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Intersection multiplicities of Noetherian functions

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Abstract

Consider a foliation defined by two commuting polynomial vector fields V_1, V_2 in $\mathbb{C}^n$, and p a non-singular point of the foliation. Denote by $\mathcal{L}$ the leaf passing through p, and let $F, G \in \mathbb{C}[X]$ be two polynomials. Assume that $F|_{\mathcal{L}} = 0, G|_{\mathcal{L}} = 0$ have several common branches. We provide an effective procedure which produces an upper bound for the multiplicity of intersection of remaining branches of $F|_{\mathcal{L}} = 0$ with $G|_{\mathcal{L}} = 0$ in terms of the dimension n and the degrees of V_1, V_2, F, G.

Keywords: Noetherian functions; Intersection multiplicity; Pfaffian functions

1. Introduction

Let $\mathcal{F}$ be the foliation generated in $X = \mathbb{C}^n$ by several commuting polynomial vector fields $V_1, \dots, V_k$. A restriction of a polynomial $F \in \mathbb{C}[X]$ to a leaf of $\mathcal{F}$ is called a *Noetherian function*. The main result of this paper is motivated by the following question: *is it possible to effectively bound the topological complexity of objects defined by Noetherian functions solely in terms of the dimensions n, k and the degrees of the vector fields and the polynomials*?

An important subclass of the class of Noetherian functions is that of Pfaffian functions. The theory of Fewnomials developed by Khovanskii provides effective upper bounds for global topological invariants (e.g., Betti numbers) of *real* varieties defined by Pfaffian equations (see [8]). Evidently, for complex varieties such bounds are impossible: the only holomorphic functions which admit "finite complexity" (e.g., finitely many zeros) in the entire complex

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domain are the polynomials. One may trace the dichotomy between the real and complex settings to the absence of a complex analog of the Rolle theorem (which is a cornerstone of the real Fewnomial theory) — the derivative of a holomorphic function with many zeros may have no zeros at all.

While the global Rolle theorem fails in the complex setting, certain local analogs still hold. Perhaps the simplest of these analogs is the statement that, if a derivative f' admits a zero of multiplicity n at some point, then f may admit a zero of multiplicity at most $n+1$ at the same point. This trivial claim, together with its ramifications, was used by Gabrielov to build a local theory of complex Pfaffian sets, and to provide effective estimates on the *local* complexity of complex Pfaffian sets [2].

The topology of global real Noetherian sets is usually infinite, as demonstrated by the Noetherian function $\sin(x)$, which admits infinitely many real zeros (and is therefore non-Pfaffian). However, an old conjecture due to Khovanskii claims that the complexity of the local topology of the Noetherian functions can be estimated through the discrete parameters of the set. This conjecture is motivated as follows: in view of Morse theory, to estimate local Betti numbers it is essentially sufficient to bound the number of critical points of Noetherian functions on germs of Noetherian sets. The latter can be bounded through a suitably defined multiplicity of a common zero of a tuple of Noetherian functions. In [4], the multiplicity of an *isolated* common zero is bounded from above. To build the general theory one has to generalize this result to non-isolated intersections.

For non-isolated intersections, even the notion of multiplicity becomes non-trivial. For a point p lying on a leaf $\mathcal{L}_0$ of the foliation $\mathcal{F}$ one can define the multiplicity of the common zero p of the functions $F_i|_{\mathcal{L}_0}$ as the maximal number of common isolated zeros of $F_i|_{\mathcal{L}}$ on neighboring leaves $\mathcal{L}$ converging to p as $\mathcal{L} \to \mathcal{L}_0$. The multiplicity defined in this manner is not intrinsic to the leaf $\mathcal{L}_0$. It depends on the foliation $\mathcal{F}$ in which $\mathcal{L}_0$ is embedded.

The case of two-dimensional leaves is the first case where the problem of non-isolated intersections presents substantial difficulties. Many of the ideas presented in this paper appear to be applicable in higher dimensions as well. However, some technical details still make the two-dimensional case easier to handle. We therefore chose in this paper to restrict attention to the case of two-dimensional leaves.

In the context of two-dimensional leaves, we consider another notion of non-isolated multiplicity, suggested by Gabrielov, which is defined intrinsically on the leaf under consideration (see Section 5.2). We prove that this multiplicity can be explicitly bounded in terms of the dimension n and the degrees of the vector fields defining the foliation.

2. Setup and notation

Let $\mathcal{F}$ be the foliation generated in $X = \mathbb{C}^n$ by two commuting polynomial vector fields V_1, V_2. For us, a Noetherian function will be the restriction of a polynomial function to a leaf of such a foliation. It is worth noting that in [4] a somewhat more general definition is used, along with a reduction between the two definitions; see [4, Section 2]. Given the reduction, our results can be carried over to the general definition as well, and we have chosen the simpler definition in order to simplify the exposition.

Given two polynomial functions F, G, we denote by $\mathrm{J}(F, G)$ the Jacobian of the map $(F, G)|_{\mathcal{L}}$ with respect to the leaves of the foliation,

$$\mathrm{J}(F, G) = V_1(F)V_2(G) - V_2(F)V_1(G). \tag{1}$$

In the entire paper $\mathcal{L}$ denotes some particular fixed leaf of the foliation, and $p \in \mathcal{L}$ a particular fixed smooth point of $\mathcal{L}$. We will denote functions defined on X using uppercase letters, and functions defined on $\mathcal{L}$ by lowercase letters. We will also denote ideals of functions on X by capital letters, and ideals of functions on $\mathcal{L}$ by calligraphic letters.

We denote by $\mathcal{O}(X)$ the ring of polynomial functions on X, and by $\mathcal{O}_p(\mathcal{L})$ the ring of germs of analytic functions on $\mathcal{L}$ at p. Given an ideal $\mathcal{I} \subset \mathcal{O}_p(\mathcal{L})$, we denote by $\operatorname{mult}_p \mathcal{I}$ the Hilbert–Samuel multiplicity of $\mathcal{I}$ on $\mathcal{O}_p(\mathcal{L})$ (see [5, Definition 11.1.5]),

$$\operatorname{mult}_p \mathcal{I} = \frac{d!}{n^d} \lim_{n\to\infty} \lambda(\mathcal{O}_p(\mathcal{L})/\mathcal{I}^n), \quad d = \dim \mathcal{O}_p(\mathcal{L}) = 2, \tag{2}$$

where $\lambda(\cdot)$ denotes the length as an $\mathcal{O}_p(\mathcal{L})$-module. This number is finite whenever $\mathcal{I}$ contains a power of the maximal ideal at the point p. We remark that, for an ideal generated by a regular sequence, this notion of multiplicity agrees with the usual geometric multiplicity of the sequence; see [5, Proposition 11.1.10].

Recall that the integral closure $\overline{\mathcal{I}}$ of an ideal $\mathcal{I} \subset \mathcal{O}_p(\mathcal{L})$ is the ideal generated by all elements $x \in \mathcal{O}_p(\mathcal{L})$ satisfying the equation $x^m + a_1 x^{m-1} + \cdots + a_m = 0$, with $a_k \in \mathcal{I}^k$. By a theorem of Rees (see [5, Theorem 11.3.1]), we have the equality

$$\operatorname{mult}_p \mathcal{I} = \operatorname{mult}_p \overline{\mathcal{I}}. \tag{3}$$

Let $I \subset \mathcal{O}(X)$ be a polynomial ideal. We will be particularly interested in the case when I is generated by two polynomial functions F, G. We denote by $V(I)$ the algebraic set associated to I. We denote by $I|_{\mathcal{L}}$ the ideal generated by the restriction of I to $\mathcal{O}_p(\mathcal{L})$.

We will say that the intersection of I (or $V(I)$) and $\mathcal{F}$ is isolated at a point $p \in V(I)$ if there exists an open neighborhood U of p such that $U \cap V(I) \cap \mathcal{L} = \{p\}$.

Suppose that $I = \langle F, G\rangle$. If the intersection of I and $\mathcal{F}$ is isolated at p, then the intersection multiplicity of F and G at p is defined to be $\operatorname{mult}_p I|_{\mathcal{L}}$. Otherwise, near p we have the following decomposition:

$$F|_{\mathcal{L}} = h_f f, \tag{4}$$
$$G|_{\mathcal{L}} = h_g g, \tag{5}$$

where h_f, h_g contain the factors which are common to $F|_{\mathcal{L}}$ and $G|_{\mathcal{L}}$ (note that they may appear with different multiplicities).

Definition 1. The (non-isolated) intersection multiplicity of F and G at p is defined to be $\operatorname{mult}_p \langle f, g\rangle|_{\mathcal{L}}$.

3. Results and motivating corollaries

Upper bounds for the multiplicity of an isolated intersection were given in [4]. The main result of this paper is as follows.

Theorem 1. *Let F, G be two polynomials. Then the non-isolated intersection multiplicity of F and G at p can be explicitly bounded in terms of the dimension n and the degrees of V_1, V_2, F, G.*

This theorem is proved by presenting an effective procedure for computing an upper bound for non-isolated intersection multiplicities. In the interest of clarity, we have chosen to present a version of this procedure which illustrates the key ideas, rather than a more technical version achieving better bounds. In light of this, while it is possible to follow the proposed algorithm and write out an explicit upper bound, we have chosen to postpone this computation to a forthcoming publication in which a more technically refined version of the algorithm will be given.

In the following subsections, we give several examples of applications of this result. In addition to their independent interest, these examples motivate Definition 1 above.

3.1. Rational Noetherian functions and blow-up transformations

Let $S = F/Q, T = G/Q$ be two rational functions. Suppose that S, T restrict to holomorphic functions on the leaf $\mathcal{L}$. This, of course, does not imply that S, T are holomorphic on nearby leaves, as illustrated by the example $\mathbb{C}^3 = (x, y, z)$, $\mathcal{F} = \langle \partial_x, \partial_y \rangle$, $\mathcal{L} = \{z = 0\}$ and $S = (xy + z)/y$.

In order to give an upper bound on the intersection multiplicity $\operatorname{mult}_p \langle S|_{\mathcal{L}}, T|_{\mathcal{L}} \rangle$ of S and T at p, we may clear the denominator Q and consider instead the intersection multiplicity of F and G. Note that, even assuming that the original intersection $S = T = 0$ is isolated on $\mathcal{L}$, clearing the denominator will create a non-isolated intersection on $Q = 0$. However, it is easy to see that the *non-isolated* intersection multiplicity of F and G is equal to the intersection multiplicity of S and T.

This construction is of particular interest in the context of blow-up transformations. If Z is a smooth blow-up locus transversal to $\mathcal{F}$ near $\mathcal{L}$, then one may show that the blow-up of a Noetherian function F remains Noetherian in each blow-up chart. However, the strict transform of F is given by F/y^k, where $\{y = 0\}$ is the equation defining the exceptional divisor in a given chart, and k is the multiplicity of the exceptional divisor in the blow-up of F. Since k may be larger on $\mathcal{L}$ than on nearby leaves, this brings us to the situation discussed in the paragraph above.

The possibility of explicitly controlling the behavior of Noetherian functions after blow-up was previously missing from the theory, and will hopefully improve our understanding of desingularization in the Noetherian category.

3.2. Desingularization of Noetherian foliations

In the two-dimensional case being considered in this paper, our results on non-isolated intersections enable us to explicitly estimate the complexity of desingularization, not only of Noetherian functions, but more generally of foliations given by Noetherian one-forms.

Indeed, suppose that, in addition to our usual Noetherian data, we are given a polynomial one-form $\omega \in \Lambda^1(\mathbb{C}^n)$. Denote by s, t the local coordinates on $\mathcal{L}$ such that $\partial_s = V_1, \partial_t = V_2$. In these local coordinates,

$$\omega|_{\mathcal{L}} = \omega(V_1)\,\mathrm{d}s + \omega(V_2)\,\mathrm{d}t. \tag{6}$$

The foliation $\mathcal{L} \cap \{\omega = 0\}$ is independent of multiplication by a scalar function, and is thus unaffected by clearing the common components of $\omega(V_1)|_{\mathcal{L}}, \omega(V_2)|_{\mathcal{L}}$. Furthermore, after clearing these common components, the foliation admits an isolated singular point. It is well known, see [6, p. 138], that in this case the number of blow-ups needed to achieve full desingularization is controlled by the Milnor number of the foliation, equal to the intersection number of $\omega(V_1)$ and $\omega(V_2)$ (with the common components removed). Thus our result on non-isolated intersection applies directly to give a bound on the complexity of desingularization.

3.3. Łojasiewicz inequalities

In [2], it is shown that Łojasiewicz exponents can be explicitly estimated for the class of Pfaffian functions. To develop an effective geometric theory of Noetherian functions, it is desirable to extend this result to the Noetherian case. Below, we provide an example of how non-isolated intersection multiplicities can be used to derive upper bounds for Łojasiewicz exponents.

Corollary 2. *Let F, G be real polynomials vanishing at $p \in \mathbb{R}^n$, let $\mathcal{F}$ be a two-dimensional foliation defined and non-singular in a neighborhood of p, and let $\mathcal{L}$ be the leaf of $\mathcal{F}$ containing p. Let $C \subset \mathcal{L}$ be a connected component of (the germ of) $\mathcal{L} \cap \{G > 0\}$. Suppose that $F > 0$ on C. Then, for $x \in C$ sufficiently close to the origin, we have*

$$F(x) \geq G(x)^d, \tag{7}$$

where d is an exponent which can be explicitly estimated in terms of the dimension n and the degrees of V_1, V_2, F, G.

Proof. Let $H = \mathrm{J}(F, G) = V_1(F)V_2(G) - V_1(G)V_2(F)$. Then the zero set of H contains the curves where F takes a minimum on the level sets $G = \varepsilon$ (unless this minimum is attained on the boundary of C, in which case one may reduce the problem to the same problem in one dimension). It will suffice to prove the claim over such curves. Furthermore, we may assume (by restricting x to a small neighborhood) that each such curve passes through the origin.

Let $\gamma \subset \{h = 0\} \subset \{H = 0\}$ be a real curve whose real part intersects C, where h is one of the irreducible components of $H|_{\mathcal{L}}$. Since $F > 0$ on C, F is not identically vanishing on $\{h = 0\}$.

Let $c(t) : (\mathbb{C}, 0) \to \gamma$ be a good parameterization of γ. Clearly, $\mathrm{ord}_0 G(c(t)) \geq 1$. On the other hand,

$$\mathrm{ord}_0(F(c(t)) = \mathrm{mult}(F, h) \leq \mathrm{mult}(F, H), \tag{8}$$

where the latter multiplicity denotes the non-isolated intersection multiplicity of F and H (which may be explicitly estimated). Thus along the curve γ we have (in a sufficiently small neighborhood of the origin)

$$F(c(t)) \geq Kt^d \geq K'G(c(t))^d \quad d = \mathrm{mult}(F, H), \tag{9}$$

with K, K' constants, and increasing d by one we may drop the constant K' to obtain the desired estimate. □

4. Noetherian pairs and controllable inclusions

When studying the restriction of algebraic functions to the leaves of a foliation, we shall often find it necessary to simultaneously keep track of functions defined locally on a particular leaf, and their algebraic counterparts defined globally. In this section, we introduce notation and terminology to facilitate the manipulation of such data.

Definition 3. A pair of ideals $(I, \mathcal{I})$ with $I \subset \mathcal{O}(X), \mathcal{I} \subset \mathcal{O}_p(\mathcal{L})$ is called a *Noetherian pair* for the leaf $\mathcal{L}$ if $I|_{\mathcal{L}} \subset \mathcal{I}$.

By an inclusion of pairs $(I, \mathcal{I}) \subset (J, \mathcal{J})$, we mean simply that $I \subset J, \mathcal{I} \subset \mathcal{J}$. The pair $(J, \mathcal{J})$ is said to *extend* the pair $(I, \mathcal{I})$.

We will introduce a number of operations which generate for a given pair $(I, \mathcal{I})$ an extension $(J, \mathcal{J})$. The goal is to form an extension in such a way that the multiplicity of $\mathcal{I}$ can be estimated from that of $\mathcal{J}$. More precisely, we introduce the following notion.

Definition 4. An inclusion $(I, \mathcal{I}) \subset (J, \mathcal{J})$ is said to be *controllable* if the following hold.

- The complexity of J can be bounded in terms of the complexity of I.
- The multiplicity of $\mathcal{I}$ can be bounded in terms of the multiplicity of $\mathcal{J}$ and complexity of I.

Here, by complexity of an ideal of $\mathcal{O}(X)$, we mean the maximal degree of its generators.

The precise estimates on the complexity and the multiplicity in the definition above will vary for the various types of inclusion we form.

We record two immediate consequences of this definition.

Proposition 5. *If each inclusion in the sequence*

$$(I_0, \mathcal{I}_0) \subset \cdots \subset (I_k, \mathcal{I}_k) \tag{10}$$

is controllable, then the inclusion $(I_0, \mathcal{I}_0) \subset (I_k, \mathcal{I}_k)$ is controllable.

Proposition 6. *Suppose that $(I, \mathcal{I}) \subset (J, \mathcal{J})$ is an controllable inclusion, and that $p \notin V(J)$. Then one can give an upper bound for* $\mathrm{mult}_p \mathcal{I}$.

Proof. By definition, one can give an upper bound for $\mathrm{mult}_p(\mathcal{I})$ in terms of $\mathrm{mult}_p(\mathcal{J})$. But $J \subset \mathcal{J}$ and $p \notin V(J)$, so $\mathrm{mult}_p \mathcal{J} = 0$, and the proposition follows. $\square$

These two proposition lay out the general philosophy of this paper. We start with a pair $(I, \mathcal{I})$ and attempt, by forming a sequence of controllable inclusions, to reach a pair $(J, \mathcal{J})$ with $V(J)$ as small as possible. If we manage to get a pair with $p \notin V(J)$, then we obtain an upper bound for $\mathrm{mult}_p \mathcal{I}$.

We now introduce three controllable inclusions which will be used in this paper.

4.1. Radical extension

Let $(I, \mathcal{I})$ be a Noetherian pair. We define a new pair $(J, \mathcal{J})$ by letting $J = \sqrt{I}$ and $\mathcal{J} = \langle \mathcal{I}, J|_{\mathcal{L}} \rangle$. The complexity of the ideal J, as well as the degree n such that $J^n \subset I$, can be computed from that of I using effective radical extraction algorithms.

The multiplicity of $\mathcal{I}$ can be estimated from that of $\mathcal{J}$ by the following simple lemma.

Lemma 7. *Let $\mathcal{K}, \mathcal{K}' \subset \mathcal{O}_p(\mathcal{L})$ be ideals of finite multiplicity, and suppose that $\mathcal{K}^n \subset \mathcal{K}'$. Then*

$$\mathrm{mult}_p \mathcal{K}' \leq n^d \mathrm{mult}_p \mathcal{K}, \quad d = \dim \mathcal{O}_p = 2. \tag{11}$$

Proof. By Huneke and Swanson [5, Proposition 11.2.9], $\mathrm{mult}_p \mathcal{K}^n = n^d \mathrm{mult}_p \mathcal{K}$. The result follows. $\square$

4.2. Jacobian extension

Let $(I, \mathcal{I})$ be a Noetherian pair, and let $F, G \in I$ be of known degrees. We define a new pair $(J, \mathcal{J})$ by letting $J = \langle I, \mathrm{J}(F, G) \rangle$ and $\mathcal{J} = \langle \mathcal{I}, \mathrm{J}(F, G) \rangle$. We claim that $(J, \mathcal{J})$ is a controllable extension of $(I, \mathcal{I})$.

Indeed, the complexity of the ideal J can clearly be estimated from that of I. Also, the multiplicity of $\mathcal{I}$ can be estimated from that of $\mathcal{J}$, as shown in the following lemma, which may be regarded as a multi-dimensional analog of the Rolle theorem.

Lemma 8. *Let $\mathcal{K} \subset \mathcal{O}_p(\mathcal{L})$ be an ideal of finite multiplicity, and suppose that $f, g \in \mathcal{K}$. Then*

$$\operatorname{mult}\mathcal{K} \leq \operatorname{mult}\langle \mathcal{K}, \mathrm{J}(f, g)\rangle + 1, \tag{12}$$

with equality only if one of f, g has a non-vanishing linear part.

Proof. According to Vasconcelos [1, p. 102, Corollary 3], the Jacobian $\mathrm{J}(f, g)$ spans the one-dimensional socle of the local algebra $\mathcal{O}_p(\mathcal{L})/\langle f, g\rangle$.

If f, g both have vanishing linear parts, then by Vasconcelos [10, Proposition 1.69] the ideal $\langle f, g\rangle$ is not integrally closed, and it follows that the socle element $\mathrm{J}(f, g)$ is integral over $\langle f, g\rangle$. In this case, by (3), we have $\operatorname{mult}\mathcal{K} = \operatorname{mult}\langle \mathcal{K}, \mathrm{J}(f, g)\ \rangle$.

In the remaining case, where f or g has a non-vanishing linear part, one can assume by an analytic equivalence that $f = y$ and $g = x^m$ for some $m \in \mathbb{N}$. In this case, we have $\operatorname{mult}\mathcal{K} = \operatorname{mult}\langle \mathcal{K}, \mathrm{J}(f, g)\rangle + 1$. $\square$

4.3. Transversal extension

Let $(I, \mathcal{I})$ be a Noetherian pair, and let $F \in I$ be of known degree. Assume that I is a radical ideal and that all intersections of $V(I)$ and $\mathcal{F}$ are non-isolated. Write $F|_{\mathcal{L}} = fh$, where h consists of the irreducible factors of $F|_{\mathcal{L}}$ which vanish on $V(I) \cap \mathcal{L}$ and f consists of the other factors. Finally, assume that $f \in \mathcal{I}$.

Let h' denote the reduced form of h. Denote by k and K respectively the minimal and the maximal multiplicities of a factor of h, so that h is divisible by $(h')^k$ (but not by $(h')^{k+1}$) and h divides $(h')^K$.

Remark 9. The order E of any function $F \in I$ of bounded degree can be effectively bounded from above. Indeed, let ℓ be a generic linear function on X vanishing at p. Then $E = \operatorname{mult}_p\langle h, \ell\rangle$, and the multiplicity of this isolated intersection can be bounded from above, either using the considerations above, or using the main result of [3] (which gives a better bound).

In particular, E is an upper bound for the total number of branches of h, counted with multiplicities, and therefore for K above.

In the notation above, we define a new Noetherian pair $(J, \mathcal{J})$ by letting

$$J = \langle I, \{V_1^{\alpha_1} V_2^{\alpha_2} F \mid \alpha_1 + \alpha_2 = k\}\rangle \tag{13}$$

$$\mathcal{J} = \overline{\langle \mathcal{I}, h'\rangle}. \tag{14}$$

We claim that $(J, \mathcal{J})$ is a controllable extension of $(I, \mathcal{I})$.

First, to prove that this is a Noetherian pair, it suffices to show that all derivatives of order k of $F|_{\mathcal{L}}$ belong to $\mathcal{J}$. This follows from the Leibnitz rule,

$$\begin{aligned} V_1^{\alpha_1} V_2^{\alpha_2} F|_{\mathcal{L}} &= V_1^{\alpha_1} V_2^{\alpha_2}(fh) \\ &= (\text{derivatives of order} \leq k-1\text{of h}) \cdot (\cdots) + (\cdots) f \in \mathcal{J}, \end{aligned} \tag{15}$$

since $f \in \mathcal{I} \subset \mathcal{J}$ and derivatives of order $\leq k-1$ of h are divisible by $h' \in \mathcal{J}$.

It is clear that the complexity of J can be estimated in terms of the complexity of I. It remains to show that the multiplicity of $\mathcal{I}$ can be estimated in terms of the multiplicity of $\mathcal{J}$. By (3), taking integral closure does not affect the multiplicity of an ideal. Therefore, by Lemma 7, it will suffice to show that, for some explicit number N, we have $(h')^N \in \overline{\mathcal{I}}$. This follows from Lemma 13, since $\overline{I|_{\mathcal{L}}} \subset \overline{\mathcal{I}}$.

To end this section, we remark that, crucially, $V(J) \subsetneq V(I)$. Indeed, the derivatives of order k of F added to I ensure that the lowest-order factors of h are not contained in $V(J)$.

Remark 10. In the particular case of h being reduced, one can omit the integral closure in the definition of $\mathcal{J}$. Lemma 17 from the Appendix shows that in this case $(h')^N \in I|_{\mathcal{L}}$ for $N = 2^K$.

5. Proofs

In this section, we prove the main result of the paper. First, we consider the multiplicities of isolated intersections. In following subsection, we extend the result to the case of non-isolated intersections.

5.1. Isolated intersection multiplicities

Let $I \subset \mathcal{O}(X)$ be a polynomial ideal. Assume that the intersection of I (or $V(I)$) and $\mathcal{F}$ is isolated at a point $p \in V(I) \cap \mathcal{L}$. We are interested in an upper bound for $\operatorname{mult} I|_{\mathcal{L}}$.

Proposition 11. *Let I be a radical ideal of known complexity, and suppose that it has isolated intersections with $\mathcal{F}$. Then there exist $F, G \in I$ of bounded degrees such that $V(\langle I, \mathrm{J}(F,G)\rangle)$ has smaller dimension than $V(I)$.*

Proof. We may assume without loss of generality that $V(I)$ is irreducible. Since the condition of having an isolated intersection is open, generic points of $V(I)$ are isolated intersections of I and $\mathcal{F}$.

We claim that, at a generic point p of $V(I)$, $T_p V(I) \pitchfork \mathcal{F}$. Indeed, otherwise the intersection of these spaces defines a line field whose integral trajectories lie in the intersection $V(I) \cap \mathcal{L}$ (where $\mathcal{L}$ is the leaf containing p), in contradiction to the assumption that p is an isolated point of intersection.

The tangent space $T_p V(I)$ is defined by differentials of functions in I. By transversality, there exist $F, G \in I$ such that $\mathrm{d}F|_{\mathcal{L}}$, $\mathrm{d}G|_{\mathcal{L}}$ are linearly independent at the point p. Moreover, these F, G can be chosen from among the generators of I, which have bounded degrees. Thus $\mathrm{J}(F,G)$ is non-vanishing at p, and the claim is proved. □

Corollary 12. *For any pair $(I, \mathcal{I})$, there exists a controllable inclusion $(I, \mathcal{I}) \subset (J, \mathcal{J})$ such that J is radical and $V(J)$ is the variety of non-isolated intersections of I and $\mathcal{F}$.*

Proof. We obtain this inclusion using Proposition 11 by forming an alternating sequence of controllable inclusions of radical and Jacobian type. Each pair of inclusions in this sequence reduces the dimension of the set of isolated intersections, so after at most n steps the sequence stabilizes on the variety of non-isolated intersections. □

Suppose now that $I = \langle F, G\rangle$ and that p is an isolated intersection point of I and $\mathcal{F}$. Consider the pair $(I, I|_{\mathcal{L}})$. Applying Corollary 12 and Proposition 6, we obtain an upper bound for $\operatorname{mult}_p I|_{\mathcal{L}}$.

5.2. Non-isolated intersection multiplicities

We now consider the case of non-isolated intersection multiplicities. Let $I = \langle F, G\rangle$, and suppose that the intersection of I and $\mathcal{F}$ at a point p is not isolated. Let $\mathcal{L}$ be the leaf of $\mathcal{F}$ containing p.

Near p, we may write

$$F|_{\mathcal{L}} = h_f f, \tag{16}$$

$$G|_{\mathcal{L}} = h_g g, \tag{17}$$

where h_f, h_g contain all the factors which are common to $F|_{\mathcal{L}}$ and $G|_{\mathcal{L}}$, as before. We stress that $h_{f,g}$, f, g are defined only on $\mathcal{L}$, and the decomposition of F and G as products of these factors need not extend outside of $\mathcal{L}$. We let

$$\mathcal{I} = \langle f, g\rangle. \tag{18}$$

We now prove Theorem 1, namely that the multiplicity $\mathrm{mult}_p\mathcal{I}$ can be bounded from above in terms of degrees of V_1, V_2, F, G and the dimension n. We begin in the same manner as in the isolated-intersection case.

Let $(I_0, \mathcal{I}_0) = (I, \mathcal{I})$. We apply Corollary 12 to obtain an controllable inclusion $(I_0, \mathcal{I}_0) \subset (I_1, \mathcal{I}_1)$ such that I_1 is radical and $V(I_1)$ is the variety of non-isolated intersections of I and $\mathcal{F}$.

We can now form a transversal extension to obtain a controllable inclusion $(I_1, \mathcal{I}_1) \subset (I_2, \mathcal{I}_2)$. Indeed, $V(I_1)$ is the locus of non-isolated intersections, and with $h_1 = h_f$ we have

- $V(I_1)|_{\mathcal{L}} = \{h_1 = 0\}$,
- $F|_{\mathcal{L}} = h_1 f$,
- $f \in \mathcal{I}_0 \subset \mathcal{I}_1$,

and the conditions of transveral extension are satisfied. By the remark at the end of Section 4.3, $V(I_2) \subsetneq V(I_1)$.

Applying Corollary 12 again, we obtain a controllable inclusion $(I_2, \mathcal{I}_2) \subset (I_3, \mathcal{I}_3)$ such that I_3 is radical and $V(I_3)$ is the variety of non-isolated intersections between I_2 and $\mathcal{F}$.

We can now again form a transversal extension to obtain a controllable inclusion $(I_3, \mathcal{I}_3) \subset (I_4, \mathcal{I}_4)$. Indeed, $V(I_3)$ has only non-isolated intersections with $\mathcal{F}$, and letting h_3 denote the factors of h_f which remain in $V(I_3)|_{\mathcal{L}}$ we have

- $V(I_3)|_{\mathcal{L}} = \{h_3 = 0\}$,
- $F|_{\mathcal{L}} = h_3 f \cdot (h_f/h_3)$,
- $f \cdot (h_f/h_3) \in \mathcal{I}_0 \subset \mathcal{I}_1$,

and the conditions of transversal extension are satisfied. Once again by the remark at the end of Section 4.3, $V(I_4) \subsetneq V(I_3)$.

Repeating these two alternating types of controllable inclusion and applying Proposition 5, we obtain an inclusion $(I_0, \mathcal{I}_0) \subset (I_{2k+1}, \mathcal{I}_{2k+1})$, where at least k factors of h_f have been removed from $V(I_{2k+1})$. In particular, for K equal to the number of factors of h_f (effectively bounded by Remark 9), we know that $p \notin V(I_{2K+1})$. Thus, applying Proposition 6, we obtain the required upper bound for $\mathrm{mult}_p\mathcal{I}_0 = \mathrm{mult}_p\mathcal{I}$.

6. Ideals of non-isolated intersections

In this section, we prove that, if a radical ideal I has only non-isolated intersections near a leaf $\mathcal{L}$, then $\overline{I|_{\mathcal{L}}}$ contains (some effectively bounded power of) the ideal of definition of $V(I) \cap \mathcal{L}$.

This is in contrast to the general situation, where the restriction of a radical ideal to an analytic set may be far from radical.

Lemma 13. *Let $I \subset \mathbb{C}[X]$ be a radical ideal of known complexity, and suppose that every intersection of I and $\mathcal{F}$ is non-isolated.*

Let $\mathcal{I} \subset \mathcal{O}_p(\mathcal{L})$ denote the ideal of functions vanishing on $V(I) \cap \mathcal{L}$. Then

$$\mathcal{I}^N \subset \overline{I|_{\mathcal{L}}} \tag{19}$$

for some N effectively bounded from above.

Let $h \in \mathcal{I}$ denote any function vanishing on $V(I) \cap \mathcal{L}$. We will prove that $h^N \in \overline{I|_{\mathcal{L}}}$ for some N effectively and uniformly bounded from above. We also let $F \in I$ denote an arbitrary function in I (since I is of bounded complexity, F can be assumed to be of bounded degree).

The proof of the lemma is based on the following characterization of the algebraic closure of an ideal [9, Theorem 2.1].

Lemma 14. *Let $f_1, \ldots, f_k$ be generators of an ideal $\mathcal{I} \subset \mathcal{O}_p(\mathcal{L})$. The function h^N belongs to $\overline{\mathcal{I}}$ if and only if, for any germ of an analytic curve $\gamma : (\mathbb{C}, 0) \to (\mathcal{L}, p)$, the ratio*

$$\frac{h^N(\gamma(t))}{\max_{i=1,\ldots,k} |f_i(\gamma(t))|} \tag{20}$$

remains bounded as $t \to 0$.

As the first step, we reformulate this statement in terms of the metric properties of $V(I)$ and $V(I) \cap \mathcal{L}$. By an effective version of the Łojasiewicz inequality, see [7], we can choose the generators of I in such a way that

$$\max_{i=1,\ldots,k} |f_i(q)| \geq \operatorname{dist}(q, V(I))^M, \quad q \in \mathbb{C}^n, \tag{21}$$

with some explicit and effective M. Also, by continuity, $|h(q)| \leq C\operatorname{dist}(q, V(I) \cap \mathcal{L})$ with some constant C for any $q \in (\mathcal{L}, p)$. Therefore, instead of (20), it is enough to prove the boundedness of

$$\frac{\operatorname{dist}(\gamma(t), V(I) \cap \mathcal{L})^N}{\operatorname{dist}(\gamma(t), V(I))^M} \tag{22}$$

on any germ of curve $\gamma : (\mathbb{C}, 0) \to (\mathcal{L}, p)$, for some explicit uniform N.

Let us denote by λ the coordinates on a germ of a transversal to $\mathcal{F}$ passing through p, with p corresponding to $\lambda = 0$, and by $\mathcal{L}_\lambda$ the leaf of $\mathcal{F}$ intersecting this transversal at the point λ (so $\mathcal{L} = \mathcal{L}_0$). We claim that it is enough to give an effective upper bound for the Łojasiewicz-type constant $\tilde{N}$ appearing in the following lemma.

Lemma 15. *There exists a constant $\tilde{C} > 0$ and a natural number $\tilde{N}$ such that, for any point $q \in \mathcal{L}_\lambda \cap V(I)$, we have*

$$\operatorname{dist}(q, V(I) \cap \mathcal{L})^{\tilde{N}} \leq \tilde{C}\|\lambda\|. \tag{23}$$

The number $\tilde{N}$ is bounded by the Gabrielov estimate for the multiplicity of an isolated zero of F on a trajectory of a vector field of degree $\max(\deg V_1, \deg V_2)$.

This lemma immediately implies (22). Indeed, let $\gamma(t) \in \mathcal{L}$, and let $q(t) \in V(I)$ be a point where the minimum of the distance from $\gamma(t)$ to $V(I)$ is achieved. Let $q(t) \in \mathcal{L}_{\lambda(t)}$. Evidently $\operatorname{dist}(\gamma(t), q(t)) \geq \|\lambda(t)\|$. Therefore

$$\frac{\operatorname{dist}(\gamma(t), V(I) \cap \mathcal{L})}{(\operatorname{dist}(\gamma(t), V(I)))^{\frac{1}{\tilde{N}}}} \leq \frac{\operatorname{dist}(\gamma(t), q(t)) + \operatorname{dist}(q(t), V(I) \cap \mathcal{L})}{(\operatorname{dist}(\gamma(t), q(t)))^{\frac{1}{\tilde{N}}}}$$
$$\leq 1 + \|\lambda(t)\|^{-\frac{1}{\tilde{N}}} \cdot \operatorname{dist}(q(t), V(I) \cap \mathcal{L}) \leq 1 + \tilde{C}^{\frac{1}{\tilde{N}}}, \tag{24}$$

where $\tilde{C}$ is the same constant as in (23). So it is enough to take $N = \tilde{N}M$ in (22).

To prove Lemma 15, we will bring F into Weierstrass form and use the following quantitative statement about the continuous dependence of a root of a polynomial on its coefficients.

Lemma 16. *Let $P(z) = z^N + \sum_{k=0}^{N-1} a_{N-k} z^k$ be a polynomial with roots at z_k, $k = 1, \ldots, N$ with $|z_k| \leq 1/2$. Let $\tilde{P}(z) = \sum_{k=0}^{N-1} b_{N-1-k} z^k$ be another polynomial of degree at most $N-1$, with $\sum |b_k| \leq \delta < 2^{-N}$. Then any root of $P + \tilde{P}$ is of distance at most $\delta^{1/N}$ from the roots of P.*

Proof. Take any point z of distance more than $\delta^{1/N}$ from the roots of P. Then evidently $|P(z)| = \prod |z - z_k| \geq \delta$. Since $|z| < 1$ by the choice of δ, we have $\left|\tilde{P}(z)\right| < \delta$. Therefore, by the argument principle, the N zeros of $P + \tilde{P}$ lie in a $\delta^{1/N}$-neighborhood of the roots of P. □

6.1. *Choosing Weierstrass coordinates on the leaf $\mathcal{L}$*

Replacing V_1, V_2 by their linear combinations, we can assume that F is not identically zero on the trajectory of V_2 passing through p on the leaf $\mathcal{L}$, and, moreover, that the branches of $F|_{\mathcal{L}}$ at p are not tangent to this trajectory.

Let us choose coordinates (x, y, λ) with the origin at p in such a way that $\partial_y = V_2$. Then, after a multiplication by a unit, $F = F(x, y, \lambda)$ becomes a Weierstrass polynomial $F_{x,\lambda}(y)$ in y. The degree N_F of F in y is effectively bounded by Gabrielov [3]: it is just a multiplicity at the origin of the restriction of F on the trajectory of V_2, as in Remark 9. This degree will be equal to the degree $\tilde{N}$ in Lemma 15.

Let $\{y = \phi_i(x)\}$ be the branches of $F = 0$ at the origin on the leaf $\lambda = 0$. Let us choose $R \in (0, 1/2]$ such that for every $\theta \in [0, 2\pi]$ the distances $|\phi_i(te^{i\theta}) - \phi_j(te^{i\theta})|$ grow monotonically as $t \in [0, R]$ grows.

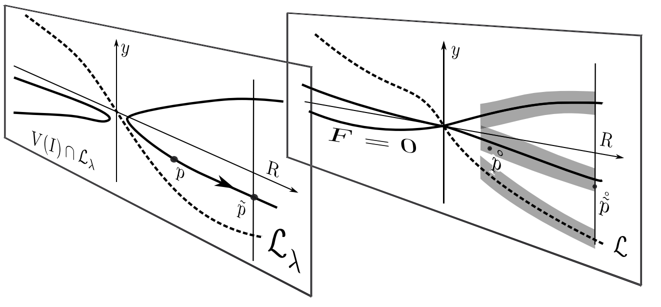
6.2. *Shadowing*

Define C as the speed of change of the coefficients $a_j(x, \lambda)$ of $F_{x,\lambda}(y) = \sum_{j=0}^{\tilde{N}} a_{\tilde{N}-j}(x, \lambda) y^j$ as λ changes,

$$C = \sup_j \sup_{|x| \leq R} \sup_{|\lambda| \leq \lambda_0} \left| \frac{a_j(x, 0) - a_j(x, \lambda)}{\lambda} \right|^{1/\tilde{N}}. \tag{25}$$

We will prove Lemma 15 by contradiction. Let $p = (x_1, y_1, \lambda) \in V(I)$ be a point where the inequality (23) with $\tilde{N} = N_F$ and the constant $\tilde{C} = (2N_F + 1)C$ is violated. By increasing $|x_0|$ and keeping λ fixed, we will construct a point $\tilde{p} = (\tilde{x}, \tilde{y}, \lambda) \in V(I)$ with $|\tilde{x}| = R$ which is $C\lambda^{1/N_F}$-close to $\{F|_{\mathcal{L}} = 0\} \setminus V(I)$.

Supposing we have constructed such a point $\tilde{p}(\lambda)$ for each sufficiently small λ, consider a point of accumulation of the set of the points $\tilde{p}(\lambda)$ as $\lambda \to 0$. By closedness, it lies in $V(I)$. On the other hand, it must lie in the set of components of $\{F|_{\mathcal{L}} = 0\} \cap \{|x| = R\}$ disjoint from $V(I)$. This is impossible, since these two sets do not intersect. This contradiction proves Lemma 15.



We proceed with the construction of $\tilde{p}$. For a point $q = (x, y, \lambda)$, we define its *shadow* as the point $\mathring{q} = (x, y, 0)$.

The projection of $\{F = 0\} \cap \mathcal{L}_\lambda$ to $\{y = 0\} \cap \mathcal{L}_\lambda$ is a ramified cover for all sufficiently small λ. Furthermore, by our assumption, $V(I) \cap \mathcal{L}_\lambda$ is a union of components of this cover, as it contains no isolated points. Therefore, there is a continuous family $p(t) = (x_t, y_t, \lambda) \in V(I)$, with $x_t = tx_0$, $1 \le t \le t_1 = R/|x_0|$, such that $\tilde{p} = p(R/|x_0|)$ lies in $|x| = R$. We claim that $\mathring{p}(t)$ is $C\lambda^{1/N_F}$-close to $\{F|_{\mathcal{L}} = 0\} \setminus V(I)$ for all t.

By Lemma 16, $\mathring{p}(t)$ is $C\lambda^{1/N_F}$-close to $S_t = \{(x_t, y, 0) | F(x_t, y, 0) = 0\}$, as $F(p(t)) = 0$ for all t. On the line $\{x = x_t, \lambda = 0\}$, consider the union $\mathcal{U}_t$ of $\tilde{N}$ discs of radius $C\lambda^{1/N_F}$ with centers at points of S_t.

For $t = 1$, the connected component of $\mathcal{U}_t$ containing the point $\mathring{p}$ does not contain points of $V(I)$. Otherwise, the distance from $\mathring{p}$ to $V(I)$ would be less than $2N_F C\lambda^{1/N_F}$, and thus the distance from p to $V(I)$ would be less than $\tilde{C}\lambda^{1/N_F}$, contrary to our assumption that p does not satisfy (23).

By choice of R, as t grows, the distances between the centers of discs grow as well, so the connected components of $\mathcal{U}_t$ cannot join. This means that for all t the point $\mathring{p}(t)$ remains in a connected component of $\mathcal{U}_t$ disjoint from $V(I)$.

Let $\tilde{p} = \mathring{p}(t)$ where $t = R/|x_0|$. By the above, $\tilde{p} \in V(I)$ is $C\lambda^{1/N_F}$-close to $\{F|_{\mathcal{L}} = 0\} \setminus V(I)$, thus concluding our construction, and with it the proof of Lemma 15.

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Appendix

The statement of Lemma 13 involves the integral closure of the ideal $I|_{\mathcal{L}}$. Integral closure does not affect the multiplicity of an ideal, so for our purposes this makes little difference. However, it is not clear that this closure is necessary — in fact, we believe that a similar result should hold without taking integral closures.

While we are currently not able to prove this result, we were able to prove it in a special case where the components of non-isolated intersection near p are simple. More accurately, we prove the following.

Lemma 17. *Let $I \subset \mathbb{C}[X]$ be a radical ideal of known complexity, and suppose that every intersection of I and $\mathcal{F}$ is non-isolated. Assume that $\langle I, V_1 I, V_2 I\rangle$ has an isolated intersection at p.*

Let $\mathcal{I} \subset \mathcal{O}_p(\mathcal{L})$ denote the ideal of functions vanishing on $V(I) \cap \mathcal{L}$. Then

$$\mathcal{I}^N \subset I|_{\mathcal{L}} \tag{26}$$

for some N effectively bounded from above.

Proof. Since $V(I) \cap \mathcal{L}$ has only non-isolated components, it follows that

$$\mathcal{I} = \langle h\rangle, \quad h \in \mathcal{O}_p(\mathcal{L}). \tag{27}$$

We will prove that there exists a function $H \in \mathcal{O}_{\mathrm{an}}(X)$ such that $H|_{\mathcal{L}}$ divides h^{2^μ} and $V(I) \subset V(H)$, where μ is the order of h at p. Since I is radical, it will follow that $H \in I_{\mathrm{an}}$, and therefore

$$\mathcal{I}^{2^\mu} = \langle h^{2^\mu}\rangle \subset \langle H|_{\mathcal{L}}\rangle \subset I|_{\mathcal{L}}, \tag{28}$$

as claimed.

We proceed with the construction of H. By the assumption that $\langle I, V_1 I, V_2 I\rangle$ has an isolated intersection at p, it follows that, for every irreducible component of h, we can choose a generator G of I such that $G|_{\mathcal{L}}$ contains this component with multiplicity one. Taking F to be a generic linear combination of the generators, we get

$$F|_{\mathcal{L}} = fh, \quad f \in \mathcal{O}_p(\mathcal{L}), \tag{29}$$

with f, h coprime in $\mathcal{O}_p(\mathcal{L})$.

Let (x, y) denote a system of coordinates on $\mathcal{L}$, and let (z) denote the coordinates parameterizing the leaves, with p being the origin and $\mathcal{L} = \{z = 0\}$. Possibly making a linear change of coordinates in (x, y), we may assume that the projection $\pi : (x, y, z) \to (x, z)$ restricted to $\{F = 0\}$ defines a *ramified* covering map in a neighborhood of the origin.

Fix an annulus A_x around the origin in (x), a sufficiently small disc D_y in (y), and a sufficiently small point $x_0 \in A_x$. Then the x_0-fibre of π restricted to $\{F = 0\} \cap \mathcal{L}$ is a discrete set B parameterizing the branches of F on $\mathcal{L}$. We write B as the disjoint union of branches corresponding to h and f, $B = B_h \amalg B_f$.

On $\mathcal{L}$, one may express the branches of $\{F = 0\}$ as multi-valued functions $y_b(x)$ for $b \in B$ defined on A_x. Furthermore, since the h-branches of F are simple, for $b \in B_h$ and z in a sufficiently small polydisc D_z these functions extend as holomorphic functions $y_b(x, z)$ on $A_x \times D_z$.

We will call a set $S \subset B$ monodromic if it is invariant under the monodromy of A_x on $\mathcal{L}$. Assume now that S in monodromic and that $S \subset B_h$, and consider the function

$$F_S(x, y, z) = \prod_{y_b : b \in S} (y - y_b(x, z)). \tag{30}$$

By continuity, the A_x-monodromy is constant for sufficiently small z. Shrinking D_z if necessary, we conclude that F_S is a single-valued function on $A_x \times D_y \times D_z$. We develop F_S as a

Puiseux series in A_x, and let F'_S denote the holomorphic part obtained by removing all negative and fractional terms. The result is a holomorphic function in a neighborhood of p.

On $\mathcal{L}$, the functions y_b actually extend from A_x to the punctured disc of the same radius, as the only ramification point occurs at the origin. Furthermore, since S is assumed to be monodromic on $\mathcal{L}$, it follows that $F_S|_{\mathcal{L}}$ is single valued and holomorphic outside the origin. Since it is bounded, it extends holomorphically at the origin well. Since $S \subset B_h$, we see that $F_S|_{\mathcal{L}}$ divides h.

On nearby leaves, the functions y_b may exhibit several ramification points (as the ramification at the origin on $\mathcal{L}$ may bifurcate). But suppose that for a certain nearby leaf $\mathcal{L}_{z_0}$ the set S happens to be monodromic with respect to the full monodromy in the (x) variable. Then, arguing just as we did for the leaf $\mathcal{L}$, we deduce that F_S is in fact holomorphic on $\mathcal{L}_{z_0}$. Hence for such leaves we have $F_S|_{\mathcal{L}_{z_0}} \equiv F'_S|_{\mathcal{L}_{z_0}}$.

Now define the function H as follows:

$$H = \prod_{S \subset B_h \text{ monodromic}} F'_S. \tag{31}$$

As a product of holomorphic functions, H is holomorphic in a neighborhood of p.

On $\mathcal{L}$, we have shown that, for monodromic $S \subset B_h$, $F'_S \equiv F_S$ and $F_S|_{\mathcal{L}}$ divides h. The number of monodromic sets is certainly bounded by $|P(B_h)| = 2^\mu$, and we deduce that $H|_{\mathcal{L}}$ divides h^{2^μ}.

It remains to show that H vanishes on $V(I)$ (at least in a sufficiently small neighborhood of the origin). Consider a fixed value of $z_0 \in D_z$. By assumption, all intersections of $V(I)$ with $\mathcal{L}_{z_0}$ are non-isolated, so $V(I)|_{\mathcal{L}_{z_0}}$ necessarily consists of some set S_{z_0} of branches. This set, being the set of branches of an analytic set on $\mathcal{L}_{z_0}$, is necessarily monodromic.

We claim that, for sufficiently small z_0, $S_{z_0} \subset B_h$. Otherwise, $V(I)|_{\mathcal{L}_{z_0}}$ contains branches from B_f for arbitrarily small values of z_0. Then, since $V(I)$ is closed, $V(I)|_{\mathcal{L}}$ contains branches of $f = 0$, which is impossible, since f and h are coprime.

To conclude, S_{z_0} is monodromic and $S_{z_0} \subset B_h$, and hence as we have seen that $F_S|_{\mathcal{L}_{z_0}} \equiv F'_S|_{\mathcal{L}_{z_0}}$. Since $F_{S_{z_0}}$ vanishes on $V(I)|_{\mathcal{L}_{z_0}}$ by definition, and $F'_{S_{z_0}}$ is a factor of H, we conclude that H vanishes on $V(I)|_{\mathcal{L}_{z_0}}$. Since this is true for any sufficiently small z_0, the claim is proved. $\square$

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