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# INTEGRAL CURVATURES, OSCILLATION AND ROTATION OF SPATIAL CURVES AROUND AFFINE SUBSPACES

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Abstract. The main result of the paper is an upper bound for oscillation of spatial curves around geodesic subspaces of the ambient space in terms of the integral geodesic curvatures of the curves.

Let $\mathbb{M}^n$ be the Euclidean space $\mathbb{R}^n$, projective space $\mathbb{P}^n$ or the sphere $\mathbb{S}^n$ equipped with the Riemannian metric of Gaussian curvature $c(\mathbb{M}) = 0, 1$ or $r^{-n} > 0$ respectively, and $\Gamma \subset \mathbb{M}$ be a smooth curve parameterized by the arc length $s \in [0, \ell]$.

For such curves the (geodesic) Frenet curvatures $\varkappa_1(s), \dots, \varkappa_{n-1}(s)$ can be defined, the last one up to the choice of sign in the non-orientable case of $\mathbb{P}^n$. The *generalized inflection points* are defined by the condition that the last curvature $\varkappa_{n-1}(s)$ vanishes.

We prove that the number of intersections of $\Gamma$ with an arbitrary affine hyperplane $L_{n-1} \subset \mathbb{R}^n$ (respectively, any equator of codimension 1 in the sphere or a projective hyperplane in $\mathbb{P}^n$) can be at most $1/\pi$ times the sum $w_0 K_n(\Gamma) + w_1 K_{n-1}(\Gamma) + \cdots + w_{n-1} K_1(\Gamma) + w_n K_0(\Gamma) + w_{n+1} K_{-1}(\Gamma)$, where:

$K_1(\Gamma), \dots, K_{n-1}(\Gamma)$ are (absolute) integral Frenet curvatures of $\Gamma$,

$K_n(\Gamma) = \pi \times$ (number of generalized inflection points),

$K_0(\Gamma) = c^{1/n}(\mathbb{M}) \cdot |\Gamma|$, where $|\Gamma|$ is the Riemannian length of $\Gamma$,

$K_{-1}(\Gamma) = 0$ or $\pi$ is $\pi/2$ times the number of endpoints of $\Gamma$,

$w_0 = w_1 = 1$, $w_2 = 2$, $w_j = j - 1$ for $j \geqslant 3$ are the universal weights.

For curves in the Euclidean case $\mathbb{M} = \mathbb{R}^n$ a similar estimate can be found for properly defined *rotation around affine subspaces* of arbitrary dimension $k$ between 0 and $n-2$. We show that this rotation can be at most $w_0 K_{k+1}(\Gamma) + \cdots + w_{k-1} K_1(\Gamma) + w_{k+1} K_{-1}(\Gamma)$, where the term $w_k K_0(\Gamma)$ is missing since $c(\mathbb{R}^n) = 0$.

The proof is based on arguments from integral geometry (alias geometric probability) and non-oscillation theory for ordinary linear equations.

## 1. Oscillation and rotation around affine subspaces

The principal question addressed in this paper, can be formulated as follows: given a smooth curve $\Gamma$ in the Euclidean space $\mathbb{R}^n$ with known integral Frenet curvatures $\int_\Gamma |\varkappa_j(s)|\, ds$, $j = 1, \dots, n-1$, give an upper bound for the number of intersections of $\Gamma$ with any affine hyperplane $A \subset \mathbb{R}^n$, $\dim A = n - 1$, and, more

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generally, estimate from above the "rotation" of $\Gamma$ around any affine subspace of codimension greater than 1 (the notion of rotation needs yet to be defined).

This problem was studied in various settings in numerous publications (especially the hyperplanar case). Our work was inspired by a beautiful paper by John Milnor [1], in which many ideas developed below, were already present.

We start with the set of definitions and list some known results concerning oscillatory properties of curves, referring to them as Facts. One of these facts, a theorem by B. Shapiro [11, 12], is a topological rather than metric assertion that implies a metric result that we formulate as Theorem 1.

Then the main result of the paper, Theorem 2, is formulated. The proof of Theorem 2 is given in §2. It rests upon two auxiliary results from integral geometry (a multidimensional generalization of Fáry theorem and the averaging property of the rotation index), and a variation on the theme of Pólya theorem. The proof of Theorem 2 is derived from these auxiliary results, which in turn are proved in §3 and §4 respectively. In §1.5 we formulate and in §2.5 prove the counterpart of Theorem 2 for spherical and projective curves. The last section §5 contains an alternative proof of Theorem 2 for curves in $\mathbb{R}^3$ using isoperimetric inequalities on the sphere.

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1.1. **Settings and definitions.** Everywhere below $|X|$ means the natural measure of a set $X$ (the number of points if $X$ is a discrete set, the length of a curve, the area of a surface *etc*).

We consider a smooth curve $\Gamma$ in the Euclidean space $\mathbb{R}^n$, parameterized as $t \mapsto x(t)$, $t \in I = [0, \ell]$, $x(t) = \big(x_1(t), \dots, x_n(t)\big) \in \mathbb{R}^n$. Sometimes we assume that the parametrization is *natural*, i.e. the parameter is the arc length measured along $\Gamma$; in this case we denote it by $s$.

1.1.1. *Integral curvatures and integral inflection.* Denote by $v_k(t)$ for $k = 1, \dots, n$, the vectors of the *osculating frame*: by definition,

$$v_k \colon I \to \mathbb{R}^n, \quad t \mapsto v_k(t) = \frac{d^k}{dt^k} x(t), \qquad k = 1, \dots, n.$$

**Definition 1** (regular curves, inflections, hyperconvexity)**.** A parameterized curve $\Gamma$ is *regular*, if the first $n-1$ vectors $v_k(t)$, $k=1,\dots,n-1$ are linear independent for all $t \in I$, while the complete set $\{v_k(t)\}_{k=1}^n$ is linear dependent only at isolated points of $\Gamma$. The points where the osculating frame degenerates, are called (generalized) *inflection points*, by analogy with the planar case. A curve without inflection points is referred to as *hyperconvex*.

*Remark* . By Thom transversality theorem, a *generic* smooth curve is regular.

For a regular curve $\Gamma$ parameterized by the natural parameter $s$ (the arc length), the osculating frame $\{v_k(s)\}_1^n$ admits orthogonalization, in other words, a tuple of smooth vector functions $e_1(s),\dots,e_n(s)$ can be constructed in such a way that

(1) for any $k$ between 1 and $n-1$ the vectors $v_1(s),\dots,v_k(s)$ and $e_1(s),\dots,e_k(s)$ span the same $k$-dimensional space, and the angle between $e_k(s)$ and $v_k(s)$ is always acute;
(2) all vectors $e_1(s),\dots,e_n(s)$ together constitute a positively oriented orthonormal frame for all $s \in I$.

Since the frame $\{e_k\}$ is orthonormal, the vectors $e_k = e_k(s)$ satisfy the system of linear equations with antisymmetric matrix of coefficients $A(s) = \{a_{ij}(s)\}_{i,j=1}^n$. The additional observation that the derivative $\frac{d}{ds}e_k(s)$ must belong to the subspace spanned by $e_1(s),\dots,e_{k+1}(s)$, implies that $a_{ij}(s) \equiv 0$ for all $i,j$ such that $|i-j| \geqslant 2$, and the system must have the form of the *Frenet formulas*

$$\frac{d}{ds}e_k(s) = -\varkappa_{k-1}(s)\,e_{k-1}(s) + \varkappa_k(s)\,e_{k+1}(s) \qquad k=1,\dots,n, \tag{1.1}$$

(assuming $\varkappa_0 = \varkappa_n \equiv 0$, $e_0 = e_{n+1} \equiv 0$). The numbers $\varkappa_j = \varkappa_j(s)$, in fact *defined* by the Frenet formulas (1.1), are called *Frenet curvatures*. For regular curves the first $n-2$ curvatures are everywhere positive, while the last one $\varkappa_{n-1}(s)$ changes sign at inflection points.

**Definition 2** (integral curvatures, integral inflection)**.** For $k = 1,\dots,n-1$ we define the $k$th integral curvature of $\Gamma$ as

$$K_k(\Gamma) = \int_0^\ell |\varkappa_j(s)|\,ds,$$

if the curve is parameterized by the arc length $s$; for an arbitrary parametrization $t$ this naturally gives the value $K_j(\Gamma) = \int_0^\ell |\varkappa_j(t)|\cdot\|v_1(t)\|\,dt$. Note that for all $k \leqslant n-2$ the Frenet curvatures $\varkappa_j$ are positive, hence $|\varkappa_j(s)| = \varkappa_j(s)$.

The *integral inflection* is defined as

$$K_n(\Gamma) = \pi \bar{K}_n(\Gamma) = \pi \cdot |\{t \in I \colon \varkappa_{n-1}(t) = 0\}|,$$

where $\bar{K}_n(\Gamma) := |\{t \colon \varkappa_{n-1}(t) = 0\}|$ is the number of inflection points on $\Gamma$.

1.1.2. *Oscillation and rotation around affine subspaces.* Let $L$ be a linear subspace in $\mathbb{R}^n$, and $A = a + L$ an affine subspace of dimension $\dim L = \dim A = k$, $0 \leqslant k \leqslant n-1$. We define oscillation of $\varGamma$ around $A$ if $\dim A = n-1$ and rotation around $A$ if $\dim A \leqslant n-2$ as the *angular length* of $\varGamma$ as seen from the origin in the direction orthogonal to $L$ (resp., as seen from the point $A \cap A^\perp$ in the direction $A^\perp$). The formal definition is given first for linear subspaces, and then for affine subspaces.

**Definition 3** (spherical indicatrice, angular length)**.** If $\varGamma \in \mathbb{R}^n$, $n \geqslant 2$ and $0 \notin \varGamma$, then the *spherical indicatrice* of $\varGamma$ is the spherical projection of $\varGamma$ on the unit sphere $\mathbb{S}^{n-1}$ from the origin. The *angular length* of $\varGamma$ is the (spherical) length of its indicatrice.

**Definition 4** (rotation around linear subspaces)**.** If $L \subset \mathbb{R}^n$ is a linear subspace of dimension $k = \dim L \leqslant n-2$, disjoint from $\varGamma$, then rotation $\Omega(\varGamma, L)$ of $\varGamma$ around $L$ is the angular length of the orthogonal projection of $\varGamma$ on $L^\perp$ along $L$.

If $\dim L = n-1$ (i.e. $L$ is a linear hyperplane) and $L$ is transversal to $\varGamma$, then $\Omega(\varGamma, L)$ is defined as $\pi \cdot |\varGamma \cap L|$.

**Definition 5** (rotation around affine subspaces)**.** If $A = L + a$ is an affine subspace transversal to $\varGamma$, then $\Omega(\varGamma, L+a)$ is defined as $\Omega(\varGamma - a, L)$, where $\varGamma - a$ is the parallel translate of $\varGamma$.

For an arbitrary affine subspace $A \subset \mathbb{R}^n$ of positive codimension we define $\Omega(\varGamma, A)$ as the upper limit of $\Omega(\varGamma, A_\varepsilon)$ as $\varepsilon \to 0^+$, where $A_\varepsilon$ is an affine subspace parallel to $A$ and $\varepsilon$-close to it.

Finally, we introduce the characteristics $\Omega_k(\varGamma)$ for all $k = 0, 1, \dots, n-1$,

$$\Omega_k(\varGamma) = \sup_{\dim A = k} \Omega(\varGamma, A),$$

where the supremum is taken over all affine subspaces of dimension $k$.

1.1.3. *Remarks.* For a planar curve the angular length is the total variation of argument along the curve. If $\varGamma \subset \mathbb{R}^2$ is closed, then $(2\pi)^{-1}\Omega(\varGamma, 0)$ gives an upper bound for the topological index of $\varGamma$ with respect to the origin.

Therefore for any subspace $A$ of codimension 2 and a closed curve $\varGamma$ the value $(2\pi)^{-1}\Omega(\varGamma, A)$ majorizes the *linking number* between $\varGamma$ and $A$.

For hyperplanes the definition of rotation is given separately and may seem somewhat artificial. The reason for this is rather simple: for the "unit sphere" $\{\pm 1\} = \mathbb{S}^0 \subset \mathbb{R}^1$ the geodesic distance between antipodal points is not defined, while for all other spheres $\mathbb{S}^k \subset \mathbb{R}^{k+1}$, $k > 0$, it is equal to $\pi$. If we extend the definition of the geodesic distance for the sphere $\mathbb{S}^0$ in the appropriate way, then the definition of $\Omega(\mathit{\Gamma}, L)$ would become more uniform. The similar reasons motivate the occurrence of the factor $\pi$ in the definition of the integral inflection.

Besides, in §3 we prove several results concerning average values of integral curvatures and rotations. It turns out that they remain valid also for the last "curvature" $K_n$ and the hyperplanar "rotation" $\Omega_{n-1}$ if the factor $\pi$ is properly placed in their definitions.

1.2. **Non-oscillation theorems: starting points.** After all the definitions are given, we list some known results in the spirit of the implication "bounded curvatures $\implies$ bounded rotation". In order to write in a uniform way the inequalities concerning both closed and non-closed curves, we introduce the notation $|\partial\mathit{\Gamma}|$ for the number of endpoints of the curve $\mathit{\Gamma}$: this number is zero if $\mathit{\Gamma}$ is closed, 2 if $\mathit{\Gamma}$ simply connected *etc.*

1.2.1. *Rotation around hyperplanes.* The simplest case, in fact an elementary exercise, concerns oscillation of planar curves $\mathit{\Gamma} \subset \mathbb{R}^2$ around straight lines.

**Fact 1** (see, e.g., [1])**.**

$$\Omega_1(\mathit{\Gamma}) \leqslant K_1(\mathit{\Gamma}) + K_2(\mathit{\Gamma}) + \pi\,|\partial\mathit{\Gamma}|. \tag{1.2}$$

The proof is based on the Rolle theorem and the following observation: if the tangent vectors at two endpoints of the curve are parallel to the same line, then either the integral curvature of the curve is at least $\pi$, or there should be an inflection point.

Most other results concerning oscillation around hyperplanes are formulated under assumption that the curve is hyperconvex. Perhaps, the most general among them is a corollary from the theorem by B. Shapiro [11], see also [12]. The Shapiro theorem is topological and concerns *oscillating curves* in $\mathbb{R}^n$. In an obvious way, for any curve in $\mathbb{R}^n$ the lower bound $\Omega_{n-1}(\mathit{\Gamma}) \geqslant \pi n$ is valid, since there always exists a hyperplane passing through $n$ arbitrary points of the curve. The curves for which $\Omega_{n-1}(\mathit{\Gamma}) = \pi n$, are called *non-oscillating* (for obvious reasons) and *oscillating* otherwise. Shapiro theorem asserts that if a curve is *oscillating*, then its osculating frame makes in some sense a full turn in the flag space (see §2.2.3 below for the exact formulation; we derive this theorem as a corollary to our main result). Since the velocity of rotation of the osculating frame is naturally measured by the Frenet curvatures, this observation implies the following sufficient condition for nonoscillation.

**Theorem 1.** *If the curve $\varGamma \subset \mathbb{R}^n$ is hyperconvex and*

$$\int_\varGamma \sqrt{\varkappa_1^2(s) + \cdots + \varkappa_{n-1}^2(s)}\, ds < \frac{1}{n\sqrt{2}}, \tag{1.3}$$

*then the curve is non-oscillating, i.e.* $\Omega_{n-1}(\varGamma) = \pi n$.

The proof of this theorem is given below in §4.3. Since any regular curve can be partitioned into $\pi^{-1}K_n(\varGamma) + 1$ hyperconvex pieces, and any such piece in turn can be subdivided into sufficiently short arcs satisfying (1.3), we conclude with the following corollary.

**Corollary 1.**

$$\Omega_{n-1}(\varGamma) \leqslant \pi n + \pi\sqrt{2}\, n^2 \cdot [K_1(\varGamma) + \cdots + K_{n-1}(\varGamma)] + n\, K_n(\varGamma). \tag{1.4}$$

This inequality, however, cannot be too accurate, since any connection between non-oscillating arcs is lost. The main result of this paper, Theorem 2, gives a better value (linear in $n$) for the coefficients (note that as $n$ grows, the largest coefficients in the right hand side have the order of magnitude of $n^2$). Some other particular results concerning oscillation of hyperconvex curves around hyperplanes, are mentioned in [11].

1.2.2. *Rotation of subspaces of codimension* 2 *and more.* This group of results is less numerous. Two main examples are the Milnor theorem on linking number between closed curves and straight lines in $\mathbb{R}^3$ [1] and a theorem by Khovanskiĭ and the second author [6] on rotation around points.

**Fact 2** (cf. Milnor [1, Theorem 3])**.**

$$\forall \varGamma \subset \mathbb{R}^3 \qquad \Omega_1(\varGamma) \leqslant K_1(\varGamma) + K_2(\varGamma) + \pi\, |\partial\varGamma|. \tag{1.5}$$

In fact, Milnor writes an upper estimate for the linking number between a closed spatial curve and any straight line in $\mathbb{R}^3$, but his arguments prove also a more strong result concerning the rotation, which can be derived using the averaging principles described below, from the estimate (1.2).

The following result gives an upper bound for rotation of $\varGamma$ around any point and is valid in all dimensions.

**Fact 3** (Khovanskiĭ and Yakovenko [6])**.**

$$\forall \varGamma \subset \mathbb{R}^n \qquad \Omega_0(\varGamma) \leqslant K_1(\varGamma) + \tfrac{\pi}{2}\, |\partial\varGamma|. \tag{1.6}$$

1.3. **Formulation of the main result.** The inequalities (1.2) compared to (1.5) and (1.6) suggest that in the general case an upper bound for $\Omega_k(\mathit{\Gamma})$ can be found in the form of a weighted sum of integral curvatures $K_1(\mathit{\Gamma})$ through $K_{k+1}(\mathit{\Gamma})$, with a constant term added if the curve is non-closed; besides, one may hope that the weights can be chosen *independently of the dimension n of the ambient space*. Our main result claims that this hope can indeed be justified.

Let $\mathit{\Gamma} \subset \mathbb{R}^n$ be a smooth regular curve with $|\partial\mathit{\Gamma}|$ endpoints (i.e. not necessarily connected).

**Theorem 2** (Main). *Oscillation of any regular curve $\mathit{\Gamma}$ satisfies the following inequalities:*

$$\begin{aligned}
&\Omega_0(\mathit{\Gamma}) \leqslant \tfrac{1}{2}\pi|\partial\mathit{\Gamma}| + K_1(\mathit{\Gamma}),\\
&\Omega_1(\mathit{\Gamma}) \leqslant \pi|\partial\mathit{\Gamma}| + K_1(\mathit{\Gamma}) + K_2(\mathit{\Gamma}),\\
&\Omega_2(\mathit{\Gamma}) \leqslant \tfrac{3}{2}\pi|\partial\mathit{\Gamma}| + 2K_1(\mathit{\Gamma}) + K_2(\mathit{\Gamma}) + K_3(\mathit{\Gamma}),\\
&\Omega_3(\mathit{\Gamma}) \leqslant 2\pi|\partial\mathit{\Gamma}| + 2K_1(\mathit{\Gamma}) + 2K_2(\mathit{\Gamma}) + K_3(\mathit{\Gamma}) + K_4(\mathit{\Gamma}),
\end{aligned}$$

*and in general for any $k \leqslant n-1$*

$$\Omega_k(\mathit{\Gamma}) \leqslant \tfrac{1}{2}\pi(k+1)|\partial\mathit{\Gamma}| + \sum_{j=1}^{k+1} w_{k+1-j}K_j(\mathit{\Gamma}), \tag{1.7}$$

*where the sequence of weights*

$$w_0 = w_1 = 1, \quad w_2 = w_3 = 2, \quad w_j = j-1 \qquad \textit{for} \;\; j = 4,5,\dots \tag{1.8}$$

*is universal.*

The inequality for $\Omega_{n-1}$ implies the sufficient condition for non-oscillation.

**Corollary 2.** *A hyperconvex curve whose integral Frenet curvatures as so small as to satisfy the inequality*

$$\sum_{j=1}^{n-1} w_{n-j}K_j(\mathit{\Gamma}) < \pi, \tag{1.9}$$

*is non-oscillating.*

This inequality is stronger than (1.3). In fact, one could well add to the left hand side the last "curvature" and drop out the hyperconvexity assumption, since for curves with inflections $K_n(\mathit{\Gamma}) \geqslant \pi$.

1.4. **Remarks.** The inequalities (1.7) give an *upper bound* for the number of intersections of $\mathit{\Gamma}$ with *any* affine hyperplane. However, most hyperplanes intersect $\mathit{\Gamma}$ by substantially smaller number of points. In particular, a random uniformly distributed hyperplane passing, say, through the origin (in order to avoid non-compactness of the set of all affine hyperplanes), intersects $\Gamma$ by at most $\pi^{-1}K_1(\Gamma)+1$ point.

It is interesting to observe that the estimate established by Theorem 2, is stable with respect to the dimension of the ambient space: if we fix $k$, then the upper bound for $\Omega_k(\Gamma)$ is *independent* of the dimension $n$. This fact suggests that an analog of Theorem 2 holds also for curves in infinite-dimensional Hilbert spaces.

The inequalities (1.7) are sharp for small $k=0,1$ and perhaps for $k=2$. All the way around, the inequality (1.3) is *not* sharp even for small $n$. From §4.3 it is rather clear how the latter can be improved, though at the price of rather sophisticated computations. However, one can relatively easily establish the implication

$$K_1(\Gamma)+K_2(\Gamma)<\tfrac{\pi}{2} \implies \Gamma \text{ is non-oscillating,} \tag{1.10}$$

for three-dimensional hyperconvex curves (see §4.4). But even this result is inferior to the inequality $2K_1(\Gamma)+K_2(\Gamma)<\pi$ guaranteeing non-oscillation of hyperconvex three-dimensional curves, the particular case of (1.9) for $n=3$ and $K_3(\Gamma)=0$.

The final remark concerns the choice of the weights $w_j$ in (1.8); obviously, without discussing this matter it is not possible to analyze the sharpness of the inequalities obtained. This choice is determined by Lemma 4 concerning roots of solutions to linear ordinary differential equations, see §4. From the geometrical point of view the integrand occurring in (1.3) seems to be more natural (it admits interpretation as the angular velocity of rotation of the osculating orthogonal frame of a curve). However, reasonably sharp estimates involving $\int_\Gamma(\varkappa_1^2(s)+\dots+\varkappa_{n-1}^2(s))^{1/2}\,ds$, are yet unavailable.

1.5. **Spherical and projective curves.** The main result admits reformulation for spherical and projective curves. Recall that for any Riemannian $n$-dimensional manifold $M^n$ and any sufficiently smooth curve $\Gamma\colon[0,\ell]\to M$ one can define the osculating frame $v_j(t)\in T_{x(t)}M$, $j=1,\dots,n$ in the same way as for curves in the Euclidean space, and this frame can be similarly orthogonalized with the only exception that the last vector $e_n(t)$ is defined modulo multiplication by $\pm1$ for non-orientable manifold $M$.

1.5.1. *Geodesic curvatures.* Denote by $\nabla$ the operator of covariant differentiation with respect to the natural (Levi-Civita) connexion compatible with the metric on $M$. Then, assuming that the curve is parameterized by the arc length, one can easily see that

$$\nabla_{e_1(t)}e_j(t)=\widetilde{\varkappa}_{j-1}(t)e_{j-1}(t)+\widetilde{\varkappa}_j(t)e_{j+1}(t) \qquad j=1,\dots,n, \tag{1.11}$$

with the standard convention that $e_0(t) \equiv e_{n+1}(t) \equiv 0$. The functions $\widetilde{\varkappa}_j(t)$, defined by (1.11), are called *geodesic Frenet curvatures* (in the non-orientable case only the modulus of the last curvature $\pm\widetilde{\varkappa}_{n-1}(t)$ is defined). However, the integral geodesic curvatures $\widetilde{K}_j(\mathit{\Gamma})$, $j = 1, \dots, n$, relative to $M$, make sense: the last one is $\pi$ times the number of points where $\widetilde{\varkappa}_{n-1}$ vanishes.

We will be interested in the two simplest Riemannian manifolds:

- the sphere $r \cdot \mathbb{S}^n$ of radius $r > 0$, that inherits its metric from the embedding in the Euclidean space $\mathbb{R}^{n+1}$, and
- the real projective space $\mathbb{P}^n$ obtained as a quotient space of the *unit* sphere $\mathbb{S}^n$ by identifying the opposite points $\pm x$. The spherical metric induces the *Fubini-Study* metric on $\mathbb{P}^n$: length of each line is equal to $\pi$.

1.5.2. *Oscillatory behavior on* $\mathbb{S}^n$ *and in* $\mathbb{P}^n$*.* For a spherical curve it makes sense to ask how many times it can intersect an *equator*, the nearest analog of a hyperplane. For projective curves one may look for an upper bound for the number of intersections with any projective hyperplane of codimension 1 in $\mathbb{P}^n$.

The following two corollaries to Theorem 2 giving answers to these questions are proved in §2.5.

**Theorem 3.** *Let $\mathit{\Gamma} \subset r \cdot \mathbb{S}^n$ be a spherical curve with the geodesic integral curvatures $\widetilde{K}_j(\mathit{\Gamma})$, $j = 1, \dots, n$ of spherical length $\widetilde{K}_0(\mathit{\Gamma}) = |\mathit{\Gamma}|$. Then $\mathit{\Gamma}$ may intersect any equator (embedded sphere $r \cdot \mathbb{S}^{n-1}$) by no more than*

$$\tfrac{1}{2} n |\partial \mathit{\Gamma}| + w_n \widetilde{K}_0(\mathit{\Gamma})/\pi r + \sum_{j=1}^{n} w_{n-j} \widetilde{K}_j(\mathit{\Gamma})/\pi \tag{1.12}$$

*isolated points, where $w_j$ are the same as before* (1.8)*.*

The upper bound (1.12) turns into (1.7) in the limit $r \to +\infty$, as one could expect. As a natural corollary, we obtain a similar result for projective curves.

**Theorem 4.** *A projective curve $\mathit{\Gamma} \subset \mathbb{P}^n$ of length $\widetilde{K}_0(\mathit{\Gamma}) = |\mathit{\Gamma}|$ intersects any projective hyperplane $P^{n-1} \subset \mathbb{P}^n$ by no more than $\frac{1}{2} n |\partial \mathit{\Gamma}| + \sum_{j=0}^{n} w_{n-j} \widetilde{K}_j(\mathit{\Gamma})/\pi$.*

*Remark .* The boundary term can be incorporated in the sum of integral curvatures without changing the law (1.8) for the weights, if we put

$$K_0(\mathit{\Gamma}) = c^{1/n} |\mathit{\Gamma}|, \qquad K_{-1}(\mathit{\Gamma}) = \tfrac{\pi}{2} |\partial \mathit{\Gamma}|, \tag{1.13}$$

where $c = c(M) > 0$ is the Gaussian curvature of the ambient manifold $M$ in each of the three cases, $M = \mathbb{R}^n$, $r \cdot \mathbb{S}^n$ or $\mathbb{P}^n$. The universal formula embracing all these cases will read then

$$\pi \times \{\text{Number of intersections with any "hyperplane"}\} \leqslant$$
$$w_{n+1}K_{-1}(\varGamma) + w_n K_0(\varGamma) + \cdots + w_0 K_n(\varGamma) = \sum_{j=-1}^{n} w_{n-j} K_j(\varGamma). \quad (1.14)$$

However, this formula is justified only *aposteriori*. It seems to be an intriguing problem to find a direct proof of (1.14) for other classes of manifolds.

## 2. Demonstration of the main result

This section is the core of the paper. Two main ingredients of the proof are the averaging lemmas for rotation and integral curvatures, belonging to the realm of Geometric Probability (alias Integral Geometry), and a variation on the theme of Pólya [8] on zeros of solutions of linear ordinary differential equations, that admits reformulation in geometric terms. In this section we derive Theorem 2 from these results. Demonstration of the two integral geometric lemmas is postponed until §3, the Pólya theorem and its generalizations are discussed in §4.

2.1. **Geometric Probability.** From now on we will deal with only linear (not affine) linear subspaces of the ambient Euclidean space $\mathbb{R}^n$. By $\mathbb{S}^{n-1}$ we denote the standard unit sphere with the Lebesgue $(n-1)$-dimensional measure $d\sigma_{n-1}$. For $p \in \mathbb{S}^{n-1}$ we denote by $\mathbb{R}p$ the line spanned by $p$; if $L \subset \mathbb{R}^n$ is a linear subspace, then $L^\perp$ is its orthogonal complement, and $P_L \colon \mathbb{R}^n \to L^\perp$ the orthogonal projection on $L^\perp$ along $L$. If $L = \mathbb{R}p$, then we write $p^\perp$ and $P_p$ instead of $(\mathbb{R}p)^\perp$ and $P_{\mathbb{R}p}$ respectively. Sometimes instead of $L^\perp$ and $p^\perp$ we will write $\mathbb{R}^{n-k}_L$ and $\mathbb{R}^{n-1}_p$, where $k = \dim L$.

2.1.1. *Averaging integral curvatures.* The first main result means that each integral curvature $K_j(\varGamma)$ can be restored by averaging the corresponding integral curvatures of orthogonal projections $P_p(\varGamma)$ along a random direction. Note that $P_p(\varGamma)$ is a hyperplanar curve, and as such possesses a complete set of integral curvatures $K_j(P_p(\varGamma))$, $j = 1, \ldots, n-1$, relative to the hyperplane $p^\perp$ (as usual, the last one is the integral inflection of the corresponding projection).

**Lemma 1.**

$$\forall j = 1, \ldots, n-1 \qquad K_j(\varGamma) = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} K_j\bigl(P_p(\varGamma)\bigr)\, d\sigma_{n-1}(p). \quad (2.1)$$

Here $|\mathbb{S}^{n-1}|$ stands for the $(n-1)$-dimensional volume of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$.

*Remark* . The factor $\pi$ was introduced in the definition of $K_n$ for this formula to remain valid for $j = n - 1$. However, the proof for this case requires separate considerations.

*Remark* . The case $n = 3$, $j = 1$ (averaging property of integral curvature for spatial curves) is known as Fáry theorem [4]. In fact, minor modifications can be made to extend the proof of Fáry for any $n$ (and $j = 1$, as before).

The case $n = 3$, $j = 2$ (integral torsion of spatial curves) was proved by Milnor in [1]. The proof given by Milnor with only minor modifications works for any $n$ and $j = n - 1$.

The averaging property for intermediate curvatures seems to be a new result.

*Remark* . In general, projection of a regular curve is *not* a regular curve: it may even be non-smooth, as simplest examples already show. However, from the Sard theorem it follows that the Lebesgue measure of directions $p \in \mathbb{S}^{n-1}$ corresponding to "bad" projections, is zero, hence one may disregard such pathologies when computing the average.

2.1.2. *Averaging rotation.* Let $L$ be a linear subspace of dimension $k \leqslant n - 2$, and $p \in \mathbb{S}^{n-1}$ a vector on the unit sphere. Then for almost all $p$ the linear sum $L + \mathbb{R}p$ is a $(k + 1)$-dimensional subspace.

**Lemma 2.**

$$\Omega(\mathit{\Gamma}, L) = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \Omega(\mathit{\Gamma}, L + \mathbb{R}p)\, d\sigma_{n-1}(p). \tag{2.2}$$

*Remark* . The factor $\pi$ is introduced in the definition of $\Omega(\mathit{\Gamma}, \cdot)$ for hyperplanes, in order for this equality to remain valid also for subspaces $L$ of dimension $n - 2$.

As before, the fact that for a metrically negligible set of directions the subspace $L + \mathbb{R}p$ degenerates, does not affect the integral.

2.2. **Flags, inflections and oscillation around hyperplanes.** The second (analytic) ingredient of the proof of Theorem 2 is introduced in this section. The principal result of it is an inequality relating the number of intersections of a spatial curve with an arbitrary affine hyperplane, with the number of inflection points of orthogonal projections of this curve on a family of (linear) subspaces of all intermediate directions.

2.2.1. *Flags.* Recall that a (complete) flag $\mathcal{L}$ in a linear $n$-dimensional space $L$ is a chain of embedded linear subspaces of $L$ of increasing dimensions:

$$\mathcal{L} = \{L_j\}_{j=0}^{n}, \qquad 0 = L_0 \subsetneq L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_{n-1} \subsetneq L_n = L, \quad \dim L_j = j.$$

If $v_1, \dots, v_n$ is any *ordered* tuple of vectors in $n$, then the flag *spanned* by this tuple is the flag whose $j$th subspace is spanned by the first $j$ vectors. Any *orthogonal* frame $e_1, \dots, e_n$ can be almost uniquely restored from the flag it spans: the map $(e_1, \dots, e_n) \mapsto (\operatorname{Span}(e_1), \operatorname{Span}(e_1, e_2), \dots, \operatorname{Span}(e_1, \dots, e_n))$ is a covering of the *flag variety* by the orthogonal group $\mathrm{SO}(n)$ with the discrete fiber of $2^{n-1}$ points.

2.2.2. *Inflections relative to flag and the Third Principal Lemma.* Let $\varGamma \subset \mathbb{R}^n$ be a regular curve and $\mathcal{L} = \{L_j\}_{j=1}^n$, $L_n = \mathbb{R}^n$, a complete flag. Denote by $\varGamma_j$ the orthogonal projection of $\varGamma$ on $L_j$ parallel to $L_j^\perp$ for all $j = 1, \dots, n$, so that $\varGamma_n = \varGamma$. Each $\varGamma_j$ is $j$-dimensional curve and as such possesses $j$ integral curvatures $K_i(\varGamma_j)$, $i = 1, \dots, j$. The last of them is the (relative) integral inflection $K_j(\varGamma_j) = \pi\nu_j$, $\nu_j = \nu_j(\varGamma, \mathcal{L}) \in \mathbb{Z}_+$, where $\nu_j$ is the number of inflection points of the projection $\varGamma_j$.

*Remark .* We have extended the notion of inflection point for parameterized "curves" in $\mathbb{R}^1$: by definition, the inflection point of a "curve" $t \mapsto x(t) \in \mathbb{R}^1$ is the point where the Wronski determinant $\langle v_1 \rangle = |x'(t)|$ vanishes, in other words, the critical point of the map $x(\cdot)\colon [0, \ell] \to \mathbb{R}^1$. The "curve" is hyperconvex, if it has no "inflections". This convention will be adopted from now on.

Now we can formulate the main result of this subsection. Let $w_0, w_1, w_2, \dots$ be the sequence of weights introduced in (1.8).

**Lemma 3.** *If $\varGamma \subset \mathbb{R}^n$ is a regular curve and $\mathcal{L} = \{L_j\}_{j=1}^n$ a complete flag such that the orthogonal projection of $\varGamma$ on $L_j$ has $0 \leqslant \nu_j = \nu_j(\varGamma, \mathcal{L}) < \infty$ inflection points, then*

(1) *the curve $\varGamma$ intersects any affine hyperplane by no more than $\frac{1}{2} n |\partial \varGamma| + \sum_{j=1}^n w_{n-j} \nu_j$ isolated points, so that*
$$\Omega_{n-1}(\varGamma) \leqslant \tfrac{1}{2}\pi n |\partial\varGamma| + \sum_{j=1}^n w_{n-j}\, \pi\nu_j(\varGamma, \mathcal{L}); \tag{2.3}$$
(2) *the velocity curve $\dot\varGamma\colon [0, \ell] \ni t \mapsto \dot x(t) \in \mathbb{R}^n$ intersects any linear hyperplane by no more than $\frac{1}{2}(n-1)|\partial\dot\varGamma| + \sum_{j=1}^n w_{n-j}\nu_j$ isolated points.*

This result is a geometric version of a theorem by G. Pólya [8, 9] on zeros of solutions of linear ordinary differential equations (1922). The proof of Lemma 3 is given in §4 together with discussion of related topics and some historical notes.

2.2.3. *Digression: Shapiro theorem.* As a corollary to Lemma 3 we obtain a condition describing *oscillating* (i.e. non-non-oscillating) curves.

**Corollary 3.** *If $\Gamma \subset \mathbb{R}^n$ is oscillating, then for any complete flag $\mathcal{L}$ the projections $\Gamma_j \subset L_j$ cannot be all hyperconvex:*

$$\nu_1(\Gamma, \mathcal{L}) + \cdots + \nu_n(\Gamma, \mathcal{L}) > 0. \tag{2.4}$$

In fact, the assertion of Corollary 3 can be formulated more naturally, using the notions of *osculating flag* and *transversality* of flags.

**Definition 6.** Two flags $\mathcal{L}$ and $\mathcal{L}' = \{L'_j\}$ in the same space are *transversal*, if for any pair of indices $i, j$ such that $i + j \geqslant n$, the subspaces $L_i$ and $L'_j$ are transversal.

The flag $\mathcal{L}'$ is said to be *orthogonal* to the flag $\mathcal{L}$, if its subspaces are orthogonal complements to the subspaces of $\mathcal{L}$ (naturally, taken in the inverse order): $L'_j = L^{\perp}_{n-j}$, $j = 1, \dots, n$. The flag orthogonal to $\mathcal{L}$ is denoted by $\mathcal{L}^{\perp}$.

If $L = \mathbb{R}^n$ and an orthogonal coordinate system is fixed by specifying an orthogonal frame $e_1, \dots, e_n$, then the *standard flag* $\mathcal{E} = \{E_j\}_{j=0}^n$ is spanned by the basis vectors. The orthogonal flag $\mathcal{E}^{\perp}$ is sometimes referred to as the *antipodal flag*.

**Definition 7.** The *osculating flag* $\mathcal{L}_\Gamma(t)$ of a regular hyperconvex curve $\Gamma$ is the (variable) flag spanned by the osculating frame.

*Remark .* If a curve has inflection points, then at these points the osculating frame does not span a flag (or, more precisely, the flag spanned by the frame, is not complete), since the vectors from the osculating frame are linear dependent. However, since we have assumed regularity of the curve, all $k$-dimensional subspaces of the osculating flag are well-defined for $k < n$, and for $k = n$ we assume that the last subspace is always $\mathbb{R}^n$.

From these definitions it is almost obvious that if $x \in \Gamma$ is a point on the curve, which becomes an inflection point of the projection $\Gamma_j$, then the $j$th subspace of the osculating flag $\mathcal{L}_\Gamma(x)$ is *non*-transversal to the subspace $L^{\perp}_j$ of the orthogonal flag $\mathcal{L}^{\perp}$. Thus we arrive to the reformulation of Corollary 3.

**Corollary 4** (Shapiro theorem [11, 12])**.** *If a hyperconvex regular curve $\Gamma \subset \mathbb{R}^n$ is oscillating, then for any complete flag $\mathcal{L}$ there exists at least one point $x \in \Gamma$ such that $\mathcal{L}_\Gamma(x) \not\pitchfork \mathcal{L}$.*

In fact, this theorem is valid also for projective curves. It makes sense to point out that Shapiro theorem generalizes to a certain extent the Rolle theorem (consider the case planar convex curves). There are some other Rolle-type theorems, see [6, 7]. Besides, the (classical) Rolle theorem is the key tool in demonstration of Lemma 3, see §4.

2.3. **Demonstration of Theorem 2 for hyperplanes.** To prove Theorem 2 for hyperplanes and estimate $\Omega_{n-1}(\Gamma)$, we construct a flag $\mathcal{L} = \{L_j\}_{j=1}^n$ in such a way that the weighted sum of integral curvatures for the projections $\Gamma_j$ of $\Gamma = \Gamma_n$ on *each* subspace $L_j$, $1 \leqslant j \leqslant n$, is bounded in terms of the weighted sum of curvatures of the original curve. This would automatically provide upper bounds for the corresponding integral inflections of the projections, and it remains only to apply Lemma 3 to estimate the number of intersections.

The construction of the flag $\mathcal{L}$ goes by induction in *co*dimension of the subspaces. Let $w_0, w_1, \dots, w_{n-1}$ be the weights (1.8), and

$$\begin{aligned} T(\Gamma) &= w_{n-1}K_1(\Gamma) + \cdots + w_1 K_{n-1}(\Gamma) + w_0 K_n(\Gamma) \\ &= w_{n-1}K_1(\Gamma) + \cdots + w_1 K_{n-1}(\Gamma) + \pi w_0 \nu_n \end{aligned}$$

the weighted sum of the integral curvatures, where $\nu_n$ is the number of inflection points of $\Gamma$.

The sum of the first $n-1$ terms admits averaging: the value $\sum_{j=1}^{n-1} w_{n-j}K_j(\Gamma)$ is equal by Lemma 1 to the average value of the function $\chi(p) = \sum_{j=1}^{n-1} w_{n-j}K_j(P_p(\Gamma))$ on the sphere $\mathbb{S}^{n-1}$, where $P_p(\Gamma)$ is he orthogonal projection of $\Gamma$ parallel to a random vector $p \in \mathbb{S}^{n-1}$ on the hyperplane $p^\perp$.

The function $\chi(p)\colon \mathbb{S}^{n-1} \to \mathbb{R}$ does not exceed its average value at some point $p \in \mathbb{S}^{n-1}$. Denote the corresponding normal hyperplane by $L_{n-1} = p^\perp$ (it will play the role of the $(n-1)$-dimensional subspace of the flag $\mathcal{L}$), the projection of $\Gamma$ on $L_{n-1}$ by $\Gamma_{n-1}$ and the number of inflections of this projection by $\nu_{n-1}$. Then

$$w_{n-1}K_1(\Gamma_{n-1}) + \cdots + w_2 K_{n-2}(\Gamma_{n-1}) + \pi w_1 \nu_{n-1} + \pi w_0 \nu_n \leqslant T(\Gamma).$$

The procedure of averaging can be repeated once again, this time applied to the truncated sum $\sum_{j=1}^{n-2} w_{n-j}K_j(\Gamma_{n-1})$, and a subspace $L_{n-2} \subset L_{n-1}$ of codimension 1 in $L_{n-1}$ hence of codimension 2 in $\mathbb{R}^n$ can be found in such a way that for the projection $\Gamma_{n-2}$ of $\Gamma$ on $L_{n-2}$ one has the inequality

$$w_{n-1}K_1(\Gamma_{n-2}) + \cdots + w_3 K_{n-3}(\Gamma_{n-2}) + \pi w_2 \nu_{n-2} + \pi w_1 \nu_{n-1} + \pi w_0 \nu_n \leqslant T(\Gamma),$$

where obviously $\nu_{n-2}$ is the number of inflections of the curve $\Gamma_{n-2} \subset L_{n-2}$.

Iterating these arguments $n$ times, we construct all subspaces $L_{n-1}$, $L_{n-2}$, $\dots$, $L_2$, $L_1$ of the flag $\mathcal{L}$, and for the number of inflection points of the corresponding projections the inequality

$$\pi(w_{n-1}\nu_1 + \cdots + w_1\nu_{n-1} + w_0\nu_n) \leqslant T(\Gamma).$$

Applying the first assertion of Lemma 3, we conclude that the number of points of intersection of $\Gamma$ with any affine hyperplane does not exceed $n|\partial\Gamma|/2 + T(\Gamma)/\pi$, hence the oscillation satisfies the inequality $\Omega_{n-1}(\Gamma) \leqslant \frac{1}{2}\pi n|\partial\Gamma| + T(\Gamma) = \frac{1}{2}\pi n|\partial\Gamma| + \sum_{j=1}^{n} w_{n-j} K_j(\Gamma)$. The proof for the codimension 1 case is complete.

2.4. **Demonstration of Theorem 2 in the general case.** The proof in the general case goes by induction in the codimension of subspaces, the hyperplanar case being the base of induction.

Suppose that the inequalities of Theorem 2 are already established for all codimension $c$ subspaces and any dimension of the ambient space ($c = 1$ corresponds to hyperplanes). Take a linear subspace $L$ of dimension $k-1$ and codimension $c+1$, so that $n = c+k$. Now let $p$ be a variable vector on the unit sphere in $\mathbb{S}^{n-1}$, and $P_p$ be the corresponding orthogonal projection on $H_p := p^{\perp}$. Then for almost all $p$ the projection $L_p := P_p(L) \subset H_p$ has dimension $k-1$, hence codimension $c$, so for the projection $\Gamma_p = P_p(\Gamma)$ we know by the assumption of induction that the (relative to $H_p$) oscillation satisfies the inequality

$$\forall p \in \mathbb{S}^{n-1} \qquad \Omega(\Gamma_p, L_p; H_p) \leqslant \tfrac{1}{2}\pi k|\partial\Gamma_p| + \sum_{j=1}^{k} w_{k-j} K_j(\Gamma_p; H_p), \tag{2.5}$$

where $\Omega(\cdot,\cdot; H_p)$ is the oscillation around $k-1$-dimensional subspace $L_p \subset H_p$ and $w_j$ are the weights introduced in (1.8).

Note that

$$\Omega(\Gamma_p, L_p; H_p) = \Omega(\Gamma, L + \mathbb{R}p; \mathbb{R}^n), \tag{2.6}$$

since both sides are by definition the angular lengths of the *same* curve in $L_p^{\perp} \cap H_p = (L + \mathbb{R}p)^{\perp}$.

The number of endpoints $|\partial\Gamma_p|$ is the same as $|\partial\Gamma|$ for almost all $p$. After averaging the inequality (2.5), from (2.6) and (2.2) we conclude that the left hand side turns into $\Omega(\Gamma, L)$ by Lemma 2, while the average of the right hand side by Lemma 1 and linearity is the weighted sum of integral curvatures of the original curve $\Gamma$. Thus the inequality

$$\Omega(\Gamma, L) \leqslant \tfrac{1}{2}\pi k|\partial\Gamma| + \sum_{j=1}^{k} w_{k-j} K_j(\Gamma),$$

is established for subspaces of codimension $c+1$ as well, hence by induction Theorem 2 is proved in full generality. □

2.5. **Oscillation of spherical and projective curves: proof of Theorems 3 and 4.** The basic case is that of curves on the unit sphere $\mathbb{S}^{n-1}$.

Consider a regular curve $\Gamma$ parameterized by the arclength. Its velocity curve $\dot\Gamma\colon t \mapsto \widetilde{x}(t) = \frac{d}{dt}x(t)$ belongs to the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. If at the end of the proof of Theorem 2 (hyperplanar case, §2.3) we replace the reference to the first claim of Lemma 3 by the second one, this will result in the upper bound for the oscillation of $\dot{\Gamma}$ in terms of integral curvatures of the primitive curve $\Gamma$. To obtain the proof of Theorem 3, it remains only re-compute integral curvatures of $\Gamma$ in terms of geodesic curvatures of $\dot{\Gamma}$.

If $e_1(t), \dots, e_n(t)$ is the Frenet frame for $\Gamma$, then $\widetilde{e}_1(t) = e_2(t), \dots, \widetilde{e}_{n-1}(t) = e_n(t)$ is the Frenet frame for $\dot{\Gamma}$ considered as a spherical curve; to obtain the full Frenet frame for the same curve considered as a spatial curve, one needs to add $\widetilde{e}_n(t) = \pm e_1(t)$, the (unitary) radius-vector of $\dot{\Gamma}$. However, the parameter $t$ *is not* the natural parameter on $\dot{\Gamma}$, since $\frac{d}{dt}\widetilde{x}(t) = \frac{d}{dt}e_1(t) = \varkappa_1(t)e_2(t) = \varkappa_1(t)\widetilde{e}_1(t)$.

The covariant derivative on a submanifold of the Euclidean space (equipped with the induced metric), admits a simple description, see [2, Chapter 2, §3.1]: one should take the (usual) derivative with respect to the ambient Euclidean space and project the result orthogonally on the tangent space to the submanifold. Since the Frenet formulas are to be written with respect to the arc length parameter $s$ on $\dot{\Gamma}$, we arrive to the set of formulas

$$\nabla_{\widetilde{e}_1(s)}\widetilde{e}_j(s) = \Pi\bigl(\tfrac{d}{dt}e_{j+1}(t)\bigr)\cdot\tfrac{dt}{ds}, \qquad j = 1, \dots, n-1,$$

where $\Pi = \Pi_t$ stands for the orthogonal projection on the tangent subspace to the sphere $\mathbb{S}^{n-1}$ (at the corresponding point $\widetilde{x}(t)$). The Frenet formulas (3.15) for the derivatives $\frac{d}{dt}e_j(t)$ together with the identity $dt/ds = 1/\varkappa_1(s)$ yield immediately

$$\nabla_{\widetilde{e}_1(s)}\widetilde{e}_j(s) = \varkappa_1^{-1}(s)\Pi(-\varkappa_j\widetilde{e}_{n-1} + \varkappa_j\widetilde{e}_{j+1}), \qquad j = 1, \dots, n-1,$$

where $\widetilde{e}_0(s) = \widetilde{x}(s)$ is the radius-vector of $\dot{\Gamma}$ and $\widetilde{e}_n \equiv 0$ by definition. The projection $\Pi$ in fact leaves all right hand sides unchanged, except for $j = 1$, where $\Pi$ kills the term proportional to $\widetilde{x}(s)$, normal to the sphere. Comparing what remains with the equalities (1.11), we conclude that the geodesic curvatures $\widetilde{k}_1, \dots, \widetilde{\varkappa}_{n-2}$ can be expressed as

$$\widetilde{\varkappa}_1(t) = \frac{\varkappa_2(t)}{\varkappa_1(t)},\ \widetilde{\varkappa}_2(t) = \frac{\varkappa_3(t)}{\varkappa_1(t)},\ \dots\ \widetilde{\varkappa}_{n-2}(t) = \pm\frac{\varkappa_{n-1}(t)}{\varkappa_1(t)}. \tag{2.7}$$

Since the arc length element is $ds = \varkappa_1(t)dt$, we arrive finally to the identities relating $K_j(\Gamma)$ to integral characteristics of $\dot{\Gamma}$ expressed in terms of the induced spherical metric on $\mathbb{S}^{n-1}$:

$$K_1(\Gamma) = |\dot{\Gamma}|, \qquad K_j(\Gamma) = \widetilde{K}_{j-1}(\dot{\Gamma}), \quad j = 2, \dots, n \tag{2.8}$$

(recall that $|\dot{\Gamma}|$ is the length of $\dot{\Gamma}$). The last equality $K_n(\Gamma) = \widetilde{K}_{n-1}(\dot{\Gamma})$ expresses the fact that vanishing points of $\varkappa_{n-1}(t)$ and $\widetilde{\varkappa}_{n-2}(t)$ are the same. The reference to the formula (1.7) completes the proof for spherical curves on the unit sphere, since *any such curve* $t \mapsto \widetilde{x}(t)$ *is the velocity curve of any its vector primitive* $t \mapsto x(t) = \int \widetilde{x}(t)\,dt$ (note that the primitive curve needs not to be closed).

If $r \neq 1$, then the obvious rescaling $x \mapsto x/r$ brings the sphere $r \cdot \mathbb{S}^{n-1}$ into the unit sphere. After this rescaling the length gets multiplied by $1/r$, while the other curvatures $\widetilde{K}_j$ remain unchanged.

Finally, if $\dot{\Gamma} \subset \mathbb{P}^{n-1}$ is a projective curve, we may consider the canonical $\mathbb{S}^{n-1} \to \mathbb{P}^{n-1}$ which is an isometric two-sheet covering. For the preimage of $\dot{\Gamma}$ on $\mathbb{S}^{n-1}$ everything will be doubled, length, integral curvatures, the number of endpoints *etc*, but the number of intersections with equators (hyperplanes) will also be. Thus one arrives to the same formula for $\mathbb{P}^{n-1}$ as for the unit sphere. □

## 3. Demonstration of averaging properties for curvatures and rotations

3.1. **Principal formula of Geometric Probability.** The proof of the two key lemmas, 2 and 1, is based on the main principle of Geometric Probability: *$k$-dimensional measure of a smooth $k$-dimensional submanifold $M \subseteq \mathbb{S}^{n-1}$ can be obtained by averaging the $(k-1)$-dimensional measures of its slices*. More precisely, the following identity holds for any smooth submanifold:

$$\frac{\sigma_k(M)}{|\mathbb{S}^k|} = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \frac{\sigma_{k-1}(M \cap \mathbb{S}^{n-2}_p)}{|\mathbb{S}^{k-1}|}\,d\sigma_{n-1}(p), \tag{3.1}$$

where $\mathbb{S}^{n-2}_p = p^\perp \cap \mathbb{S}^{n-1}$ is the $(n-2)$-dimensional equator orthogonal to the direction $p$, $\sigma_k(M)$ is the Lebesgue $k$-measure of $M$ and $\sigma_{k-1}(M \cap \mathbb{S}^{n-2}_p)$ is the $(k-1)$-measure of the slice cut from $M$ by $\mathbb{S}^{n-2}_p$.

The general discussion of this fact can be found in [10]. We will need this formula in two particular cases, where it can be justified by one-line arguments.

**Proposition 1.** *The length $|\gamma|$ of a spherical curve $\gamma \subset \mathbb{S}^{n-1}$ is equal $\pi$ times the average number of intersections with the random equator $\mathbb{S}^{n-2}_p$:*

$$|\gamma| = \frac{\pi}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} |\gamma \cap \mathbb{S}^{n-2}_p|\,d\sigma_{n-1}(p), \tag{3.2}$$

*which, taken into account the definitions of rotation around the origin and the hyperplane $p^\perp$, is the same as the identity*

$$\Omega(\Gamma, 0) = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \Omega(\Gamma, p^\perp)\,d\sigma_{n-1}(p). \tag{3.3}$$

*Sketch of the proof.* The proof of (3.1) for $k = 1$, which coincides with (3.2), is obvious if $\gamma$ is a piece of a large circle: then the integrand in the right hand side is the function equal to 1 in a spherical sector between two meridians with the opening proportional to the length of $\gamma$. Hence this result is valid for spherical polygons, and the case of a general smooth curve is obtained by approximation. $\square$

**Proposition 2.** *For any measurable function* $\chi\colon \mathbb{S}^{n-1}\to\mathbb{R}$

$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\chi(r)\,d\sigma_{n-1}(r) = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}d\sigma_{n-1}(p)\left\{\frac{1}{|\mathbb{S}^{n-2}|}\int_{\mathbb{S}^{n-2}_p}\chi(q)\,d\sigma_{n-2}(q)\right\}. \tag{3.4}$$

*Sketch of the proof.* The formula (3.4) coincides with (3.1) for $\chi$ being the indicator function of a full-dimensional submanifold and $k=n-1$. To justify (3.1) in this extreme case, it is sufficient to note that the measure given by the right hand side of (3.1) is $(n-1)$-dimensional and rotational invariant. Thus it must coincide with the Lebesgue area $\sigma_{n-1}(\cdot)$ modulo a constant factor. By taking $M=\mathbb{S}^{n-1}$ one can easily check that this factor is in fact equal to 1. $\square$

3.2. **Rotation around random subspaces and the proof of Lemma 2.** We start with elementary properties of rotation. In order to avoid confusion, we use the extended notation $\Omega(\Gamma,L;L')$ for the rotation of $\Gamma\subset L'$ around $L\subset L'$.

3.2.1. *Rotation and projections.* Rotation of $\Gamma$ along any subspace $L$ is equal to the rotation around $P^{-1}(P(L))$ for any projection $P=P_p$. This follows from the following identity.

**Proposition 3.** *If $L\subset\mathbb{R}^n$ is a linear subspace, $p\in L$ a direction (as usual, identified with a point on the sphere $\mathbb{S}^{n-1}$) and $P=P_p$ the orthogonal projection from $\mathbb{R}^n$ onto $\mathbb{R}^{n-1}_p=p^\perp$, then*

$$\Omega(\Gamma,L;\mathbb{R}^n)=\Omega(P(\Gamma),P(L);\mathbb{R}^{n-1}_p). \tag{3.5}$$

*Remark* . This construction can be iterated as many times, as necessary, so that for any pair of subspaces $L\subseteq L'\subsetneq\mathbb{R}^n$ we have

$$\Omega(\Gamma,L';\mathbb{R}^n)=\Omega(P(\Gamma),P_L(L');L^\perp). \tag{3.6}$$

Assertion of (3.6) in the case $L=L'$ coincides with the *definition* of $\Omega(\Gamma,L)$.

*Proof.* Indeed, both parts of (3.5) are the angular (spherical) length of the projection of $\Gamma$ onto $L^\perp\subset\mathbb{R}^{n-1}_p$, since $P_L=P_{P(L)}\circ P$ (recall that $P_L\colon\mathbb{R}^n\to L^\perp$ stands for the orthogonal projection along $L$). $\square$

3.2.2. *Random one-dimensional extensions of subspaces.* There are two equivalent ways to parameterize $(k+1)$-dimensional linear subspaces containing a given $k$-dimensional subspace $L$. This implies an integral identity that will be used later.

**Proposition 4.** *For any $\varGamma \subset \mathbb{R}^n$ and any $L \subset \mathbb{R}^n$, $\dim L = k < n-1$,*

$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \Omega(\varGamma, L+\mathbb{R}p)\, d\sigma_{n-1}(p) = \frac{1}{|\mathbb{S}^{n-k-1}|}\int_{\mathbb{S}_L^{n-k-1}} \Omega(\varGamma, L\oplus\mathbb{R}q)\, d\sigma_{n-k-1}(q), \tag{3.7}$$

*where, as usual, $\mathbb{S}_L^{n-k-1} = \mathbb{S}^{n-1}\cap L^\perp$.*

*Proof.* Note first that for almost all $p \in \mathbb{S}^{n-1}$ the subspace $L+\mathbb{R}p$ is $(k+1)$-dimensional, while $L\oplus\mathbb{R}q$ is *always* $(k+1)$-dimensional for $q \in L^\perp$. Obviously, $L+\mathbb{R}p = L\oplus\mathbb{R}\cdot(P_L(p))$ for $p \notin L$, and for $p$ uniformly distributed over $\mathbb{S}^{n-1}$ the normalized projection $P_L(p)/\|P_L(p)\|$ is uniformly distributed in $\mathbb{S}_L^{n-k-1}$. □

3.2.3. *Proof of Lemma 2 for zero-dimensional subspaces.* Let $L=\{0\}$ be a zero-dimensional subspace. Then the assertion of Lemma 2 can be formulated as follows:

$$\Omega(\varGamma, 0) = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \Omega(\varGamma, \mathbb{R}p)\, d\sigma_{n-1}(p), \tag{3.8}$$

Substitute into (3.4) the function $\chi(r) = \Omega(\varGamma, r^\perp)$, the oscillation around the hyperplane $r^\perp$. Then by virtue of (3.3) the left hand side is just $\Omega(\varGamma, 0)$. On the other hand, denoting $P = P_p$, we have

$$\begin{aligned}
\Omega(\varGamma, \mathbb{R}p) &= \Omega(P(\varGamma), 0; \mathbb{R}_p^{n-1}) && \text{by (3.5)}\\
&= \frac{1}{|\mathbb{S}^{n-2}|}\int_{\mathbb{S}_p^{n-2}} \Omega(P(\varGamma), q^\perp\cap\mathbb{R}_p^{n-1}; \mathbb{R}_p^{n-1})\, d\sigma_{n-2}(q) && \text{by (3.3)}\\
&= \frac{1}{|\mathbb{S}^{n-2}|}\int_{\mathbb{S}_p^{n-2}} \Omega(\varGamma, q^\perp; \mathbb{R}^n)\, d\sigma_{n-2}(q) && \text{by (3.5),}
\end{aligned}$$

which after integration over all $p \in \mathbb{S}^{n-1}$ yields the right hand side of (3.4). The proof in the case $\dim L = 0$ is over.

3.2.4. *Proof of Lemma 2 in the general case.* Consider the orthogonal projection $P_L\colon \mathbb{R}^n \to \mathbb{R}_L^{n-k} = L^\perp$. We have

$$\begin{aligned}
\Omega(\varGamma, L; \mathbb{R}^n) &= \Omega(P_L(\varGamma), 0; L^\perp) && \text{by definition}\\
&= \frac{1}{|\mathbb{S}^{n-k-1}|}\int_{\mathbb{S}^{n-k-1}} \Omega(P_L(\varGamma), \mathbb{R}p; L^\perp)\, d\sigma_{n-k-1}(p) && \text{by (3.8)}\\
&= \frac{1}{|\mathbb{S}^{n-k-1}|}\int_{\mathbb{S}^{n-k-1}} \Omega(\varGamma, L\oplus\mathbb{R}p; \mathbb{R}^n)\, d\sigma_{n-k-1}(p) && \text{by (3.6)}\\
&= \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \Omega(\varGamma, L+\mathbb{R}p; \mathbb{R}^n)\, d\sigma_{n-1}(p) && \text{by (3.7).}
\end{aligned}$$

Thus the assertion of Lemma 2 is proved in full generality. □

3.3. **Integral curvature of a random projection and the proof of Lemma 1.** Let $\varGamma$ be a curve with an arbitrary (not necessary natural) parametrization $[0,\ell]\ni t\mapsto x(t)\in\mathbb{R}^n$. We start with an analytic expression for Frenet curvatures.

3.3.1. *Analytic expression for curvatures.* Let $v_1(t),\dots,v_n(t)$ be successive vector derivatives $v_1(t)=\dot x(t)$, $v_2(t)=\ddot x(t)$, …, $v_n(t)=\frac{d^n}{dt^n}x(t)$. Taken together they constitute the osculating frame.

Denote by $V_k(t)=\langle v_1(t),\dots,v_k(t)\rangle$ for $k=1,\dots,n$ the $k$-dimensional volume of the tuple $v_1,\dots,v_k$:

$$\langle v_1\rangle=\|v_1\|,\quad \langle v_1,v_2\rangle=\det{}^{1/2}\begin{bmatrix}(v_1,v_1) & (v_1,v_2)\\ (v_2,v_1) & (v_2,v_2)\end{bmatrix},$$

and in general for any $k=1,2,\dots,n$

$$\langle v_1,\dots,v_k\rangle=\det{}^{1/2}\begin{bmatrix}(v_1,v_1) & (v_1,v_2) & \cdots & (v_1,v_k)\\ (v_2,v_1) & (v_2,v_2) & \cdots & (v_2,v_k)\\ \vdots & \vdots & \ddots & \vdots\\ (v_k,v_1) & (v_k,v_2) & \cdots & (v_k,v_k)\end{bmatrix}$$

(obviously, $(\cdot,\cdot)$ is the Euclidean scalar product in $\mathbb{R}^n$).

The Frenet curvatures $\varkappa_j(t)$, originally defined via orthogonalization of the osculating frame, admit the following representation:

$$\begin{gathered}\varkappa_k(t)=\frac{\langle v_1,\dots,v_{k-1}\rangle\ \langle v_1,\dots,v_{k+1}\rangle}{\langle v_1,\dots,v_k\rangle^2\ \langle v_1\rangle}=\frac{V_{k-1}(t)\,V_{k+1}(t)}{V_k^2(t)\,V_1(t)},\\ v_j=v_j(t),\qquad k=1,\dots,n-1.\end{gathered}\tag{3.9}$$

Indeed, let $L_k=L_k(t)=\operatorname{Span}(v_1,\dots,v_k)=\operatorname{Span}(e_1,\dots,e_k)$ be the $k$th subspace of the osculating flag. Since the frame $\{e_k\}$ is obtained from the frame $\{v_k\}$ by orthogonalization, $v_k=\frac{V_k}{V_{k-1}}\,e_k \bmod L_{k-1}$. Differentiating this equality and denoting the natural parameter by $s$, we obtain from Frenet formulas (3.15)

$$v_{k+1}=\frac{d}{dt}v_k=\frac{V_k}{V_{k-1}}\cdot\frac{de_k}{ds}\cdot\frac{ds}{dt}\bmod L_k=\frac{V_k}{V_{k-1}}\,\varkappa_k e_{k+1}\,|v_1|\bmod L_k.$$

On the other hand, we should have $v_{k+1}=\frac{V_{k+1}}{V_k}\,e_{k+1}\bmod L_k$. This implies immediately that $V_1\varkappa_k=|v_1|\varkappa_k=\frac{V_{k+1}}{V_k}:\frac{V_k}{V_{k-1}}$, which coincides with (3.9).

After integration the identity (3.9) takes the form independent of the choice of the parametrization:

$$K_k(\varGamma)=\int_0^\ell\frac{\langle v_1,\dots,v_{k-1}\rangle\ \langle v_1,\dots,v_{k+1}\rangle}{\langle v_1,\dots,v_k\rangle^2}\,dt=\int_0^\ell\frac{V_{k-1}(t)\,V_{k+1}(t)}{V_k^2(t)}\,dt.\tag{3.10}$$

If $P=P_p\colon\mathbb{R}^n\to\mathbb{R}^{n-1}_p=p^\perp$ is the orthogonal projection along the direction $p\in\mathbb{S}^{n-1}$, then by linearity the osculating frame of $P(\varGamma)$ is the frame $P(v_1),\dots,P(v_n)$, and this identity yields explicit formulas for Frenet curvatures of the projected curve. If we denote by $\mathcal{L}_\Gamma(t)=\{L_j(t)\}_{j=1}^{n-1}$ the osculating flag of $\Gamma$ spanned by the frame $\{v_j\}$, then for any $k=1,\dots,n-1$

$$\begin{gathered}\langle P(v_1),\dots,P(v_k)\rangle=\sin(p,L_k)\cdot\langle v_1,\dots,v_k\rangle\,,\\ P=P_p,\ v_j=v_j(t),\ L_k=L_k(t)=\operatorname{Span}(v_1(t),\dots,v_k(t)),\end{gathered}\tag{3.11}$$

and $\sin(p,L_k)$ is the sine of the angle between $p$ and $L_k$, defined as the Euclidean angle in $\mathbb{R}^n$ between $p$ and its orthogonal projection on $L_k$.

3.3.2. *Demonstration of Lemma 1 for $k\leqslant n-1$.* Substituting (3.11) into (3.10) for any $k$ between 1 and $n-2$, we obtain

$$K_k(P(\Gamma))=\int_0^\ell\frac{V_{k+1}(t)V_{k-1}(t)}{V_k^2(t)}\cdot\frac{\sin(p,L_{k+1}(t))\sin(p,L_{k-1}(t))}{\sin^2(p,L_k(t))}\,dt.$$

Denoting $\kappa_k=V_{k+1}V_{k-1}/V_k^2=\|v_1\|\cdot\varkappa_k$ and averaging this equality over $p\in\mathbb{S}^{n-1}$, we obtain

$$\begin{aligned}&\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}K_k(P_p(\Gamma))\,dp\\ &=\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}d\sigma_{n-1}(p)\left\{\int_0^\ell\kappa_k(t)\cdot\frac{\sin(p,L_{k+1}(t))\cdot\sin(p,L_{k-1}(t))}{\sin^2(p,L_k(t))}\,dt\right\}\\ &=\int_0^\ell\kappa_k(t)\,dt\left\{\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\frac{\sin(p,L_{k+1}(t))\cdot\sin(p,L_{k-1}(t))}{\sin^2(p,L_k(t))}\,d\sigma_{n-1}(p)\right\}\\ &\overset{(!)}{=}\int_0^\ell\kappa_k(t)\,dt\left\{\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\frac{\sin(p,E_{k+1})\cdot\sin(p,E_{k-1})}{\sin^2(p,E_k)}\,d\sigma_{n-1}(p)\right\}\\ &=\left(\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\frac{\sin(p,E_{k+1})\cdot\sin(p,E_{k-1})}{\sin^2(p,E_k)}\,d\sigma_{n-1}(p)\right)\times\left(\int_0^\ell\kappa_k(t)\,dt\right)\\ &=\operatorname{const}_{k,n-1}\cdot K_k(\Gamma).\end{aligned}\tag{3.12}$$

where $E_j$ are subspaces of the standard flag $\mathcal{E}=\{E_j\}_1^{n-1}$. The transformation marked by (!), the key point of all the computation, holds by the rotational symmetry: any three subspaces of the flag $\mathcal{L}_\Gamma$ can be simultaneously transformed into three subspaces of any other flag by an appropriate rotation of $\mathbb{R}^n$.

In order to achieve the proof, it remains only to show that the constant factor denoted by $\operatorname{const}_{k,n-1}$ in (3.12), converges and is in fact equal to 1. This is done by the straightforward computation in the spherical coordinates on $\mathbb{R}^{n-1}$. For convenience we replace the sphere $\mathbb{S}^{n-1}\subset\mathbb{R}^n$ by $\mathbb{S}^n\subset\mathbb{R}^{n+1}$ endowed with the Euclidean coordinates $p=(p_1,\dots,p_{n+1})$. Our goal is to prove the identity

$$\operatorname{const}_{k,n}=\frac{1}{|\mathbb{S}^n|}\int_{\mathbb{S}^n}\frac{\sin(p,E_{k+1})\cdot\sin(p,E_{k-1})}{\sin^2(p,E_k)}\,d\sigma_n(p)\tag{3.13}$$

The sphere can be parameterized by the angles $\phi_1, \dots, \phi_n$ as

$$
\begin{aligned}
p_1 &= \sin\phi_1, \\
p_2 &= \cos\phi_1 \sin\phi_2, \\
p_3 &= \cos\phi_1 \cos\phi_2 \sin\phi_3, \\
&\dots \\
p_n &= \cos\phi_1 \cdots \cos\phi_{n-1} \sin\phi_n, \\
p_{n+1} &= \cos\phi_1 \cdots \cos\phi_{n-1} \cos\phi_n,
\end{aligned}
\qquad
\begin{gathered}
\phi_i \in (-\pi/2, \pi/2), \\
\forall\ i = 1, \dots, n-1, \\
\phi_n \in (-\pi, \pi)
\end{gathered}
$$

so that the $n$-volume element on the sphere has the form

$$d\sigma_n(p) = \cos^{n-1}\phi_1 \cos^{n-2}\phi_2 \cdots \cos\phi_{n-1}\, d\phi_1 \cdots d\phi_{n-1} d\phi_n.$$

Introducing the notation

$$B_k = \int_{-\pi/2}^{\pi/2} \cos^k\theta\, d\theta = \sqrt{\pi}\,\frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)},$$

where $\Gamma$ is the Euler gamma-function, we have the identity

$$1 = \frac{2\pi}{|\mathbb{S}^n|}\, B_{n-1}B_{n-2}\cdots B_2B_1, \tag{3.14}$$

following from the definition $\int_{\mathbb{S}^n} d\sigma_n(p) = |\mathbb{S}^n|$.

The angle between a point $p$ and the coordinate plane $E_k \subset \mathbb{R}^{n+1}$ spanned by the first $k$ coordinate vectors, can be easily measured: the squared sine of this angle is equal to the squared length of projection of $p$ on the remaining (complementary) coordinate subspace $E_k^\perp$. In other words, we have

$$
\begin{aligned}
\sin^2(p, E_k) &= \cos^2\phi_1 \cdots \cos^2\phi_k \times \\
&\qquad (\sin^2\phi_{k+1} + \cos^2\phi_{k+1}\sin^2\phi_{k+2} + \cdots + \cos^2\phi_{k+1}\cdots\cos^2\phi_n^2) \\
&= \cos^2\phi_1 \cdots \cos^2\phi_k.
\end{aligned}
$$

Since $k \leqslant n-1$ (the case we are interested in), all cosines are positive in the domain of parametrization, hence for the integral (3.13) we have the expression

$$
\begin{aligned}
\operatorname{const}_{n,k} &= \frac{1}{|\mathbb{S}^n|}\int_{\mathbb{S}^n} \frac{(\cos\phi_1\cdots\cos\phi_{k-1})(\cos\phi_1\cdots\cos\phi_{k+1})}{(\cos\phi_1\cdots\cos\phi_k)^2}\, d\sigma_n(p) \\
&= \frac{2\pi}{|\mathbb{S}^n|} \underset{-\pi/2\leqslant\phi_j\leqslant\pi/2}{\int\cdots\int} \frac{\cos\phi_{k+1}}{\cos\phi_k}\cdot\cos^{n-1}\phi_1\cdots\cos\phi_{n-1}\, d\phi_1\cdots d\phi_{n-1}.
\end{aligned}
$$

Now it is immediately clear that the integral converges for $n-1 \geqslant k \geqslant 1$ and is equal to the same product (3.14) with the terms $B_k$ and $B_{k+1}$ transposed. Hence we have proved the identity $\operatorname{const}_{n,k} = 1$, so that the assertion of Lemma 1 is proved for all integral curvatures $K_k$ except for the last one, the integral inflection.

3.3.3. *Averaging integral inflection.* The idea of the proof is the same as in [1].

Let $e_1(s), \dots, e_n(s)$ be the orthogonal osculating frame of a naturally parametrized regular curve $\Gamma \subset \mathbb{R}^n$, obtained by orthogonalization of the frame $v_1, \dots, v_n$. The last Frenet formula takes the form

$$\dot{e}_n(s) = -\varkappa_{n-1}(s)\, e_{n-1}(s). \tag{3.15}$$

Consider the curve $\Gamma^*$, parameterized as $s \mapsto e_n(s)$. Since $\|e_n(s)\| \equiv 1$, $\Gamma^*$ is a spherical curve, and from (3.15) it follows that

$$|\Gamma^*| = \int_0^\ell |\varkappa_{n-1}(s)|\, ds = K_{n-1}(\Gamma). \tag{3.16}$$

Applying the formula (3.2), we conclude that

$$K_{n-1}(\Gamma) = \frac{\pi}{|\mathbb{S}^{n-1}|} \cdot |\Gamma^* \cap p^\perp|\, d\sigma_{n-1}(p).$$

Now it remains only to note that if at some point $s \in [0, \ell]$ the vector $e_n(s)$ is orthogonal to the vector $p \in \mathbb{S}^{n-1}$, then the curve $P_p(\Gamma) \subset \mathbb{R}^{n-1}_p = p^\perp$ has an inflection point at $P_p(x(s))$, since the projection $P_p$ restricted on the subspace $L_{n-1}$ of the osculating flag $\mathcal{L}_\Gamma$, is *degenerate* (the rank is not full). Therefore $|\Gamma^* \cap p^\perp| = \cancel{K}_{n-1}(P_p(\Gamma))$, and taking the coefficient $\pi$ into account, we obtain the equality

$$K_{n-1}(\Gamma) = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} K_{n-1}(P_p(\Gamma))\, d\sigma_{n-1}(p).$$

The proof of Lemma 1 is complete. □

## 4. Pólya theorem and demonstration of Lemma 3

4.1. **Roots of linear combinations.** We start with a seemingly irrelevant question. Given a tuple of sufficiently smooth functions $f_1(t), \dots, f_n(t)$ all defined on a common interval $I = [\alpha, \beta] \subset \mathbb{R}^1$, how many isolated zeros may have a (nontrivial) linear combination $\lambda_1 f_1 + \dots + \lambda_n f_n$ on that interval for an arbitrary choice of the coefficients $\lambda_j \in \mathbb{R}$? We shall address this problem in *periodic* and *non-periodic* context, the former meaning that all $f_j$ extend as $(\beta - \alpha)$-periodic functions on $\mathbb{R}^1$. To formulate the results in the uniform way, we introduce the number $\delta \in \{0, 1\}$, equal to 0 if the functions are periodic and 1 otherwise. The periodic context corresponds in fact to functions defined on the circle $\mathbb{S}^1 = I/(\alpha \sim \beta)$ rather than on the interval $I$, thus in some sense $2\delta = |\partial I|$ is the number of endpoints of $I$.

To that end, we fix the order of functions $f_j$ and introduce (following G. Pólya [8, 9]) $n + 1$ functions $W_k \colon I \to \mathbb{R}$ as the Wronski determinants of the first $k$

functions $f_1, \dots, f_k$:

$$W_0(t) \equiv 1, \quad W_1(t) = f_1(t), \quad W_2(t) = f_1'(t)f_2(t) - f_1'(t)f_2(t), \quad \dots$$

$$W_k(t) = \det \begin{bmatrix} f_1(t) & f_2(t) & \cdots & f_k(t) \\ f_1'(t) & f_2'(t) & \cdots & f_k'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(k-1)}(t) & f_2^{(k-1)}(t) & \cdots & f_k^{(k-1)}(t) \end{bmatrix}, \quad k = 1, 2, \dots, n. \tag{4.1}$$

4.1.1. *Chebyshev systems and Pólya theorem.* In the simplest case nonvanishing of the Wronskians implies the Chebyshev property (nonoscillation) of the linear spaces of functions.

**Theorem 5** (Pólya theorem [8])**.** *If all Wronskians $W_1, \dots, W_n$ are nonvanishing on $I$, then any linear combination $\sum_{j=1}^n \lambda_n f_n$ may have at most $n-1$ isolated root on $I$, counting with multiplicities.*

4.1.2. *The general case.* If the Wronskians $W_j$ have zeros on $I$, not vanishing identically, then Theorem 5 holds no more. However, in this case one could find an upper bound for the number of isolated zeros occurring in linear combinations. Denote by $\nu_j \geqslant 0$ the number of zeros of $W_j$ on $I$.

The simplest way is to consider partition of $I$ by the roots of Wronskians into $\delta + \sum_{j=1}^n \nu_j$ subintervals. Then Theorem 5 can be applied to each of them, yielding an upper bound of $(n-1)(\delta + \sum \nu_j)$ for the number of zeros *outside* the partition points. Adding the number of zeros eventually occurring at these points, we arrive to the upper bound $(n-1)\delta + n \sum \nu_j$. However, this estimate can be substantially improved. The idea of such improvement was given in [7], where a result, though inferior to the inequality below, but still sufficient for the purposes of [7], was proved.

**Lemma 4.** *If $\nu_j$ is the number of zeros of $W_j$ on the segment $I$, then the number of isolated zeros occurring in any nontrivial linear combination of the functions $f_1, \dots, f_n$, does not exceed*

$$(n-1) \cdot \delta + \sum_{j=1}^n w_{n-j} \nu_j, \tag{4.2}$$

*where the sequence of $w_j$ is the same as in* (1.8)*.*

Note that if $\nu_1 = \cdots = \nu_n = 0$, then the assertion of Lemma 4 coincides with that of Theorem 5.

*Proof.* The proof is based on the fact that all linear combinations satisfy the following $n$th order linear ordinary differential equation with variable coefficients:

$$D_n D_{n-1} \cdots D_1 \, y = 0, \qquad D_i = \frac{W_i}{W_{i-1}} \, \partial \, \frac{W_{i-1}}{W_i}, \quad \partial = \frac{d}{dt} \tag{4.3}$$

(this form is due to Frobenius, and a simple proof of this fact can be found in [7]). The equation (4.3) can be transformed to the form

$$\Delta_{n-1} \cdot \frac{1}{W_{n-3}} \cdot \Delta_{n-2} \cdots \Delta_3 \cdot \frac{1}{W_1} \cdot \Delta_2 \cdot \frac{1}{W_0} \cdot \Delta_1 \, y = \text{const} \cdot W_{n-2} W_n, \tag{4.4}$$

where $\Delta_j = W_j^2 \cdot \partial \cdot (W_j)^{-1}$ is the differential operator transforming a function $\varphi$ into $W_j^2(\varphi/W_j)'$, and without loss of generality one may assume that const $\neq 0$ (otherwise the order of the equation can be further reduced). We need first to modify the Rolle theorem to allow for functions with poles and differential operators other than $\partial$.

If $g \colon I \to \mathbb{R}$ is a smooth function and $N(g) < \infty$ the number of its zeros on $I$, then $\Delta_j g$ is also smooth, and

$$N(g) \leqslant N(\Delta_j g) + \nu_j + \delta, \tag{4.5}$$

where $\delta$ is 0 or 1, depending on whether $g$ is periodic or not. Indeed, $g/W_j$ is a function that is smooth on $\nu_j + \delta$ intervals between zeros of $W_j$. Application of $\partial$ may decrease the number of zeros on each interval at most by one (Rolle theorem), and multiplication by $W_j^2$ restores the smoothness.

If the function $g$ itself has $p$ poles (e.g., $g$ is a fraction whose denominator has $p$ isolated zeros), then (4.5) should be replaced by

$$N(g) \leqslant N(\Delta_j g) + \nu_j + p + \delta, \tag{4.6}$$

since the number of intervals of continuity will be in this case $\nu_j + p + \delta$.

Let $f = \sum_{j=1}^n \lambda_j f_j$ be a nontrivial linear combination of functions $f_j$. Consider the sequence of functions occuring in evaluation of the left hand side of (4.4):

$$\begin{aligned} &F_0 = f,\ F_1 = \Delta_1 F_0,\ F_2 = \Delta_2 W_0^{-1} F_1,\ F_3 = \Delta_3 W_1^{-1} F_2,\ \ldots, \\ &\qquad F_{n-2} = \Delta_{n-2} W_{n-4}^{-1} F_{n-3},\ F_{n-1} = \Delta_{n-1} W_{n-3}^{-1} F_{n-2} = \text{const} \cdot W_{n-2} W_n. \end{aligned}$$

The number of poles of $F_k$ is at most $\nu_{n-2} + \nu_{n-3} + \nu_{n-4} + \cdots$, assuming that $\nu_0 = \nu_{-1} = \cdots = 0$.

Iterating the inequalities (4.6), we obtain the chain of $(n-1)$ inequalities

$$\begin{aligned} N(F_1) &\geqslant N(F_0) - \nu_1 - \delta, \\ N(F_2) &\geqslant N(F_1) - \nu_2 - \nu_0 - \delta, \\ N(F_3) &\geqslant N(F_2) - \nu_3 - (\nu_0 + \nu_1) - \delta, \\ N(F_4) &\geqslant N(F_3) - \nu_4 - (\nu_0 + \nu_1 + \nu_2) - \delta, \end{aligned}$$

and so on. Adding up all these inequalities, we arrive to the estimate

$$N(W_nW_{n-2}) = N(F_{n-1}) \geqslant N(f) - (n-1)\delta - (\nu_1 + \nu_2 + \cdots + \nu_{n-1}) \\ - (n-2)\nu_0 - (n-3)\nu_1 - \cdots - \nu_{n-3}.$$

The left hand side of this inequality is equal to $\nu_{n-2} + \nu_n$ by (4.4), while $\nu_0 = 0$. Therefore the number of zeros $N(f)$ is estimated from above by the combination

$$\nu_n + \nu_{n-1} + 2\nu_{n-2} + 2\nu_{n-3} + 3\nu_{n-4} + \cdots + (n-3)\nu_2 + (n-2)\nu_1 + (n-1)\delta.$$

The proof is complete. $\square$

*Remark* . In the above proof we have assumed that the roots of $W_j$ are disjoint from the roots of the corresponding factor, so that each division by $W_j$ increases the number of intervals of continuity by $\nu_j$ but does not change the number of zeros. As a matter of fact, the coincidence may happen, so that cancellation occurs, but then the number of intervals of continuity will be smaller on *each* step from that moment on. One can easily check that the overall estimate in this case will be even *better*. Alternatively, one can use small perturbation to move zeros of $W_j$ away and then use the semicontinuity arguments for the number of zeros.

4.2. **Demonstration of Lemma 3.** Let $\mathcal{L} = \{L_j\}_{j=1}^n$ be the complete flag and $\nu_j = \nu_j(\Gamma, \mathcal{L}) < \infty$ the number of inflection points of the orthogonal projection of $\Gamma$ on $L_j$.

Choose the orthogonal coordinate system in such way that the frame spans the flag $\mathcal{L}$ (this choice is essentially unique, modulo change of signs of the coordinates). Then the curve $\Gamma$ corresponds to a smooth vector-function $t \mapsto x(t) = (x_1(t), \dots, x_n(t))$. Denote $f_j(t) = \frac{d}{dt}x_j(t)$ and consider the Wronskians corresponding to the ordered tuple $f_1, \dots, f_n$.

Then vanishing of the $j$th Wronskian $W_j$ corresponds to the inflection point of the projection of $\Gamma$ on $L_j$ (the first $j$ coordinates), see §3.3.1. Note also that the velocity curve $\dot\Gamma$ is closed if and only if the functions $f_j$ are periodic.

By Lemma 4, the number of isolated roots of any linear combination $\sum_{j=1}^n \lambda_j f_j(t)$ can be at most $N = \frac{1}{2}(n-1)|\partial\dot\Gamma| + \sum_{j=1}^n w_{n-j}\nu_j$. In geometric terms this means that the *velocity vector* $\dot x(t) = (\dot x_1(t), \dots, \dot x_n(t))$ intersects any *linear* hyperplane $\{\sum \lambda_j x_j = 0\} \subset \mathbb{R}^n$ at most by $N$ points. This is exactly the second claim of Lemma 3. But then by the Rolle theorem, the curve $\Gamma$ itself can intersect any *affine* hyperplane at most by $N + \frac{1}{2}|\partial\dot\Gamma|$ points, which is the inequality asserted by the Lemma, since $|\partial\dot\Gamma| \leqslant |\partial\Gamma|$ (the velocity curve $\dot\Gamma$ may be closed, while $\Gamma$ not, but the inverse is impossible).

An alternative (direct) way to prove the second assertion of Lemma 3 would be to consider the system of $n+1$ functions $1, x_1(t), \dots,\ x_n(t)$ and expand the corresponding Wronskians in elements of the first row, proving thus an upper bound for the number of intersections of $\varGamma$ with any affine hyperplane $\lambda_0+\sum \lambda_j x_j = 0$. $\square$

4.3. **Demonstration of Theorem 1.** The idea of demonstration is straightforward: if a hyperconvex curve $\varGamma$ parameterized by the arclength $t \in [0,\ell]$ is oscillating, then its osculating flag $\mathcal{L}_\varGamma(t)$ must become non-transversal to any other flag, in particular, to $\mathcal{L}' = \mathcal{L}_\varGamma(0)^\perp$. But $\mathcal{L}_\varGamma(0)$ is in some sense "maximally transversal" to $\mathcal{L}'$, and to take a non-transversal position, $\mathcal{L}_\varGamma(t)$ must make a sufficiently long path in the flag variety. On the other hand, the velocity of the "curve" $t \mapsto \mathcal{L}_\varGamma(t)$ in this variety is controlled by the instant curvatures $\varkappa_j(t)$, $j = 1,\dots,n-1$, which means that the integral curvatures cannot bee too small.

For practical reasons it is more convenient to work not in the flag variety, but rather in its covering space identified with the orthogonal group.

Let $\mathfrak{G}\colon I = [0,\ell] \to \mathrm{SO}(n)$, $t \mapsto E(t)$ be the *associated curve*: the point $t$ is mapped into the orthogonal matrix whose columns are the vectors $e_j(t)$ of the orthogonalized Frenet frame. Then the Frenet formulas take the form $\dot E(t) = A(t)E(t)$, where $A(t)$ is an antisymmetric matrix function with the entries $\pm\varkappa_j(t)$, $j = 1,\dots,n-1$, occurring on the principal sub- (resp., super)-diagonal. Without loss of generality we can assume that $E(0)$ is the identity matrix (the corresponding flag is the coordinate one $\mathcal{E}$).

We embed the orthogonal group $\mathrm{SO}(n)$ into the Euclidean space $\mathbb{R}^{n^2}$ of square matrices with the norm $\|X\|^2 = \sum_{i,j=1}^n x_{ij}^2$, where $x_{ij}$ are the entries of $X$.

By the Cauchy–Bunyakovskii inequality, $\|AX\| \leqslant \|A\|\cdot\|X\|$. Evidently, $\|A(t)\|^2 = 2\varkappa_1^2(t) + \dots + 2\varkappa_{n-1}^2(t)$, and $\|E(t)\|^2 \equiv n$, since $E(t)$ is an orthogonal matrix. Therefore

$$\|\dot E(t)\| \leqslant \sqrt{2n}\cdot\sqrt{\textstyle\sum_{j=1}^{n-1}\varkappa_j^2(t)}, \qquad |\mathfrak{G}| \leqslant \sqrt{2n}\cdot\int_0^\ell \sqrt{\textstyle\sum_{j=1}^{n-1}\varkappa_j^2(t)}\,dt, \tag{4.7}$$

where $|\mathfrak{G}|$ is the length of $\mathfrak{G}$ in $\mathbb{R}^{n^2}$.

On the other hand, all matrices sufficiently close to the identity matrix, correspond to flags transversal to the antipodal flag $\mathcal{E}^\perp$. More precisely, if all upper-left $(k\times k)$-minors of $E(t)$ are nonzero, then the above transversality holds. In particular, if $\|X\| < 1/\sqrt{n}$, then $E(0)+X$ corresponds to the flag still transversal to $\mathcal{E}^\perp$. Indeed, in this case for any $i = 1,\dots,n$ we have $\bigl(\sum_{j=1}^n |x_{ij}|\bigr)^2 \leqslant n\sum_{j=1}^n x_{ij}^2 \leqslant n\|X\|^2 < 1$, and any row of the matrix $E(0)+X$ has a dominant diagonal element. Thus all minors are nonzero, and the required transversality holds.

In other words, we have proved that the ball $\{E(0)+X \in \mathbb{R}^{n^2} : \|X\| < 1/\sqrt{n}\}$ of radius $1/\sqrt{n}$ centered at $E(0)$, consists of matrices that span flags transversal to $\mathcal{E}^\perp$. If the curve $\varGamma$ is oscillating, then by Shapiro theorem the associated curve $\mathfrak{G}$

should leave this ball, hence its length should be at least $1/\sqrt{n}$. Taken into account the inequality (4.7), we arrive to the final estimate (1.3). $\square$

It is clear that measuring lengths in $\mathbb{R}^{n^2}$ rather than in the group $\mathrm{SO}(n)$ results in the loss of sharpness. To get the best results from this approach, one should use left-invariant metric on $\mathrm{SO}(n)$ and estimate the distance from $E(0)$ to the nearest non-transversal matrix in this metric. However, we do not want to discuss the general case, but rather consider three-dimensional curves where similar computation is relatively easy.

4.4. **Non-oscillating curves in $\mathbb{R}^3$ via Shapiro theorem.** As an illustration of the Shapiro theorem (Corollary 4), we prove that a hyperconvex (non-closed) curve with $K_1(\Gamma) + K_2(\Gamma) < \frac{\pi}{2}$ is non-oscillating, $\Omega_2(\Gamma) \leqslant 3\pi$.

Consider the osculating flag $\mathcal{L}(s) = \mathcal{L}_\Gamma(s)$. Without loss of generality we may assume that $\mathcal{L}(0) = \mathcal{E}$ (the standard flag). If the curve is oscillating, then $\mathcal{L}(s)$ should become non-transversal to the antipodal flag $\mathcal{E}^\perp$ at some point, by virtue of Corollary 4. This may happen in one of the two possible scenarios:

- either the tangent $e_1(s)$ intersects the plane spanned by $e_2(0)$ and $e_3(0)$,
- or the vector $e_3(s)$ intersects the plane spanned by $e_1(0)$ and $e_2(0)$.

In both cases the length of the path made by the corresponding vector on the sphere before the intersection occurs, is at least $\frac{\pi}{2}$ (the spherical distance from the north pole to the equator).

On the other hand, from Frenet formulas $\dot e_1(s) = \varkappa_1(s)e_2(s)$, $\dot e_3(s) = -\varkappa_2(s)e_2(s)$, it follows that the path made by $e_1(s)$ for $s \in [0,\ell]$, is exactly $K_1(\Gamma)$, while that made by $e_3(s)$ is $K_2(\Gamma)$. Therefore the inequality $K_1 + K_2 < \frac{\pi}{2}$ excludes either possibility, and the contradiction thus achieved proves that the curve $\Gamma$ is non-oscillating.

## 5. Isoperimetric inequalities on $\mathbb{S}^2$ and non-oscillation

In this section we consider *three-dimensional hyperconvex* curves, primarily the closed ones, and prove the inequality (1.7) for $\Omega_2(\Gamma)$, based on a completely different set of arguments. We will always assume that $\Gamma$ is parameterized by the arc-length $s \in [0,\ell]$, so the velocity curve (*godograph*) $\dot\Gamma\colon s \mapsto \dot x(s)$ is a spherical curve. We also return locally to the classical terminology, referring to $\varkappa(s) = \varkappa_1(s) > 0$ as the curvature and $\theta(s) = \varkappa_2(s)$ as the torsion.

As follows from the identities (2.7), the arc-length element on $\dot\Gamma$ is $\varkappa(s)\,ds$, while the (first and unique) geodesic curvature of $\dot\Gamma$ is $\widetilde\varkappa(s) = \theta(s)/\varkappa(s)$. If $\Gamma$ is hyperconvex (without inflection points), then $\dot\Gamma$ is *geodesically convex*.

5.1. **Isoperimetric inequalities on the sphere.** First we consider *hemispheric convex lobes*, closed piecewise smooth curves formed by a piece $A$ of a smooth geodesically convex curve and an arc of a large circle (equator) $E$, entirely belonging to one hemisphere. Denote by $\alpha$ and $\alpha'$ the exterior angles at the vertices of the lobe, and let $\widetilde{K}(A)$ be the integral *geodesic* curvature of the arc $A$ (integral of the geodesic curvature $\widetilde{k}(s)$ against the arc length). Our local goal is to prove the inequality

$$\alpha + \alpha' + \widetilde{K}(A) + 2|A| \geqslant 2\pi. \tag{5.1}$$

Note that $\alpha + \alpha' + \widetilde{K}(A)$ is the integral geodesic curvature of the *entire lobe*, since $\widetilde{K}(E) = 0$. To prove (5.1), we first note that by the spherical excess theorem (Gauss–Bonnet formula),

$$\alpha + \alpha' + \widetilde{K}(A) = 2\pi - S, \tag{5.2}$$

where $S$ is the area of the lobe [2, Chapter 1, §2.7]. Now the problem is to majorize $S$ in terms of $|A|$.

The standard isoperimetric inequality between the length $|\gamma|$ of a simple closed spherical curve and $S$, the area bounded by this curve, has the form

$$|\gamma|^2 \geqslant 4\pi S - S^2, \tag{5.3}$$

the equality being attained only if $\gamma$ has a constant geodesic curvature. In a similar way the *Dido problem* of finding a shortest curve with endpoints on equator, bounding together with the piece of equator the largest possible area, has only constant curvature solutions, normally crossing the equator. For the corresponding solution $A$ one has the isoperimetric inequality

$$|A|^2 \geqslant 2\pi S - S^2, \tag{5.4}$$

where $S$ is now the area bounded by the curve and the equator together. From (5.4) it follows that within the range $0 < S \leqslant \frac{8}{5}\pi$ the length and the area are related by the inequality

$$|A| \geqslant \tfrac{1}{2} S, \qquad 0 \leqslant S \leqslant \tfrac{8}{5}\pi. \tag{5.5}$$

Unfortunately, (5.4) without additional considerations does not imply any lower bound for $|A|$, if $S$ approaches the area of the hemisphere $2\pi$. In order to analyze the region of large areas, we apply instead the inequality (5.3) to the closed arc formed by $A$ and $E$ together: since $S \leqslant 2\pi$ (the lobe belongs to a hemisphere), the inequality $(|A| + |E|)^2 \geqslant 4\pi S - S^2$ implies

$$|A| + |E| \geqslant \pi + \tfrac{1}{2} S, \qquad \tfrac{2}{5}\pi \leqslant S \leqslant 2\pi. \tag{5.6}$$

Now it remains to point out that for a *convex* lobe $|E| \leqslant \pi$. Indeed, otherwise rotating a half-equator inside the lobe while keeping its endpoints fixed, we can obtain an *inner* tangency between $|A|$ and a geodesic curve, which is impossible. Substituting this into (5.6), we conclude that the inequality $|A| \geqslant S$ holds on two overlapping intervals, $[0, \frac{8}{5}\pi]$ and $[\frac{2}{5}\pi, 2\pi]$, covering together the entire range of admissible areas $[0, 2\pi]$. Together with (5.2) this proves (5.1).

Our next goal is to extend the inequality (5.1) for geodesically convex hemispheric curves with endpoints on an equator, but eventually self-intersecting. It turns out to be even easier.

If a curve $A$ is self-intersecting, forming a number of "petals", then one can break $A$ into (oriented) smooth pieces and re-connect them in such a way that together with the arc $E$ of the equator, they will form $\nu > 1$ closed curves, only one of them containing $E$ (the case $\nu = 1$ corresponds to simple $A$). The domains bounded by these curves, may overlap, but in any case their areas $S_i$ will not exceed $2\pi$, and at least one of them will be convex disjoint from $E$.

The spherical excess theorem can be applied to each domain; if we add the corresponding equalities together, then the resulting equality will take the form

$$\alpha + \alpha' + \widetilde{K}(A) = 2\pi\nu - (S_1 + \cdots + S_\nu).$$

For each domain with the area $S_j$ and bounded by the arc of the length $|A_j|$ (except for one of them whose boundary whose the length $|A_j| + |E|$), the isoperimetric inequality (5.3) gives the inequality $|A_j|^2 \geqslant 4\pi S_j - S_j^2$ which, since all $S_j$ are less than $2\pi$, implies that $|A_j| \geqslant S_j$ (for the exceptional domain the latter takes the form $|A_j| + |E| \geqslant S_j$). Adding these inequalities together and noting that $|A_1| + \cdots + |A_\nu| = |A|$, we arrive to the inequality $\alpha + \alpha' + \widetilde{K}(A) + |A| + |E| \geqslant 2\pi\nu$. It remains only to observe that $|E| \leqslant 2\pi$ and $\nu > 1$ to conclude that

$$\alpha + \alpha' + \widetilde{K}(A) + |A| \geqslant 2\pi(\nu - 1) \geqslant 2\pi,$$

which is even stronger than asserted by (5.1).

5.2. **Length–curvature inequality for geodesically convex spherical curves oscillating around an equator.** Now assume that $A$ is a geodesically convex spherical curve intersecting some equator at $n \geqslant 2$ points. We show that in this case

$$\widetilde{K}(A) + 2|A| \geqslant \pi(n - |\partial A|). \tag{5.7}$$

To prove this, we break $A$ into smooth pieces between subsequent intersections with the equator; their number is $n$ if $A$ is closed and $n-1$ otherwise. Denote by $\alpha_i$ and $\alpha_i'$ the exterior angles of the lobes formed by $A_i$ together with the corresponding arcs of the equator. Then, as one can easily see,

$$\alpha_{i+1} + \alpha'_i = \pi \qquad i = 1, \dots, n-1$$

(if $A$ is closed, then the index $i$ is cyclical modulo $n$).

Apply the inequality (5.1) to each of these lobes, and add the results together. Then, since all curvatures $\widetilde{K}(A_i)$ are of the same sign, the resulting inequality will take the form

$$\begin{cases} \pi n + \widetilde{K}(A) + 2|A| \geqslant 2\pi n, & \text{if } A \text{ is closed,} \\ \alpha_1 + \alpha'_{n-1} + \pi(n-2) + \widetilde{K}(A) + 2|A| \geqslant 2\pi(n-1), & \text{otherwise.} \end{cases}$$

Since both $\alpha_1, \alpha'_{n-1}$ are less than $\pi$, we arrive to the inequality (5.7).

5.3. **Hyperconvex curves in $\mathbb{R}^3$ and on $\mathbb{S}^2$.** The inequality (5.7) is in fact the upper bound for oscillation of spherical curves, that coincides with (1.12) for $n = 2$ and $r = 1$. To prove the inequality $\Omega_2(\Gamma) \leqslant 2K_1(\Gamma) + K_2(\Gamma) + \frac{3}{2}|\partial\Gamma|$ which is a particular case of Theorem 2 for three-dimensional hyperconvex curves, we apply once again the Rolle theorem to the spherical curve $A = \dot{\Gamma}$, as in the proof of Theorem 2 for hyperplanes: the resulting estimate will be then

$$\begin{aligned} \Omega_2(\Gamma) &\leqslant \tfrac{\pi}{2}|\partial\Gamma| + \pi|\partial\dot{\Gamma}| + 2|\dot{\Gamma}| + \widetilde{K}(\dot{\Gamma}) \\ &\leqslant \tfrac{3}{2}\pi|\partial\Gamma| + 2K_1(\Gamma) + K_2(\Gamma) \end{aligned}$$

by virtue of the formulas (2.8). □

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