

SYSTEMS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH BOUNDED COEFFICIENTS MAY HAVE VERY OSCILLATING SOLUTIONS

D. NOVIKOV

ABSTRACT. An elementary example shows that the number of zeroes of a component of a solution of a system of linear ordinary differential equations cannot be estimated through the norm of coefficients of the system.

Bounds for oscillations. In [1] it was shown that a linear ordinary differential equation of order n , with real analytic coefficients bounded in a neighborhood of the interval $[-1, 1]$, admits a uniform upper bound for the number of isolated zeros of a solution defined on this interval. The analyticity condition can be relaxed; only the boundedness of the coefficients matters. Probably, the simplest result in this spirit is the following theorem for the linear ordinary differential equation

$$y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \dots + a_n(t)y(t) = 0 \quad (1)$$

with continuous coefficients on $[\alpha, \beta] \subset \mathbb{R}$.

Theorem 1 ([3, 4]). *If the coefficients of the differential equation (1) are uniformly bounded by the constant $C \geq 1$ (that is, $\max\{|a_i(t)| : i = 1, \dots, n\} \leq C$), then a solution defined on $[\alpha, \beta]$ cannot have more than $n - 1 + \frac{n}{\ln 2}C|\beta - \alpha|$ isolated zeros.*

An analog of this result for a *system* of ordinary differential equations, viewed as a vector field in space, would concern the number of isolated intersections between integral trajectories of the vector field and hyperplanes (or, more generally, hypersurfaces). For *polynomial* systems of degree d on \mathbb{R}^n of the form

$$\dot{x}_i = v_i(t, x), \quad i = 1, \dots, n, \quad v_i(t, x) = \sum_{k+|\alpha| \leq d} v_{ik\alpha} t^k x^\alpha, \quad (2)$$

and algebraic hypersurfaces given by $\{P = 0\}$ where $P = P(t, x)$ is a polynomial of degree d , the following theorem, proved in [3] (see also [2]), gives a bound for the number of isolated intersections in case the magnitude of the domain of the solution and the amplitude of the solution are controlled by the *height* of the polynomial system, that is, the number $\max\{|v_{ik\alpha}| : k + |\alpha| \leq d, i = 1, \dots, n\}$.

Theorem 2. *Suppose that the height of system (2) is bounded by the positive constant C . If γ is an orbit of the system contained in the box $B_C = \{(t, x) \in \mathbb{R}^{n+1} : |t| < C, |x_i| < C\}$, then the number of isolated intersections of γ and $\{P = 0\}$ is at most $(2 + R)^B$ where $B = B(n, d)$ is an explicit elementary function of d and n whose growth rate is smaller than $\exp \exp \exp \exp(4n \ln d + O(1))$ as $d, n \rightarrow \infty$.*

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As mentioned in [3], Theorem 2 is nontrivial even for linear systems

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad A(t) = \sum_{k=0}^d A_k t^k \quad (3)$$

and linear hyperplanes $\{\sum_{i=1}^n p_i x_i = 0\}$. In this case, the box condition reduces to the requirement that $t \in [-C, C]$; the height condition reduces to the uniform boundedness of the norms of the matrix coefficients $A_k \in \text{Mat}_{n \times n}(\mathbb{R})$.

Corollary 3. *If, for system (3), $\max\{\|A_k\| : k = 0, \dots, d\} < C$, then there is a uniform bound (expressible as an elementary function of C) for the number of isolated zeros of every component of every (vector) solution defined on the interval $[-C, C]$.*

A comparison of Theorems 1 and 2 suggests the following question: Can the height condition on the polynomial vector field in Theorem 2 be replaced, for instance, by a bound on the norm $\max_{i=1, \dots, n, (t,x) \in B_C} |v_i(t, x)|$; or, in Corollary 3, can it be replaced by a bound on the norm $\max_{t \in [-C, C]} \|A(t)\|$? We will show that *this is impossible*.

The example. For each integer d (no matter how large), there is a linear 2×2 system (3) of degree $2d$ with $\max_{t \in [-1, 1]} \|A(t)\| \leq 1$ such that a component of one of its solutions in the box B_1 has d isolated zeros in the interval $[-1, 1]$.

Let t_1, \dots, t_d be distinct numbers in the interval $[-1, 1]$ and let $a(t) := \lambda(t - t_1) \cdots (t - t_d)$ where λ is a number chosen so small that $|a(t)| + |\dot{a}(t) + a^2(t)| < 1$ whenever $t \in [-1, 1]$. While the solution $\phi_1(t) = \exp(\int_0^t a(s) ds)$ of the differential equation $\dot{x}_1 = a(t)x_1$ has no zeroes, its derivative $\phi_2 = \dot{\phi}_1 = a(t)\phi_1$ has d zeros and also satisfies the equation $\dot{\phi}_2 = (\dot{a} + a^2)\phi_1$. Hence, the supremum over $[-1, 1]$ of the coefficient matrix of the degree $2d$ polynomial linear system

$$\dot{x}_1 = a(t)x_1, \quad \dot{x}_2 = (\dot{a}(t) + a(t)^2)x_1 \quad (4)$$

is bounded by 1, and the second component of the solution $t \mapsto (\phi_1(t), \phi_2(t))$ has d isolated zeros in this interval. Moreover, because the system is linear, a constant multiple of this solution is in the box B_1 .

Remark 1. The example shows that the bound stated in Theorem 1 cannot be extended to derivatives of solutions. Also, by choosing λ sufficiently small, the coefficients of system (4) can be made uniformly small in every preassigned *complex* neighborhood of the real segment $[-1, 1]$. Hence, the bounds for oscillation with respect to hyperplanes cannot be achieved in the spirit of [1] by imposing bounds for analytic coefficients in the complex domain.

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DEPARTMENT OF MATHEMATICS, TORONTO UNIVERSITY, TORONTO, CANADA
E-mail address: `dmitry@math.toronto.edu`