MULTIPLICITY OPERATORS

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Abstract. For functions of a single complex variable, zeros of multiplicity greater than k are characterized by the vanishing of the first k derivatives. There are various quantitative generalizations of this statement, showing that for functions that are in some sense close to having a zero of multiplicity greater than k, the first k derivatives must be small.

In this paper we aim to generalize this situation to the multi-dimensional setting. We define a class of differential operators, the *multiplicity operators*, which act on maps from $\mathbb{C}^n$ to $\mathbb{C}^n$ and satisfy properties analogous to those described above. We demonstrate the usefulness of the construction by applying it to some problems in the theory of Noetherian functions.

1. Introduction

The derivative operator is an invaluable tool in the study of zeros of functions of a single variable. In the real setting, the Rolle theorem roughly states that if a function admits k zeros on an interval, its derivative admits at least $k-1$ zeros. This statement and its numerous ramifications have found applications in the qualitative theory of ordinary differential equations (see for instance the survey [15] and references therein), and in the theory of Fewnomials and Pfaffian functions (see the monograph [11] and the paper [5]).

In the complex setting, the literal analog of the Rolle theorem fails. However, certain local analogs still hold and can be used to study the zeros of holomorphic functions (in one variable). The goal of this paper is to provide multi-dimensional analogs of the derivative operator in this context, and to illustrate their application in the study of Noetherian systems of equations, their multiplicities and the number of their solutions. In the remainder of this section we review some of the classical one-dimensional results in this spirit and reference results in the body of the paper which generalize them to several variables.

We fix some notations used throughout the paper. We let D_r denote the closed complex disc of radius r and D_r^n (resp. B_r^n) denote the corresponding polydisc (resp. ball) in $\mathbb{C}^n$. When r is omitted it is assumed that $r=1$. For a complex domain U (in any dimension) we let $\mathcal{O}(U)$ denote the space of analytic functions on U, and $\|\cdot\|$ the maximum norm on $\mathcal{O}(U)$. Finally, we let $\mathcal{O}^n(U)$ denote the space of analytic maps into $\mathbb{C}^n$, and $\|\cdot\|$ the direct sum norm.

A function of a single complex variable admits a zero of multiplicity greater than k if and only if its derivatives vanish up to order k. It is natural to expect that a quantitative generalization of this statement also holds – namely, that if a function

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is "close" to having a zero of multiplicity greater than k then its derivatives up to order k must be small. Below we give two quantitative statements of this form.

Theorem (I)**.** *Let $f \in \mathcal{O}(D)$ and assume that $\|f\| \leqslant 1$. There exists an absolute positive constant $C^Z_{1,k}$ such that if f has more than k zeros in D_r then*

$$C^Z_{1,k}\,|f^{(k)}(0)| < r \tag{1}$$

This theorem shows that if a function has a k-th derivative of size s then the function cannot have more than k zeros in a disc whose radius is proportional to s. It therefore gives control on the number of zeros of a function in terms of its derivatives. A generalization of this theorem to several variables is given in Theorem 1.

Theorem (II)**.** *Let $f \in \mathcal{O}(D)$ and assume that $\|f\| \leqslant 1$. There exist absolute positive constants $A_{1,k}, B_{1,k}$ with the following property:*

For every $r < s = |f^{(k)}(0)|$ there exists $A_{1,k}r < \tilde{r} < r$ such that

$$|f(z)| \geqslant B_{1,k} s \tilde{r}^k \text{ for every } |z| = \tilde{r} \tag{2}$$

This theorem shows that if a function has a k-th derivative of size s then up to a constant, the function grows at least like sr^k on spheres of radius $r < s$. This is not true for every such sphere (for instance, the function might have zeros on some of them), but it is true if one is allowed to move the sphere within some constant multiplicative range. A generalization of this theorem to several variables is given in Theorem 2.

In conjunction with Rouché's theorem, this theorem allows to conclude that a perturbation of size $\varepsilon < s$ doesn't change the number of zeros of f in a disk of radius $(\varepsilon/s)^{1/k}$ (up to a multiplicative constant). A precise statement of this type is given in Corollary 14.

The derivative operator also has a controlled decreasing effect on the multiplicities of functions and ideals. We summarize various simple formulations of this fact below.

Theorem (III)**.** *The following statements hold:*

1. *Let $f \in \mathbb{C}((z))$. Then* $\operatorname{mult}_0 f^{(k)} \geqslant \operatorname{mult}_0 f - k$.
2. *Let f be a real analytic function on* $[0,1]$. *Then* $\operatorname{ord}_0 f^{(k)} \geqslant \operatorname{ord}_0 f - k$.
3. *Let $I \subset \mathbb{C}((z))$ be an ideal. Denote by $I^{(k)}$ the ideal generated by derivatives up to order k of elements of I. Then $t^k I^{(k)} \subset I$ and* $\operatorname{mult} I^{(k)} \geqslant \operatorname{mult} I - k$.

Various generalizations of these statements to several variables are given in Theorems 3 and 5 and in Corollary 18.

2. Multiplicity Operators

Let $R_n = \mathbb{C}[[x_1, \ldots, x_n]]$ denote the ring of formal power series in n variables, and $f_1, \ldots, f_n \in R_n$. Denote $F = (f_1, \ldots, f_n)$. We denote by $I(F)$ the ideal generated by $f_1, \ldots, f_n$. Recall that the multiplicity of the common zero $f_1 = \ldots = f_n = 0$, denoted $\operatorname{mult} F$, is defined to be $\dim_{\mathbb{C}} R_n / I(F)$.

Since our estimates involve many constants, we use the convention that constants appearing with a numeric superscript, e.g. C^1, may appear in different proofs to denote different numbers. Constants with non-numeric superscripts are meant to be defined throughout the paper.

Let k be a non-negative integer. Let $\mathfrak{m}$ denote the maximal ideal in R_n, let $J_{n,k} = R_n/\mathfrak{m}^{k+1}$ denote the ring of k-jets in n variables, and let $j : R_n \to J_{n,k}$ denote the canonical homomorphism. We also fix an arbitrary norm, for instance the l_1 norm, on the space of polynomials.

We are interested in writing down differential-algebraic conditions on F that are satisfied if and only if $\operatorname{mult} F > k$. We begin by recalling the following standard lemma.

Lemma 1. *We have* $\operatorname{mult} F > k$ *if and only if* $\dim_{\mathbb{C}} J_{n,k}/j(I(F)) > k$

Proof. First, since

$$J_{n,k}/j(I(F)) \simeq R_n/(I(F) + \mathfrak{m}^{k+1}) \tag{3}$$

it is clear that $\dim_{\mathbb{C}} J_{n,k}/j(I(F)) > k$ implies $\operatorname{mult} F > k$. Conversely, suppose that $\dim_{\mathbb{C}} J_{n,k}/j(I(F)) \leqslant k$. Then it known that $j(\mathfrak{m}^k) \subset j(I(F))$ (see, for instance, [1, Lemma, Section 5.5]). Since R_n is complete with respect to the $\mathfrak{m}$-filtration, it follows from [3, Proposition 7.12] that $\mathfrak{m}^k \subset I(F)$. Then it follows from (3) that $\dim_{\mathbb{C}} J_{n,k}/j(I(F)) = \operatorname{mult} F$. □

2.1. **Construction of the multiplicity operators.** For $\mathbf{i} \in \mathbb{N}^n$ let $x^{\mathbf{i}}$ denote the corresponding monomial in R_n. For any co-ideal $B \subset \mathbb{N}^n$ we call the set $x^B := \{x^{\mathbf{b}} : b \in B\}$ a standard monomial set. We recall another standard fact from Gröbner base theory.

Lemma 2. *For any ideal $I \subset J_{n,k}$, the ring $J_{n,k}/I$ has a standard monomial base.*

Proof. Let $\operatorname{LT}(I) \subset \mathbb{N}^n$ denote the ideal generated by monomials which are leading terms of members of I, with respect to some monomial ordering. Then the monomials outside of $\operatorname{LT}(I)$ form a base for the ring $J_{n,k}/I$. □

A standard monomial set x^B of size k spans $J_{n,k}/j(I(F))$ if and only if $J_{n,k} = \mathbb{C}\left\langle x^B \right\rangle + j(I(F))$. We encode this information in the linear map

$$T_B^F : \mathbb{C}^{|B|} \oplus J_{n,k}^{\oplus n} \to J_{n,k} \qquad (c_{\mathbf{i}})_{\mathbf{i}\in B} \oplus u_1 \oplus \ldots \oplus u_n \to \sum_{\mathbf{i}\in B} c_{\mathbf{i}} x^{\mathbf{i}} + \sum_i u_i j(f_i). \tag{4}$$

For explicitness, we represent T_B^F as a matrix with respect to the basis consisting of $c_{\mathbf{i}}, \mathbf{i} \in B$ followed by the basis of monomials for $J_{n,k}^{\oplus n}$. The set x^B spans $J_{n,k}/j(I(F))$ if and only if one of the maximal minors containing the first $|S|$ columns is non-zero.

The entries of the first k columns of T_B^F are constants (in fact, each has one component 1 and zero elsewhere). For the remaining columns, each component is a Taylor coefficient of the map F of order up to k. We thus view minors of the matrix as differential operators of order bounded by k.

Definition 3. *Let $\{M_B^\alpha\}$ be the set of maximal minors of T_B^F which contain the first $|B| = k$ columns. We call each M_B^α a* basic multiplicity operator *of order k. We call any operator $M^{(k)}$ in the convex hull of this set a* multiplicity operator *of order k.*

Each multiplicity operator $M^{(k)}$ is a differential operator of order k of homogeneity degree $\dim J_{n,k} - k = \binom{n+k}{k} - k$. Therefore when studying the behavior of multiplicity operators on analytic maps $F \in \mathcal{O}^n(D^n)$, there will be no loss of generality in assuming $\|F\| \leqslant 1$, and we will usually restrict to this case.

Example 4. *For $F:(\mathbb{C},0)\to(\mathbb{C},0)$, $F(x)=\eta x+x^2$ the multiplicity operators are $M^\alpha_{\{1\}}=\eta$ and $M^\alpha_{\{1,x\}}=1$ for $x^B=\{1\}$ and $x^B=\{1,x\}$ correspondingly.*

To simplify the notations, we denote by $M^{(k)}_p$ the differential functional obtained by applying $M^{(k)}$, followed by evaluation at the point p.

The following proposition, a direct conclusion of the discussion above, is the basic property of the multiplicity operators.

Proposition 5. *We have* $\operatorname{mult} F_p > k$ *if and only if $M^{(k)}_p(F)=0$ for all multiplicity operators of order k.*

2.2. **Effective decomposition in the local algebra.** The following proposition gives the main technical property of the multiplicity operators that will be used throughout this paper.

Proposition 6. *Let $F\in\mathcal{O}^n(D^n)$ with $\|F\|\leqslant 1$. Let M^α_B be a basic multiplicity operator of order k. Assume $s=|M^\alpha_B(F)(0)|>0$. Then for any $P\in J_{n,k}$ we have a decomposition*

$$P=\sum_{\mathbf{i}\in B}c_{\mathbf{i}}x^{\mathbf{i}}+\sum U_i f_i+E,\quad E\in\mathfrak{m}^{k+1} \tag{5}$$

with

$$\max(|c_{\mathbf{i}}|,\|U_i\|,\|E\|)\leqslant C^D_{n,k}s^{-1}\|P\|,$$

where $C^D_{n,k}$ is some universal constant.

Proof. The existence of such a decomposition with the corresponding estimates on $|c_{\mathbf{i}}|$ and $\|U_i\|$ is a direct consequence of Cramer's rule applied to the maximal minor of the matrix T^F_B corresponding to M^α_B (this gives (5) without the E term in the ring $J_{n,k}$).

Since $\|F\|\leqslant 1$, the Taylor coefficients of order k of F are bounded by some universal constant (for instance by the Cauchy formula). Thus we may also assume that s is bounded by a universal constant independent of F. The estimate on $\|E\|$ then follows, with an appropriate universal constant $C^D_{n,k}$, from

$$E=P-\sum_{\mathbf{i}\in B}c_{\mathbf{i}}x^{\mathbf{i}}-\sum U_i f_i \tag{6}$$

using the triangle inequality. □

The following is an important corollary of Proposition 6.

Lemma 7. *Let $F\in\mathcal{O}^n(D^n)$ with $\|F\|\leqslant 1$. Suppose a k-th multiplicity operator $M^{(k)}_0(F)$ is nonzero. Let $p_0,...,p_k\in J_{n,k}$ be some linearly independent polynomials of unit norm. Then there exists their linear combination $P=\sum c_ip_i$ of degree k such that $\sum|c_i|=1$ and*

$$P=\sum_i U_i f_i+E$$

where

$$\max(\|U_i\|,\|E\|)<C^D_{n,k}\,|M^{(k)}_0(F)|^{-1}\quad\text{and } E\in\mathfrak{m}^{k+1}.$$

Proof. It is clearly sufficient to prove the statement for every basic multiplicity operator $M^{(k)}=M^\alpha_B$.

According to Proposition 6 we have the following decompositions:

$$p_j = \sum_{\mathbf{i}\in B} c^j_{\mathbf{i}} x^{\mathbf{i}} + \sum U^j_i f_i + E_j, \quad j = 0, ..., k,$$

with $\left\|U^j_i\right\| \leqslant C^D_{n,k} \left|M^{(k)}_0(F)\right|^{-1}$, $\|E_j\| \leqslant C^D_{n,k} \left|M^{(k)}_0(F)\right|^{-1}$ and $E_j \in \mathfrak{m}^{k+1}$.

Since $|B| = k$, there is a non-trivial linear combination of these equations for which all of the $x^{\mathbf{i}}$ terms vanish. Normalizing the coefficients to $\sum |c_i| = 1$, we obtain the required decomposition and estimates. □

Remark 8. *Note that, in general, even if $g \in I(F)$ one cannot claim that there is a decomposition $g = \sum u_i f_i$ with bounds as in Lemma 7. E.g. for $F = \eta x + x^2$ one has $M^\alpha_{\{1,x\}}(F) = 1$, but for $g = x$ the coefficients of the decomposition $g = (\eta^{-1} - \eta^{-2}x)F$ explode as $\eta \to 0$, whereas the coefficients of the decomposition $g = 0 \cdot 1 + 1 \cdot x + 0 \cdot F$ of Lemma 6 remain bounded.*

When $M^\alpha_B(F)(0) \neq 0$, Proposition 6 gives estimates on the corresponding decompositions in the ring of k-jets $J_{n,k}$. However, by Lemma 1 in is natural to expect that similar decompositions exist in the local ring of germs of holomorphic functions. Our next goal is the corresponding quantitative result.

2.3. Effective division by F with remainder in x^B.

Let $R_{n,t}$ be the linear subspace of R_n consisting of series with finite $\| \, \|_t$ norm, with $\|\sum c_\alpha x^\alpha\|_t = \sum t^{|\alpha|}|c_\alpha|$.

Note that if $f \in \mathcal{O}(D^n)$ and $\|f\| \leqslant 1$ then absolute values of all Taylor coefficients of f are bounded by 1, and therefore $\|f\|_{1/2} \leqslant 2^n$. Therefore, if $j^k(f) = 0$ then

$$\|f\|_t \leqslant (2t)^{k+1}\|f\|_{1/2} \leqslant t^{k+1}2^{k+n+1}\|f\| \tag{7}$$

for $0 < t < 1/2$.

Lemma 9. *Let $F \in \mathcal{O}^n(D^n)$ with $\|F\| \leqslant 1$. Assume $s = |M^\delta_B(F)(0)| > 0$.*

There exist positive universal constants $\epsilon_{n,k}, \epsilon'_{n,k}, C_{n,k} > 0$ such that for some t, $\epsilon'_{n,k}s \leqslant t \leqslant \epsilon_{n,k}s$, and for every $P \in R_{n,t}$ there exists a decomposition

$$P = \sum_{i=1}^n u_i f_i + \sum_{\mathbf{i}\in B} c_{\mathbf{i}} x^{\mathbf{i}}, \tag{8}$$

with

$$\sum \|u_i\|_t + \|\sum_{\mathbf{i}\in B} c_{\mathbf{i}} x^{\mathbf{i}}\|_t \leqslant C_{n,k}s^{-k-1}\|P\|_t. \tag{9}$$

The strategy of the proof is as follows. Let $A > 2$ be fixed (eventually $A = 3$).

We begin by showing that for some t in the prescribed range, and for every monomial $x^\alpha, |\alpha| = k$, there exists a decomposition

$$x^\alpha = P_\alpha(x) + \sum u_{i,\alpha} f_i + E'_\alpha, \qquad \deg P_\alpha(x) < k, \quad E'_\alpha \in \mathfrak{m}^{k+1} \tag{10}$$

with

$$\|P_\alpha\|_t + \|E'_\alpha\|_t < A^{-1}\|x^\alpha\|_t. \tag{11}$$

By multiplying this decomposition by an arbitrary monomial and applying (5) we get for any monomial $x^\alpha \in \mathfrak{m}^{k+1}$, and, by linearity, for any $f \in \mathfrak{m}^{k+1}_t$ the decomposition

$$f = \pi_B(f) + U(f) \cdot (f_1, ..., f_n)^T + E(f), \tag{12}$$

where

$$\pi_B : \mathfrak{m}_t^{k+1} \to \bigoplus_{\mathbf{i}\in B} \mathbb{C}x^{\mathbf{i}}, \quad U : \mathfrak{m}_t^{k+1} \to R_{n,t}^{\oplus n}, \quad E : \mathfrak{m}_t^{k+1} \to \mathfrak{m}_t^{k+1} \tag{13}$$

are some explicitly bounded linear operators, and the "remainder operator" E has norm $2A^{-1} < 1$. Therefore

$$f = \pi_B(1-E)^{-1}(f) + U(1-E)^{-1}(f) \cdot (f_1, ..., f_n)^T, \tag{14}$$

which is exactly (8) for elements of $\mathfrak{m}^{k+1}$. The general case then follows easily from Proposition 6.

Proof. Let $\epsilon_{n,k}^{-1} = 2^{n+k+1}C_{n,k}^D$ and let $C_{n,k}(A)^{-1} = (2A+1)^{\binom{n+k-1}{k}2(k+1)}(k+1)$.

Lemma 10. *For any $A > 1$ there exists some t,*

$$C_{n,k}(A)\epsilon_{n,k}s \leqslant t \leqslant \epsilon_{n,k}s, \tag{15}$$

such that any monomial x^α, $|\alpha| = k$ can be represented as

$$x^\alpha = P_\alpha(x) + \sum u_{i,\alpha} f_i + E'_\alpha, \qquad \deg P_\alpha(x) < k, \quad E'_\alpha \in \mathfrak{m}^{k+1}. \tag{16}$$

with

$$\|P_\alpha\|_t + \|E'_\alpha\|_t < A^{-1}\|x^\alpha\|_t, \quad \|u_{i,\alpha}\|_t \leqslant 2C_{n,k}^D s^{-1} t^{-k} \|x^\alpha\|_t. \tag{17}$$

Proof. Let $\{x^{\alpha_i}\}_{i=0}^k$ be a chain of length $k+1$ of monomials ending with x^α, such that each monomial is divisible by the previous one (so $\deg x^{\alpha_i} = i$).

By Lemma 7 there exists some linear combination $\sum_{i=0}^k c_{\alpha,i}x^{\alpha_i}$ with $\sum |c_i| = 1$ such that

$$\sum_{i=0}^k c_{\alpha,i}x^{\alpha_i} = \sum u_{\alpha,i} f_i + E_\alpha, \quad E_\alpha \in m^{k+1}, \ \deg u_{\alpha,i} \leqslant k \tag{18}$$

and

$$\max_{i,\alpha} \|u_{\alpha,i}\| \leqslant C_{n,k}^D s^{-1}, \quad \|E_\alpha\| \leqslant C_{n,k}^D s^{-1}. \tag{19}$$

Here the norm is the supremum norm on the unit polydisc.

Apply Lemma 12 below to the set of the sequences $\{|c_{\alpha,0}|, ..., |c_{\alpha,k}|, 2^{n+k+1}\|E_\alpha\|\}$, with $M = 2^{n+k+1}C_{n,k}^D s^{-1}$ and $t_0 = \epsilon_{n,k}s$. As

$$M(k+1) > (\epsilon_{n,k}s)^{-1} \tag{20}$$

we conclude existence of some t satisfying (15) and some $j(\alpha)$ such that

$$\begin{aligned} \|c_{\alpha,j(\alpha)}x^{\alpha_{j(\alpha)}}\|_t &\geqslant A\left[\|\sum_{i\neq j} c_{\alpha,i}x^{\alpha_i}\|_t + t^{k+1}2^{k+n+1}\|E_\alpha\|\right] \\ &\geqslant A\left[\|\sum_{i\neq j} c_{\alpha,i}x^{\alpha_i}\|_t + \|E_\alpha\|_t\right], \end{aligned} \tag{21}$$

and

$$\begin{aligned} \|u_{\alpha,i}\|_t \leqslant \|u_{\alpha,i}\|_1 \leqslant C_{n,k}^D s^{-1} \|\sum c_{\alpha,i}x^{\alpha_i}\|_1 \leqslant C_{n,k}^D s^{-1}t^{-k}\|\sum c_{\alpha,i}x^{\alpha_i}\|_t \\ \leqslant C_{n,k}^D t^{-k}s^{-1}(1+A^{-1})\|c_{\alpha,j(\alpha)}x^{\alpha_{j(\alpha)}}\|_t. \end{aligned} \tag{22}$$

Dividing by $c_{\alpha,j(\alpha)}$ and multiplying by $x^{\alpha-\alpha_{j(\alpha)}}$ we get the required decomposition. □

Lemma 11. *For any monomial $x^\alpha \in \mathfrak{m}^{k+1}$ there exists a representation*

$$x^\alpha = \sum_{\mathbf{i}\in B} c_{\mathbf{i},\alpha} x^{\mathbf{i}} + \sum \tilde{u}_{i,\alpha} f_i + \tilde{E}_\alpha, \quad E_\alpha \in \mathfrak{m}^{k+1} \tag{23}$$

with

$$\|\tilde{E}_\alpha\|_t \leqslant 2A^{-1}\|x^\alpha\|_t, \tag{24}$$

$$\|\sum_{\mathbf{i}\in B} c_{\mathbf{i},\alpha} x^{\mathbf{i}}\|_t \leqslant (k+1)C^D_{n,k}C_{n,k}(A)^{-k}s^{-k-1}A^{-1}\|x^\alpha\|_t, \tag{25}$$

$$\|\tilde{u}_{i,\alpha}\|_t \leqslant 2^{n+1}C^D_{n,k}C_{n,k}(A)^{-k}s^{-k-1}\|x^\alpha\|_t, \tag{26}$$

where $A > 2$ and t is as in Lemma 10.

Proof. Multiplying (16) by a monomial x^β, we get a similar representation for $x^{\alpha+\beta}$. If $|\beta| \geqslant k+1$ then the product $x^\beta\,(P_\alpha(x) + E'_\alpha)$ is already in $\mathfrak{m}^{k+1}$ and the bounds follow trivially as soon as $A > 2$.

If, however, $|\beta| \leqslant k$, then some monomials from the product could have degree smaller than $k+1$. Denote the sum of these monomials by R. Evidently, $\deg R \leqslant k$ and $\|R\|_t < A^{-1}\|x^\alpha\|_t$. Using Proposition 6, one can write

$$R = \sum_{\mathbf{i}\in B} c_{\mathbf{i}} x^{\mathbf{i}} + \sum U_i f_i + E, \quad E \in \mathfrak{m}^{k+1}. \tag{27}$$

We now express the bounds of Proposition 6 in terms of $\|\cdot\|_t$ norms. We have

$$\|\sum_{\mathbf{i}\in B} c_{\mathbf{i}} x^{\mathbf{i}}\|_t \leqslant k\|\sum_{\mathbf{i}\in B} c_{\mathbf{i}} x^{\mathbf{i}}\| \leqslant kC^D_{n,k}s^{-1}\|R\| \leqslant kC^D_{n,k}t^{-k}s^{-1}\|R\|_t, \tag{28}$$

$$\|U_i\|_t \leqslant 2^n\|U_i\| \leqslant 2^nC^D_{n,k}t^{-k}s^{-1}\|R\|_t, \tag{29}$$

and

$$\begin{aligned}\|E\|_t &\leqslant t^{k+1}2^{k+n+1}\|E\| \leqslant C^D_{n,k}2^{k+n+1}t^{k+1}s^{-1}\|R\| \\ &\leqslant C^D_{n,k}2^{k+n+1}ts^{-1}\|R\|_t.\end{aligned} \tag{30}$$

Together with bounds (15) and (17) of Lemma 10 this ends the proof. □

Extending (23) by linearity to $\mathfrak{m}^k$, we get the operator identity (12) with operators π_B, U bounded. Choosing $A = 3$ and $\epsilon'_{n,k} = C_{n,k}(3)\epsilon_{n,k}$, we get $\|E\| \leqslant 2/3$ and $\|(1-E)^{-1}\| \leqslant 3$, which implies (14) for $f \in \mathfrak{m}^{k+1}$.

For arbitrary $f \in R_{n,t}$, we first decompose its k-th jet $j^k(f)$ as in Proposition 6, and apply (14) to the remainder E. One can see as above that $\|E\|_t \leqslant C'_{n,k}\|f\|_t$ for some absolute constant $C'_{n,k}$, and the result follows. □

2.4. Lemma à la Valiron.

Lemma 12. *Let $\vec{a^j} = (a_{j,0}, ..., a_{j,k+1})$ be N sequences of non-negative numbers satisfying*

$$a_{j,0} + ... + a_{j,k} = 1, \quad a_{j,k+1} \leqslant M. \tag{31}$$

For any $A > 1, t_0 > 0$ there exist $t \in (0, t_0]$, and $i(j), 0 \leqslant i(j) \leqslant k$ for $j = 1, ..., N$ such that

$$t^{i(j)}a_{i(j)} \geqslant A\sum_{i\neq i(j)} t^i a_{j,i} \tag{32}$$

and

$$t \geqslant (2A+1)^{-2N(k+1)}\min\left[t_0, (M(k+1))^{-1}\right]. \tag{33}$$

Proof. Let $B = 2A + 1$. Consider the functions $\tilde{\phi_j}(i) = \log_B a_{j,i}$ and let $\phi_j(i)$ be their smallest concave majorates, $j = 1, ..., N$. From the bounds on $a_{j,i}$ we conclude that

$$\min_i \frac{\phi_j(k+1) - \phi_j(i)}{k - i} \leqslant \log_B (M(k+1)).$$

Note that replacing a_i by $t^i a_i$ adds the linear function $i \log_B t$ to ϕ_j. It is enough to find t such that the graphs of the piece-wise linear concave functions $\phi_j(i) + i \log_B t$ have maximum point $i(j)$ not at $k+1$ and have edges with absolute values of edge slopes at least 1. Indeed, this slope condition means that $t^i a_i \leqslant B^{-|i-i(j)|} t^{i(j)} a_{i(j)}$, where j is the maximum point of $\phi + k \log_B t$, and (32) follows by the sum of geometric progression formula.

In other words, we have to find some t such that, first, $\log_B t$ lies outside of 1-neighborhood of $-\Delta_{ji}$, where $\Delta_{ji} = \phi_j(i+1) - \phi_j(i)$ are slopes of the edges of all graphs of $\phi_j(i)$, and, second, $\log_B t$ is smaller than both $\log_B t_0$ and $-\Delta_{j,k}$ for all $j = 1, ..., N$. But this 1-neighborhood has length at most $2N(k+1)$, and one can see from (31) that

$$-\Delta_{j,k} \geqslant -\log_B (M(k+1)). \tag{34}$$

so one can find the required t with

$$\log_B t \geqslant \min(\log_B t_0, -\log_B (M(k+1))) - 2N(k+1)$$

and (33) follows. □

3. Multiplicity operators, growth and zeros

In this section we collect several theorems showing how the multiplicity operators can be used to control the number of zeros and the asymptotic growth of a map F in small polydiscs or balls.

3.1. **Number of zeros in a small polydisc.**

The following is a simple consequence of Lemma 9.

Theorem 1. *Let $F \in \mathcal{O}^n(D^n)$ with $\|F\| \leqslant 1$. Assume F has $k+1$ zeros (counted with multiplicities) in the polydisc D^n_r. Then for every k-th multiplicity operator $M^{(k)}$,*

$$C^Z_{n,k} \, |M^{(k)}_0(F)| \leqslant r. \tag{35}$$

where $C^Z_{n,k} > 0$ is a universal constant.

Proof. It is clearly sufficient to prove the statement for every basic multiplicity operator M^α_B. Let $C^Z_{n,k} = \epsilon'_{n,k}$.

Suppose that $C^Z_{n,k} s > r$. Making an arbitrarily small perturbation, we may also assume that F admits $k+1$ *distinct* zeros in D^n_r.

Applying Lemma 9 we obtain $t > r$ such that the image of the restriction map $\mathcal{O}(D^n) \to \mathcal{O}(D^n_r)/\langle F \rangle$ is at most k-dimensional (and generated by x^B). However, since F admits $k+1$ distinct zeros in D^n_r, the restrictions of the polynomial functions already span a $k+1$ dimensional subspace of $\mathcal{O}(D^n_r)/\langle F \rangle$, leading to a contradiction. □

3.2. **Growth in small balls.** We begin this section with a simple result giving a lower bound for univariate polynomials of unit norm. The following simple lemma appears as [2, Lemma 7].

Lemma 13. *Let $P = \sum_{i=0}^{k} c_i z^i$ be a polynomial of degree d and $\|P\|_1 = \sum |c_i| = 1$. Let $Z = \{P = 0\}$ denote the zero-locus of P. Then for any $z \in D$,*

$$|P(z)| \geqslant C_d \operatorname{dist}(z, Z)^d \tag{36}$$

where C_d is some universal constant.

The following result is a local multidimensional analog of this lemma, stated in terms of multiplicity operators.

Theorem 2. *Let $F \in \mathcal{O}^n(D^n)$ with $\|F\| \leqslant 1$. Let $M^{(k)}$ be a multiplicity operator of order k. Assume $s = |M_0^{(k)}(F)| \neq 0$. There exist positive universal constants $A_{n,k}, B_{n,k}$ with the following property:*

For every $r < s$ there exists $A_{n,k} r < \tilde{r} < r$ such that

$$\|F(z)\| \geqslant B_{n,k} s \tilde{r}^k \text{ for every } \|z\| = \tilde{r} \tag{37}$$

Proof. Let ℓ denote one of the standard coordinates on $\mathbb{C}^n$. According to Lemma 7 there exists a polynomial P, with $\|P\|_1 = 1$, such that

$$P(\ell) = \sum_i U_i f_i + E \tag{38}$$

where

$$\max(\|U_i\|, \|E\|) < C_{n,k}^D s^{-1} \text{ and } E = O(|z|^{k+1}). \tag{39}$$

Let $\rho = C_{n,k}^1 r$ and $\tilde{\rho} = C_{n,k}^2 r$ where the universal constants $C_{n,k}^2 < C_{n,k}^1 < 1$ will be chosen later.

From the Schwartz lemma it follows that for $\|z\| < \rho$,

$$\begin{aligned} |E(z)| &\leqslant C_{n,k}^D s^{-1} |z|^{k+1} \leqslant C_{n,k}^D s^{-1} (C_{n,k}^1)^{k+1} r^{k+1} \\ &\leqslant C_{n,k}^D (C_{n,k}^1)^{k+1} r^k \end{aligned} \tag{40}$$

Let $U_{\tilde{\rho}} \subset \mathbb{C}$ denote the union of discs of radius $\tilde{\rho}$ around the zeros of P. By Lemma 13 we have

$$|P(\ell(z))| \geqslant C_k \tilde{\rho}^k = C_k (C_{n,k}^2)^k r^k \text{ for } \ell(z) \notin U_{\tilde{\rho}}. \tag{41}$$

Now choose constants $C_{n,k}^2 < C_{n,k}^1 < 1$ such that

$$\begin{aligned} &C_{n,k}^D (C_{n,k}^1)^{k+1} < (1/2) C_k (C_{n,k}^2)^k \\ &C_{n,k}^2 < (1/2)(4n)^{-1/2} k^{-n} C_{n,k}^1. \end{aligned} \tag{42}$$

Then for any $z \in D_\rho^n$, if $\ell(z) \notin U_{\tilde{\rho}}$ we have

$$\left| \sum U_i(z) f_i(z) \right| = |P(\ell(z)) - E(z)| \geqslant (1/2) C_k \tilde{\rho}^k \tag{43}$$

and since $\|U_i\| \leqslant C_{n,k}^D s^{-1}$ it follows that

$$\|F(z)\| \geqslant (1/2) C_k C_{n,k}^D s \tilde{\rho}^k. \tag{44}$$

To summarize, in B_ρ^n we have $\|F(z)\| \geqslant C_{n,k}^3 s \rho^k$ unless $\ell(z) \in U_{\tilde{\rho}}$ for some universal constant $C_{n,k}^3$. Since this is true for each coordinate ℓ, we have in total a union U of k^n polydiscs of radius $\tilde{\rho}$ such that in $B_\rho^N \setminus U$ the inequality $\|F(z)\| \geqslant$

$C^3_{n,k} s\rho^k$ holds. By (42) U has total diameter smaller than $\rho/2$, and hence there exists a sphere of radius $\rho/2 < \tilde r < \rho$ which is disjoint from U. This concludes the proof. □

In the proof of Lemma 1 we used the fact that if $\operatorname{mult} F \leqslant k$ then $\mathfrak{m}^{k+1} \subset I(F)$. In particular, this implies that if G is a map whose components lie in $\mathfrak{m}^{k+1}$ then $\operatorname{mult} F = \operatorname{mult}(F+G)$. We can now give a quantitative version of this fact.

Corollary 14. *Let $F, G \in \mathcal{O}^n(D^n)$ with $\|F\|, \|G\| \leqslant 1$. Let $M^{(k)}$ be a multiplicity operator of order k. Denote $s := |M^{(k)}_0(F)| \neq 0$. Let $0 < \varepsilon < 1$. There exist positive universal constants $A'_{n,k}, B'_{n,k}$ with the following property:*

If

$$\left| \frac{1}{\alpha!} \frac{\partial^\alpha G_i}{\partial z^\alpha}(0) \right| \leqslant \varepsilon^{k+1-|\alpha|} \qquad \text{for all } i = 1, \dots, n,\ |\alpha| \leqslant k \tag{45}$$

then for every $\varepsilon < r < B'_{n,k} s$ there exists $A'_{n,k} r < \tilde r < r$ such that

$$\|F(z)\| > \|G(z)\| \ \text{ for every } \ \|z\| = \tilde r, \tag{46}$$

and in particular

$$\#\{z : F(z) = 0, \|z\| < \tilde r\} = \#\{z : F(z) + G(z) = 0, \|z\| < \tilde r\}. \tag{47}$$

Proof. Indeed, by Theorem 2 there exists $A_{n,k} r < \tilde r < r$ such that

$$\|F(z)\| \geqslant B_{n,k} s \tilde r^k \text{ for every } \ \|z\| = \tilde r. \tag{48}$$

On the other hand we have

$$G(z) = \sum_{|\alpha| \leqslant k} \frac{1}{\alpha!} \frac{\partial^\alpha G_i}{\partial z^\alpha}(0) z^\alpha + G_{k+1}(z) \tag{49}$$

where $\|G_{k+1}\| < C^1_{n,k}$. From (45) and the Schwartz lemma applied to G_{k+1} we conclude that

$$\|G(z)\| \leqslant C^2_{n,k} \tilde r^{k+1} \text{ for every } \ \|z\| = \tilde r. \tag{50}$$

Choosing an appropriate $B'_{n,k}$ we can guarantee that $\|F(z)\| > \|G(z)\|$ for every $\|z\| = \tilde r$, and the conclusion follows by an argument of Rouché type (using topological degree theory). □

We also record a convenient special case of Corollary 14.

Corollary 15. *Let $F, G \in \mathcal{O}^n(D^n)$ with $\|F\|, \|G\| \leqslant 1$. Let $M^{(k)}$ be a multiplicity operator of order k. Denote $s := |M^{(k)}_0(F)| \neq 0$. Let $0 < \varepsilon < 1$. There exist universal constants $A''_{n,k}, B''_{n,k}$ with the following property:*

If $\|G(0)\| < \varepsilon$ then for every $\varepsilon < r < B''_{n,k} s$ there exists $A''_{n,k} r < \tilde r < r$ such that

$$\|F(z)\| > \left\|G^{k+1}(z)\right\| \ \text{ for every } \ \|z\| = \tilde r, \tag{51}$$

and in particular

$$\#\{z : F(z) = 0, \|z\| < \tilde r\} = \#\{z : F(z) + G^{k+1}(z) = 0, \|z\| < \tilde r\}. \tag{52}$$

The proof is analogous to the proof of Corollary 14.

4. Multiplicity operators and ideals

4.1. **Growth along analytic curves.** Let $\gamma \subset (\mathbb{C}^n, 0)$ be a germ of a real analytic curve. Let $G(t) : (\mathbb{R}_{\geqslant 0}, 0) \to \mathbb{C}^n$ be a length parametrization for γ,

$$\gamma = \{G(t) : t \in (\mathbb{R}, 0)\} \qquad \|G(t)\| = t. \tag{53}$$

The components of $G(t)$ are convergent Puiseux series in t.

We define the *order* of an analytic function along γ as

$$\operatorname{ord}_\gamma : \mathcal{O}(\mathbb{C}^n, 0) \to \mathbb{Q}_{\geqslant 0} \qquad \operatorname{ord}_\gamma(f) = \lim_{t\to 0^+} \frac{\log|f(G(t))|}{\log t}. \tag{54}$$

In terms of Puiseux series, the order of f along γ is equal to the leading exponent in the Puiseux expansion of the function $f(G(t))$.

We will require the following standard lemma.

Lemma 16. *Let $k \in \mathbb{N}$. Suppose that* $\operatorname{ord}_\gamma f_i \geqslant k$. *Then* $\operatorname{mult} F \geqslant k$.

Proof. Let ℓ be a functional on $\mathbb{C}^n$ which is transversal to the tangent vector to γ at $t = 0$. Then one can easily check that the functions $1, \ell, \ldots, \ell^{k-1}$ are linearly independent (over $\mathbb{C}$) in the local ring of F. Hence the dimension of the local ring is at least k. □

We now examine the order of a multiplicity operator along an analytic curve.

Theorem 3. *Let $M^{(k)}$ be a multiplicity operator of order k. Then*

$$\operatorname{ord}_\gamma M^{(k)}(F) \geqslant \min\{\operatorname{ord}_\gamma f_i : i = 1, \ldots, n\} - k \tag{55}$$

Proof. In this proof it is more convenient to let t denote a Puiseux coordinate (i.e. a coordinate transversal to γ at the origin) rather than the arc-length parameter. This does not change the notion of the order along γ.

Let $\tau \in (\mathbb{R}_{\geqslant 0}, 0)$ and consider the functions

$$\begin{aligned} f_1^\tau &= f_1 - E_1^\tau \qquad & E_1^\tau &= f_1(G(\tau)) + \ldots + \frac{1}{k!}\frac{\mathrm{d}}{\mathrm{d}t^k} f_1(G(t))|_{t=\tau}(t-\tau)^k \\ &\vdots \\ f_n^\tau &= f_n - E_n^\tau \qquad & E_n^\tau &= f_n(G(\tau)) + \ldots + \frac{1}{k!}\frac{\mathrm{d}}{\mathrm{d}t^k} f_n(G(t))|_{t=\tau}(t-\tau)^k \end{aligned} \tag{56}$$

If we let γ_τ denote the germ of γ through the point $G(\tau)$, then it is clear by construction that $\operatorname{ord}_{\gamma_\tau} f_i^\tau > k$ for $i = 1, \ldots, n$. Therefore by Lemma 16 we have $\operatorname{mult} F^\tau \geqslant k$, where $F^\tau = (f_1^\tau, ..., f_n^\tau)$, and hence $M^{(k)}(F^\tau)|_{G(\tau)} = 0$.

Since $M^{(k)}$ depends polynomially on the k-jet of its parameter, its derivative is uniformly bounded by a constant C around the k-jet $j(F) \in (J_{n,k}(G(\tau)))^n$ of F at $G(\tau)$. Thus for every sufficiently small τ,

$$\begin{aligned} \left|M^{(k)}(F)|_{G(\tau)}\right| &= \left|M^{(k)}(F^\tau + E^\tau)|_{G(\tau)}\right| \\ &\leqslant \left|M^{(k)}(F^\tau)|_{G(\tau)}\right| + C\,\|E^\tau\| = C\,\|E^\tau\| \end{aligned} \tag{57}$$

Finally, since E^τ is defined in terms of derivatives up to order k of $f_i(G(t)), i = 1, \ldots, n$ and since each derivative can decrease the order by at most 1, we conclude that

$$\operatorname{ord}_\gamma \left\|E^t\right\| \geqslant \min\{\operatorname{ord}_\gamma f_i : i = 1, \ldots, n\} - k \tag{58}$$

as claimed. □

4.2. **Integral closure and Multiplicity.** We recall the notions of *multiplicity* and *integral closure* of an ideal and some known results concerning them.

Let p be a point in $\mathbb{C}^n$ and denote by $\mathcal{O}_p$ the local ring of holomorphic functions in a neighborhood of p. We denote by $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_p$. Let $I \subset \mathcal{O}_p$ be an $\mathfrak{m}$-primary ideal. The *Hilbert-Samuel multiplicity* of I on $\mathcal{O}_p(\mathbb{C}^n, 0)$ is defined to be

$$\operatorname{mult} I = d! \lim_{n\to\infty} \frac{\lambda(\mathcal{O}_p/I^n)}{n^d} \tag{59}$$

where λ denotes the length as an $\mathcal{O}_p$-module. When I is generated by a regular sequence, $I = \langle f_1, \ldots, f_n\rangle$ this notion agrees with the geometric notion of $\operatorname{mult}(f_1, \ldots, f_n)$ [8, Proposition 11.1.10].

The Hilbert-Samuel multiplicity satisfies a Brunn-Minkowski type inequality ([4, Appendix by Tessier], see also [10]). Namely, for any two $\mathfrak{m}$-primary ideals $I, J \subset \mathcal{O}_p$ we have

$$\operatorname{mult}^{1/n}(IJ) \leqslant \operatorname{mult}^{1/n} I + \operatorname{mult}^{1/n} J \tag{60}$$

Recall that the integral closure $\overline{I}$ of an ideal $I \subset \mathcal{O}_p$ is the ideal generated by all elements $x \in \mathcal{O}_p$ satisfying an equation $x^n + a_1 x^{n-1} \ldots + a_n = 0$ with $a_k \in I^k$. By a theorem of Rees, we have $\operatorname{mult} I = \operatorname{mult} \overline{I}$ [8, Theorem 11.3.1].

Finally recall the following characterization of the integral closure (see [13, Theorem 2.1] or [7, Theorem 5.A.1]).

Theorem 4. *Let $f_1, \ldots, f_q$ be generators of I and $f \in \mathcal{O}_p$. Then $f \in \overline{I}$ if and only if for any germ of an analytic curve $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^n, p)$ the ratio*

$$\frac{f(\gamma(t))}{\max\{|f_1(\gamma(t))|, \ldots, |f_q(\gamma(t))|\}} \tag{61}$$

remains bounded as $t \to 0$.

To begin our study of the relation between the multiplicity operators and the notions of multiplicity and integral closure we make the following definition.

Definition 17. *The k-th* multiplicity operator ideal *of I is defined to be*

$$M_{n,k}(I) = I + \langle M^{(k)}(f_1, \ldots, f_n)\rangle \tag{62}$$

where $M^{(k)}$ varies over the multiplicity operators of order k, and $(f_1, \ldots, f_n)$ varies over all n-tuples of elements in I.

We now present a result relating the multiplicity operator ideal to the integral closure and multiplicity of an ideal.

Theorem 5. *Let $I \subset \mathcal{O}_p$ be an ideal. Then*

$$\mathfrak{m}^k M_{n,k}(I) \subset \overline{I}. \tag{63}$$

In particular, if I is $\mathfrak{m}$-primary then

$$\operatorname{mult}(I + \mathfrak{m}^k M_{n,k}(I)) = \operatorname{mult} I. \tag{64}$$

Proof. This follows immediately from Theorems 3 and 4. □

And the following corollary.

Corollary 18. *Let $I \subset \mathcal{O}_p$ be an $\mathfrak{m}$-primary ideal. Then*

$$\operatorname{mult}^{1/n} M_{n,k}(I) \geqslant \operatorname{mult}^{1/n} I - k \tag{65}$$

Proof. By (63) we have

$$\operatorname{mult}(\mathfrak{m}^k M_{n,k}(I)) \geqslant \operatorname{mult} \overline{I} = \operatorname{mult} I. \tag{66}$$

On the other hand, by (60) we have

$$\begin{aligned} \operatorname{mult}^{1/n}(\mathfrak{m}^k M_{n,k}(I)) &\leqslant \operatorname{mult}^{1/n} \mathfrak{m}^k + \operatorname{mult}^{1/n} M_{n,k}(I) \\ &= k + \operatorname{mult}^{1/n} M_{n,k}(I), \end{aligned} \tag{67}$$

and the conclusion easily follows. □

5. Applications for Noetherian Functions

A subring $K \subset \mathcal{O}(U)$ for a domain $U \subset \mathbb{C}^n$ is called a *ring of Noetherian functions* if

(1) It contains the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ and is finitely generated as a ring over it.
(2) It is closed under the partial derivatives $\frac{\partial}{\partial x_i}, i = 1, \dots, n$.

This notion is due to Khovanskii. A related notion where the assumption of finite generation is replaced by a topological noetherianity condition was introduced by Tougeron in [14].

More explicitly, we consider the polynomial ring $R = \mathbb{C}[x_1, \dots, x_n, f_1, \dots, f_m]$ along with the system of differential equations

$$\frac{\partial f_i}{\partial x_j} = P_{ij}, \qquad P_{ij} \in R \quad \text{for} \quad i = 1, \dots, n, \quad j = 1, \dots, m. \tag{68}$$

For simplicity we will assume that this system is integrable everywhere, and therefore $\operatorname{Spec} R$ is foliated by graphs of solutions of satisfying (68).

Remark 19. *In general it may happen that the locus of integrability of the system forms only an algebraic subvariety. This extra generality is studied in* [6] *and our arguments can be extended using their approach. We restrict attention to the completely integrable case to simplify the presentation.*

For every $p \in \operatorname{Spec} R$ we denote by $\mathcal{L}_p$ the graph of the solution of the system (68) around the point p. On $\mathcal{L}_p$ we may think of $f_1, \dots, f_m$ as functions of the variables $x_1, \dots, x_n$.

Let $p \in \operatorname{Spec} R$ and let $P_1, \dots, P_n \in R$ be an n-tuple of polynomials. Suppose that $\deg P_{ij} \leqslant \delta$ and $\deg P_i \leqslant d$. If

$$\operatorname{mult}_p(P_1|_{\mathcal{L}_p}, \dots, P_n|_{\mathcal{L}_p}) < \infty, \tag{69}$$

then it is natural to ask for an explicit upper bound for this multiplicity in terms of the parameters n, m, d, δ. This problem was studied in [6] using topological methods related to the study of Milnor fibers, leading to the following estimate

Theorem 6. *The multiplicity* $\operatorname{mult}_p(P_1|_{\mathcal{L}_p}, \dots, P_n|_{\mathcal{L}_p})$*, assuming it is finite, does not exceed the maximum of the following two numbers:*

$$\tfrac{1}{2}Q\left((m+1)(\delta-1)[2\delta(n+m+2)-2m-2]^{2m+2} + 2\delta(n+2)-2\right)^{2(m+n)}$$

$$\tfrac{1}{2}Q\left(2(Q+n)^n(d+Q(\delta-1))\right)^{2(m+n)}, \quad \textit{where} \quad Q = en\left(\tfrac{e(n+m)}{\sqrt{n}}\right)^{\ln n+1}\left(\tfrac{n}{e^2}\right)^n$$

In subsection 5.1 we present a simple approach to this problem using multiplicity operators. While the estimates obtained by this method are not as sharp as those of [6], the proof is simple and illustrates the manner in which multiplicity operators can be used in the study of multiplicities.

Having obtained a bound on the multiplicity of a Noetherian intersection, it is natural to move beyond the local context, and ask whether one can estimates the number of solutions of a system of equations $P_1|_{\mathcal{L}} = \ldots = P_n|_{\mathcal{L}} = 0$ on a compact piece of the graph $\mathcal{L}$. In subsection 5.2 we use the multiplicity operators to prove a result in this direction.

5.1. **An estimate for the intersection multiplicity of Noetherian functions.** For any point $p \in \operatorname{Spec} R$, one can consider the polynomials $P_1, \ldots, P_n$ as functions of $x_1, \ldots, x_n$ by restricting them to the graph $\mathcal{L}_p$. Then their derivatives with respect to $x_1, \ldots, x_n$ are again given by polynomial functions according to (68).

Let $M^{(k)}$ be a k-th multiplicity operator in the variables $x_1, \ldots, x_n$. By the above, it follows that $M^{(k)}(P_1, \ldots, P_n)$ (viewed, at any point $p \in \operatorname{Spec} R$, as the multiplicity operator of $P_1|_{\mathcal{L}_p}, \ldots, P_n|_{\mathcal{L}_p}$) is a polynomial, and moreover

$$\deg M^{(k)}(P_1, \ldots, P_n) \leqslant \binom{n+k}{k}(d + k\delta). \tag{70}$$

The following lemma shows that if an algebraic variety meets a graph $\mathcal{L}_p$ in an isolated point, then generic nearby intersections are transversal.

Lemma 20. *Let $V \subset \operatorname{Spec} R$ be an algebraic variety, and $p \in V$ a point in V. Suppose that $V \cap \mathcal{L}_p = \{p\}$ in a small neighborhood of p. Then at a generic point $q \in V$ near p, the graph $\mathcal{L}_q$ is transversal to V.*

Proof. Let D be a small ball around p, and consider the projection $\pi : (D \cap V) \to (\mathbb{C}^m, 0)$ from $D \cap V$ to the m-dimensional space parameterizing the leafs around $\mathcal{L}_p$. By assumption π is a proper finite-to-one map, and hence $\dim D \cap V = \dim \operatorname{Im} \pi$. Then by Sard's theorem, a generic point $q \in D \cap V$ is a non-critical point of π, which means that V is transversal to $\mathcal{L}_q$ at q as stated. $\square$

We now show that when the variety of an ideal I meets a graph $\mathcal{L}_p$ transversally, the multiplicity of $I|_{\mathcal{L}_p}$ at p can be bounded in algebraic terms.

Lemma 21. *Let $I = \langle F_1, \ldots, F_k \rangle \subset R$ and $p \in V(I)$ a point where $V(I)$ is smooth of codimension k and transversal to $\mathcal{L}_p$. Suppose $\deg F_i \leqslant D$ for $i = 1, \ldots, k$. Then*

$$\operatorname{mult}_p I|_{\mathcal{L}_p} \leqslant D^{kn}. \tag{71}$$

Proof. Let $J = \sqrt{I}$. Since $V(I)$ is smooth at p, J contains functions whose differentials define the tangent space to $V(I)$ at p. Since this tangent space is transversal to $\mathcal{L}_p$, one can find n functions in J whose differentials are linearly independent on $\mathcal{L}_p$. Therefore $\operatorname{mult} J|_{\mathcal{L}_p} = 1$.

It remains to note that by the effective Nullstellensatz of [12], $J^{D^k} \subset I$, and therefore

$$\operatorname{mult} I|_{\mathcal{L}_p} \leqslant \operatorname{mult} J^{D^k} = D^{kn} \operatorname{mult} J = D^{kn}. \tag{72}$$

$\square$

We are now ready to state and prove an upper bound for intersection multiplicities of Noetherian functions.

Theorem 7. *Let $p \in \operatorname{Spec} R$ and let $P_1, \dots, P_n \in R$ be an n-tuple of polynomials with $\deg P_i \leqslant d$. Assume*

$$\operatorname{mult}_p(P_1|_{\mathcal{L}_p}, \dots, P_n|_{\mathcal{L}_p}) < \infty. \tag{73}$$

Then

$$\operatorname{mult}_p(P_1|_{\mathcal{L}_p}, \dots, P_n|_{\mathcal{L}_p}) < (2d\delta)^{n(n+1)^{2m}(m+n)^m}. \tag{74}$$

Proof. Let $I_0 = \langle P_1, \dots, P_n \rangle$ be the ideal generated by $P_1, \dots, P_n$ and V_0 the corresponding variety. By the assumption, V_0 has codimension $k = n$ around p. By Lemmas 20 and 21, at a generic point q close to p we have

$$\operatorname{mult}_q I_0|_{\mathcal{L}_q} \leqslant d^{nk} < d^{n(m+n)}. \tag{75}$$

Let M_0 denote a generic multiplicity operator of order $d^{n(m+n)}$ of I_0. Then M_0 does not vanish at generic points q close to p. By (70),

$$\deg M_0 \leqslant \binom{n + d^{n(m+n)}}{d^{n(m+n)}} (d + d^{n(m+n)}\delta) \tag{76}$$

Let $\ell \in R$ denote a generic affine functional vanishing on p. Denote

$$M_0' = \ell^{d^{n(m+n)}} M_0 \tag{77}$$

and $I_1 = I_0 + \langle M_0' \rangle$. Then

$$\deg M_0' \leqslant 2d^{(n+1)^2(m+n)}\delta, \tag{78}$$

and by Theorem 5 we have

$$\operatorname{mult}_p I_0|_{\mathcal{L}_p} = \operatorname{mult}_p I_1|_{\mathcal{L}_p}. \tag{79}$$

Since M_0' does not vanish at generic points of V_0 near p, $V(I_1)$ has codimension $k = n + 1$ in a neighborhood of p. One can then repeat the argument above to obtain a sequence of ideals $I_0 \subset \cdots \subset I_m$, where $V(I_k)$ has codimension $n + k$ and

$$\operatorname{mult}_p I_0|_{\mathcal{L}_p} = \cdots = \operatorname{mult}_p I_m|_{\mathcal{L}_p}. \tag{80}$$

By induction using (78), all generators of I_m have degrees bounded by

$$\deg M_m' \leqslant (2d\delta)^{(n+1)^{2m}(m+n)^m}. \tag{81}$$

Finally, I_m is a zero-dimensional complete intersection near p, and from the Bezout theorem we deduce that

$$\operatorname{mult}_p I_0|_{\mathcal{L}_p} = \operatorname{mult}_p I_m|_{\mathcal{L}_p} \leqslant \operatorname{mult}_p I_m \leqslant (2d\delta)^{n(n+1)^{2m}(m+n)^m} \tag{82}$$

as claimed. □

5.2. **An estimate for the number of solutions of a Noetherian system.** Let $\mathcal{P}$ denote an algebraic family of Noetherian equations in n variables and m functions in the origin, i.e. an algebraic variety where each point consists of a tuple of coefficients $\{P_{ij}\}$ of a Noetherian system (68), plus a tuple of polynomials $\{P_i\}$. We think of each point in the variety as representing the systems of equations $\{P_i = 0\}$ on the solution of (68) through the origin, $\mathcal{L}_0$.

Remark 22. *The choice of the origin is a matter of technical convenience. One can of course add the coordinates of an arbitrary point to the description of $\mathcal{P}$.*

Let $\mathcal{P}_\Sigma \subset \mathcal{P}$ denote the subvariety of equations having a non-isolated solution at the origin, i.e. such that

$$\operatorname{mult}_0(P_1|_{\mathcal{L}_p}, \dots, P_n|_{\mathcal{L}_p}) = \infty. \tag{83}$$

Let K denote an upper bound for the multiplicity of an isolated Noetherian intersection in the class $\mathcal{P}$. For instance, one could use the estimate given by Theorem 7 or the better estimates given in [6]. For specific classes $\mathcal{P}$ one may have better a-priori bounds K.

Let I_K denote the ideal generated by all multiplicity operators of order K of the tuple $P_1, \dots, P_n$. Then the common zeros of I_K correspond to Noetherian intersections of multiplicity greater than K, and by the definition of K this implies $V(I_K) = \mathcal{P}_\Sigma$.

Theorem 8. *Suppose that $\mathcal{P}$ is defined as an affine subvariety of $\mathbb{C}^N$ by equations of degree bounded by D. Let $\|\cdot\|$ denote the usual Euclidean norm for some choice of linear coordinates in $\mathbb{C}^N$ (the choice of different norms will only affect the constant C below). Let K denote an upper bound for the multiplicity of an isolated Noetherian intersection in $\mathcal{P}$.*

Let $p \in \mathcal{P}$, and suppose that the corresponding Noetherian functions $P_1, \dots, P_n$ are defined in the unit ball and have norm bounded by 1. Then the number of solutions of the Noetherian system $P_1, \dots, P_n = 0$ in a ball of radius r around the origin (in the x variables) does not exceed K, where

$$r = C \left(\frac{\operatorname{dist}(p, \mathcal{P}_\Sigma)}{1 + \|p\|^2} \right)^e, \quad e = \max \left(D, \binom{n+K}{k} (d + K\delta) \right)^N \tag{84}$$

and C is a constant depending only on $\mathcal{P}$.

Proof. By the effective Łojasiewicz inequality given in [9], there exists a constant $C' > 0$ such that

$$\max_{\alpha, |B|=K} \left| M_p^{(k)}(P_1, \dots, P_n) \right| \geqslant C' \left(\frac{\operatorname{dist}(p, \mathcal{P}_\Sigma)}{1 + \|p\|^2} \right)^e. \tag{85}$$

The result now follows by applying Theorem 1. □

Remark 23. *One can estimate the norms of $f_1, \dots, f_m$, and hence $P_1, \dots, P_n$, in terms of the coefficients of the Noetherian system (in some fixed ball) using Gronwall's inequality. In some cases one may have better a-priori estimates due to the structure of the particular Noetherian system.*

When the variety $\mathcal{P}$ is defined by equations involving only rational coefficients, one can also obtain estimates for the constant C in terms of the complexity of the description of the system by using the appropriate effective Łojasiewicz inequality results.

We leave the details of these explicit computations for the reader.

5.3. **Concluding remarks.** The arguments of subsection 5.1, as well as the finer arguments of [6], establish upper bounds for multiplicities of isolated solutions of Noetherian equations. They do not apply when studying degenerate systems with non-isolated solution. This limitation is reflected in Theorem 8 as we demonstrate below.

Recall that we estimate the number of solutions in a ball of a specified radius, and this radius tends to zero as p tends to infinity or to $\mathcal{P}_\Sigma$. As p tends to infinity, one can indeed construct examples where the number of zeros in a ball of any fixed radius also tends to infinity. For instance one can consider the system

$$\frac{\partial f}{\partial x} = pg \tag{86}$$

$$\frac{\partial g}{\partial x} = -pf \tag{87}$$

where $p \in \mathbb{C}$, with the solution $f = \sin(px), g = \cos(px)$, along with the equation $f = 0$.

On the other hand, we have no similar examples corresponding to the case where p tends to $\mathcal{P}_\Sigma$. In fact, it is an old conjecture of Khovanskii (which is given precisely in [6]) that this scenario is impossible. However, all known results apply only to multiplicities of isolated solution — and this is reflected in our estimates, which become weaker and weaker as we approach the locus of non-isolated solutions.

The study of deformations of systems of Noetherian equations with non-isolated solutions has been one of our principal motivations in developing the theory of multiplicity operators. Our intention in this section has been to develop a simple representative application of this theory in the context of Noetherian functions. A more systematic application for the study of of non-isolated solutions will appear in a separate paper.

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