 1$ , and for any segment  $[a, b]$  of the line  $L$  we define the set  $P_{[a,b]}(A)$  by the following condition: a section  $P_{[a,b]}(A) \cap N$  of this set by any two-dimensional plane  $N$  containing  $L$  is

obtained from the section  $A \cap N$  of the set  $A$  by operation  $F_{[a,b]}$  in the plane  $N$ :  $P_{[a,b]}(A) \cap N = F_{[a,b]}(A \cap N)$ .

**Theorem.** *For any  $L$ -convex-concave set  $A$ ,  $\dim L = 1$ , and any segment  $[a, b] \subset L$  on the line  $L$  the set  $P_{[a,b]}(A)$  is  $L$ -convex-concave.*

**Proof.** To any  $L$ -convex-concave set  $A$  in  $\mathbb{R}P^n$  corresponds its  $L$ -dual  $D = (A_L^\perp)$  in the dual projective space  $(\mathbb{R}P^n)^*$ . The set  $D$  is a  $L^*$ -convex-concave set, and  $\dim L^* = n - 2$ . The line  $L$  is dual to the set of parameters  $(\mathbb{R}P^n)^*/L^*$ . To the segment  $[a, b] \subset L$  corresponds a dual interval  $\langle a^*, b^* \rangle \subset (\mathbb{R}P^n)^*/L^*$ . By the segment  $[a^*, b^*] \subset (\mathbb{R}P^n)^*/L^*$  and the  $L^*$ -convex-concave set  $D = A_L^\perp$  we define a new  $L^*$ -convex-concave set  $S_{[a^*, b^*]}(D)$ . To prove the theorem it is enough to check that the set  $P_{[a,b]}(A)$  is  $L^*$ -dual to the set  $S_{[a^*, b^*]}(D)$ , where  $D = A_L^\perp$ . This is proved below.

**Proposition.** *The set  $P_{[a,b]}(A)$  is  $L^*$ -dual to the set  $S_{[a^*, b^*]}(D)$ .*

**Proof.** We proved above that if the set  $D$  is  $L^*$ -convex-concave, then the set  $S_{[a^*, b^*]}(D)$  is also  $L^*$ -convex-concave and described how to obtain the planar projections of the set  $S_{[a^*, b^*]}(D)$  from the planar projections of the set  $D$ .

Consider the sets  $D_{L^*}^\perp = A$  and  $S_{[a^*, b^*]}(D)_{L^*}^\perp$   $L^*$ -dual to the sets  $D$  and  $S_{[a^*, b^*]}(D)$  correspondingly. Planar projections of the sets  $D$  and  $S_{[a^*, b^*]}(D)$  are dual to the planar sections of the sets  $A$  and  $(S_{[a^*, b^*]}(D))_{L^*}^\perp$ . Looking on the planar pictures, one easily sees that sections of the set  $(S_{[a^*, b^*]}(D))_{L^*}^\perp$  are obtained from the sections of the set  $A$  by the surgery  $F_{[a,b]}$ . Therefore  $(S_{[a^*, b^*]}(D))_{L^*}^\perp = P_{[a,b]}(A)$ .

If the complements  $\langle a, b \rangle_0$  and  $\langle c, d \rangle_0$  to the segments  $[a, b]$  and  $[c, d]$  do not intersect, then the operations  $P_{[a,b]}$  and  $P_{[c,d]}$  commute. Divide the line  $L$  into a finite number of intervals  $\langle a_1, a_2 \rangle_0, \dots, \langle a_{k+1}, a_1 \rangle_0$ , complementary to segments  $[a_1, a_2], \dots, [a_{k+1}, a_1]$  (the segments intersect one another) and apply to the  $L$ -convex-concave set  $A$  the operations  $P_{[a_i, a_{i+1}]}(A)$  corresponding to all these segments. As a result we will get a  $L$ -convex-concave set  $D$ , whose section by any two-dimensional plane  $N$  containing the line  $L$ , is a polygon with  $2k$  sides circumscribed around the section  $A \cap N$  (some of the sides of the resulting polygons can degenerate into points). To each point  $a_i$  correspond two parallel sides of the polygon passing through the point  $a_i$  and lying on the support lines to the section  $(A \cap N)$ .

**Remark.** To a three-dimensional set  $A \subset \mathbb{R}P^3$ ,  $L$ -convex-concave with respect to a line  $L$ , both surgeries are applicable, since  $\dim L = 1 = n - 2$  for  $n = 3$ . Let  $[a, b]$  be a segment on the line  $L$ , and  $[c, d]$  be a segment on the line  $\mathbb{R}P^3/L$ . Then, as can be easily proved, the surgeries  $P_{[a,b]}$  and  $S_{[c,d]}$  commute.

**A space intersecting support half-planes to sections.** Let, as before,  $A$  be a  $L$ -convex-concave subset of  $\mathbb{R}P^n$ , and  $\dim L = 1$ . Consider the following problem. Suppose that a certain set  $\{N_\alpha\}$ ,  $\alpha \in I$ , of two-dimensional planes containing the line  $L$ , is fixed, and suppose that on each affine plane  $N_\alpha \setminus L$  some supporting to a convex section  $N_\alpha \cap A$  half-plane  $p_\alpha^+ \subset N_\alpha$  is fixed. We want to find an  $(n - 2)$ -dimensional subspace of  $\mathbb{R}P^n$ , intersecting all half-planes  $p_\alpha^+$ ,  $\alpha \in I$ .

**Theorem.** *Suppose that the set  $Q = \{\partial p_\alpha^+ \cap L\}$ ,  $\alpha \in I$ , contains at most four points, where  $\partial p_\alpha^+$  denotes the boundary line of half-plane  $p_\alpha^+$  supporting to the*

section  $N_\alpha \cap A$  of a  $L$ -convex-concave set  $A \subset \mathbb{R}P^n$ . Then there exists an  $(n-2)$ -dimensional subspace of  $\mathbb{R}P^n$  intersecting all supporting half-planes  $p_\alpha^+$ ,  $\alpha \in I$ .

**Proof.** Suppose that the set  $Q$  contains exactly four points  $a_1, \dots, a_4$  (if not, add to  $Q$  a necessary amount of some other points). The points  $a_i$  divide the projective line into four segments  $\langle a_1, a_2 \rangle$ ,  $\langle a_2, a_3 \rangle$ ,  $\langle a_3, a_4 \rangle$ ,  $\langle a_4, a_1 \rangle$ . Denote by  $I_1, \dots, I_4$  the complementary segments (these segments intersect one another). Apply to the set  $A$  the four surgeries  $P_{I_i}$  and denote the resulting set by  $D$ .

By the very definition of the set  $D$  the half-planes  $p_\alpha^+ \subset N_\alpha$  are supporting half-planes for the sections  $D \cap N_\alpha$ , so any space lying inside  $D$ , will intersect the half-planes  $p_\alpha^+$ . According to the theorem of §5 there exists an  $(n-2)$ -dimensional subspace of  $\mathbb{R}P^n$ , lying inside the set  $D$ . This space intersects all half-planes  $\pi_\alpha^+$ .

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