Lectures on meromorphic flat connections

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Abstract

These notes form an extended version of a minicourse delivered in Université de Montréal (June 2002) within the framework of a NATO workshop "Normal Forms, Bifurcations and Finiteness Problems in Differential Equations".

The focus is on Poincaré–Dulac theory of "Fuchsian" (logarithmic) singularities of integrable systems, with applications to problems on zeros of Abelian integrals in view.

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Instead of introduction: Infinitesimal Hilbert Problem

One of the challenging problems that recently attracted much of attention, is the question about the number of zeros of complete Abelian integrals of algebraic 1-forms over closed ovals of plane algebraic curves. This problem arises as a "linearization" of the Hilbert sixteenth problem on the number of limit cycles of polynomial vector fields, see [Yak01].

There were numerous attempts to solve the infinitesimal Hilbert problem in various, sometimes relaxed, settings. The approach recently suggested by the authors of these notes, suggests to exploit the fact that Abelian integrals satisfy a system of linear ordinary differential equations with rational coefficients, the so called Picard–Fuchs equations.

The general claim concerning such systems, was established in [NY02], see also [Yak01]. Consider a *Fuchsian linear* $(n\times n)$*-system* on the Riemann sphere $\mathbb{C}P^1$, written in the Pfaffian (coordinate-free) form as $dX = \Omega X$, where Ω is a meromorphic (rational) matrix 1-form on $\mathbb{C}P^1$ having only simple poles. After choosing an affine coordinate $t \in \mathbb{C}$ on the Riemann sphere so that infinity is not a pole of $\Omega = A(t)\,dt$, the Pfaffian system can be reduced to a system of *linear ordinary differential equations with rational coefficients* of the form

$$\tfrac{d}{dt}X = A(t)X, \qquad A(t) = \sum_{j=1}^{r} \frac{A_j}{t-t_j}, \quad \sum_{1}^{r} A_j = 0, \tag{0.1}$$

with the $(n\times n)$-matrix residues $A_1, \dots, A_r \in \operatorname{Mat}_n(\mathbb{C})$ and the singular points $t_1, \dots, t_r \in \mathbb{C}$.

Under certain assumption on the spectra of monodromy operators (linear transformations of solutions after analytic continuation along closed loops avoiding singular points), solutions of the system (0.1) possess the following property making them remotely similar to algebraic functions. Namely, the number of zeros of any rational combination $f \in \mathbb{C}(X)$ of components of any fundamental matrix solution $X = X(t)$ in any semialgebraic domain $T \subset \mathbb{C} \smallsetminus \{t_1, \dots, t_r\}$ can be explicitly majorized by a computable (though enormously large) function of the dimension n and the degree r of the system (0.1), the degree of f in $\mathbb{C}(X)$ and the *height* $h(\Omega)$ of the system (0.1) defined as $\sum_1^r |A_j|$. It is important to stress that the bound is *uniform* over configurations of the singular points, provided that the condition imposed on the monodromy persists.

The Picard–Fuchs system for Abelian integrals was explicitly derived in [NY01] in the form that can be easily reduced to (0.1). However, the direct application of results from [NY02] is in general impossible (exception occurs in the hyperelliptic case when the methods rather than results of [NY02] can be made to work, see [NY99]). The reason is the *explosion of residues*.

This phenomenon occurs when two or more singular points of (0.1) collide when the additional parameters defining system, change. In general system (0.1) the parameters t_j, A_j are completely independent and can be restricted by the condition on the height $h(\Omega) \leqslant \text{const}$. However, in the Picard–Fuchs systems for Abelian integrals, which depend on the coefficients of the equation of the algebraic curve as the parameters, the norms of the matrix residues $|A_i|, |A_j|$ usually tend to infinity if the corresponding singular points t_i and t_j tend to each other. This makes impossible application of the principal theorem from [NY02] in order to obtain uniform bounds on the number of zeros of Abelian integrals.

The reason behind this phenomenon is the fact that the system (0.1) for Abelian integrals is an *isomonodromic* family (deformation) if considered as depending on the natural parameters of the problem. This means that the monodromy group of the system remains “the same” even though the parameters vary. While this isomonodromy makes it much easier to verify the spectral condition required in [NY02], it very often forces the residues explode even when the parametric family of Fuchsian systems (0.1) is not related to Abelian integrals.

The natural way to investigate isomonodromic families of linear systems is to consider them in the multidimensional setting when the the independent variable t and the parameters play the same role. Then the isomonodromy property takes the form of the *flatness condition* (integrability) of the corresponding *meromorphic connection.* The main goal of these notes is to provide the reader with the necessary background in the theory of integrable linear Pfaffian systems. In particular, we discuss briefly the possibility of getting rid of exploding residues by a suitable meromorphic gauge transformation, which (if successfully implemented for the Picard–Fuchs systems) would entail applicability of the technique from [NY02].

1 Integrable Pfaffian systems

Let U be a complex m-dimensional manifold (in most cases U will be an open subset of $\mathbb{C}^m$ or even more specifically, a small polydisc centered at the origin). Consider a *matrix Pfaffian 1-form* $\Omega = \|\omega_{ij}\|_{i,j=1}^n$, an $n \times n$-matrix whose entries are holomorphic 1-forms $\omega_{ij} \in \Lambda^1(U)$. This matrix defines a systems of linear Pfaffian equations

$$dx_i = \sum_{j=1}^{n} \omega_{ij} x_j, \quad \text{or in the vector form} \quad dx = \Omega x. \tag{1.1}$$

Solution of the system (1.1) is a tuple $x = (x_1, \dots, x_n)$ of holomorphic functions on U, $x(\cdot)\colon U \to \mathbb{C}^n$. Any number of (column) vector solutions of (1.1) can be organized into a matrix. A *fundamental matrix solution* $X = \|x_{ij}\|$ is the holomorphic *invertible* square $n \times n$-matrix solution of the *matrix Pfaffian equation*

$$dX = \Omega X, \qquad X\colon U \to \mathrm{GL}(n, \mathbb{C}), \tag{1.2}$$

where dX is the matrix 1-form with the entries dx_{ij}. Columns of the fundamental matrix solution are linear independent vector solutions of the system (1.1).

If $m = 1$, after choosing any local coordinate $t = t_1$ on U the system (1.1) becomes a system of linear *ordinary* differential equations with respect to the unknown functions $x_i(t)$. For $m > 1$, (1.1) is a system of partial linear differential equations with respect to any coordinate system $t = (t_1, \dots, t_m)$, in both cases with holomorphic coefficients.

The "ordinary" case is very well known. In many excellent textbooks, e.g., [Har82, For91, AI88, Bol00a] one can find a detailed treatment of the following issues: local and global existence of solutions, local classification of singularities, local and global theory of systems having only simplest (Fuchsian) singular points.

The main reference for the "partial" case, no less important (in particular, because of its connections with *deformations* of the "ordinary" systems), is the classical book by P. Deligne [Del70]. Yet this book is written in rather algebraic language which naturally determines the choice of the questions addressed therein. The primary goal of these notes was to supply an "analytic translation" of parts of the book [Del70] roughly along the lines characteristic for the "ordinary" point of view. However, the contents of these notes is not limited to re-exposition of [Del70], as we include (with what can be considered as more or less complete proofs) some of the results announced in [YT75, Tak79].

In what follows we will usually assume that the reader is familiar with the "ordinary" case, however, the proofs supplied for the "partial" case will be mostly independent and working also for $m = 1$.

Notations. We will systematically use the matrix notation in these notes. The wedge product $\Omega \wedge \Theta$ of two matrix 1-forms $\Omega = \|\omega_{ij}\|$, $\Theta = \|\theta_{ij}\|$ will denote the matrix 2-form with the entries $\sum_{k=1}^{n} \omega_{ik} \wedge \theta_{kj}$. If $A = \|a_{ij}(t)\|$ is a matrix *function*, then $A\Omega$ is a matrix 1-form with the entries $\sum_j a_{ij}\omega_{jk}$ (sometimes we use the dot notation for this product). The commutator $[A, \Omega]$ means $A\Omega - \Omega A$; similarly, $[\Omega, \Theta] = \Omega \wedge \Theta - \Theta \wedge \Omega$. Note that in the matrix case the bracket is *not* antisymmetric, so that $\Omega \wedge \Omega$ needs not necessarily be zero. On the other hand, the associativity $\Omega \wedge (A\Theta) = (\Omega A) \wedge \Theta$ holds.

1.1 Local existence of solutions. Integrability

For $m = 1$ the linear system (1.1) (respectively, (1.2)) always admits solution (resp., fundamental matrix solution) at least locally, near each point $a \in U$. When $m > 1$, additional *integrability condition* is required.

Indeed, if X is a fundamental matrix solution for (1.2), then $\Omega = dX \cdot X^{-1}$. Using the formula
$$d(X^{-1}) = -X^{-1} \cdot dX \cdot X^{-1}$$
for the differential of the inverse matrix X^{-1}, the Leibnitz rule and the associativity, we obtain
$$d\Omega = -dX \wedge d(X^{-1}) = dX \wedge (X^{-1} \cdot dX \cdot X^{-1}) = \Omega \wedge \Omega.$$
This shows that the integrability condition
$$d\Omega = \Omega \wedge \Omega \tag{1.3}$$
is necessary for existence of fundamental matrix solutions of (1.2).

Theorem 1.1. *The necessary integrability condition* (1.3) *is sufficient for local existence of fundamental matrix solutions of the system* (1.2) *near each point of holomorphy of the Pfaffian matrix* Ω.

As soon as the local existence theorem is established, it implies in the standard way the structural description of all solutions of (1.1) and (1.2). Any solution x of (1.1) has the form Xc, where $c \in \mathbb{C}^n$ is a constant column vector and X is a fundamental matrix solution of the matrix equation (1.2). Any two fundamental matrix solutions X, X' of the same system (1.2) differ by a locally constant invertible right matrix factor, $X' = XC$, $C \in \mathrm{GL}(n, \mathbb{C})$. For any point $a \in U$ and any vector v (resp., any matrix V) the *initial value problem* (the Cauchy problem) $x(a) = v$ (resp., $X(a) = V$) has a unique local solution near a. These assertions are proved by differentiation of $X^{-1}x$, resp., $X^{-1}X'$.

Proof. The assertion follows from the Frobenius theorem [War83]. Though it is formulated there in the real smooth category, neither the formulation nor the proof need not any change for the complex analytic settings.

Consider the complex analytic manifold $M = U \times \mathrm{Mat}_n(\mathbb{C})$ of dimension $m + n^2$ with the coordinates (t, X) and n^2 Pfaffian equations on this manifold, written in the matrix form as
$$\varTheta = 0, \qquad \text{where} \quad \varTheta = \|\theta_{ij}\| = dX - \Omega X. \tag{1.4}$$
By the Frobenius integrability theorem, these Pfaffian equations admit an integral manifold of codimension n^2 (i.e., of dimension m) through any point $(t_*, X_*) \in M$ if the exterior differentials $d\theta_{ij}$ of the (scalar) forms $\theta_{ij} \in \varLambda^1(M)$ constituting the Pfaffian matrix $\varTheta$, belong

to the ideal $\langle\theta_{ij}\rangle \subset \Lambda^2(M)$ generated by the forms θ_{ij} in the exterior algebra. The latter condition can be immediately verified:

$$\begin{aligned} d\Theta &= -d(\Omega X) = -d\Omega\cdot X + \Omega\wedge dX = -d\Omega\cdot X + \Omega\wedge(\Theta+\Omega X)\\ &= (-d\Omega+\Omega\wedge\Omega)X + \Omega\wedge\Theta. \end{aligned}$$

If the integrability condition (1.3) holds, then $d\Theta = \Omega\wedge\Theta$ in $\Lambda^2(M)$, so that each $d\theta_{ij}$ is expanded as $\sum_k \omega_{ik}\wedge\theta_{kj} \in \langle\theta_{1j},\dots,\theta_{nj}\rangle$ and the assumptions of the Frobenius theorem are satisfied. □

1.2 Global solutions. Monodromy and holonomy

For a globally defined system (1.2) on the manifold U represented as the union of local neighborhoods, $U = \bigcup_\alpha U_\alpha$ the corresponding local solutions X_α can be sometimes adjusted to form a global solution. The possibility of doing this depends on the topology of U. Indeed, one may look for suitable constant matrices C_α such that the solutions $X'_\alpha = X_\alpha C_\alpha$ will coincide on the pairwise intersections $U_{\alpha\beta} = U_\alpha \cap U_\beta$. For this, the correction terms C_α must satisfy the identities $C_\alpha C_\beta^{-1} = C_{\alpha\beta}$ on any nonempty intersection $U_{\alpha\beta}$, where $C_{\alpha\beta} = X_\alpha^{-1}X_\beta$ are the *constant* matrices arising on these intersections from the local solutions. In topological terms, the problem is reduced to *solvability of constant matrix cocycle*.

One particular case when this solvability is immediate, appears when U is a small neighborhood of a sufficiently regular (say, compact smooth non-selfintersecting) curve γ. Then the coordinate neighborhoods U_α can be linearly ordered (indexed by an integer variable $\alpha = 1,\dots,N$ increasing along γ) and chosen so small that $U_\alpha\cap U_\beta$ is non-void if and only if they have consecutive numbers, $|\alpha-\beta| \leqslant 1$. The solution $\{C_\alpha\}$ of the respective cocycle $\{C_{\alpha\beta}\}$ is obtained by taking appropriate products,

$$C_\alpha = C_{\alpha,\alpha-1}C_{\alpha-1,\alpha-2}\cdots C_{32}C_{21}C_1$$

for any choice of C_1 and any $\alpha \leqslant N$.

As a corollary, we conclude that any collection of local solutions $\{X_\alpha\}$ along a simple (smooth non-selfintersecting) parameterized curve γ can be modified into a solution X defined and holomorphic in a sufficiently small neighborhood of γ. This fact is usually stated as the *possibility of unlimited continuation of solutions of any integrable Pfaffian system along any curve*. Moreover, a small variation of the curve γ with fixed endpoints results in the same solution, which means that *the result of continuation depends only on the homotopy class of the curve* γ.

However, if the curve γ is a closed non-contractible loop, the result of continuation of any solution X along γ in general differs from the initial solution. Let $a\in U$ be a point and $\pi_1(U,a)$ the *fundamental group* of U with the base point a. If X_0 is a fundamental matrix solution near the point a, then for any $\gamma\in\pi_1(U,a)$ the result X_γ of analytic continuation of X_0 along γ differs from X_0 by a constant right matrix factor M_γ, called the *monodromy matrix*: $X_\gamma = X_0\cdot M_\gamma$. Thus in case U is topologically nontrivial (not simply connected), solutions of integrable Pfaffian system exist on U globally but only as multivalued matrix functions. By construction, every branch of any fundamental matrix solution is everywhere nondegenerate.

The correspondence $\gamma \mapsto M_\gamma$ is an (anti)*representation* of the fundamental group: $M_{\gamma\gamma'} = M_{\gamma'} M_\gamma$ for any two loops $\gamma, \gamma' \in \pi_1(U, a)$. The image of this representation is the *monodromy group* of the integrable system (1.2). It is defined modulo a simultaneous conjugacy of all monodromy matrices: if another fundamental solution $X_0' = X_0 C$ is used for continuation, then M_γ will be replaced by $C^{-1} M_\gamma C$. Considered in the invariant terms as an automorphism of the space of solutions of the system (1.1), induced by continuation along γ, M_γ is the *monodromy operator*.

A closely related notion of *holonomy* does not depend on the choice of the fundamental solution. For a curve γ connecting any two points $a, b \in U$, the holonomy is the linear transformation $F_{ab\gamma} \colon \{a\} \times \mathbb{C}^n \to \{b\} \times \mathbb{C}^n$, sending (a, v) to $(b, x_\gamma(b))$, where x_γ is the unique solution of (1.1) over γ, defined by the initial condition $x(a) = v$. If γ is closed and $a = b$, then $F_{aa\gamma}$ is an automorphism of $\{a\} \times \mathbb{C}^n$; choosing a different base point a' close to a results in a conjugate automorphism $F_{a'a'\gamma}$ close to $F_{aa\gamma}$.

Remark 1.2. To reinstate the ultimate rigor, note that the groups $\pi_1(U, a)$ and $\pi_1(U, a')$ are canonically isomorphic for all a' sufficiently close to a. This allows us to identify the same loop γ in the above two groups and write $F_{a'a'\gamma}$ instead of $F_{a'a'\gamma'}$.

1.3 Geometric language: connections and flatness

Starting from the Pfaffian matrix Ω, for any vector field v on U one can define the *covariant derivation* ∇_v, a differential operator on the linear space of vector functions $\{x(\cdot) \colon U \to \mathbb{C}^n\}$, by the formula

$$\nabla_v x = i_v(dx - \Omega x), \tag{1.5}$$

where the right hand side is a vector function, the value taken by the vector-valued 1-form $dx - \Omega x$ on the vector v tangent to U. The derivative $\nabla_v x$ depends linearly on v and satisfies the Leibnitz rule with respect to x: for any holomorphic function f, $\nabla_{fv} x = f \nabla_v x$ and $\nabla_v(fx) = f \nabla_v x + (\nabla_v f) \cdot x$, where $\nabla_v f = i_v\, df$ is the Lie derivative of f along v and i_v the antiderivative (substitution of v as the argument of a differential form).

The vector function x is called ∇-*horizontal* (or simply horizontal when the connection is defined by the context) along the smooth (parameterized) curve $\gamma \colon [0, 1] \to U$, $s \mapsto t(s)$, if $\nabla_{\dot\gamma} x = 0$, where $\dot\gamma = \frac{d\gamma}{ds} \in \mathbb{C}^m$ is the velocity vector of γ. For any curve γ there exists a unique function $x(\cdot)$ horizontal along this curve with any preassigned initial condition $x(\gamma(0)) = c \in \mathbb{C}^n$. This follows from the existence/uniqueness of solutions of *ordinary* differential equations with the real time variable $s \in [0, 1]$.

Horizontal functions realize *parallel transport*, a collection of linear maps between different spaces $\{t(s)\} \times \mathbb{C}^n$, $s \in [0, 1]$, in particular, between the points $a = t(0)$ and $b = t(1)$. By definition, $T_{ss'}$ is a map sending $c \in \mathbb{C}^n$ into $x(s') \in \mathbb{C}^n$, where $x(\cdot)$ is horizontal along γ and satisfies the boundary condition $x(s) = c$. In general (i.e., without the integrability assumption (1.3)), the result of the parallel transport between two points a and b in U depends on the curve γ connecting them. In particular, the parallel transport over a small closed loop may well be nontrivial. An example of the parallel transport is the holonomy construction introduced above.

Consider two *commuting* vector fields v, w and the parallel transport $T_\varepsilon \colon \mathbb{C}^n \to \mathbb{C}^n$ along four sides of an ε-small curvilinear parallelogram formed by flow curves of v and w. One can show, using standard "calculus of infinitesimals", that the difference between T_ε and the identity is ε^2-proportional to the commutator of the two differential operators $[\nabla_v, \nabla_w] = \nabla_v\nabla_w - \nabla_w\nabla_v$. If this commutator is identically zero for any two commuting vector fields, then the parallel transport along any sufficiently small loop is identical and therefore the global transport from $\{a\} \times \mathbb{C}^n$ to $\{b\} \times \mathbb{C}^n$ depends only on the homotopy class of the path γ connecting a with b.

In differential geometry the rule associating a differential operator ∇_v with any vector field v on U is called a *connection* (more precisely, affine connection). The *curvature* of the connection is the tensor $\nabla_v\nabla_w - \nabla_w\nabla_v - \nabla_{[v,w]}$, coinciding with the above commutator $[\nabla_v, \nabla_w]$ when $[v, w] = 0$. Connections with zero curvature are called *flat*. As follows from this definition, for flat connections the horizontal vector-functions can be defined without reference to any specific curve.

One can immediately verify by the direct computation that the connection (1.5) constructed from a Pfaffian matrix Ω satisfying (1.3), is flat. Any local solution $x(\cdot)$ of the corresponding system (1.1) is horizontal along any sufficiently short curve γ entirely belonging to the domain of this local solution.

Though we will not explore further the geometric aspects, the Pfaffian linear systems satisfying the integrability condition, will be often referred to as *flat connections*. The matrix Ω is the *connection matrix* or *connection form* in this language. The language of connections becomes especially convenient when discussing *holomorphic vector bundles* that only locally have the cylindric structure $U \times \mathbb{C}^n$ but may be globally nontrivial. However, we will not discuss the global questions in these notes, see [For91, Bol00a].

1.4 Gauge transform, gauge equivalence

Two matrix 1-forms Ω and Ω', both holomorphic on U, are called (holomorphically) *gauge equivalent*, or simply equivalent or conjugate, if there exists a holomorphic and holomorphically invertible matrix function $H \colon U \to \mathrm{GL}(n, \mathbb{C})$, such that

$$\Omega' = dH \cdot H^{-1} + H\,\Omega\,H^{-1}. \tag{1.6}$$

This definition reflects the change $x \mapsto H(t)x$ of the dependent variables in the respective linear system (1.1) or (1.2): if Ω is integrable and X its (local) fundamental matrix solution of the first system, then $X' = HX$ is a fundamental matrix solution for the second system which is therefore also automatically integrable, $\Omega' = dX' \cdot (X')^{-1}$.

Clearly, this is an equivalence relationship between Pfaffian systems. The fact that H is single-valued on U means that the monodromy of solutions of gauge equivalent systems is the same: after continuation along any loop both X and $X' = HX$ acquire the same right matrix factor M_γ. The converse statement is also trivially true.

Theorem 1.3. *Two linear integrable holomorphic Pfaffian systems* (1.2) *on the same manifold are holomorphically gauge equivalent if and only if their monodromy groups coincide.*

Proof. Without loss of generality we may assume that the base point a and the two fundamental (multivalued) solutions X, X' of the two systems are chosen so that their monodromy matrices M_γ are the same for all loops γ. But then the matrix ratio $H = X' \cdot X^{-1}$ is single-valued (unchanged by continuation along any loop). Being holomorphic and holomorphically invertible, it realizes the holomorphic gauge equivalence between the two systems. $\square$

2 Meromorphic flat connections

Integrable Pfaffian systems on multiply connected manifolds naturally appear if the Pfaffian matrix Ω is meromorphic (e.g., in a polydisc $\{|t_1| < 1, \dots, |t_m| < 1\}$ in $\mathbb{C}^n$). However, the corresponding theory of flat connections with singularities is much richer than that of connections holomorphic on $U \smallsetminus \Sigma$, where Σ is the polar locus of Ω.

In what follows we will assume that the reader is familiar with some very basic facts from the local theory of analytic sets (dimension, irreducibility *etc.*). The books [GR65] and [Chi89] contain all necessary information.

2.1 Polar locus. First examples

Let f be a holomorphic function on U and $\Sigma = \{f = 0\}$ the null set, an analytic hypersurface. We do not assume that Σ is smooth (i.e., that df is nonvanishing on Σ). However, the standing assumption will be that the locus $\Sigma_0 = \{f = 0,\ df = 0\} \subset \Sigma$ of *non-smooth* points is an analytic variety of codimension at least 2 in U. In particular, f must be *square-free*, i.e., *not* divisible, even locally, by a square (or any higher power) of a holomorphic function. One may show that this condition is also sufficient to guarantee that the non-smooth part of Σ is small in the above sense.

A function g is said to be meromorphic in U with the polar set in Σ, if g is holomorphic outside Σ and for any $a \in \Sigma$ the product $f^r g$ is holomorphic near a for a sufficiently large natural $r \in \mathbb{N}$. The same construction defines also meromorphic differential forms. The matrix 1-form Ω is meromorphic with the polar locus Σ, if all its entries are meromorphic 1-forms and Σ is the minimal analytic hypersurface with this property. The minimal natural r such that $f^r\Omega$ is holomorphic near $a \in \Sigma$, is called the order of pole of Ω at a. If $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \cdots$ is reducible, the order of poles along different components Σ_j may be different.

While analytic hypersurfaces of one-dimensional manifolds are very easy to describe (they are locally finite unions of points), the geometry of analytic hypersurfaces is much richer even locally.

Example 2.1 (Smooth hypersurface, smooth point). If df does not vanish on $\Sigma = \{f = 0\}$, then the latter is smooth. By the implicit function theorem, locally near any its point Σ can be represented as $\{t_1 = 0\}$. The fundamental group of the complement is the free cyclic group $\mathbb{Z}$ generated by a small loop around zero in the t_1-line, intersecting Σ transversally.

Example 2.2 (Isolated singularity). If df vanishes only at an isolated point of Σ, this case is called that of an isolated singularity. Its treatment is different for $m = 2$ and $m > 2$. The hierarchy of isolated singularities of functions and their null hypersurfaces is well known (at least, the first several steps of classification), see [AGV85].

Example 2.3 (Cuspidal point). The cuspidal point (*cusp*) is the simplest isolated singularity of a holomorphic curve, which is locally irreducible. It corresponds to the function $f(t_1, t_2) = t_1^2 - t_2^3$. The fundamental group of the complement $(\mathbb{C}^2, 0) \smallsetminus \Sigma$ is the *group of trefoil* generated by two loops γ_1, γ_2 with the identity $\gamma_1^2\gamma_2^3 = \mathrm{id}$, see [Ful95, §22b].

Example 2.4 (*Normal crossing*). This example, by far most important for applications, corresponds to few (no greater than m) smooth hypersurfaces intersecting transversally. Locally near each point Σ can be represented as the locus $\{t_1 \cdots t_k = 0\}$ with $1 < k \leqslant m$

(the case $k = 1$ is nonsingular). The complement $U \smallsetminus \Sigma = \{0 < |t_i| < 1,\ i = 1, \dots, k\}$ is contractible on the k-torus $\mathbb{T}^k = \prod_{i=1}^k \{|t_i| = 1\}$. Therefore the fundamental group $\pi_1(U \smallsetminus \Sigma)$ is the cyclic group $\mathbb{Z}^k$ with k commuting generators corresponding to small loops around the local components $\Sigma_i = \{t_i = 0\}$.

This example is especially important since for any analytic hypersurface Σ one can construct a *resolution* (blow-up), a holomorphic map $F \colon U' \to U$ between two holomorphic manifolds, such that $\Sigma' = F^{-1}(\Sigma)$ has only normal crossings and F is bijective between $U \smallsetminus \Sigma$ and $U' \smallsetminus \Sigma'$. This is the famous Hironaka desingularization theorem, see §7 below.

Remark 2.5. Speaking in more abstract language, one should describe the singular locus Σ in terms of sheaves of ideals in the rings of holomorphic germs. However, we prefer to use less invariant but more elementary language and always consider Σ together with its square-free local equations. It may cause some technical language problems when dealing with blow-ups (cf. with §7.2), but in most cases simplicity of the exposition justifies our choice.

2.2 Monodromy. Euler system

The monodromy of a meromorphic system in U with polar locus Σ is defined as the monodromy of its restriction on $U \smallsetminus \Sigma$ where it is holomorphic. In particular, if U is polydisk and $\Sigma = \{t_1 \cdots t_k = 0\}$ is the standard "coordinate cross", the monodromy group is generated by k *commuting* matrices $M_1, \dots, M_k$.

Theorem 2.6. *Any collection of commuting invertible matrices $M_1, \dots, M_k$ can be realized as the monodromy group of a Pfaffian system of the special form*

$$\Omega_0 = A_1 \frac{dt_1}{t_1} + \cdots + A_k \frac{dt_k}{t_k} \tag{2.1}$$

with constant pairwise commuting matrices $A_1, \dots, A_k \in \mathrm{Mat}_n(\mathbb{C})$.

The system (2.1) will be referred to as the (generalized) *Euler system*; the usual Euler system corresponds to $k = m = 1$.

Proof. The matrix function $t_j^{A_j} = \exp(A_j \ln t_j)$ depending only on the variable t_j, has the monodromy matrix $M_j = \exp 2\pi \mathrm{i} A_j$. This identity can be resolved with respect to A_j by several methods. One method suggests to reduce M_j to the Jordan block-diagonal form and for each block of the form $\lambda(E + N)$ with $\lambda \in \mathbb{C} \smallsetminus \{0\}$ and nilpotent N, define the logarithm by the formula $\ln[\lambda(E + N)] = (\ln \lambda)E + [N - \frac{1}{2}N^2 + \frac{1}{3}N^3 - \cdots]$. Since N is nilpotent, the series converges. One can choose arbitrarily the branch of logarithm $\ln \lambda$ for all eigenvalues of M_j independently of each other.

An alternative approach is based on the matrix Cauchy integral formula for the logarithm, see [Gan59],

$$A_j = \frac{1}{2\pi \mathrm{i}} \oint_{\partial D} (\zeta E - M_j)^{-1} \ln \zeta \, d\zeta, \tag{2.2}$$

where $D \subseteq \mathbb{C}$ is any simply connected domain containing all eigenvalues of all matrices $M_1, \dots, M_k$, but *not* containing the origin $\zeta = 0$. This approach has the advantage that commutativity of the matrices A_j is transparent when M_j commute.

Regardless of the choice of the matrices A_j the commutative product $Y(t) = t_1^{A_1} \cdots t_k^{A_k}$ has all monodromy matrices equal to M_j as required. It remains to verify by the direct computation that $dY \cdot Y^{-1}$ has the required form (2.1). $\square$

2.3 Regular singularities

In the “ordinary” (univariate) case the *regular* singular point is defined as a point $a \in \Sigma$ of the polar locus of Ω, such that any fundamental solution $X(t)$ grows *moderately* (i.e., no faster than polynomially in $|t-a|^{-1}$ as $t \to a$ along any non-spiraling curve in the t-plane $\mathbb{C}^1$, for instance, a ray). If the monodromy around the point a is trivial (identical), this is tantamount to requirement that the fundamental solution is meromorphic at a. Note that if the function is multivalued, say, $f(t) = \ln t$, then its growth along spirals slowly approaching $a = 0$, e.g., $s \mapsto s\exp(\mathrm{i}/s)$, $s \in [1, 0)$, may well be exponential in $1/|t|$ despite the moderate growth of $|f(t)|$ along all rays.

In the multivariate case the definition may be given in parallel terms. An integrable Pfaffian system (1.2) is said to have a *regular singularity* (on its polar locus Σ), if for any simply connected bounded *semianalytic* subset $S \subseteq U \smallsetminus \Sigma$ any fundamental solution $X(t)$ satisfies the estimate

$$|X(t)| \leqslant c\,|f(t)|^{-r}, \qquad t \in S.$$

The constants $c > 0$ and $r < +\infty$ may depend on the solution and the domain S, while the semianalyticity assumption is aimed to exclude uncontrolled spiraling of S around Σ.

Alternatively, one can use holomorphic parameterized *probe curves* $z\colon (\mathbb{C}^1, 0) \to (U, \Sigma)$ not contained entirely in the polar locus. Each such curve defines an “ordinary” system with the matrix 1-form $z^*\Omega$ (the pullback) on $(\mathbb{C}^1, 0)$. For a system regular in the sense of the previous definition, all such probe restrictions will exhibit regular singularities. It turns out that the converse is also true and even in the stronger sense: it is sufficient to consider only probe curves transversal to Σ and only at smooth points of the latter.

Theorem 2.7 (see [Del70]). *If the pullback $z^*\Omega$ is regular for any probe curve $z\colon (\mathbb{C}, 0) \to (U, \Sigma)$ transversal to Σ at any smooth point $a = z(0)$ of the latter, then the integrable Pfaffian system* (1.2) *has a regular singularity on Σ.*

Proof. We sketch proof only in the particular case when Σ is a normal crossing $\{t_1 \cdots t_k = 0\}$. It was shown (Theorem 2.6) that one can always find an Euler system (2.1) and its solution $Y(t)$ with the same monodromy matrix factors as the fundamental solution $X(t)$ of the system. Clearly, Y is regular on Σ. The matrix ratio $H(t) = X(t)Y^{-1}(t)$ is hence single-valued. By the assumption of the theorem, for any smooth point $a \in \Sigma$ the function H may have pole of some finite order $r_a \in \mathbb{N}$. By the Baire category theorem, there exists $r \in \mathbb{N}$ such that f^rH is holomorphic at all smooth points of Σ. Since the non-smooth points constitute a thin set (of analytic codimension at least 2), f^rH is in fact holomorphic everywhere in U. It remains to note that since both H and Y grow moderately on semianalytic sets off Σ, so their product $X = HY$ does. □

Remark 2.8. The proof in the general case differs only in one instance: one has to construct explicitly just one matrix function Y with moderate growth on semianalytic subsets, having the same monodromy as X. This can be done using the resolution of Σ to normal crossings, see [Del70] for details.

2.4 Holomorphic and meromorphic gauge equivalence

For systems with singularities one can consider two different gauge equivalence relationships. We say that two integrable Pfaffian systems with meromorphic matrices Ω and Ω' having *the same* polar locus Σ, are *holomorphically* gauge equivalent if (1.6) holds for some matrix function H holomorphic and holomorphically invertible everywhere *including* Σ. The two systems are said to be *meromorphically* gauge equivalent, if H is holomorphic and holomorphically invertible only outside Σ, while having at worst poles of finite order on Σ for both H and H^{-1}. Note that the meromorphic equivalence is stronger than holomorphic gauge equivalence on the holomorphy set $U \setminus \Sigma$, occurring in Theorem 1.3. However, for regular systems the difference disappears.

Theorem 2.9. *Two integrable Pfaffian systems* (1.2) *having only regular singularities, are meromorphically equivalent if and only if their monodromy groups are the same.*

Proof. The matrix ratio of two fundamental solutions is a single-valued matrix function having moderate growth on the polar locus, hence meromorphic. □

Corollary 2.10. *Any integrable Pfaffian system* (1.2) *with a regular singularity on the coordinate cross* $\{t_1 \cdots t_k = 0\} \subset \mathbb{C}^m$ *is meromorphically gauge equivalent to a suitable Euler system* (2.1). □

The holomorphic gauge classification is considerably more difficult. It will be addressed in more details in §6 below.

2.5 Formal and convergent solutions

For future purposes we need to introduce formal gauge equivalence; here we explain it and discuss some basic convergence results.

Assume that Σ is the coordinate cross $\Sigma = \{t_1 \cdots t_k = 0\}$ and consider the algebra of formal Laurent series in the variables $t = (t_1, \dots, t_k)$ involving only non-negative powers of $t_{k+1}, \dots, t_m$. Over this algebra one can naturally define formal matrix functions, formal meromorphic differential forms, formal vector fields *etc.*

Everywhere in these notes we will grade formal Taylor or Laurent series and forms, assigning the same weights $w_j \in \mathbb{R}_+$ to the differentials dt_j of the independent variables and to the variables themselves so that $\frac{dt_j}{t_j}$ always has weight zero. In most cases (though not always) it will be sufficient to choose $w_1 = \cdots = w_m = 1$. In such cases we try to distinguish between the degree and the weighted (quasihomogeneous) degree.

Any meromorphic Pfaffian system with the polar locus on $\Sigma = \{t_1 \cdots t_k = 0\}$, can be expanded as a formal Laurent series, $\Omega = \sum_{s \geqslant -r} \Omega_s$, where each Ω_s is a *homogeneous* matrix form of degree s.

Solutions of (integrable) Pfaffian systems can be also constructed recurrently as infinite sums of homogeneous terms. In general, it is not sufficient to consider only Laurent polynomials: already in the "ordinary" case $n = 1$ one has to introduce fractional powers t^a, $a \notin \mathbb{Z}$, and logarithms $\ln t$. However, the important fact is that in the regular case any such formal solution always converges. We will use only the simplest version of the corresponding result, dealing only with Laurent formal solutions not involving fractional powers.

Theorem 2.11. *If* Ω *is an integrable Pfaffian form with a regular singularity on the coordinate cross* $\{t_1 \cdots t_k = 0\} \subset (\mathbb{C}^m, 0)$, *then any formal* (*Laurent*) *solution of the system* (1.1) *converges, i.e., is meromorphic.*

Proof. The assertion of the Theorem is invariant by meromorphic gauge equivalence: two equivalent forms Ω, Ω' both satisfy or do not satisfy simultaneously the property that any formal solution necessarily converges.

By Corollary 2.10, any regular system is meromorphically equivalent to an Euler system (2.1), hence it is sufficient to prove the Theorem for Euler systems only.

A (vector) formal Laurent series $\sum h_r x_r$ with vector coefficients $x_r \in \mathbb{C}^n$ and scalar (quasi)homogeneous terms h_r of degree r satisfies the system $dx = \Omega_0 x$ with Ω_0 as in (2.1), if and only if each term $h_r x_r$ satisfies this system. This follows from the fact that both d and multiplication by Ω_0 preserve the weight (quasihomogeneous degree) of vector-valued 1-forms. If we choose the weights of the variables t_j independent over $\mathbb{Z}$, the only quasihomogeneous Laurent polynomials will be monomials, $h_r = t^{\mathbf{a}} = t_1^{a_1} \cdots t_m^{a_m}$, $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{Z}^m$, $a_1 w_1 + \cdots + a_m w_m = r$, modulo a constant multiplier.

The product $x(t) = t^{\mathbf{a}} x_r$ satisfies the linear system $dx = \Omega_0 x$ if and only if

$$t^{\mathbf{a}} x_r \cdot \sum_1^m a_j \frac{dt_j}{t_j} = t^{\mathbf{a}} \cdot \sum_1^m (A_j x_r) \frac{dt_j}{t_j},$$

which is possible only if

$$a_j x_r = A_j x_r, \qquad j = 1, \dots, m$$

(we assume that $A_{k+1} = \cdots = A_m = 0$. But these identities imply that the exponents $a_j \in \mathbb{Z}$ are bounded (they range over the spectra of the residue matrices A_j). Thus there may be only finitely many nonzero terms in the apriori infinite sum $x = \sum h_r x_r$ which is thus automatically convergent. This proves that the solution $x = x(t)$ must be meromorphic. $\square$

This Theorem will always guarantee that there are no differences between formal and convergent (analytic) results as far as only regular singularities are allowed.

2.6 Flat connections and isomonodromic deformations

It was already noted in §2.3 that a meromorphic flat connection Ω on U with the polar locus $\Sigma \subset U$ induces a (necessarily flat) connection on any embedded holomorphic curve $Z \subset U$. If Z is parameterized by a chart $z \colon \mathbb{C} \to Z$, the condition describing horizontal sections of the induced connection takes the form of an ordinary differential equation. This equation exhibits singularities at the points of intersection $Z \cap \Sigma$; they are regular if Σ was a regular singularity of the initial connection.

The flatness of the restricted connection follows automatically from the assumption that Z is of (complex) dimension 1. Integrability (flatness) of the initial multivariate connection Ω results in the fact that the monodromy of the restriction of Ω on Z is in some sense *independent of* Z.

More precisely, assume that the complex m-dimensional analytic manifold U on which the flat meromorphic connection Ω lives, is fibered over another complex $(m-1)$-dimensional manifold P (the "parameter space") by a full rank holomorphic map $p \colon U \to P$. Assume that for some value $\lambda_0 \in P$ the map p restricted on $U \smallsetminus \Sigma$ is topologically trivial over a small neighborhood $V \ni \lambda_0$: it is sufficient to require that the fiber $p^{-1}(\lambda_0)$ transversally intersects Σ only at finitely many smooth points. Under this assumption the nearby fibers $Z_\lambda = p^{-1}(\lambda)$ are all homeomorphic to Z_{λ_0} (hence to each other); though this homeomorphism is not canonical, it allows for canonical identification of the fundamental groups $\pi_1(Z_\lambda, a_\lambda)$

with $\pi_1(Z_{\lambda_0}, a_{\lambda_0})$ for any continuous section $a\colon P \to U \smallsetminus \Sigma$ of the bundle p. Needless to say, the topology of the fibers Z_λ may be arbitrary; in particular, they can be all conformally equivalent to the Riemann sphere or any other Riemann surface. Everything depends on the global structure of U and P.

After all these technical precautions, it is obvious that the monodromy groups of the restricted connections $\Omega_\lambda = \Omega|_{Z_\lambda}$ are conjugate to each other. It is sufficient to choose a (multivalued) fundamental solution X on $p^{-1}(V)$ of the integrable system (1.2) and restrict this solution on each one-dimensional fiber.

Conversely, one may show (cf. with [Bol97]), that a reasonably defined *isomonodromic parametric family*, or *isomonodromic deformation* of systems of linear ordinary differential equations can be obtained from a suitable flat connection on some multidimensional manifold, e.g., the cylinder $\mathbb{C}^1 \times$ (polydisk).

This link between meromorphic flat connections on higher dimension manifolds and isomonodromic deformations is (for us) a primary source of motivation for studying the former. It is important to stress that speaking about isomonodromic deformation means *introducing the additional bundle structure* on U. However, at least locally (near a smooth point $a \in \Sigma$) one can always introduce a local coordinate system $(z, \lambda) \in (\mathbb{C}^1 \times \mathbb{C}^{m-1}, 0)$ on U so that $p(z, \lambda) = \lambda$ and $\Sigma = \{z = 0\}$. When performing calculations, we will always use such *adapted* local coordinates.

3 Logarithmic forms, logarithmic poles, residues

In the "ordinary" univariate case, there is a simple sufficient condition guaranteeing regularity of a meromorphic connection matrix Ω on its polar locus Σ. The point $a \in \Sigma \subset \mathbb{C}^1$ is called *Fuchsian*, if Ω has a first order pole at a. In the local chart $t = t_1$ the point $t = a$ is Fuchsian, if its principal polar (non-holomorphic) part is an Euler system,

$$\Omega = A\frac{dt}{t-a} + \text{holomorphic matrix 1-form.} \tag{3.1}$$

The constant matrix $A \in \operatorname{Mat}(n, \mathbb{C})$ is referred to as the *residue* of Ω at $a \in \Sigma$.

In the multivariate case the order condition on the pole *does not* guarantee that its principal part is an Euler system (2.1) similarly to (3.1). Before proceeding with the definition of an analog of Fuchsian poles for the multivariate case in §4, we need some analytic preparatory work.

3.1 Preparation: de Rham division lemma

Consider the exterior algebra $\Lambda^\bullet(U)$ of holomorphic forms on $U \subset \mathbb{C}^m$. If $\xi = \sum_1^m c_i(t)\,dt_i \in \Lambda^1(U)$ is a given nonzero 1-form, then a k-form ω, divisible by ξ, i.e., representable as

$$\omega = \theta \wedge \xi, \qquad \theta \in \Lambda^{k-1}(U), \tag{3.2}$$

necessarily satisfies the condition

$$\omega \wedge \xi = 0 \in \Lambda^{k+1}(U). \tag{3.3}$$

It turns out that the necessary divisibility condition (3.3) is very close to be sufficient. Note that the identity (3.2) can be considered as a system of linear algebraic (non-homogeneous) equations on the coefficients of the form θ in the basis $dt_1, \dots, dt_m$; the condition (3.3) is a necessary condition of solvability of this system. G. de Rham [dR54] described the forms ξ for which (3.3) implies divisibility. We will need only a simple particular case of his result.

Lemma 3.1. *If one of the coefficients of the form ξ, e.g., $c_1(t)$, is non-vanishing (invertible) at some point $a \in U$, then any holomorphic k-form ω, $k < m$, satisfying (3.3), is divisible by ξ and the ratio θ may be chosen holomorphic at this point.*

More generally, if c_1 is allowed to vanish but not identically, $c_1(t) \not\equiv 0$, and ω is holomorphic or meromorphic with the poles only on $\Sigma_1 = \{c_1 = 0\} \subset U$, then θ can also be chosen meromorphic with the polar locus on Σ_1.

Proof. If c_1 is holomorphically invertible, then without loss of generality we may assume that $\xi = dt_1 + \sum_2^m c_i\, dt_i$ with the other coefficients c_i holomorphic at $a \in U$. Consider the 1-forms $\xi_1 = \xi$, $\xi_i = dt_i$, $i = 2, \dots, m$. These holomorphic forms and their exterior products constitute a basis in the exterior algebra, so that any holomorphic k-form can be uniquely written as the sum of wedge monomials

$$\sum_{i_1 < \cdots < i_k} c_{i_1 \cdots i_k}(t)\, \xi_{i_1} \wedge \cdots \wedge \xi_{i_k}$$

with holomorphic coefficients.

For $\xi = \xi_1$ the assertion of the Lemma is obvious. The condition (3.3) means that in the expansion of the k-form ω may appear only monomials involving ξ_1 as a factor. Each such monomial $c_{1 i_2 \cdots i_k} \xi_1 \wedge \xi_{i_2} \wedge \cdots \wedge \xi_{i_k}$ is divisible by ξ_1, the ratio being (among other possibilities) a monomial $\pm c_{1 i_2 \cdots i_k}(t)\, \xi_{i_2} \wedge \cdots \wedge \xi_{i_k}$ with the holomorphic coefficient. This completes the proof when c_1 is invertible.

The case when c_1 vanishes but admits meromorphic inverse, is treated in the same way, with the only exception: the form $\xi_1 = dt_1 + \sum_2^m c_i\, dt_i$ will have meromorphic coefficients with the polar locus Σ_1. □

Remark 3.2. The natural question appears, whether one can use the non-uniqueness of the division to ensure that the polar locus of the meromorphic coefficients of ξ is smaller than a hypersurface.

Theorem 3.3 (R. Moussu [Mou75]). *If $\xi = \sum_1^m c_i(t)\, dt_i$ is a holomorphic 1-form on $(\mathbb{C}^m, 0)$ with an isolated singularity at the origin (i.e., all the coefficients $c_1, \dots, c_m$ vanish simultaneously only at the origin $t = 0$), then each holomorphic k-form with $k < m$, satisfying the necessary condition (3.3), is divisible by ξ with a holomorphic "ratio" form θ.* □

The assertion of Theorem 3.3 for m-forms fails. Any such form trivially satisfies (3.3), but $\omega = c(t) dt_1 \wedge \cdots \wedge dt_m$ is holomorphically divisible by ξ if and only if the holomorphic function $c(t)$ belongs to the ideal $\langle c_1, \dots, c_m \rangle$. In particular, $c(\cdot)$ must vanish at the singular point $t = 0$ of ξ.

3.2 Logarithmic poles. Residues

A meromorphic k-form $\omega \in \Lambda^k(U \smallsetminus \Sigma)$ is said to have a *logarithmic pole* on the analytic hypersurface $\Sigma = \{f = 0\}$, if *both* ω and $d\omega$ have a first order pole on Σ:

$$f\omega,\ f\, d\omega \qquad \text{both extend holomorphically on } \Sigma. \tag{3.4}$$

By our standing assumption on f, we may assume that one of the partial derivatives, say, $\partial f/\partial t_1$, is vanishing on an analytic hypersurface

$$\Sigma_1 = \{\partial f/\partial t_1 = 0\} \subset U$$

such that $\Sigma \cap \Sigma_1$ has (complex) codimension at least 2 in U.

Lemma 3.4. *If a k-form ω has a logarithmic pole on Σ, then it can be represented as*

$$\omega = \theta \wedge \frac{df}{f} + \eta, \tag{3.5}$$

where $\theta \in \Lambda^{k-1}(U \smallsetminus \Sigma_1)$ and $\eta \in \Lambda^k(U \smallsetminus \Sigma_1)$ are both holomorphic on $\Sigma \smallsetminus \Sigma_1$.

The (meromorphic) $(k-1)$-form θ is called the *residue* of ω on Σ and denoted by $\operatorname{res}_\Sigma \omega$. While the representation (3.5) is not unique, the restriction of θ on Σ is unique.

Proof. The form $df \wedge \omega = d(f\omega) - f\,d\omega$ is holomorphic on Σ by definition of the logarithmic pole, and vanishes after exterior multiplication by df (obviously outside Σ, hence everywhere). By the Division Lemma 3.1,

$$df \wedge \omega = df \wedge \eta, \tag{3.6}$$

where η is a k-form holomorphic everywhere outside Σ_1.

The difference $f\omega - f\eta$ is holomorphic on $\Sigma \smallsetminus \Sigma_1$ since $f\omega$ is, and by (3.6), this difference vanishes after exterior multiplication by df. Again by virtue of the Division Lemma, we have

$$f\omega - f\eta = \theta \wedge df \tag{3.7}$$

with $\theta \in \Lambda^{k-1}(U \smallsetminus \Sigma_1)$ as required.

To show the uniqueness of the restriction of θ on $\Sigma \smallsetminus \Sigma_1$, it is sufficient to show that $\omega \equiv 0$ implies vanishing of this restriction. In this case $\theta \wedge df = -f\eta$ and multiplying this identity by df we see that $df \wedge \eta \equiv 0$. Again by the Division Lemma, η must be divisible by df, $\eta = \zeta \wedge df$. We conclude then that $(\theta - f\zeta) \wedge df = 0$. But $df \neq 0$ on $\Sigma \smallsetminus \Sigma_1$, while the restriction of $f\zeta$ on $\Sigma \smallsetminus \Sigma_1$ vanishes since ζ was holomorphic there. This implies that the restriction of θ must vanish on $\Sigma \smallsetminus \Sigma_1$, as asserted. $\square$

Note that $\operatorname{res}\omega$ does not depend on the choice of f used to describe the logarithmic pole Σ: multiplying f by an invertible factor contributes a holomorphic additive term to the logarithmic derivative $f^{-1}df$ and nothing to the residue.

Recall that for a (scalar) meromorphic 1-form ω on a Riemann surface, locally represented as $\omega = g\frac{dt}{t-a} + \cdots$, the residue $g(a) = \operatorname{res}_a \omega$ at the pole $t = a$ can be defined via the Cauchy integral $\frac{1}{2\pi i}\oint_\gamma \omega$, where γ is a small loop encircling the pole, and does not depend on the choice of the chart $t \in \mathbb{C}$. In the multidimensional case if ω has a logarithmic pole on Σ and $\omega = g\,\frac{df}{f} + \cdots$, then for any probe curve $z\colon (\mathbb{C}^1, 0) \to (U, a)$ through a smooth point $a \in \Sigma$, the residue of the restricted form $z^*\omega$ at the origin is $g(a)$ independently of the choice of the probe curve. In other words, the residue can be detected by studying probe curves, if and only if the pole on Σ is logarithmic.

3.3 Logarithmic complex

Denote by $\Lambda^k(\log\Sigma)\subseteq\Lambda^\bullet(U\smallsetminus\Sigma)$ the collection of meromorphic k-forms having a logarithmic pole on the given polar locus $\Sigma\subset U$, and let

$$\begin{aligned}\Lambda^\bullet(\log\Sigma)&=\bigcup\nolimits_{k\geqslant 0}\Lambda^k(\log\Sigma)\\ &=\{\omega\colon\ f\omega \text{ and } f\,d\omega \text{ in } \Lambda^\bullet(U)\}\\ &=\{\omega\colon\ f\omega \text{ and } df\wedge\omega \text{ in } \Lambda^\bullet(U)\}.\end{aligned}\tag{3.8}$$

We will refer to $\Lambda^\bullet(\log\Sigma)$ as *logarithmic forms* if the locus Σ is clear from the context.

Clearly, logarithmic forms constitute a (graded) module over the ring of holomorphic functions on U. It is obvious that the exterior derivative $d\colon\Lambda^k\to\Lambda^{k+1}$ preserves logarithmic forms: indeed, if both $f\omega$ and $f\,d\omega$ are holomorphic (on Σ), then $f\,d\omega$ and $f\,d^2\omega=0$ also are. It is less obvious that logarithmic forms are closed by taking exterior (wedge) products.

Lemma 3.5. *If $\omega,\omega'\in\Lambda^\bullet(\log\Sigma)$ are two logarithmic forms (of different degrees, in general), then their product $\omega\wedge\omega'$ is also a logarithmic form.*

Proof. By Lemma 3.4, each of these forms can be represented as $\omega=\theta\wedge df/f+\eta$, $\omega'=\theta'\wedge df/f+\eta'$, where the forms $\theta,\theta',\eta,\eta'$ of appropriate degrees are holomorphic everywhere outside the auxiliary locus Σ_1 intersecting the polar Σ by a thin set (of codimension 2 at least). Computing the wedge product, we conclude that $\omega\wedge\omega'$ has at most a first order pole on $\Sigma\smallsetminus\Sigma_1$ (i.e., that $f\,\omega\wedge\omega'$ extends holomorphically on $\Sigma\smallsetminus\Sigma_1$). On the other hand, $f\,\omega\wedge\omega'$ is holomorphic outside Σ, since both ω and ω' are holomorphic there. Thus $f\,\omega\wedge\omega'$ is holomorphic outside $\Sigma\cap\Sigma_1$. Since the intersection $\Sigma\cap\Sigma_1$ is thin, $f\,\omega\wedge\omega'$ is holomorphic everywhere in U.

The wedge product $df\wedge(\omega\wedge\omega')=df\wedge\eta\wedge\eta'$ is also holomorphic outside $\Sigma_1\cap\Sigma$ hence everywhere for exactly the same reason. The second line of (3.8) allows to conclude that $\omega\wedge\omega'\in\Lambda^\bullet(\log\Sigma)$. ☐

Together with the previously remarked d-closeness, Lemma 3.5 implies that *logarithmic forms constitute a graded exterior algebra* which is made into a cochain complex by restriction of the exterior derivative d.

Logarithmic forms forms exhibit the mildest singularity along Σ. In particular, a logarithmic 0-form (function) $g\in\Lambda^0(\log\Sigma)$ is necessarily holomorphic. The term "logarithmic pole" is motivated by the fact that for any meromorphic function g holomorphic and invertible outside Σ, its logarithmic derivative $g^{-1}\,dg$ belongs to $\Lambda^1(\log\Sigma)$. Moreover, one can show [Del70] that $\Lambda^\bullet(\log\Sigma)$ is in fact generated by logarithmic derivatives of meromorphic functions as an exterior algebra over the ring of holomorphic functions in U.

3.4 Holomorphy of residues. Saito criterion

The local representation established in Lemma 3.4 means that the residue of a logarithmic 1-form ω is a function $g=\operatorname{res}\omega$ which is apriori only meromorphic on Σ, eventually exhibiting poles on an auxiliary hypersurface $\Sigma_1\cap\Sigma\subsetneq\Sigma$. The polar set $\Sigma_1\cap S$ depends only on Σ and not on the form(s).

By the stronger form (Theorem 3.3) of the Division lemma, the residue of a logarithmic 1-form ω is *holomorphic* on the polar locus Σ of this form, if $m>2$ and Σ has an isolated singularity.

This is, in particular, the case of smooth divisors $\varSigma = \{f = 0\}$ when $df|_\varSigma$ is non-vanishing (in this case $\varSigma_1 = \varnothing$). However, for other analytic hypersurfaces the residue may indeed exhibit singularities (in particular, to be locally unbounded) near non-smooth points of $\varSigma$.

Example 3.6. The 1-form

$$\omega = \frac{1}{y-x}\left(\frac{dx}{x} - \frac{dy}{y}\right), \qquad (x,y) \in \mathbb{C}^2 \tag{3.9}$$

on the complex 2-plane $\mathbb{C}^2$, has the logarithmic singularity on the union of three lines through the origin $\varSigma = \{xy(x-y) = 0\}$. The residues on each of the lines $\{x = 0\}$, $\{y = 0\}$ and $\{y = x\}$ are equal to $1/y$, $1/x$ and $-2/x$ correspondingly: all three exhibit a "polar" singularity at the origin.

However, similar example with only two lines in the plane $\mathbb{C}^2$ is impossible. We will show that if $\varSigma$ is a transversal intersection of smooth analytic hypersurfaces, the residue of any logarithmic form is necessarily holomorphic. The exposition below is based on the seminal paper by K. Saito [Sai80].

Lemma 3.7. *Let $\varSigma \subset U$ be an analytic hypersurface in an m-dimensional analytic manifold, and $\omega_1, \dots, \omega_m \in \varLambda^1(\log \varSigma)$ logarithmic 1-forms.*

If the (holomorphic by Lemma 3.5) m-form $f\,\omega_1 \wedge \cdots \wedge \omega_m$ is non-vanishing on $\varSigma$, then the forms ω_i form a basis of $\varLambda^1(\log \varSigma)$ over the ring of holomorphic functions: any logarithmic 1-form ω can be expanded as

$$\omega = h_1\,\omega_1 + \cdots + h_m\omega_m, \qquad h_i \in \varLambda^0(U), \tag{3.10}$$

with the coefficients $h_1(t), \dots, h_m(t)$ holomorphic on $\varSigma$.

Proof. The forms ω_i are necessarily linear independent outside $\varSigma$, so the decomposition (3.10) exists with the coefficients h_i that are at worst meromorphic on $\varSigma$. The same argument shows that h_i are uniquely defined.

To prove that each h_i admits holomorphic extensions on $\varSigma$, we multiply (3.10) by the wedge product $\varXi_i \in \varLambda^{m-1}(\log \varSigma) = \bigwedge_{j\neq i} \omega_j$ of all the forms except ω_i. The coefficient h_i can be obtained by the Cramer rule as a ratio of two m-forms, logarithmic by Lemma 3.5,

$$h_i = \frac{\omega \wedge \varXi_i}{\omega_j \wedge \varXi_i} = \pm \frac{f\,\omega \wedge \varXi_i}{f\,\omega_1 \wedge \cdots \wedge \omega_m}.$$

Both forms in the right hand side are holomorphic by definition of logarithmic poles and the denominator is nonvanishing on $\varSigma$ near a by assumption. □

As a corollary, we can immediately prove that for a divisor with normal crossings, the residues must be holomorphic functions along each component, though *not necessarily coinciding on the intersections.*

Theorem 3.8. *If $\varSigma = \{t_1 \cdots t_k = 0\}$, $k \leqslant m$ is a normal crossing divisor, than any 1-form ω with logarithmic pole on $\varSigma$ can be expressed as*

$$\omega = \sum_{i=1}^{k} g_i(t)\,\frac{dt_i}{t_i} + \eta, \tag{3.11}$$

where the 1-form η and the functions g_i are holomorphic on U including the polar locus $\varSigma$.

Proof. The 1-forms $\frac{dt_1}{t_1}, \dots, \frac{dt_k}{t_k}, dt_{k+1}, \dots, dt_m$ satisfy the assumptions of Lemma 3.7: their wedge product is $(1/f)$ times the standard "volume form" $dt_1 \wedge \dots \wedge dt_m$. $\square$

3.5 Closed logarithmic forms

One particular case is worth being singled out. Let $\Sigma = \bigcup_1^k \Sigma_j$ be a polar locus represented as the union of *irreducible components*. The irreducibility implies, among other, that the smooth part of each hypersurface $\Sigma_j \smallsetminus \bigcup_{i \neq j} \Sigma_i$ is connected.

Lemma 3.9. *If $\omega \in \Lambda^1(\log \Sigma)$ is a closed logarithmic form, $d\omega = 0$, then the residue of ω is constant on any component of the polar locus Σ. Any such form with at least one nonzero residue admits representation*

$$\omega = \sum_{j=1}^{k} a_j \frac{df_j}{f_j}, \qquad a_1, \dots, a_k \in \mathbb{C}, \tag{3.12}$$

where f_j are suitably chosen equations of the components Σ_j.

Proof. After differentiation and multiplication by f, the representation $\omega = g\,df/f + \eta$ (3.5) yields the identity between the holomorphic forms

$$0 = dg \wedge df + f\,d\eta, \qquad g \in \Lambda^0(U),\ \eta \in \Lambda^1(U).$$

It implies immediately that $dg \wedge df$ vanishes on Σ near every smooth point, so that dg is everywhere proportional to df and vanishes on vectors tangent to Σ. Therefore $g = \text{const}$ along the smooth part of Σ. Subtracting from ω the principal part $\sum_j a_j\,df_j/f_j$ which is already a closed logarithmic form, we obtain a nonsingular closed form η which can be always expressed as df_0/f_0 for a suitable holomorphically invertible function f_0. This term f_0 can be incorporated into any of the equations corresponding to a nonzero residue, $a_j\,df_j/f_j + df_0/f_0 = a_j\,df'_j/f'_j$, if we put $f'_j = f_j \exp(f_0/a_j)$. $\square$

4 Flat connections with logarithmic poles

Rather naturally, a meromorphic flat connection with the connection matrix Ω is said to have a logarithmic pole on $\Sigma = \{f = 0\}$, if both $f\Omega$ and $f\,d\Omega$ extend holomorphically on Σ. Near any smooth point $a \in \Sigma$ it can be represented as

$$\Omega = A(t)\frac{df}{f} + \text{holomorphic matrix 1-form}, \qquad A = \operatorname{res}_\Sigma \Omega,$$

where the *residue matrix function* $A(\cdot)$ is uniquely defined and holomorphic on the smooth part Σ' of Σ. Abusing the notation, we will write $\Omega \in \Lambda^1(\log \Sigma)$ meaning that Ω has a logarithmic pole on Σ.

If the pole of Ω is known to be logarithmic, then for any probe curve $z\colon (\mathbb{C}^1, 0) \to (U, a)$, passing through a smooth point $a \in \Sigma$ and not entirely embedded in Σ, the induced connection $z^*\Omega$ has a Fuchsian singularity and its residue can be immediately computed,

$$\operatorname{res}_0 z^*\Omega = (\operatorname{res}_\Sigma \Omega)(a), \qquad a = z(0).$$

Indeed, if (locally) $f = t_1$ and $a = 0$, then a form having first order pole on the hyperplane $\{t_1 = 0\}$ can be represented as

$$\omega = A_1(t_2, \dots, t_m)\,\frac{dt_1}{t_1} + \sum_{i=2}^{m} A_i(t_2, \dots, t_m)\,\frac{dt_i}{t_1} + \cdots$$

(the dots denote holomorphic terms). The pole is logarithmic only if $A_2 = \cdots = A_m \equiv 0$, and $A_1(0)$ is the residue in this case.

In this section we will study behavior of the residue(s) of *flat* meromorphic connections satisfying the integrability condition (1.3).

4.1 Residues and holonomy

Consider a flat logarithmic connection $\Omega \in \Lambda^1(\log \Sigma)$. Let $a \in \Sigma'$ be a point on the smooth part Σ' of the polar locus Σ. For any point $t \notin \Sigma$ sufficiently close to a, one can unambiguously define a small positive simple loop γ_t around Σ' beginning and ending at t so that the holonomy operator F_t corresponding to the loop γ_t analytically depends on $t \notin \Sigma$.

Lemma 4.1. *The operator F_t has a limit $F_a = \lim_{t \to a} F_t$ related to the residue $A(a) = \operatorname{res}_a \Omega$ of the flat logarithmic connection Ω by the identity $F_a = \exp 2\pi \mathrm{i} A(a)$.*

Proof. The assertion is almost obvious in the "ordinary" case $m = 1$, where it is sufficient to compute the principal asymptotic term of the expansion of solutions of a Fuchsian system. The multivariate assertion follows now from comparing restrictions of Ω on various probe curves $z \colon (\mathbb{C}^1, 0) \to (U, a)$ and the independence of the residue of the choice of the probe curve. $\square$

It should be remembered, however, that while all operators F_t, $t \notin \Sigma$, are conjugate to each other, it may well happen (even in the "ordinary" case) that the limit F_a does *not* belong to the same conjugacy class. The examples can be found in [Del70, Gan59]. What can be asserted is that the characteristic polynomials for F_t and F_a coincide.

4.2 Conjugacy of the residues

The following result can be considered as a matrix generalization of Lemma 3.9.

Theorem 4.2. *If $\Omega \in \Lambda^1(\log \Sigma)$ is a flat connection matrix with logarithmic singularity on Σ, then the residue matrix $A(a) = \operatorname{res}_a \Omega$ varies in the same conjugacy class of* $\mathrm{GL}(n, \mathbb{C})$ *as soon as the point a varies continuously along the smooth part of Σ.*

Proof. We give two different proofs of this Theorem, one based on topological considerations, the other achieved by direct explicit computation.

First proof. For any two sufficiently close smooth points $a, a' \in \Sigma'$ on Σ, one can construct two small loops γ, γ' as in §4.1. Moreover, we choose the base points t, t' for these loops on the same "distance" from Σ' so that $f(t) = f(t')$. Then these loops are conjugate in the fundamental group by a path σ connecting t with t'; without loss of generality we may assume that the whole path σ belongs to the level curve $f = \text{const}$.

The corresponding holonomy operators are conjugated by the holonomy operator $F_{tt'\sigma}$. The singular term $A\,\frac{df}{f}$ vanishes on σ, thus the operator $F_{tt'\sigma}$ has a uniform invertible limit

$C = C(a, a') \in \mathrm{GL}(n, \mathbb{C})$ as $t \to a$, $t \to a'$ respecting the above assumption $f(t) = f(t')$. By Lemma 4.1, both F_t and $F_{t'}$ have uniform limits F_a and $F_{a'}$ and the operator C conjugates them. It remains to notice that the matrix exponential map $\exp\colon A \mapsto \exp A$ is a covering of $\mathrm{GL}(n, \mathbb{C})$, so the two close conjugated exponentials correspond to two conjugated logarithms $A(a)$ and $A(a')$.

Second proof. The same assertion may be derived directly from the integrability condition. Consider the $(d, \wedge)$-closed subalgebra

$$\mathcal{V} = f\Lambda^\bullet(U) + df \wedge \Lambda^\bullet(U)$$

of holomorphic matrix k-forms vanishing after restriction on Σ.

Then the logarithmic connection form Ω can be written as

$$\Omega = A\,\frac{df}{f} + \Theta + f(\cdots) + df \wedge (\cdots) = A\,\frac{df}{f} + \Theta \mod \mathcal{V},$$

where Θ is a holomorphic matrix 1-form on the smooth part $\Sigma' \subseteq \Sigma$ and A a holomorphic matrix function on Σ'.

The integrability condition $d\Omega = \Omega \wedge \Omega$ yields the identity

$$dA \wedge \frac{df}{f} + d\Theta = [\Theta, A] \wedge \frac{df}{f} + \Theta \wedge \Theta \mod \mathcal{V}.$$

From this identity we derive, equating forms of different type (tangent to Σ' and having a normal component df), the following two identities,

$$dA = [\Theta, A] \mod \mathcal{V}, \tag{4.1}$$

$$d\Theta = \Theta \wedge \Theta \mod \mathcal{V}. \tag{4.2}$$

The condition (4.2) means that the connection induced on Σ' by restriction of the holomorphic matrix form Θ, is integrable. Hence there exists a holomorphic matrix solution $H\colon \Sigma' \to \mathrm{GL}(n, \mathbb{C})$ of the equation $dH = \Theta H|$ on Σ', normalized by the condition $H(a) = E$. We claim that if Σ' is connected, then the only solution of the equation (4.1) with the initial condition $A(a) = C$ is $A = HCH^{-1}$, where $C = A(a)$ is the *constant matrix*, the residue at the point $a \in \Sigma'$. This would imply that $A(\cdot)$ remains in its conjugacy class along connected components of Σ', as asserted.

Direct verification shows that $A = HCH^{-1}$ is indeed a solution of (4.1):

$$\begin{aligned} dA &= dH \cdot CH^{-1} + HC \cdot d(H^{-1}) \\ &= dH \cdot H^{-1}HCH^{-1} - HC(H^{-1}\,dH \cdot H^{-1}) = \Theta A - A\Theta = [\Theta, A]. \end{aligned}$$

The initial condition $A(a) = C$ is satisfied since $H(a) = E$. $\square$

Theorem 4.2 guarantees that for a logarithmic connection, all eigenvalues of the residue matrix (and hence all coefficients of its characteristic polynomial) are locally constant along the smooth part of the singular locus. This will allow us to say about resonances later in §6.

4.3 Residues on normal crossings

If $\varSigma = \{f_1 \cdots f_k = 0\}$ is a normal crossing of $k \leqslant m$ smooth hypersurfaces $\varSigma_i = \{f_i = 0\}$ and $\Omega \in \varLambda^1(\log \varSigma)$, then

$$\Omega = \sum_{i=1}^{k} A_i \frac{df_i}{f_i} + \text{holomorphic terms}, \tag{4.3}$$

where the residue A_i associated with the ith component $\varSigma_i$ is a matrix function on $\varSigma_i$ holomorphic also on the intersections $\varSigma_i \cap \varSigma_j$, though not necessarily coinciding with the respective A_j there (thus, strictly speaking, the residue is *not* a matrix function defined on the whole of $\varSigma$ but rather on its *normalization*).

Theorem 4.3. *If $\varSigma = \bigcup \varSigma_i$ is a hypersurface with normal crossings, $\Omega \in \varLambda^1(\log \varSigma)$ and $A_i = \operatorname{res}_{\varSigma_i} \Omega$, then on $\varSigma_i \cap \varSigma_j$ the residues A_i commute,*

$$[A_i, A_j] = 0 \qquad \text{on} \quad \varSigma_i \cap \varSigma_j. \tag{4.4}$$

Proof. Substituting the representation for Ω into the integrability condition (1.3), we obtain

$$\sum_s dA_s \wedge \frac{df_s}{f_s} + \cdots = \sum_{i,j} A_i A_j \frac{df_i \wedge df_j}{f_i f_j} + \text{poles of first order.}$$

Collecting the principal polar (non-holomorphic) terms of order 2, i.e., multiplying both parts of $f_i f_j$ and restricting on $(\varSigma_i \cap \varSigma_j) \smallsetminus \bigcup_{k \neq i,j} \varSigma_k$, we see that

$$0 = A_i A_j \, df_i \wedge df_j + A_j A_i \, df_j \wedge df_i = [A_i, A_j] \, df_j \wedge df_i.$$

Since df_i and df_j are linear independent, $[A_i, A_j] = 0$ on $\varSigma_i \cap \varSigma_j$ outside $\varSigma_0 = \bigcup_{k \neq i,j} \varSigma_k$. Since the "bad locus" $\varSigma_i \cap \varSigma_j \cap \varSigma_0$ is thin and A_i, A_j are holomorphic on $\varSigma_i$ and $\varSigma_j$ respectively, A_i and A_j commute everywhere on $\varSigma_i \cap \varSigma_j$. □

One can prove this theorem also by using the commutativity of the holonomy operators corresponding to small loops around $\varSigma_i$ and $\varSigma_j$ and passing to limit as in the first proof of Theorem 4.2.

The arguments proving Theorem 4.2 can be almost literally applied to the form (4.3) proving that each residue matrix A_j holomorphic on the respective component $\varSigma_j$, remains within the same conjugacy class. Moreover, the conjugacy matrix function $H \colon \varSigma \to \mathrm{GL}(n, \mathbb{C})$ defined as the solution of the system $dH = \varTheta H$, integrable on each $\varSigma_j$, after extension from $\varSigma$ to U conjugates the form (4.3) with the form with constant matrices A_j, normalizing the principal part of Ω as follows.

Theorem 4.4. *A flat connection with logarithmic poles on a normal crossing $\varSigma = \varSigma_1 \cup \cdots \cup \varSigma_k$, $k \leqslant m$, is holomorphically gauge equivalent to a system* (4.3) *with the constant residues $A_j|_{\varSigma_j} = \mathrm{const}_j \in \mathrm{Mat}_n(\mathbb{C})$ pairwise commuting with each other.* □

If $\varSigma$ is not a normal crossing, then the commutativity of the residues is no longer valid.

Example 4.5. Let $l_1, \dots, l_k$, $k \geqslant 3$ be any number of pairwise different *linear forms* on the plane $\mathbb{C}^2 = \{(t_1, t_2)\}$ and $A_1, \dots, A_k$ any collection of constant matrices. Consider the matrix 1-form $\Omega = \sum_1^k A_j \frac{dl_j}{l_j}$. Assume that none of the factors l_j is proportional to t_2, so that without loss of generality we may assume

$$l_j = t_1 - \lambda_j t_2, \qquad \lambda_j \in \mathbb{C},\ j = 1, \dots, k.$$

Using the function $z(t) = t_2/t_1$, and the identity $l_j = t_1(z - \lambda_j)$, so that $dl_j/l_j = dt_1/t_1 + d(z - \lambda_j)/(z - \lambda_j)$, the matrix Ω can be written

$$\Omega = B\frac{dt_1}{t_1} + \sum_{j=1}^{k} A_j \frac{dz}{z - \lambda_j}, \qquad B = \sum_{j=1}^{k} A_j.$$

This transformation is in fact a blow-up considered in more details in §7.1.

If the matrix $B = \sum A_j$ commutes with each of the matrices A_j, the form Ω is flat. Indeed, $d\Omega = 0$ since all A_j are constant, whereas $\Omega \wedge \Omega$ reduces to a sum of commutators involving B only,

$$\Omega \wedge \Omega = \sum_j [B, A_j] \frac{dt_1 \wedge dz}{t_1(z - \lambda_j)}$$

(pairwise commutators $[A_i, A_j]$ disappear since $dz \wedge dz = 0$). Therefore both sides of (1.3) are zeros.

Clearly, when $k > 2$, one can construct an example of a flat connection with non-commuting residues: for example, it is sufficient to take an arbitrary collection of matrices $A_1, \dots, A_{k-1}$ and put $A_k = -(A_1 + \cdots + A_{k-1})$ to ensure that $B = 0$. For $k = 2$ this is impossible: $0 = [A_1, B] = [A_1, A_1 + A_2] = [A_1, A_2]$.

Remark 4.6. Note that for $k > 2$ the fundamental group of $U \smallsetminus \varSigma$ is indeed noncommutative. Indeed, the projectivization map $\mathbb{C}^2 \smallsetminus \{0\} \to \mathbb{C}P^1$ restricted on $U \smallsetminus \varSigma$ is a topological bundle over $K = \mathbb{C}P^1 \smallsetminus \{\lambda_1, \dots, \lambda_k\}$ with the fiber $\mathbb{C} \smallsetminus \{0\}$ which is homotopically equivalent to the circle $\mathbb{S}^1$. The exact homotopy sequence

$$0 = \pi_2(K) \to \pi_1(\mathbb{S}^1) \to \pi_1(U \smallsetminus \varSigma) \to \pi_1(K) \to \pi_0(\mathbb{S}^1) = 0$$

implies that the factor of $\pi_1(U \smallsetminus \varSigma)$ by $\mathbb{Z} = \pi_1(\mathbb{S}^1)$ is $\pi_1(K)$ which is isomorphic to the free group with $k - 1$ generators.

4.4 Schlesinger equations

Assume that $U = \mathbb{C}^1 \times \mathbb{C}^k = \{(z, \lambda_1, \dots, \lambda_k)\}$ is the affine space equipped with the natural projection $p\colon (z, \lambda) \mapsto \lambda$, and let $\varSigma = \bigcup_{j=1}^k \{z - \lambda_j = 0\}$ be the union of bisector hyperplanes, all of them *transversal* to the fibers $\{\lambda = \text{const}\}$ of this projection. A flat connection Ω with poles on $\varSigma$ can be identified, according to §2.6, with an isomonodromic deformation of a linear "ordinary" system with singularities at the points $\lambda_1, \dots, \lambda_k \in \mathbb{C}$, parameterized by the location of these points.

We want to construct a flat meromorphic connection with poles on $\varSigma$: any such connection will automatically correspond to an isomonodromic deformation, as explained in §2.6. The extra requirement is that after restriction on each fiber $\lambda = \text{const}$, the connection would have only simple (Fuchsian) poles at all finite points $\lambda_1, \dots, \lambda_k$ *and* at the point $z = \infty$ after the compactification.

We start with the formal expression

$$\Omega = \sum_1^k A_j \frac{df_j}{f_j}, \qquad f_j(z, \lambda) = z - \lambda_j. \tag{4.5}$$

with holomorphic matrix functions $A_j(z, \lambda)$. However, if the restriction of Ω on each fiber $\lambda = \text{const}$ is to have a Fuchsian singularity after compactification at $z = \infty$, then necessarily A_j must be *constant* along this fiber. Actually, to *avoid* appearance of singularities at infinity, we will assume that that the residue $\operatorname{res}_{z=\infty} \Omega|_{\lambda=\text{const}} = -\sum A_j(\lambda)$ vanishes identically in λ,

$$A_j = A_j(\lambda), \quad j = 1, \dots, k, \qquad \sum_1^k A_j(\lambda) \equiv 0. \tag{4.6}$$

We claim that there exists at least one flat connection of the form specified by (4.5)–(4.6). By construction, this connection has logarithmic singularities on the hyperplane Σ_j. However, it will in general have additional singularities on a union of hyperplanes *parallel* to the z-direction.

Theorem 4.7. *The connection* (4.5)–(4.6) *is flat if and only if the matrix functions* $A_1(\lambda), \dots, A_k(\lambda)$ *satisfy the system of quadratic partial differential equations*

$$dA_s = -\sum_{j \neq s} [A_s, A_j] \frac{d\lambda_s - d\lambda_j}{\lambda_s - \lambda_j}, \qquad \forall s = 1, \dots, k. \tag{4.7}$$

Proof. Indeed, the integrability condition (1.3) for the system (4.5) implies the identity between matrix 2-forms

$$\sum_s dA_s \wedge \frac{df_s}{f_s} = \sum_{i,j} A_i A_j \frac{df_i \wedge df_j}{f_i f_j} = \sum_{i<j} [A_i, A_j] \frac{df_i \wedge df_j}{f_i f_j}. \tag{4.8}$$

Let $v = \partial/\partial z$ be the unit vector field on U tangent to the vertical fibers $\lambda = \text{const}$ and i_v the corresponding antiderivation (substitution of v as the first argument in a differential form, scalar or matrix). Our assumptions on A_j and f_j imply that

$$i_v dA_j = 0, \qquad i_v df_j = 1, \qquad j = 1, \dots, k,$$

so that

$$i_v(dA_s \wedge df_s) = (i_v \, dA_s) df_s - dA_s (i_v \, df_s) = -dA_s, \qquad i_v(df_i \wedge df_j) = df_i - df_j.$$

Applying i_v to both parts of (4.8) and using these identities, we arrive to the identity between (matrix) 1-forms

$$-\sum_s \frac{dA_s}{f_s} = \sum_{i,j} A_i A_j \frac{df_i - df_j}{f_i f_j}.$$

Equating the principal polar parts of both sides on each of the hyperplanes Σ_s (more precisely, on $\Sigma_s \smallsetminus \bigcup_{j\neq s}\Sigma_j$), we conclude that after restriction on each hyperplane Σ_s the following identities are satisfied,

$$-dA_s = \sum_j A_s A_j \frac{df_s - df_j}{f_j} + \sum_i A_i A_s \frac{df_i - df_s}{f_i} = \sum_j [A_s, A_j] \frac{df_s - df_j}{f_j}.$$

It remains to note that $df_s - df_j = d\lambda_s - d\lambda_j$ and the restriction of $f_j = z - \lambda_j$ on $\Sigma_s = \{z - \lambda_s = 0\}$ is equal to $\lambda_s - \lambda_j$. This shows the necessity of the conditions (4.7).

To prove sufficiency of the condition (4.7), we rewrite it in the form

$$dA_s = -\sum_{j\neq s} [A_s, A_j] \frac{df_j - df_s}{f_j - f_s},$$

wedge multiply each equation by df_s/f_s from the right and add the results together. Then the left hand side of the sum coincides with the left hand side of the equality (4.8) which is the flatness condition for Ω. It remains to show that the right sides also coincide, i.e., that after obvious transformations and renaming the summation variable in (4.8),

$$\sum_{s\neq j} [A_s, A_j] \frac{df_s \wedge df_j}{(f_j - f_s) f_s} = \sum_{s<j} [A_s, A_j] \frac{df_s \wedge df_j}{f_i f_j}. \tag{4.9}$$

Note that the sum in the left involves unordered pairs (s, j), while the sum in the right is extended only on the ordered pairs. After grouping the terms (s, j) and (j, s) together, the identity (4.9) follows from the identity

$$\frac{1}{(f_j - f_s) f_s} + \frac{1}{(f_s - f_j) f_j} = \frac{1}{f_s f_j}.$$

Thus (4.7) implies (4.8) as asserted. □

A computation similar to that proving Theorem 1.1, shows that the system (4.7), known as the *Schlesinger equations*, is also *integrable* so that *locally* (off the *collision locus* $C = \bigcup_{i\neq j}\{\lambda_i - \lambda_j = 0\} \subset \mathbb{C}^k$) it admits solutions $(\lambda_1, \dots, \lambda_k) \mapsto \big(A_1(\lambda), \dots, A_k(\lambda)\big)$. However, since this system is *nonlinear* (quadratic), its solution in general explode. The *movable poles* of solutions depend on the initial condition. In [Bol00b] it is proved that if the initial configuration of the singular points and residue matrices corresponded to a Fuchsian equation with an irreducible monodromy, then the singularities of solutions of the Schlesinger equation off the collision locus occur on an analytic hypersurface $M \subset \mathbb{C}^m$ and have second order poles on M.

Thus the attempt to construct a flat connection on $\mathbb{C}P^1 \times \mathbb{C}^m$ with logarithmic singularities on the hyperplanes Σ_j, implicitly encoded in the representation (4.5), leads to creation of additional singular locus $C \cup M$ on which it is in general non-logarithmic.

4.5 First order non-logarithmic poles and isomonodromic deformations of Fuchsian systems

The gap between flat *logarithmic* connections and flat meromorphic connections exhibiting a *first order pole* on a smooth analytic hypersurface Σ, is not very wide and disappears in the *non-resonant* case (see below).

Indeed, assuming that the hypersurface Σ in suitable local coordinates z, λ has the form $z = 0$, any form with a first order pole on it, can be written as

$$\Omega = \frac{A(\lambda)\,dz + \Theta}{z} + \cdots, \qquad \Theta = \sum_{1}^{m-1} B_i(\lambda)\,d\lambda_i, \tag{4.10}$$

the dots as usual denoting holomorphic terms. For every point λ the operator $A(\lambda)$ can be described as the residue of the restriction of Ω on the "normal" curve $Z = \{\lambda = \mathrm{const}\}$ to Σ. In absence of the additional structure of "isomonodromic deformation" this normalcy makes no particular sense, yet for any other curve Z' transversal to Σ at the same point, the residue of the restriction will be conjugate to $A(\lambda)$ (cf. with the first proof of Theorem 4.2; recall that Ω is assumed to be flat). Thus the spectrum (and more generally the characteristic polynomial) of the matrix $A(\lambda)$ make an invariant sense and are locally constant along the smooth connected components of Σ.

Theorem 4.8. *If no two eigenvalues of the matrix $A|_\Sigma$ of a flat connection* (4.10) *differ by* 1*, then necessarily $\Theta = 0$ and the connection is in fact logarithmic.*

Proof. Keeping polar terms of second order from the integrability condition for (4.10) yields

$$-\frac{dz}{z^2} \wedge \Theta = -\frac{1}{z^2}(A\,dz \wedge \Theta + \Theta \wedge A\,dz) + \cdots,$$

where the dots denote the terms having the first order pole (or holomorphic) on Σ, so that

$$\Theta = A\Theta - \Theta A = [A, \Theta].$$

For a diagonal $A = \operatorname{diag}\{\alpha_i\}$ this means the identity $\theta_{ij} = \theta_{ij}(\alpha_i - \alpha_j)$ for the matrix elements θ_{ij} of Θ. If none of the differences $\alpha_i - \alpha_j$ is equal to 1, then this is possible only when all θ_{ij} vanish so that $\Theta = 0$.

In the general case when A may have coinciding eigenvalues and be non-diagonalizable, we use the well known fact [Lan69]: *the matrix equation of the form $AX - XA' = B$ has a unique solution for any square matrix B if and only if A and A' have no common eigenvalues.* The equation $\Theta = [A, \Theta]$ reduces to the equation $(A - E)\Theta - \Theta A = 0$ which has only the trivial solution, since adding the identity matrix E to A shifts all eigenvalues by 1. $\square$

The same argument can be applied to the meromorphic matrix form

$$\Omega = A(\lambda)\frac{dz}{z} + \frac{1}{z^r}(\Theta + \cdots) \tag{4.11}$$

having a pole of order $r \geqslant 2$. In order for this assumption to make sense, we have to assume the structure of isomonodromic deformation (i.e., the bundle over λ-space making the z-direction exceptional, see §2.6). Then (4.11) means that the singular point at $z = 0$ remains Fuchsian for all values λ of the parameters of the deformation. The residue $A(\lambda)$ in this case becomes unambiguously defined.

Exactly the same computation as above leads to the equation $r\Theta = [A, \Theta]$, $2 \leqslant r \in \mathbb{Z}$ and proves the following claim.

Theorem 4.9. *If no two eigenvalues of the residue matrix $A(\lambda)$ differ by a nonzero integer number, then any Fuchsian isomonodromic deformation* (4.11) *is in fact logarithmic.* $\square$

The condition that no two eigenvalues of the residue differ by a nonzero integer, is very important. It will be referred to as the *non-resonance condition*.

It is the assertion of Theorem 4.9 that explains the apriori choice (4.5) of the connection form when discussing isomonodromic deformations of Fuchsian systems in §4.4. The connection form (4.5) has logarithmic singularities on each hyperplane $\{z - \lambda_j = 0\}$, at least off the diagonal (on the open set $\{\lambda_i \neq \lambda_j,\ i \neq j\} \subseteq \mathbb{C}P^1 \times \mathbb{C}^k$).

Indeed, if all residues of a Fuchsian system on $\mathbb{C}P^1$ are non-resonant, then the connection (4.5–4.6) satisfying the Schlesinger equations (4.7), is essentially the only possible isomonodromic deformation of this system. In the resonant case one may well have poles of higher order. This order is nevertheless bounded by the maximal integer difference $\max_{i,j} |\alpha_i - \alpha_j|$ between the eigenvalues. The complete proof of this result and the description of isomonodromic deformations can be found in [Bol97].

5 Abelian integrals and Picard–Fuchs systems

One of the most important examples of flat meromorphic connections exhibiting only logarithmic poles, is the *Gauss–Manin connection* described by the *Picard–Fuchs system* of differential equations involving *Abelian integrals* $\int_\Gamma \sigma$ of polynomial 1-forms along 1-cycles on nonsingular algebraic curves $\Gamma = \{H = 0\} \subset \mathbb{C}^2$, considered as functions of the parameters (coefficients of the form σ and the polynomial equation $H = 0$ of the curve).

5.1 Definitions

Let $P = P(w_1, w_2) \in \mathbb{C}[w] = \mathbb{C}[w_1, w_2]$ be a homogeneous complex polynomial of degree $r = \deg P$ in two complex variables which has an *isolated critical point* at the origin $w = 0$. For this, it is necessary and sufficient to require that P were *square-free*, i.e., has no repeating linear factors in the complete factorization.

Consider a *general bivariate polynomial* with the principal part P,

$$H(w, t) = P(w) + \sum_{\deg w^{\mathbf{a}} < r} t_{\mathbf{a}} w^{\mathbf{a}}, \tag{5.1}$$

where by $t_{\mathbf{a}}$, $\mathbf{a} \in \mathbb{Z}^2_+$, are denoted the complex coefficients before the lower degree monomials $w^{\mathbf{a}} = w_1^{a_1} w_2^{a_2}$.

The list of these parameters can be flattened to simplify our notation: if m is the dimension of the parameter space (which can be easily computed knowing the degree r), then we write $t = (t_1, \dots, t_m)$ and use the notation $H_t = H(\cdot, t)$ for $t \in \mathbb{C}^m$ to denote the bivariate polynomial with the corresponding non-principal part. The coefficients will be always ordered so that t_1 is the *free term* of H.

For most of the values of $t \in \mathbb{C}^m$ the zero level curve

$$\Gamma_t = \{H_t = 0\} \subset \mathbb{C}^2, \qquad t \in \mathbb{C}^m, \tag{5.2}$$

is non-singular (smooth) algebraic curve. The exceptional values belong to the *discriminant*, the algebraic hypersurface

$$\begin{aligned} \Sigma &= \{t \in \mathbb{C}^m \colon 0 \text{ is a critical value for } H_t\} \\ &= \{t \in \mathbb{C}^m \colon \exists w_* \in \mathbb{C}^2 \text{ such that } dH_t(w_*) = 0,\ H_t(w_*) = 0\}. \end{aligned} \tag{5.3}$$

It can be easily verified, see e.g., [Yak01], that the assumption on the principal part P ensures the local *topological triviality* of the family of the curves $\{\Gamma_t\}$ for $t \notin \Sigma$. This means that the curves Γ_t vary continuously with t outside Σ so that any 1-cycle δ (represented by a closed curve) on a nonsingular affine curve Γ_t, $t \notin \Sigma$, can be locally continued as a uniquely defined element $\delta(t)$ in the first homology group of Γ_t. In other words, a *flat connection* on the *homology bundle* is defined.

Remark 5.1. If the principal part P is not square-free (has one or several lines entirely consisting of critical points), the affine curves Γ_t may change their topological type for some values $t \notin \Sigma$. The reason is that after projective compactification, Γ_t may be non-transversal to the infinite line and thus undergo "bifurcations at infinity". These exceptional *atypical values* form an algebraic subvariety in $\mathbb{C}^m$.

The constructed connection, albeit flat, still has a nontrivial *monodromy.* If $\gamma : [0,1] \to \mathbb{C}^m$ is a closed path in the t-space, avoiding the discriminant locus Σ, then the result of continuation of $\delta(t)$ along γ may well be nontrivial, $\delta(\gamma(0)) \neq \delta(\gamma(1))$. It will be described in more details later.

Any polynomial 1-form $\sigma \in \Lambda^1(\mathbb{C}^2)$ on $\mathbb{C}^2$ can be restricted on the curve Γ_t, $t \notin \Sigma$, and integrated along any cycle on this curve. A (complete) *Abelian integral* is by definition the multivalued analytic function on $\mathbb{C}^m \smallsetminus \Sigma$ obtained by integration of σ along a continuous (horizontal) section $\delta(t)$ of the homological bundle.

Denote by n the rank of the first homology group of a generic level curve Γ_t: this means that there can be found n cycles $\delta_1, \dots, \delta_1$ generating all other cycles over $\mathbb{Z}$. Again by the topological triviality, the result of continuation $(\delta_1(t), \dots, \delta_n(t))$ will be a *framing* of the (first homology groups of) level curves $\{\Gamma_t\}$. A collection of polynomial 1-forms $\sigma_1, \dots, \sigma_n$ is called *coframe*, if the *period matrix*

$$X(t) = \begin{pmatrix} \oint\limits_{\delta_1(t)} \sigma_1 & \cdots & \oint\limits_{\delta_n(t)} \sigma_1 \\ \vdots & \ddots & \vdots \\ \oint\limits_{\delta_1(t)} \sigma_n & \cdots & \oint\limits_{\delta_n(t)} \sigma_n \end{pmatrix} \tag{5.4}$$

is not identically degenerate, $\det X(t) \not\equiv 0$. This definition in fact does not depend on the choice of the frame $\{\delta_i(t)\}_1^n$.

It is worth mentioning that in many sources the Abelian integrals are considered as analytic functions of the variable $t_1 \in \mathbb{C}^1$ only (the free term), being defined as integrals $\int_{H(w)=t_1} \sigma$ for a given *fixed* bivariate polynomial $H(w_1, w_2)$. By this definition, the integral is ramified over the set of complex critical values of H (and only over this set under the assumption on the principal homogeneous part of H). An abundant wealth of information is accumulated about behavior of Abelian integrals defined this way; some of it will be used to derive multivariate properties.

5.2 Ramification of Abelian integrals and Picard–Lefschetz formulas

The part of the discriminant locus Σ corresponding to polynomials $H(w,t) = P(w) + \sum t_{\mathbf{a}} w^{\mathbf{a}}$ which have only one nondegenerate (Morse) critical point on the zero level curve $\Gamma_t = \{H_t = 0\}$, is smooth. If $t_* \in \Sigma$ is such a point, then for all sufficiently close $t \in \mathbb{C}^m$ the polynomial

H_t has a unique critical value $f(t)$ which is close to zero and holomorphically depends on t. The equation $\{f=0\}\subset\mathbb{C}^m$ locally defines Σ near the smooth point t_*.

All these assertions are easy to verify. Denote by $w_*\in\mathbb{C}^2$ the critical point of $H_*=H(\cdot,t_*)$. For nearby values of t the critical point $w_t\in\mathbb{C}^2$ of $H_t(\cdot,\cdot)$ is determined by the two equations $\frac{\partial H}{\partial w_i}(w_t)=0$, $i=1,2$. By virtue of the nondegeneracy condition

$$\det\frac{\partial^2 H}{\partial w_i\partial w_j}(w_*,t_*)\neq 0.$$

By the implicit function theorem, the point w_t analytically depends on t and therefore the value $f(t)=H(w_t,t)$ also does. To see that $df(t_*)\neq 0$, notice that

$$\frac{\partial f}{\partial t_1}=\sum_{i=1}^{2}\frac{\partial H}{\partial w_i}\cdot\frac{\partial w_i}{\partial t_1}(w_t,t)+\frac{\partial H}{\partial t_1}(w_t,t)=\frac{\partial H}{\partial t_1}(w_t,t)=1\neq 0.$$

This computation shows that projection of Σ along t_1 is smooth at smooth points of the former and in part explains the exceptional role played by the coefficient t_1.

To describe branching of the period matrix near smooth points of Σ, a special frame of cycles has to be chosen. If $t_*\in\Sigma$ is a smooth point on the discriminant and $w_*=(w_{*1},w_{*2})\in\mathbb{C}^2$ the corresponding critical point of $H_*=H(\cdot,t_*)$, then by the Morse lemma one can introduce local coordinates on an open set $U\subset\mathbb{C}^2$ near w_* so that

$$H(w,t)=(w_1-w_{*1})^2+(w_2-w_{*2})^2+f(t),\qquad w\in U,\ t_*\text{ near }t_*.$$

There is a unique (modulo orientation and homotopy equivalence) simple loop on the "bottleneck", the local level curve $\{H_t=0\}$ (topologically a cylinder if $f(t)\neq 0$) that is not contractible and shrinks as $f(t)\to 0$. This loop is called the *vanishing cycle* $\delta_1(t)$, and by its construction the integral $\oint_{\delta_1(t)}\sigma$ of *any* polynomial 1-form $\sigma\in\Lambda^1(\mathbb{C}^2)$ is single-valued and bounded near t_*. Being thus holomorphic on Σ, this integral in fact vanishes on Σ, whence comes the term "vanishing".

Other cycles on Γ_t, not reducible to a multiple of δ_1, can be ramified over Σ. To describe ramification, recall that on all nonsingular level curves Γ_t, $t\notin\Sigma$, the *intersection index* between cycles is well defined. This intersection index, an integer number, is well-defined and locally constant as t varies outside Σ. It turns out, see [AGV88], that after t goes around Σ near a smooth point of the latter, any other cycle $\delta(t)$ is transformed by adding an integer multiple of the vanishing cycle δ_1, the multiplier being the intersection index $(\delta,\delta_1)\in\mathbb{Z}$ between δ and δ_1:

$$\delta\mapsto\delta+(\delta,\delta_1)\,\delta_1.\tag{5.5}$$

This formula is known by many names, from *Dehn twist* to (most frequently) *Picard–Lefschetz formula*. Its appearance is rather natural: indeed, on the level of homology the impact of the parallel transport must be an additive contribution supported by a small neighborhood of the critical point only (since far away from w_* the curves Γ_t must form a topologically trivial family). On the other hand, it was already remarked that the only nontrivial cycle(s) living in a small neighborhood of a Morse critical point, are (multiples) of the vanishing cycle. The accurate proof can be found in [AGV88].

5.3 Nondegeneracy of the period matrix

The Picard–Lefschetz formula has numerous corollaries. First, the determinant of the period matrix $\det X(t)$ is a holomorphic function on $\mathbb{C}^m$ vanishing on the hypersurface Σ. Indeed, after going around any smooth component of Σ, independently of the choice of the coframe forms σ_i, the first column of X remains intact whereas any other column is transformed by adding an integer multiple of the first column. It remains to notice that all integrals in the first column vanish on Σ, while all other entries are bounded as $t \to \Sigma$ along non-spiraling domains.

This description means that if the frame $\delta(t)$ of the homology bundle is chosen so that the first cycle $\delta_1(t)$ is vanishing, then locally near each smooth point $a \in \Sigma$, the period matrix can be represented as

$$X(t) = V(t)Y(t), \qquad Y(t) = \begin{pmatrix} f & c_2\, f \ln f & \cdots & c_n\, f \ln f \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \tag{5.6}$$

where $f(t) = 0$ is the local equation for Σ, the complex constants $c_2, \dots, c_n \in \mathbb{C}$ are proportional to the respective intersection indices, $c_j = (\delta_j, \delta_1)/2\pi\mathrm{i}$, $j = 2, \dots, n$, and $V(t)$ is a holomorphic matrix function near $a \in \Sigma$ (depending, in general, on the point a).

For an arbitrary choice of the coframe $\{\sigma_i\}_{i=1}^n$, the matrix function $V(t)$ may be degenerate on some analytic hypersurfaces eventually intersecting Σ. However, in the particular case when the forms are chosen of minimal possible degrees compatible with the nondegeneracy of X, the matrix function $V(t)$ is holomorphically invertible. More precisely, the "proper" choice of the coframe can be described as follows.

Recall that $r = \deg H = \deg P$ denotes the degree of the homogeneous polynomial P, the principal homogeneous part of $H(\cdot, t)$. In the case when the weights of both w_1 and w_2 are equal to 1, one can compute the rank n of the (co)homology of the affine curves Γ_t:

$$n = (r-1)^2.$$

Not accidentally, this number coincides with the number of critical points of H_t counted with multiplicity which in turn is equal to the (algebraic) dimension of the quotient algebras

$$n = \dim_{\mathbb{C}} \Lambda^2(\mathbb{C}^2)/dP \wedge \Lambda^1(\mathbb{C}^2) = \dim_{\mathbb{C}} \Lambda^2(\mathbb{C}^2)/dH_t \wedge \Lambda^1(\mathbb{C}^2). \tag{5.7}$$

The following result concerning Abelian integrals as functions of one complex variable, belongs to the folklore. With various degrees of generality, completeness and precision it appeared in many sources, among them the recent paper [Gav98]. Other sources, together with the complete demonstrations, can be found in [Nov02].

Lemma 5.2 (see [Nov02] and references therein). *Assume that the monomial 1-forms $\sigma_1, \dots, \sigma_n \in \Lambda^1(\mathbb{C}^2)$ satisfy the condition*

$$\sum_1^n \deg \sigma_i = n \deg H. \tag{5.8}$$

Then the determinant of the corresponding period matrix X (5.4) *as a function of* t_1 *is a polynomial of degree no greater than* n.

This polynomial is not identically zero if the differentials $d\sigma_1, \dots, d\sigma_n$ *are linear independent modulo the ideal* $dP \wedge \Lambda^1(\mathbb{C}^2)$, *i.e., if no nontrivial linear combination of* $d\sigma_i$ *is divisible by the 1-form* dP. $\square$

Any collection of monomial 1-forms meeting the condition (5.8) and whose differentials $d\sigma_i$ are linear independent modulo $dP \wedge \Lambda^1(\mathbb{C}^2)$, will be referred to as the *basic coframe*.

Remark 5.3. The 1-forms constituting a basic coframe can always be chosen *monomial*. Moreover, the (monomial) primitives of $(r-1)^2$ monomial 2-forms $w_1^{a_1} w_2^{a_2}\, dw_1 \wedge dw_2$ with $0 \leqslant a_{1,2} \leqslant r-2$, constitute a basic coframe for *almost all* principal homogeneous polynomials of the same degree r.

Application of Lemma 5.2 immediately shows that if the forms σ_i constitute a basic coframe, then in the local representation (5.6) of the period matrix (5.4) the matrix term $V(t) = X(t)Y^{-1}(t)$, $Y = Y_f(t)$ as in (5.6), must be holomorphically invertible on Σ for any choice of the local equation f. Indeed, almost any complex 1-dimensional line in $\mathbb{C}^m$ parallel to the t_1-direction, intersects Σ by exactly $n = (r-1)^2$ smooth points corresponding to n (in general, distinct) critical values of the polynomial H_t. The determinant $\det X$ restricted on this line, necessarily has roots at the points of this intersection. These roots are simple if and only if $\det V \neq 0$ at these points, since $\det Y(t) = f(t)$ already has a simple root there. On the other hand, the number of roots cannot exceed the degree of $\det X$ in t_1 since the latter is different from identical zero. Thus the number of roots of $\det X(t_1, \mathrm{const})$ is exactly n and all of them are simple. Therefore $\det V(t)$ is nonvanishing at almost all (except for a thin set of complex codimension 2 or more) smooth points of Σ.

5.4 Global equation of Σ

If the monomial forms σ_i constitute a basic coframe, then any polynomial 2-form on $\mathbb{C}^2$ can be divided by dH_t with the remainder that is a linear combination of $d\sigma_i$. In particular, this refers to the forms $H_t\, d\sigma_i$:

$$H_t\, d\sigma_i = \eta_i \wedge dH_t + \sum_{j=1}^{n} R_{ij}\, d\sigma_j, \qquad \eta_i \in \Lambda^1(\mathbb{C}^2),\ R_{ij} \in \mathbb{C}. \tag{5.9}$$

where $t \in \mathbb{C}^m$ is considered as a parameter. The process of division is analyzed in details in [Yak02] where it is shown, in particular, that the result (both the incomplete ratios η_i and the coefficients of the remainders) depends *polynomially* on t, giving rise to a matrix polynomial $R(t) = \|R_{ij}(t)\|$. This is a consequence of the fact that H_t depends on t polynomially while its principal homogeneous part does not depend on t at all.

Lemma 5.4. *The singular locus* $\Sigma \subset \mathbb{C}^m$ *is given by one polynomial equation* $\det R(t) = 0$.

Proof. The identity (5.9) evaluated at a critical point $w_t \in \mathbb{C}^2$ of H_t with the critical value $0 = H_t(w_t) = H(w_t, t)$, means that the nonzero vector of coefficients of the forms $d\sigma_i$ at this point belongs to the null space of the matrix $R(t)$, since both dH_t and H_t vanish at w_t. $\square$

In fact, if $\{\sigma_i\}_{i=1}^n$ is a basic coframe, then the determinant of the corresponding period matrix $X(t)$ is proportional to $\det R(t)$:

$$\det X(t) = \mathrm{const} \cdot \det R(t),$$

the constant depending on the principal part P of the polynomial H and on the coframe. This constant was computed in [Glu00] for a special choice of the basic coframe.

5.5 Finite singularities of the Abelian integrals are logarithmic

Theorem 5.5. *Assume that $H(w,t) = P(w) + \sum t_{\mathbf{a}} w^{\mathbf{a}}$, $t \in \mathbb{C}^m$, is a general polynomial with the square-free principal homogeneous part P of degree r, and $\sigma_1, \dots, \sigma_n$, $n = (r-1)^2$, are monomial forms constituting a basic coframe as described in Lemma 5.2.*

Then the period matrix $X(t)$ satisfies an integrable Pfaffian system (1.2) *with the matrix $\Omega = dX \cdot X^{-1}$ having logarithmic poles on the algebraic hypersurface $\Sigma \subset \mathbb{C}^m$. The residue matrix $A = \operatorname{res}_\Sigma \Omega$ is conjugate to* $\operatorname{diag}\{1, 0, \dots, 0\}$.

Proof. It is sufficient to show that the "logarithmic derivative" $dX \cdot X^{-1}(t)$ has a logarithmic pole near any smooth point of $\Sigma = \{f = 0\}$, $f = \det R$, where R is the matrix polynomial described in Lemma 5.4.

By (5.6), locally near Σ there exists a representation $X = VY$, where V is holomorphically invertible and Y explicitly given by

$$Y(t) = \operatorname{diag}\{f, 1, \dots, 1\} \cdot \begin{pmatrix} 1 & c_2 \ln f & \cdots & c_n \ln f \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

Since V is holomorphically invertible, Ω is holomorphically gauge equivalent to

$$\Omega' = dY \cdot Y^{-1} = \begin{pmatrix} \frac{df}{f} & c_2\, df & \cdots & c_n\, df \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

which obviously has a logarithmic pole on $\{f = 0\}$: $f\Omega'$ is clearly holomorphic on Σ, while $d\Omega' = \Omega' \wedge \Omega' = 0$. As Ω and Ω' are holomorphically gauge equivalent near Σ, the form Ω is also logarithmic on smooth parts of $\{f = 0\}$. Since both Ω and f are globally defined, the standard arguments involving thin sets of codimension $\geqslant 2$ in $\mathbb{C}^m$ imply then that Ω has a logarithmic singularity on the whole of $\Sigma \subset \mathbb{C}^m$. □

It is worth a remark that for a different choice of the coframe the period matrix $X'(t)$ differing from the period matrix for a basic coframe $X(t)$ by a *polynomial* matrix factor, $X' = C(t)X(t)$, $C \in \mathrm{GL}(n, \mathbb{C}[t])$. This factor is in general non-invertible along an algebraic hypersurface Σ' different from Σ (and depending on the choice of the coframe), and the corresponding "logarithmic derivative" $\Omega' = dX' \cdot (X')^{-1}$ has no reasons to be logarithmic on $\Sigma' \smallsetminus \Sigma$.

The elementary arguments proving Theorem 5.5, can be considerably generalized. A. Bolibruch established in [Bol77] necessary and sufficient conditions for a matrix function of the form $X(t) = V(t)\,f(t)^A$ to have a logarithmic singularity on $\{f = 0\}$. As a particular case of his result, using elementary arguments one can show that the "logarithmic derivative" $dX \cdot X^{-1}$ of a *holomorphic* matrix function $X(t)$ has a logarithmic pole on $\varSigma = \{\det X = 0\}$ at those smooth points of the latter hypersurface, where the determinant has a simple (first order) zero.

Concluding this section, note that a large part of the results described here can be modified for the multivariate quasihomogeneous case where the polynomial $H = H(w_1, \dots, w_p)$ in $p \geqslant 2$ complex variables is a sum of the principal quasihomogeneous part $P(w_1, \dots, w_p)$ (subject to the nondegeneracy condition that dP has an isolated singularity) and lower degree monomials with indeterminate coefficients $t_{\mathbf{a}}\,w^{\mathbf{a}}$, $\deg w_{\mathbf{a}} < \deg P$. The weights assigned to the independent variables w_j need not necessarily be equal to each other. Some of the details can be found in [Yak02], where it is shown that the system (1.2) for the period matrix of a basic coframe has the form

$$R(t)\,dX = R'(t)X,$$

where $R(t)$ is as above and $R'(t)$ is another matrix polynomial.

6 Poincaré–Dulac theorem for flat logarithmic connections

In this section we make the first step towards holomorphic gauge classification of integrable connections with logarithmic poles. While the result itself is not new (see [YT75]; it can also be found in [Bol77]), the proof that we give below, close to [YT75] is more in vein with the traditional "ordinary" approach based on the Poincaré–Dulac technique. It admits immediate generalizations for some types of polar loci different from normal crossings.

6.1 "Ordinary" normalization of univariate systems

It is well-known that a system of linear ordinary differential equations

$$\dot{x} = A(t)x, \qquad A(t) = t^{-1}(A_0 + tA_1 + \cdots) \tag{6.1}$$

near the singular point $t = 0 \in \mathbb{C}^1$, by an appropriate *holomorphic* gauge transformation $x \mapsto H(t)x$, $H = E + tH_1 + \cdots$, can be simplified to keep only *resonant* terms. The latter are defined by the residue matrix A_0 of the system depending on the arithmetic properties of this residue. In particular, if A_0 is *nonresonant*, i.e., *no two eigenvalues of the residue matrix differ by a nonzero integer*, then there are no resonant terms and the system (6.1) by a suitable holomorphic gauge transformation can be made into the Euler system $\dot{x} = t^{-1}A_0x$; the corresponding Pfaffian matrix is $A_0\frac{dt}{t}$, see (2.1).

The proof consists in application of the Poincaré–Dulac method of construction of a *formal gauge transformation* conjugating (6.1) with its normal form. Then it can be proved (relatively easy) that the formal gauge transformation in fact always converges.

We implement the same construction in the general Pfaffian case and prove the following theorem. Consider a flat connection with the holomorphic pole on the normal crossing $\varSigma =$

$\{t_1 \cdots t_k = 0\} \subset U = (\mathbb{C}^m, 0)$,

$$\Omega = \sum_{s=1}^{k} A_s(t)\,\frac{dt_s}{t_s} + \text{holomorphic terms.} \tag{6.2}$$

By Theorem 4.4, the holomorphic residues A_s corresponding to the smooth components $\Sigma_s = \{t_s = 0\}$, can without loss of generality be assumed constant, $A_s = \text{const} \in \operatorname{Mat}_n(\mathbb{C})$. By Theorem 4.3, the residue matrices A_s commute with each other.

Following the persistent pedagogical tradition, we assume for simplicity that all the matrices $A_1, \dots, A_k$ are diagonal: it will be automatically the case if one of them has only simple (pairwise different) eigenvalues. In this case the matrix 1-form

$$\Omega_0 = \sum_{s=1}^{k} A_s\,\frac{dt_s}{t_s} = \operatorname{diag}\{\alpha_i\}_{i=1}^{n} \tag{6.3}$$

will be diagonal with the entries $\alpha_i = \sum_{s=1}^{k} a_i^s \frac{dt_s}{t_s} \in \Lambda^1(\log \Sigma)$ on the diagonal. Note that *any* diagonal flat connection is necessarily *closed*, $d\Omega = 0$, since $d\alpha_i = \alpha_i \wedge \alpha_i = 0$. The numbers a_i^s are (constant by Lemma 3.9) *residues* of closed logarithmic forms α_i on the irreducible components $\Sigma_s \subset \Sigma$ of the polar locus.

The closed logarithmic diagonal matrix form $\Omega_0 = \operatorname{diag}\{\alpha_1, \dots, \alpha_n\}$ will be called *resonant*, if some difference $\alpha_i - \alpha_j \in \Lambda^1(\log \Sigma)$ has *all* integer residues. This happens if and only if this difference is the logarithmic derivative dh/h of a meromorphic function h holomorphically invertible outside Σ. If we denote by $\mathbf{a}_i = (a_i^1, \dots, a_i^k, 0, \dots, 0) \in \mathbb{C}^n$ the vector of residues of the form α_i, then the diagonal matrix Ω is resonant if and only if $\mathbf{a}_i - \mathbf{a}_j \in \mathbb{Z}^n$ for some $i \neq j$. Otherwise Ω is *non-resonant.*

Non-resonance is a rather weak arithmetic condition: for example, if only one of the matrices A_s is nonresonant (no two eigenvalues differ by an integer), the entire collection will be nonresonant in the sense of this definition.

Theorem 6.1. *Assume that the closed diagonal matrix form Ω_0 with the logarithmic pole on the normal crossing $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_k$ is non-resonant.*

Then any flat logarithmic form $\Omega_0 +$ (holomorphic terms) *is holomorphically gauge equivalent to the principal part Ω_0.*

The proof of this Theorem occupies sections §6.2 through §6.4

6.2 Poincaré–Dulac method: derivation and simplification of the homological equation

Recall that we grade meromorphic functions and differential forms on $(\mathbb{C}^m, 0)$, assigning the same weights to differentials dt_j and the corresponding variables t_j. Then the exterior derivative d will be degree-compatible: for any monomial form ω we have $\deg d\omega = \deg \omega$ unless $d\omega = 0$. In principle the weights assigned to different t_j, can (and will) be chosen unequal, so instead of homogeneous forms it is more accurate to speak about *quasi*homogeneity. Regardless of the choice of the weights, the logarithmic residual terms $A_j \frac{dt_j}{t_j}$ always have degree 0.

The connection form Ω can be explicitly expanded as a sum of (quasi)homogeneous terms,

$$\Omega = \Omega_0 + \Omega_r + \cdots, \qquad \Omega_0 = \sum_{j=1}^{k} A_j \frac{dt_j}{t_j}, \quad \deg \Omega_r = r,$$

where the "additive dots" denote (as customary) terms of degree higher than that of the terms explicitly specified, the first nontrivial term being Ω_r. Consider a formal matrix function $H = E + H_r + \cdots$, $\deg H_r = r$, whose inverse and differential expand as

$$H = E + H_r + \cdots, \qquad H^{-1} = E - H_r + \cdots, \qquad dH = dH_r + \cdots,$$

Applying to Ω the formal gauge transform with the matrix H, we obtain the new matrix form

$$\Omega' = \Omega_0 + \Omega_r + dH_r + H_r\Omega_0 - \Omega_0 H_r + \cdots.$$

If the homogeneous term H_r can be found such that

$$dH_r - \Omega_0 H_r + H_r \Omega_0 = -\Omega_r, \tag{6.4}$$

then the homogeneous term Ω_r will be eliminated from the expansion of Ω. If this step can be iterated for all higher degrees, then the proof of Theorem 6.1 will be achieved on the formal level. In analogy with the "ordinary" case, the equation (6.4) is called the *homological equation*.

The term Ω_r in the right hand side of the homological equation cannot be arbitrary: the condition of flatness of Ω imposes a restriction on Ω_r. Indeed, from (1.3) and the identity $d\Omega_0 = 0$ it follows that

$$d\Omega_r + \cdots = \Omega_0 \wedge \Omega_r + \Omega_r \wedge \Omega_0 + \cdots.$$

This implies that on the level of r-homogeneous terms

$$d\Omega_r = \Omega_0 \wedge \Omega_r + \Omega_r \wedge \Omega_0. \tag{6.5}$$

One can immediately see that the equation (6.4) can be solved only for Ω_r meeting the condition (6.5). Indeed, any connection obtained by a holomorphic gauge transform $H = E + H_r + \cdots$ from the flat Euler connection Ω_0, must necessarily be flat. We will show that in fact the homological equation (6.4) is always solvable with respect to H_r for Ω_r satisfying the necessary condition (6.5).

Note the similarity between the equation (6.4) and the necessary condition (6.5) for its solvability: both can be written using the operator $[\Omega_0, \bullet]$ of commutation with the principal part Ω_0, if we agree that the "commutator" of two matrix 1-forms has the properly adjusted sign $[\Omega, \Theta] = \Omega \wedge \Theta + \Theta \wedge \Omega$. Using this "extended" commutator, (6.4) and (6.5) take the respective forms

$$dH - [\Omega_0, H] = -\Omega_r, \qquad d\Omega_r - [\Omega_0, \Omega_r] = 0. \tag{6.6}$$

While this similarity may be still regarded as artificial, the fact that Ω_0 is diagonal allows to express the identities (6.6) using the same *scalar* differential operator. Indeed, since $\Omega_0 = \operatorname{diag}\{\alpha_1, \ldots, \alpha_n\}$, both the homological equation and its solvability condition (6.5) split into scalar equations on the matrix elements $\omega = \omega_{ij} \in \Lambda^1(U)$ of Ω_r and $h = h_{ij} \in \Lambda^0(U)$ of H_r. If we denote by

$$\alpha_{ij} = \alpha_i - \alpha_j \in \Lambda^1(\log \Sigma)$$

the corresponding closed logarithmic forms, then (6.4) and (6.5) will take the scalar form which requires to find a solution $h = h_{ij}$ of the equation

$$dh_{ij} - \alpha_{ij} h_{ij} = -\omega_{ij}, \tag{6.7}$$

for any homogeneous 1-form ω_{ij} meeting the condition

$$d\omega_{ij} - \alpha_{ij} \wedge \omega_{ij} = 0. \tag{6.8}$$

6.3 Solvability of the scalar homological equation

Denote provisionally by $\hat{\Lambda}^p$ the collection of p-forms with coefficients being Laurent polynomials in $t_1, \dots, t_k$ and the (usual) Taylor polynomials in $t_{k+1}, \dots, t_m$. Let $\alpha \in \Lambda^1(\log \Sigma)$ be a closed logarithmic form with the residues $a_s = \operatorname{res}_{\Sigma_s} \alpha$, $s = 1, \dots, k$.

Consider the formal differential operator ∇ acting on forms of all ranks,

$$\nabla \colon \hat{\Lambda}^p \to \hat{\Lambda}^{p+1}, \qquad \nabla = d - \alpha \wedge \cdot, \qquad \alpha = \sum_1^k a_s \frac{dt_s}{t_s}. \tag{6.9}$$

Note that ∇ is degree-preserving for any quasihomogeneous grading of $\Lambda^\bullet$.

It can be immediately verified that

$$\nabla^2 = d^2 - d(\alpha \wedge \cdot) - \alpha \wedge d + (\alpha \wedge \alpha) \wedge \cdot = -(d\alpha) \wedge \cdot = 0,$$

that is, the diagram

$$0 \longrightarrow \hat{\Lambda}^0 \xrightarrow{\nabla} \hat{\Lambda}^1 \xrightarrow{\nabla} \hat{\Lambda}^2 \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \hat{\Lambda}^m \xrightarrow{\nabla} 0 \tag{6.10}$$

is a cochain complex. In algebraic terms, the question on solvability of the equation (6.7) under the assumption (6.8) reduces essentially to proving that this complex is exact in the term Λ^1, i.e., $\ker \nabla = \nabla(\hat{\Lambda}^0)$ in $\hat{\Lambda}^1$. Somewhat more accurately, one has also to verify that the solution h will be a Taylor polynomial whenever $\omega \in \ker \nabla$ is. The need to consider Laurent polynomials is caused by appearance of denominators in α so that in general ∇ takes Taylor polynomials to Laurent polynomials only.

The answer depends on the 1-form $\alpha \in \hat{\Lambda}^1$, more precisely, on arithmetic properties of its residues $a_s \in \mathbb{C}$.

Lemma 6.2. *Assume that the vector of residues* $\mathbf{a} = (a_1, \dots, a_k, 0, \dots, 0) \in \mathbb{C}^m$ *of* α *is not integer,* $\mathbf{a} \notin \mathbb{Z}^m$.

Then the equation $\nabla h = \omega$ *has a unique solution* $h \in \hat{\Lambda}^0$ *if and only if the form* $\omega \in \hat{\Lambda}^1$ *satisfies the condition* $\nabla \omega = 0$. *This solution is a quasihomogeneous Laurent polynomial of the same degree as* ω, *if the latter is a quasihomogeneous Laurent polynomial* 1*-form.*

Conditions for holomorphic solvability are slightly less stringent.

Lemma 6.3. *If the vector of residues* $\mathbf{a} = (a_1, \dots, a_m)$ *is not* nonnegative integer*,*

$$\mathbf{a} \notin \mathbb{Z}_+^m, \tag{6.11}$$

then any holomorphic 1-form $\omega \in \Lambda^1$ *satisfying the condition* $\nabla\omega = 0$*, is uniquely representable as* $\omega = \nabla h$*, with* $h \in \Lambda^0$ *being a holomorphic function.*

In other words, under the assumption $\mathbf{a} \notin \mathbb{Z}^m$ the cohomology of the complex (6.10) is trivial in the terms $\hat\Lambda^0$ and $\hat\Lambda^1$. Both results are proved by the same simple computation.
Proof. of both Lemmas. Because of the choice of the form α, the operator ∇ preserves the degrees of all forms *independently of the choice of the weights* w_j assigned to the individual variables t_j. Thus it is sufficient to prove the Lemma only for quasihomogeneous forms.

Choose the weights linear independent over $\mathbb{Z}$ (as in the proof of Theorem 2.11). Then the only quasihomogeneous holomorphic forms are those that have the structure

$$\omega = t^{\mathbf{b}}\,\gamma, \qquad \gamma = \sum_1^m c_j \frac{dt_j}{t_j}, \quad c_j \in \mathbb{C}, \quad \mathbf{b} = (b_1, \dots, b_m) \in \mathbb{Z}^m$$

(in the holomorphic case we necessarily have $\mathbf{b} \in \mathbb{Z}_+^m$). Since $d(t^{\mathbf{b}}) = t^{\mathbf{b}}\,\beta$, the condition $\nabla\omega = 0$ means that

$$t^{\mathbf{b}}\,\beta \wedge \gamma - t^{\mathbf{b}}\,\alpha \wedge \gamma = t^{\mathbf{b}}(\beta - \alpha) \wedge \gamma = 0, \qquad \beta = \sum_1^m b_j \frac{dt_j}{t_j}.$$

The equality $(\beta - \alpha) \wedge \gamma = 0$ is possible if and only if $(\mathbf{b} - \mathbf{a}) \wedge \mathbf{c} = 0$, where $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{C}^n$. If

$$\mathbf{b} - \mathbf{a} \neq 0 \in \mathbb{C}^n, \tag{6.12}$$

the last equality is possible if and only if $\mathbf{c}$ is proportional to $\mathbf{a} - \mathbf{b}$, that is, only if $\gamma = \alpha - \beta$ up to a scalar multiplier. Then

$$\omega = t^{\mathbf{b}}(\alpha - \beta) = \alpha t^{\mathbf{b}} - d(t^{\mathbf{b}}) = \nabla h, \qquad h = -t^{\mathbf{b}} \in \Lambda^0,$$

as asserted. The solvability condition (6.12) for any $\mathbf{b} \in \mathbb{Z}^m$ (resp., for any $\mathbf{b} \in \mathbb{Z}_+^m$) follows from the non-resonance conditions on $\mathbf{a}$ from the assumptions of each Lemma.

To see that the equation $\nabla h = 0$ has a *unique* solution $h = 0$, if α is non-resonant, it is sufficient again to consider only the monomial case $h = t^{\mathbf{b}}$, $\mathbf{b} \in \mathbb{Z}_+^m$. Then $0 = d(t^{\mathbf{b}}) - t^{\mathbf{b}}\,\alpha = t^{\mathbf{b}}(\beta - \alpha)$ is possible only if $\mathbf{b} - \mathbf{a} = 0$, that is, never (unless $h = 0$). □

In the resonant case the above computation allows to describe in simple terms the (co)kernel of the operator ∇.

6.4 Convergence of the formal gauge transform

Application of Lemma 6.3 proves that in the nonresonant case on each step r the homological equation (6.4) can be solved an a formal gauge transform conjugating the system (6.2) to its principal (Euler) part Ω_0, see (6.3).

In order to complete the proof of Theorem 6.1, we have to prove that the formal series converges.

Note that the gauge transform H conjugating Ω with Ω_0, if it exists, satisfies the Pfaffian system
$$dH = \Omega_0 H - H\Omega. \tag{6.13}$$
Though this is *not* a matrix Pfaffian equation (1.2), it is still a system of n^2 Pfaffian equations on the components of the matrix function $H = H(t)$ arranged in any order. The "Pfaffian matrix" $\boldsymbol{\Omega}$ of this system of size $n^2 \times n^2$ is built from the entries of the matrix forms Ω and Ω_0. Therefore, $\boldsymbol{\Omega}$ has a logarithmic pole on Σ. Flatness (integrability) of $\boldsymbol{\Omega}$ outside Σ follows either from a straightforward computation or from the fact that the corresponding Pfaffian system has locally n^2 linear independent solutions near any nonsingular point.

If the holomorphic matrix form Ω is formally gauge equivalent to the Euler matrix Ω_0, then the holomorphic Pfaffian system (6.13) admits a formal solution. By Theorem 2.11, this formal solution is automatically convergent. This proves that in the assumptions of Theorem 6.1, Ω and Ω_0 are holomorphically gauge equivalent. □

6.5 Resonant normal form

In the resonant case, when some of the differences $\alpha_i - \alpha_j$ have all residues nonnegative integers, Theorem 6.1 is no longer valid, since not all homological equations are solvable.

In the same way as with the "ordinary" Poincaré–Dulac theorem, one can remove from the expansion of Ω all *non-resonant* terms, leaving only those that correspond to the resonances: if
$$\alpha_i - \alpha_j = \sum_{s=1}^{m} a_s \frac{dt_s}{t_s}, \qquad a_s \in \mathbb{Z}_+, \tag{6.14}$$
then the (i,j)th matrix element of Ω is allowed to contain a linear combination
$$t^{\mathbf{a}}\,\gamma, \qquad \gamma = \sum_{s=1}^{m} c_s \frac{dt_s}{t_s}, \qquad c_s \in \mathbb{C}, \tag{6.15}$$
with any complex coefficients c_s, different from zero if $a_s \geqslant 1$.

The corresponding *resonant normal form*, a full analog of the Poincaré–Dulac normal form known in the "ordinary" case, is obtained by exactly the same arguments as those proving Theorem 6.1. Convergence of the formal gauge transform follows again from Theorem 2.11. Modulo minor technical details (assuming Ω_0 diagonal rather than upper-triangular, which does not change the assertion), this is the main result announced in [YT75].

6.6 Uniqueness of the normalizing gauge transform

Theorem 6.4. *In the assumptions of Theorem 6.1, the only holomorphic gauge transform conjugating the normal form Ω_0 with itself, is a constant diagonal matrix.*

Consequently, the gauge transformation putting Ω into the normal form Ω_0, is uniquely defined by the normalizing condition $H(0) = E$.

Proof. If in (6.4) $\Omega_r = 0$, then the matrix elements of the homogeneous matrix monomial $H_r = \|h_{ij}\|$ satisfy the equations
$$dh_{ij} - h_{ij}(\alpha_i - \alpha_j) = 0, \qquad i \neq j,$$
$$dh_{ii} = 0, \qquad i = 1, \dots, n, \qquad \deg h_{ii} = r.$$

The off-diagonal terms h_{ij}, $i \neq j$, must be zero by Lemma 6.3. The diagonal terms must be constant, which is possible only if $r = 0$. $\square$

7 Polar loci that are not normal crossings

Unlike in the "ordinary" case, in the multivariate case there exists a possibility of locally nontrivial changes of the independent variable, in particular, *blow-ups*. Iteration of such transformations allows to simplify the local structure of the polar locus Σ.

7.1 Blow-up

By a *blow-up* of an analytic set $S \subset U$ of codimension $\geqslant 2$ in an analytic manifold U we broadly mean a holomorphic map $F\colon U' \to U$ between two analytic manifolds, which is biholomorphically invertible outside S and such that the preimage $S' = F^{-1}(S)$ of S is an analytic (eventually singular) hypersurface in U'. The most important example is "compactification" of the map

$$F\colon \mathbb{C}^2 \mapsto \mathbb{C}^2, \qquad F(t_1, s) = (t_1, t_2), \qquad t_2 = st_1. \tag{7.1}$$

Consider the domains ("complex strips")

$$U_1' = \{(t_1, s)\colon |t_1| < \varepsilon,\ s \in \mathbb{C}^1\}, \qquad U_2 = \{(s', t_2)\colon |t_2| < \varepsilon,\ s' \in \mathbb{C}^1\},$$

together with the maps $F_{1,2}$,

$$U_1' \xrightarrow{F_1} (\mathbb{C}^2, 0) \xleftarrow{F_2} U_2', \qquad F_1(t_1, s) = (t_1, st_1), \quad F_2(s', t_2) = (s't_2, t_2).$$

If we identify points (t_1, s) and (s', t_2) of the disjoint union $U_1' \sqcup U_2'$ satisfying the identities $s = t_2/t_1 = 1/s'$, the result will be an (abstract) analytic manifold U' which can be described as the "complex Möbius band". The maps F_1, F_2 induce a well-defined map $F\colon U' \to U = (\mathbb{C}^2, 0)$. This map will be a blow-up of the origin $(0, 0) \in U$: the preimage $S' = F^{-1}(0)$ in each chart will be a (smooth) analytic hypersurface $\{t_1 = 0\}$, resp., $\{t_2 = 0\}$, called the *exceptional divisor*.

Note that complexified Möbius band U' obtained as patching of two "complex strips" $\{|t_i| < \varepsilon\} \times \mathbb{C}$, is not a cylinder (even topologically), and the exceptional divisor *can not* be globally described as the zero locus of a function holomorphic on U'. Indeed, such function when restricted on $U' \smallsetminus S'$ and pushed forward on $U = (\mathbb{C}^2, 0)$, would be holomorphic outside the origin and bounded, hence holomorphic at the origin also. But the zero locus of a holomorphic germ cannot consist of an isolated point. Having in mind these global complications, we will nevertheless proceed by doing computations in only one of the charts U_i', assuming that the map F has the form (7.1).

Remark 7.1. Consider the *tautological line bundle* over $\mathbb{C}P^1$ whose fiber over a point $(t_1 : t_2) \in \mathbb{C}P^1$ is identified with the line $\mathbb{C} \cdot (t_1, t_2) \subset \mathbb{C}^2$ on the plane. Then the blow-up U' can be described in these terms as the neighborhood of the zero section of this bundle. After deleting the zero section itself the total space of this bundle corresponds to the punctured neighborhood $(\mathbb{C}^2, 0) \smallsetminus \{0\}$ equal to the union of all punctured lines $\mathbb{C}^* \cdot (t_1, t_2)$, $\mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$.

The above construction admits a multidimensional analog, whose explicit description we will not use. Importance of these bow-ups is immense. Iterating them, one may, among other things, resolve singularities of any analytic hypersurface to normal crossings.

Theorem 7.2 (Particular case of Hironaka theorem). *For any germ of an analytic hypersurface $\Sigma \subset (\mathbb{C}^m, 0)$ there exists a holomorphic blow-up map $F \colon U' \to (\mathbb{C}^m, 0)$, holomorphically invertible outside an analytic set of codimension $\geqslant 2$ in $\mathbb{C}^m$, such that $F^{-1}(\Sigma)$ is an analytic hypersurface having only normal crossings in U'.* □

Unfortunately, there is no way to predict what the result of the desingularization be globally. Implementation of the known algorithms is very labor-consuming in general. Therefore we will restrict further discussion entirely to the two-dimensional case.

Example 7.3 (continuation of Example 4.5). Blowing up the logarithmic matrix form $\Omega = \sum_{j=1}^{k} A_j \frac{dl_j}{l_j}$ on the plane $(\mathbb{C}^2, 0)$ with constant residues A_j and the linear functions l_j satisfying the assumption $dl_i \wedge dl_j(0) \neq 0$, we obtain a connection form on the Möbius band which is flat provided that the sum $B = \sum_j A_j$ commutes with each A_j. The polar locus of this form consists of the exceptional divisor Σ_0 and k "parallel" lines $\Sigma_1, \dots, \Sigma_k$ normally crossing it. The computation carried out in Example 4.5, shows that the blow-up of Ω is logarithmic, B being the residue along Σ_0. If this blow-up is flat, Theorem 4.3 ensures that $[B, A_j] = 0$. This explains in geometric terms why the sufficient condition of flatness, obtained in Example 4.5, is also necessary.

However, blow-up of a logarithmic form needs not necessarily be logarithmic.

7.2 Blow-up of logarithmic poles may be not logarithmic

The reason why definition of logarithmic poles is *not invariant* by transformations that are not locally biholomorphic, is rather simple. If $F \colon U' \to U$ is a holomorphic map and $\Sigma \subset U$ a (singular) analytic hypersurface defined by an equation $\{f = 0\}$, then the preimage $\Sigma' = F^{-1}(\Sigma)$ is *not* defined by the equation $\{f' = 0\}$, where $f' = F^* f$. More precisely, the function f' may violate our standing assumption that all equations should be locally square-free, as shows the following example.

Example 7.4. Let $f(t_1, t_2) = t_1 t_2 (t_2 - t_1)$ the equation of three lines through the origin. Then $F^* f(t_1, s) = t_1^3 s(s-1)$ defines the union of three lines and the exceptional divisor $\{t_1 = 0\}$, the latter with multiplicity 3 (the third line $\{s = \infty\}$ is visible only in the second chart U_2', see §7.1). Thus the holomorphy of $f'\omega' = F^*(f\,\omega)$ that holds if $\omega \in \Lambda^\bullet(\log \Sigma)$, does not imply that $F^*\omega$ has a first order pole on Σ'.

Indeed, the pullback of the logarithmic form (3.9) on $\mathbb{C}^2$ is (modulo the obvious change of notation)

$$\frac{1}{t_1} \frac{ds}{s(s-1)}$$

which has logarithmic poles on $\{s = 0\}$ and $\{s = 1\}$ but a non-logarithmic pole along $\{t_1 = 0\}$.

7.3 Blow-up of *closed* logarithmic forms is logarithmic

However, if the logarithmic form $\omega \in \Lambda^1(\log \Sigma)$ is closed, then $F^*\omega$ is logarithmic, $F^*\omega \in \Lambda^1(\log \Sigma')$, $\Sigma' = F^{-1}(\Sigma)$.

For a *logarithmic derivative* df/f of a *holomorphic* function this follows from the fact that F^*f is again holomorphic and the discussion in §3.3 explaining that df/f has always a logarithmic residue regardless of the order of vanishing of f on Σ. Note that the residues of logarithmic derivatives of meromorphic functions are *always integer*: $\operatorname{res}_{\Sigma_i}(df/f) \in \mathbb{Z}$ for any smooth component Σ' of the polar locus Σ.

A general closed logarithmic form can be represented by virtue of Lemma 3.9 as $\omega = \sum_j a_j \, df_j/f_j +$ (holomorphic terms), with constant residues $a_j \in \mathbb{C}$. Its pullback is

$$F^*\omega = \sum_j a_j \frac{df_j'}{f_j'} + \text{(holomorphic terms)}, \qquad f_j' = F^* f_j,$$

which immediately means that $\omega' = F^*\omega$ has logarithmic pole. This representation implies also that the residue of ω' on the exceptional divisor Σ_0 is an integer combination of the residues,

$$\operatorname{res}_{\Sigma_0} \omega' = \sum_j \nu_j \, a_j, \qquad \nu_j = \operatorname{res}_{\Sigma_0} \frac{df_j'}{f_j'} \in \mathbb{Z}_+. \tag{7.2}$$

In fact, the above arguments admit generalization for any logarithmic form with *analytic residues*: if $\Sigma = \{f_1 \cdots f_k = 0\}$, f_j being irreducible equations for components of Σ, and $\omega = \sum a_j(t) \frac{df_j}{f_j} +$ (holomorphic terms) with analytic functions $a_j(\cdot)$ defined everywhere in U, then $F^*\omega$ will also have logarithmic poles.

7.4 Solvability of the general ∇-equation

As an example of application of blow-up technique, we can prove the following theorem generalizing Lemma 6.3.

Assume that $U = (\mathbb{C}^m, 0)$ and $\Sigma = \bigcup_1^k \Sigma_j$ is the representation of an analytic hypersurface as a union of irreducible components (among other, this assumption means that the smooth part of Σ lying inside each Σ_j, is locally connected). Consider a closed logarithmic 1-form $\alpha = \sum_1^k a_j \frac{df_j}{f_j}$ and the corresponding ∇-equation

$$\nabla h = \omega, \qquad \nabla = d - \alpha \wedge \cdot, \qquad \omega \in \Lambda^1(U). \tag{7.3}$$

Theorem 7.5. *Assume that $\alpha \in \Lambda^1(\log \Sigma)$ is a closed logarithmic form whose residues $a_j = \operatorname{res}_{\Sigma_j} \alpha \in \mathbb{C}$ on each irreducible component are independent over nonnegative integers: $\sum \nu_j a_j \notin \mathbb{Z}_+$ for any $\nu_1, \dots, \nu_k \in \mathbb{Z}_+$.*

Then the necessary condition $\nabla\omega = 0$ for solvability of the equation (7.3) *is also sufficient, and the holomorphic solution h is unique.*

Proof. Consider the blow-up $F \colon U' \to U$ of Σ such that $\Sigma' = F^{-1}(\Sigma)$ is a hypersurface with normal crossings in U' which is a neighborhood of Σ'.

The pullback $\alpha' = F^*\alpha$ will be again a closed logarithmic form with the residues a_j on the strict transforms $\hat{\Sigma}_j$ of Σ_j. The residues of α' on components of the exceptional divisor are nonnegative integral combinations $\sum \nu_j a_j$, $\nu_j \in \mathbb{Z}_+$, of the residues a_j by (7.2).

In such situation Lemma 6.3 allows to assert that near any point $a \in \Sigma'$ the ∇-equation $dh' - h'\alpha' = \omega'$, $\omega' = F^*\omega$, meeting the necessary condition of solvability $d\omega' - \alpha' \wedge \omega' = 0$, admits a unique holomorphic solution $h'_a \in \Lambda^0(U', a)$. Being unique, all these solutions are patches of a globally defined solution $h' \in \Lambda^0(U')$ in a neighborhood of Σ'.

The holomorphic function $h = (F^{-1})^* h'$ can be pushed forward on $(\mathbb{C}^m, 0)$ everywhere outside the critical locus S of the blow-up F. Since S is thin, h extends as a holomorphic solution of the initial ∇-equation (7.3). $\square$

7.5 Example: normalization on the union of pairwise transversal curves

Application of one or several bow-ups allows to reduce many questions about flat logarithmic connections with arbitrary polar loci to those with normal crossings. As an instructive example, consider the following situation.

Assume that the polar locus of a flat meromorphic connection Ω is a finite union of smooth pairwise transversal curves $\Sigma_j = \{f_j = 0\}$, $j = 1, \dots, k$, passing through the origin in $(\mathbb{C}^2, 0)$:

$$df_i \wedge df_j(0) \neq 0.$$

Assume that Ω has logarithmic pole on $\Sigma = \bigcup_{j=1}^k \Sigma_j$.

Theorem 7.6. *If the residues $A_j = \operatorname{res}_{\Sigma_j} \Omega$ are bounded on Σ_j near the origin and no two eigenvalues of the sum $A_0 = \sum A_j(0)$ differ by an integer number, then the form Ω is holomorphically conjugate to a form "with constant coefficients"*

$$\Omega_0 = \sum_{j=1}^k A_j(0)\,\frac{df_j}{f_j}.$$

Remark 7.7. The existence of the limits $A_j(0) = \lim_{t\to 0,\ t\in\Sigma_j} A_j(t)$ and hence their sum $A_0 = \sum A_j(0)$ follow from the removable singularity theorem.

Proof. First we notice that the residue matrix functions, holomorphic on $\Sigma_j \setminus \{0\}$ and bounded, are forced to remain holomorphic on Σ_j and can be extended as holomorphic functions on $(\mathbb{C}^2, 0)$. This means that

$$\Omega = \sum_1^k A_j(t)\,\frac{df_j}{f_j} + (\text{holomorphic terms}).$$

Consider the standard blow-up $F\colon U' \to (\mathbb{C}^2, 0)$ described in §7.1. Since $df_j(0) \neq 0$, $f'_j = F^* f_j$ is a holomorphic function that vanishes on the union $\Sigma'_j = \Sigma_0 \cup \hat\Sigma_j$, where Σ_0 is the exceptional divisor and $\hat\Sigma_j$ the *strict preimage* of Σ_j, a smooth curve transversal to Σ_0, obtained as the closure of the preimage $F^{-1}(\Sigma_j \setminus \{0\})$. The logarithmic derivative df_j/f_j has residues equal to 1 on both Σ_0 and $\hat\Sigma_j$.

The pullback $\Omega' = F^*\Omega$ is another matrix form, also flat and having logarithmic poles on Σ' which now has only normal crossings. The residues of $\Omega' = F^*\Omega$ on $\hat\Sigma_j$ coincide with the pullback $F^* A_j$, while the residue on Σ_0 is equal to the constant matrix $A_0 = \sum A_j(0)$. By our assumption, A_0 is diagonal(izable), and by Theorem 4.4, after a suitable holomorphic gauge transform the residues A_j can also be assumed constant (equal to their limit values

$A_j(0)$). Commuting with the diagonal matrix A_0 with distinct eigenvalues, all $A_j(0)$ are also diagonal.

The "complex Möbius band" U' can be covered by the union of local charts U'_s such that in each chart the preimage $F^{-1}(\Sigma)$ is the union of one or more coordinate hyperplanes. By Theorem 6.1, the connection form Ω' is holomorphically gauge equivalent to *diagonal* flat logarithmic connection Ω'_s. The corresponding local gauge transforms H_s, being uniquely defined by the normalizing condition $H'_s|_{\Sigma'} = E$ (Theorem 6.4), coincide on the intersections of the charts and hence define a globally defined on U' holomorphic gauge transformation $H' \colon U' \to \mathrm{GL}(n, \mathbb{C})$. This transformation descends as a holomorphic holomorphically invertible matrix function $H \colon (\mathbb{C}^2, 0) \smallsetminus \{0\} \to \mathrm{GL}(n, \mathbb{C})$ by F and, since the origin has codimension 2 on the plane, H extends as a holomorphic gauge equivalence on the entire neighborhood of the origin.

By construction, action of H on Ω is a diagonal flat logarithmic connection, $\Omega = \mathrm{diag}\{\alpha_1, \dots, \alpha_n\}$, which means that each $\alpha_j \in \Lambda^1(\log \Sigma)$ is a closed logarithmic form. By Lemma 3.9, this completes the proof. $\square$

A very similar theorem is proved by different arguments by K. Takano [Tak79]. He deals with logarithmic connections on $(\mathbb{C}^m, 0)$ with any $m \geqslant 2$ rather than with two-dimensional case, but imposes more stringent conditions on the smooth components Σ_j that have to be linear hyperplanes. Actually, the arguments given above can be modified to prove also the Takano theorem in its original formulation: the knowledge that Σ_j are hyperplanes ensures that the complete desingularization of Σ to normal crossings, has especially simple structure.

7.6 Concluding remarks

Theorem 7.6 apparently admits generalizations for other types of polar loci, different from normal crossings, also in more dimensions $m > 2$. However, the assumption that the residues are holomorphic, is crucial and in general does not hold for an arbitrary flat connection with logarithmic poles only. An interesting problem is to develop holomorphic and meromorphic gauge classification of singularities with non-holomorphic (either locally unbounded or bounded but not extendible holomorphically from singular components Σ_j onto $(\mathbb{C}^m, 0)$) residues. One can list here two problems that, if solved, would advance considerably several problems, in particular related to the infinitesimal Hilbert problem (see the Introduction).

Problem 7.8. For a given germ of an analytic hypersurface $\Sigma \subset (\mathbb{C}^m, 0)$, find necessary and sufficient conditions on the linear representation $M. \colon \pi_1((\mathbb{C}^m, 0) \smallsetminus \Sigma) \to \mathrm{GL}(n, \mathbb{C})$ to be realizable as the monodromy group of a flat connection having only logarithmic poles on Σ.

This local problem is a counterpart of the Hilbert 21st problem. Quite surprisingly, its global analog is nontrivial already when Σ is a union of two smooth hypersurfaces with normal crossings in $U = \mathbb{C}P^2$ and $n = 1$ (A. Bolibruch, [Bol80]).

Problem 7.9. When the germ Ω of a flat connection with logarithmic poles on Σ is meromorphically gauge equivalent to a flat logarithmic connection Ω' having locally bounded residues, eventually on a larger hypersurface Σ'?

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