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QUASIALGEBRAICITY OF PICARD–VESSIOT FIELDS

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To Vladimir Igorevich Arnold with admiration

ABSTRACT. We prove that under certain spectral assumptions on the monodromy group, solutions of Fuchsian systems of linear equations on the Riemann sphere admit explicit global uniform bounds on the number of their isolated zeros in a way remotely resembling algebraic functions of one variable.

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1. INTRODUCTION

1.1. Fuchsian systems, monodromy. Consider a system of first order linear ordinary differential equations with rational coefficients on the complex Riemann sphere $\mathbb{C}P^1$. In the matrix form such a system can be written as

$$\dot{X}(t) = A(t)X(t), \qquad X(t) \in \operatorname{Mat}_{n\times n}(\mathbb{C}), \quad t \in \mathbb{C}, \tag{1.1}$$

where t is an affine chart on $\mathbb{C}P^1$, $A(t)$ is the rational *coefficients matrix* and $X(t)$ a multivalued *fundamental matrix solution*, $\det X(t) \not\equiv 0$.

Assume that the system is *Fuchsian*, that is, the matrix-valued differential 1-form $\Omega = A(t)\,dt$ has only simple poles on $\mathbb{C}P^1$, eventually including the point $t = \infty$. Then the coefficients matrix $A(t)$ has the form

$$A(t) = \sum_{j=1}^{m} \frac{A_j}{t - t_j}, \qquad A_j \in \operatorname{Mat}_{n\times n}(\mathbb{C}), \tag{1.2}$$

with the constant *matrix residues* $A_1, \dots, A_m$ at the singular points $t_1, \dots, t_m$. The point $t = \infty$ is singular if $A_1 + \dots + A_m \neq 0$, and then the residue at $t = \infty$ is the negative of the above sum of finite residues.

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Definition 1. The *height* of the rational matrix function $A(t)$ as in (1.2), is the sum of norms of the residues,

$$\|A(\cdot)\| = \|A_1\| + \cdots + \|A_m\| + \|A_\infty\|, \qquad A_\infty = -(A_1 + \cdots + A_m). \tag{1.3}$$

The *singular*, or *polar locus* $\Sigma = \{t_1, \ldots, t_m\}$ of the system (1.1) is the union of all *finite* singular points; if $t = \infty$ is a singular point also, then we will denote $\Sigma^* = \Sigma \cup \{\infty\} \subset \mathbb{C}P^1$.

Any fundamental matrix solution $X(t)$ of the linear system (1.1) with a rational matrix of coefficients is an analytic function on $\mathbb{C}$ ramified over the polar locus Σ. If γ is a closed loop avoiding Σ, then analytic continuation $\Delta_\gamma X(t)$ of the fundamental matrix solution $X(t)$ along γ produces another fundamental solution, necessarily of the form $X(t)M_\gamma$, where the constant invertible matrix $M_\gamma \in \operatorname{Mat}_{n\times n}(\mathbb{C})$ is called the *monodromy factor*. Choosing a different fundamental solution results in replacing M_γ by a matrix $CM_\gamma C^{-1}$ conjugate to M_γ.

Definition 2. The linear system (1.1) with a rational matrix $A(t)$ and the singular locus $\Sigma \subset \mathbb{C}$, is said to satisfy the *simple-loop spectral condition*, if all eigenvalues of each monodromy factor M_γ associated with any simple loop (non-selfintersecting closed Jordan curve), belong to the unit circle:

$$\text{for any simple loop } \gamma \in \pi_1(\mathbb{C} \smallsetminus \Sigma), \quad \operatorname{Spec} M_\gamma \subset \{|\lambda| = 1\}. \tag{1.4}$$

Clearly, this condition does not depend neither on the choice of a fundamental solution, nor on the parametrization or even the orientation of the loop.

Example 1. Let $x(t) = (x_1(t), \ldots, x_n(t))$ be the maximal linear independent collection of branches of an *algebraic function* of t. Analytic continuation of the vector function $x(t)$ results in a linear monodromy operator M_γ for *any* loop γ avoiding ramification locus Σ of the function. Since the number of branches is finite, all operators M_γ are roots of unity, therefore all their eigenvalues are roots of unity, verifying in particular the simple loop spectral condition (1.4).

Note that by the Riemann theorem, the vector function $x(t)$ satisfies a linear system $\dot x = A(t)x$ with some rational matrix function $A(t)$, not necessarily Fuchsian and in general having singularities also outside the ramification locus.

1.2. Principal result. Let $T \subset \mathbb{C} \smallsetminus \Sigma$ be a simply connected domain in $\mathbb{C} \smallsetminus \Sigma$. Then any fundamental matrix solution $X(t)$ is analytic in T and, moreover, the linear space spanned by all entries $x_{ij}(t)$, $i, j = 1, \ldots, n$, is independent of the choice of X.

The main problem addressed in the article is to place an explicit upper bound on the number of isolated zeros of an arbitrary function from this linear space.

One can rather easily see that this bound must necessarily depend on the dimension n of the system and the degree m of the rational matrix function $A(t)$. Simple examples suggest that the height r is also a relevant parameter: solutions of systems with large height can have many zeros. A more careful analysis shows that to exclude accumulation of zeros of solutions, one should impose certain restrictions on the spectra of the residue matrices A_i and exclude from consideration simply connected domains that are "too spiralling" around one or more singularities (all these examples are discussed in details in the lecture notes [Yak01]).

It turns out that besides those already mentioned, there are no other parameters that may affect the number of isolated zeros. Moreover, an upper bound can be explicitly *computed* in terms of the parameters n, m, r. Recall that an integer valued function of one or more integer arguments is *primitive recursive*, if it can be defined by several inductive rules, each involving induction in only one variable. This is a strongest form of computability, see [Man77], [Yak01].

Theorem 1. *There exists a primitive recursive function $\mathfrak{N}(n, m, r)$ of three natural arguments, with the following property.*

If the Fuchsian system (1.1)–(1.2) of dimension $n \times n$ with m finite singular points satisfies the simple-loop spectral condition (1.4) and its height is no greater than r, then any linear combination of entries of a fundamental solution in any triangular domain $T \subset \mathbb{C} \smallsetminus \Sigma$ has no more than $\mathfrak{N}(n, m, r)$ isolated roots there.

The "existence" assertion concerning the primitive recursive counting function $\mathfrak{N}$, is constructive. The proof of Theorem 1 in fact yields an algorithm for computing $\mathfrak{N}(n, m, r)$ for any input $(n, m, r) \in \mathbb{N}^3$.

It is important to stress that the bound established in Theorem 1 is uniform over all triangles, all possible configurations of m distinct singular points, and all combinations of residues A_j of total norm $\leqslant r$, provided that the simple-loop spectral condition holds.

The assumption of triangularity of the domain T, as well as the focus on linear combinations only, are not essential. The general assertion, Theorem 3 on *quasialgebraicity* of Picard–Vessiot fields for Fuchsian systems meeting the spectral condition (1.4), will be formulated in Section 2.2 after introducing all technical definitions.

1.3. The Euler system. The simplest example of a Fuchsian system is the *Euler system*,

$$\dot{X} = t^{-1}AX, \qquad A \in \operatorname{Mat}_{n\times n}(\mathbb{C}). \tag{1.5}$$

The system (1.5) can be explicitly solved: $X(t) = t^A = \exp(A \ln t)$. The monodromy group of the Euler system is cyclic and generated by the monodromy operator $\exp 2\pi i A$ for the simple loop encircling the origin. Taking A in the Jordan normal form (without losing generality) allows to compute the matrix exponent and verify that the linear space spanned by the components of $X(t)$, consists of *quasipolynomials*, functions of the form

$$f(t) = \sum_{\lambda\in\Lambda} t^\lambda\, p_\lambda(\ln t), \qquad \Lambda = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{C}, \qquad p_\lambda \in \mathbb{C}[\ln t]. \tag{1.6}$$

The *spectrum* Λ of these quasipolynomials coincides with the spectrum of the residue matrix A and the *degree* $d = \sum_\Lambda (1 + \deg p_\lambda)$ is equal to the dimension n of the initial system.

Distribution of complex zeros of quasipolynomials depends on their degree and spectrum. For example, if $\Lambda = \{\pm i\}$, then the quasipolynomial $f(t) = t^i + t^{-i} = 2\cos \ln t$ has an infinite number of positive real roots accumulating to the origin.

On the other hand, if $\Lambda \subset \mathbb{R}$, then at least for quasipolynomials with real coefficients, i. e., when $p_\lambda \in \mathbb{R}[\ln t]$, the number of real positive zeros is no greater than $d - 1$ (similarly to the usual polynomials corresponding to the case $\Lambda \subset \mathbb{N}$).

It turns out that if $\Lambda \subset \mathbb{R}$ then not only real, but also all complex roots can be counted. The following result which is a particular case of Theorem 1, will be used as a basis for the inductive proof of the general case as well.

Lemma 1 (see [KY96], Theorem 2)**.** *The number of isolated roots of any quasipolynomial* (1.6) *of degree* $\leqslant d$ *with arbitrary complex coefficients but the* real *spectrum* $\Lambda \subset [-r, r] \subset \mathbb{R}$*, in any triangular domain* T *not containing the origin* $t = 0$*, is explicitly bounded in terms of* d *and* r*.*

More precisely, by [KY96] *this number never exceeds* $4r + d - 1$*.* □

1.4. Systems with pairwise distant singular points. The most difficult part of the proof of Theorem 1 is to treat *confluent singularities*, ensuring that the bounds on the number of zeros would remain uniform even when the distances $|t_i - t_j|$ are arbitrarily small. If this is not the case, more precisely, if the bounds are allowed to depend on the configuration of the singular points, then both the formulation and the proof can be considerably simplified.

The corresponding result, formulated in a proper context below (Theorem 4, see Section 7.3), differs from its unrestricted counterpart, Theorem 2, by several instances.

(1) Fewer restrictions are imposed on the monodromy group: only *small loops* around singular points must satisfy the spectral condition (1.4).
(2) As an immediate consequence and in contrast with the general case, the spectral condition for such small loops can be immediately verified by inspection of the eigenvalues of A_j of the system: it is sufficient to require that all these eigenvalues must be real.
(3) Since the proof is considerably simplified, the bounds in this case are much less excessive, though still unlikely to be accurate.
(4) The price one has to pay is that the bounds for the number of zeros depend explicitly on the minimal distance between the singular points on $\mathbb{C}P^1$. These bounds explode as this distance tends to zero.

This last observation on explosion of bounds notwithstanding, any three points of $\mathbb{C}P^1$ can always be placed by an appropriate conformal isomorphism to $0, 1, \infty$. This implies computability of the number of zeros for systems having only three Fuchsian singular points on the sphere $\mathbb{C}P^1$, provided that the three residue matrices have only real eigenvalues (cf. with Section 1.3). Such result may be of interest for the theory of special functions, many of which are defined by this type of systems.

1.5. Applications to tangential Hilbert problem. The problem on zeros of functions defined by Fuchsian equations, is intimately related to the *tangential*, or *infinitesimal Hilbert problem* [AO79], [Arn94], see [NY01] and the notes [Yak01] for the detailed exposition and bibliography. Recall that the problem concerns the maximal possible number of isolated real zeros of Abelian integrals of the form $I(t) = \oint_{H=t} \omega$, where H is a bivariate polynomial and ω a polynomial 1-form of a given degree d. The key circumstance is the fact that Abelian integrals satisfy a more or less explicitly known linear system of Picard–Fuchs differential equations [AGV88], [NY01].

The results formulated below, imply *computability* of this bound for polynomials H whose critical points are sufficiently distant from each other (Theorem 4). The general case still requires additional efforts to treat confluent singularities, as explained in [NY01].

Yet there is one specific *hyperelliptic* case when $H(x, y) = y^2 + F(x)$, $F \in \mathbb{C}[x]$. Computability of the bound in this case under the additional assumption that all critical values of F are real, was proved in [NY99a] by almost the same construction as exposed below. The difference concerns only two technical instances. The preliminary folding of the Fuchsian system (Appendix A) is made obsolete by the above additional assumption. On the other hand, the isomonodromic surgery explained in Section 6 and Appendix C in this case can be bypassed by a suitably adapted application of Lyashko–Looijenga theorem [Loo74].

2. Quasialgebraicity

2.1. Strategy. The proof of Theorem 1 is organized as follows. First, we reformulate the assertion on zeros of linear combinations of components of a fundamental solution, as a general claim (Theorem 2) on *quasialgebraicity* of *Picard–Vessiot fields*. Speaking loosely, this quasialgebraicity means algorithmic computability of bounds for the number of zeros of any function belonging to the function field $\mathbb{C}(X)$ generated by the entries $x_{ij}(t)$ of any fundamental solution $X(t)$.

The proof of quasialgebraicity goes essentially by induction in the number of finite singular points of the system (1.1). Actually, the assumptions on the system that are required (and reproduce themselves in the induction) slightly differ from those mentioned in Theorem 1. Instead of requiring all singularities to be Fuchsian, we allow one of them (at infinity) to be regular non-Fuchsian, with bounded negative Laurent matrix coefficients. On the contrary, we require that all finite singularities be aligned along the real axis and moreover belong to the segment $[-1, 1]$. Section 3 contains the initial reduction, the proof of the fact that this additional condition can be always achieved by suitable transformations preserving the quasialgebraicity.

The hard core of the inductive step is provided by the following general *isomonodromic reduction principle*: if two given Fuchsian systems have the same monodromy in a simply connected polygonal domain $U \subset \mathbb{C}$ whose boundary is sufficiently distant from all singularities, then the Picard–Vessiot fields for these two systems, after restriction on U, are both quasialgebraic or both not quasialgebraic simultaneously. The accurate formulation of this principle is given in Section 5.

The proof of the isomonodromic reduction rests upon another quite general fact, the *bounded meandering principle* [NY97], [NY99b]. It provides a possibility to majorize explicitly the variation of argument of functions from Picard–Vessiot fields, along segments distant from the singular locus of a rational system. The accurate formulations are given in Section 4 while the proofs (completely independent from the rest of the paper) are moved to Appendix B.

To carry out the inductive process, it is necessary to construct a system with fewer singular points, isomonodromic with a given system in a prescribed domain U. This *isomonodromic surgery* is closely related to the Hilbert 21st problem (the *Riemann–Hilbert problem*), see [Bol00]. Its analytic core is the quantitative version of the matrix factorization problem, see Appendix C. It is this step that actually requires extending the class of Fuchsian systems to that allowing one regular non-Fuchsian point (see above).

2.2. Picard–Vessiot fields: definitions, notations. Consider a linear system (1.1) with rational coefficients matrix $A(t)$ and let $X(t)$ be a fundamental matrix solution of this system.

Consider the extension of the polynomial ring $\mathbb{C}[t]$ obtained by adjoining all entries $x_{ij}(t)$ of the chosen fundamental solution X: we will denote it by $\mathbb{C}[X]$ instead of a more accurate but certainly more cumbersome notation $\mathbb{C}[t, x_{11}(t), \dots, x_{nn}(t)]$. Moreover, from now on we will assume (to simplify the language) that the list of generators $X = \{x_{ij}\}$ of any Picard–Vessiot extension contains the independent variable t.

The ring $\mathbb{C}[X]$ can be identified with a subring of the ring $\mathcal{O}(t_*)$ of germs of analytic functions at any nonsingular point $t_* \notin \varSigma$. The field of fractions of $\mathbb{C}[X]$ will be denoted by $\mathbb{C}(X)$ and can be identified with a subfield of the field $\mathcal{M}(t_*)$ of germs meromorphic at t_*. The following properties of the ring $\mathbb{C}[X]$ and $\mathbb{C}(X)$ are obvious.

(1) The construction of $\mathbb{C}(X)$ does not depend on the choice of the fundamental solution X. For any two different nonsingular points the “realizations” of $\mathbb{C}(X)$ by fields of meromorphic germs at these points, are isomorphic to each other. However, this isomorphism (e. g., by analytic continuation along a path connecting these points, see below) is not canonical.

(2) Any element $f \in \mathbb{C}[X]$ (respectively, from $\mathbb{C}(X)$), if identified with a germ at $t_* \notin \varSigma$, can be analytically continued as a holomorphic (resp., meromorphic) function along any path avoiding $\varSigma$. If the path is a closed loop beginning and ending at t_*, then the result of such continuation again belongs to $\mathbb{C}[X]$ (resp., $\mathbb{C}(X)$). The corresponding *monodromy operator* is an automorphism of the ring (field), extending the linear transformation $X \mapsto XM_\gamma$ on the set of generators of $\mathbb{C}(X)$.

(3) For any simply connected domain U free from singular points of the system (1.1), $\mathbb{C}[X]$ and $\mathbb{C}(X)$ can be identified with a subring (resp., subfield) of the ring $\mathcal{O}(U)$ of holomorphic (resp., the field $\mathcal{M}(U)$ of meromorphic) functions in U. These subring and subfield will be denoted respectively by $\mathbb{C}[X|U]$ and $\mathbb{C}(X|U)$ and referred to as *restrictions* of $\mathbb{C}[X]$ and $\mathbb{C}(X)$ on the domain U.

(4) The field $\mathbb{C}(X)$ is closed by differentiation, and its subfield of constants is $\mathbb{C}$. By definition this means that $\mathbb{C}(X)$ is the *Picard–Vessiot extension* of $\mathbb{C}(t)$, see [Mag94]. Somewhat surprisingly, we will not use this fact directly.

(5) The ring $\mathbb{C}[X]$ and the field $\mathbb{C}(X)$ are naturally filtered by the degree. By definition, degree of an element $f \in \mathbb{C}[X]$ is the lowest possible degree d of any polynomial expression $\sum c_{k,\alpha} t^k X^\alpha$, $c_{k,\alpha} \in \mathbb{C}$, $|\alpha| + k \leqslant d$, representing $f(t)$ (there may be several such combinations, as we do not assume algebraic independence of the entries $x_{ij}(t)$ of X). Degree of a fraction $f = g/h$, $g, h \in \mathbb{C}[X]$, $f \in \mathbb{C}(X)$, is the maximum of $\deg g$, $\deg h$ for the “most economic” representation of f.

(6) The monodromy operators are degree-preserving.

2.3. Computability in the Picard–Vessiot fields. To say about *computability* of bounds on the number of isolated zeros of functions from Picard–Vessiot fields, one has in addition to the degree of functions introduce one or several natural parameters characterizing the field itself (i. e., the corresponding system (1.1) for that matter), in terms of which the bound should be expressed. Besides, when counting zeros one should be aware of the multivaluedness of functions from $\mathbb{C}(X)$.

The latter problem is resolved by the agreement to count zeros of functions only in triangular domains free from singular points. To address the former problem, we introduce two admissible *standard classes* of systems (1.1) with rational coefficients and in each case we list explicitly the natural parameters on which the bounds are allowed to depend.

2.3.1. *Fuchsian class.* Let n, m, r be three natural numbers.

Definition 3. The *Fuchsian class* $\mathcal{F}(n, m, r)$ consists of all linear $n \times n$-systems (1.1) with rational matrices $A(t)$ of the form (1.2), having no more than m singular points (including the one at infinity) and the height no greater than r,

$$\|A_1\| + \cdots + \|A_m\| + \|A_1 + \cdots + A_m\| \leqslant r.$$

This is the basic and the most important class. Its definition is invariant by conformal changes of the independent variable (Möbius transformations). However, for technical reasons the theorem on zeros has to be formulated and proved for Picard–Vessiot fields built from another type of linear systems.

2.3.2. *The special class.* This class consists of systems having only one regular eventually non-Fuchsian point at infinity. On the other hand, position of finite Fuchsian singularities is subject to an additional restriction.

Let again n, m, r be three natural numbers.

Definition 4. The *special* class $\mathcal{S}(n, m, r)$ consists of linear systems (1.1) with the matrix function $A(t)$ of the following form,

$$A(t) = \sum_{j=1}^{m} \frac{A_j}{t - t_j} + \sum_{k=0}^{r} A'_k\, t^k, \qquad A_j,\, A'_k \in \operatorname{Mat}_{n\times n}(\mathbb{C}), \tag{2.1}$$

such that:

(1) the finite singular points $t_1, \ldots, t_m$ are all on the real interval $[-1, 1] \subset \mathbb{C}$,

$$t_j \in \mathbb{R}, \quad |t_j| \leqslant 1, \qquad \forall j = 1, \ldots, m, \tag{2.2}$$

(2) all coefficients A_j, A'_k are real matrices, A_j, $A'_k \in \operatorname{Mat}_{n\times n}(\mathbb{R})$,

(3) the total norm of Laurent coefficients of the matrix $A(t)$ (including the singularity at infinity), is at most r,

$$\sum_{j=1}^{m} \|A_j\| + \sum_{k=0}^{r} \|A'_k\| \leqslant r, \tag{2.3}$$

(4) the singular point $t = \infty$ is regular, and any fundamental matrix solution $X(t)$ grows no faster than $|t|^r$ as $t \to \infty$ along any ray:

$$\|X(t)\| + \|X^{-1}(t)\| \leqslant c\,|t|^r, \qquad c > 0,\ \ |t| \to +\infty,\ \ \operatorname{Arg} t = \text{const}\,. \tag{2.4}$$

The parameter r will be referred to as the *height* of a system from the special class.

Remark 1. The choice of parameters may seem artificial, in particular, the assumption that r simultaneously bounds so diverse things as the residue norms, the growth exponent at infinity and the order of pole of $A(t)$ at the regular singular point $t = \infty$. Yet since our goal is only to prove computability without striving for reasonable explicit formulas for the bounds, it is more convenient to minimize the number of parameters, lumping together as many of them as possible.

In both cases, the tuple of natural numbers describing the class, will be referred to as *the parameters* of the class or even simply *the parameters* of a particular system belonging to that class. Accordingly, we will refer to Picard–Vessiot fields built from solutions of the corresponding systems, as Picard–Vessiot fields of Fuchsian or special type respectively, the tuple $(n,\ m,\ r)$ being labelled as *the parameters of the field* (dimension, number of ramification points and the height respectively). For the same reason we call Σ the ramification locus of the Picard–Vessiot field $\mathbb{C}(X)$.

2.3.3. *Definition of quasialgebraicity.* Let $\mathbb{C}(X)$ be a Picard–Vessiot field of a Fuchsian or special type. Recall that by $\mathbb{C}(X|T)$ we denote the restriction of the field $\mathbb{C}(X)$ on a domain $T \subset \mathbb{C}P^1$.

Definition 5. The field $\mathbb{C}(X)$ is called *quasialgebraic*, if the number of isolated zeros and poles of any function $f \in \mathbb{C}(X|T)$ in any *closed* triangular domain $T \Subset \mathbb{C} \smallsetminus \Sigma$ free from singular points of the corresponding system, is bounded by a primitive recursive function $\mathfrak{N}'(d,\ n,\ m,\ r)$ of the degree $d = \deg f$ and the parameters $n,\ m,\ r$ of the field.

Remark 2. It is important to stress that the bound for the number of zeros is allowed to depend only on the parameters of the class as a whole, and *not* on specific choice of the system. In other words, precisely as in Theorem 1, in any assertion on quasialgebraicity of certain Picard–Vessiot fields the upper bound for the number of isolated zeros and poles should be uniform over:

(1) all configurations of m singular points $t_1,\ \dots,\ t_m$,
(2) all triangles $T \subset \mathbb{C} \smallsetminus \{t_1,\ \dots,\ t_m\}$,
(3) all matrix coefficients $A_j,\ A'_k$ meeting the restrictions on the total norm that occur in the definition of the standard classes,
(4) all functions f of degree $\leqslant d$ from $\mathbb{C}(X)$.

Remark 3. To avoid overstretching of the language, everywhere below outside the principal formulations to be "computable" is synonymous to be bounded by a primitive recursive function of the relevant parameters (usually clear from the context). The triangles will be always closed.

2.4. Formulation of the results. The principal result of this paper is the following apparent generalization of Theorem 1.

Theorem 2 (quasialgebraicity of Fuchsian fields)**.** *A Picard–Vessiot field $\mathbb{C}(X)$ of Fuchsian type is quasialgebraic if the corresponding system* (1.1)–(1.2) *satisfies the spectral condition* (1.4)*.*

Actually this theorem is equivalent to Theorem 1. In the case $m = 1$ (i. e., for Euler systems) it follows from Lemma 1. In the general case Theorem 2 is obtained as a corollary to the similar statement concerning the special class.

Theorem 3 (main)**.** *A Picard–Vessiot field $\mathbb{C}(X)$ of special type is quasialgebraic if the corresponding system* (1.1), (2.1) *satisfies the spectral condition* (1.4)*.*

2.5. Notes, remarks. Several obvious remarks should be immediately made in connection with these definitions and formulations.

2.5.1. *Zeros versus poles.* To prove quasialgebraicity, it is sufficient to verify that any function f from the *polynomial ring* $\mathbb{C}[X]$ admits a computable bound for the number of zeros only. Indeed, for a rational function written as a ratio of two polynomials, the poles may occur only at the roots of the denominator, since the generators X are always analytic in T.

2.5.2. *Independence on generators.* Quasialgebraicity of the Picard–Vessiot fields is essentially independent of the choice of generators. Indeed, if $Y = \{y_1, \dots, y_k\} \subset \mathbb{C}(X)$ is another collection of multivalued functions such that $\mathbb{C}(Y) = \mathbb{C}(X)$ and the degrees $\deg y_i$ are bounded by $k \in \mathbb{N}$, then any function $f \in \mathbb{C}(Y)$ of degree d in Y is a rational combination of X of degree $\leqslant d' = 2kd$. Computability in terms of d or d' is obviously equivalent.

Example 2. The field $\mathbb{C}(X^{-1})$ generated by entries of the inverse matrix $X^{-1}(t)$, coincides with $\mathbb{C}(X)$. Indeed, $\det X \in \mathbb{C}[X]$ and therefore $\mathbb{C}(X^{-1}) \subseteq \mathbb{C}(X)$, and the role of X and X^{-1} is symmetric. Since X^{-1} is analytic outside Σ (has no poles), to verify quasialgebraicity of $\mathbb{C}(X)$ it is sufficient to study zeros of functions from $\mathbb{C}[X^{-1}]$. Note that in this case $\deg X^{-1} = n$, hence the two degrees $\deg_X$ and $\deg_{X^{-1}}$ on the same field $\mathbb{C}(X) = \mathbb{C}(X^{-1})$ are easily re-computable in terms of each other.

2.5.3. *Triangular and polygonal domains.* The restriction by triangular domains is purely technical. As follows from Proposition 1 below, one can instead choose to count zeros in any semialgebraic domains in $\mathbb{C} \simeq \mathbb{R}^2$ of known "complexity" (e. g., polygonal domains with a known number of edges). However, in this case the bound on zeros necessarily depends on the complexity of the domains. For instance, polygonal domains with a large number of edges may wind around singularities, spreading thus through many sheets of the Riemann surface.

Proposition 1. *Let Σ be a point set of m distinct points in $\mathbb{R}^2$ and $D \Subset \mathbb{R}^2 \smallsetminus \Sigma$ a compact simply connected semialgebraic domain bounded by finitely many real algebraic arcs of the degrees $d_1, d_2, \dots, d_k$ of total degree $d = d_1 + \dots + d_k$.*

Then D can be subdivided into at most $(d + \frac{1}{2})m(m + 1) + 2$ connected pieces such that each of them lies inside a triangular domain free from points of Σ.

Proof. Consider a partition of $\mathbb{R}^2 \smallsetminus \Sigma$ into triangles, for instance, slitting the plane $\mathbb{R}^2$ along some of the straight line segments connecting points of Σ. It can be also considered as a spherical graph G_1 with $m+1$ vertices (one "at infinity") and $M \leqslant m(m+1)/2$ algebraic edges of degree 1.

The domain D together with its complement $\mathbb{R}^2 \smallsetminus D$ can be also considered a spherical graph G_2 with k vertices, k edges and 2 faces.

Superimposing these two graphs yields a new graph with new vertices added at the points of intersection between the edges, and the number of edges increased because the newly added vertices subdivide some old edges. The number of such new vertices on each edge, straight or "curved", can be immediately estimated from Bézout theorem. As a result, the joint graph will have at most $M(d_1+\cdots+d_k+1) = M(d+1)$ straight edges, at most $(Md_i+1)+\cdots+(Md_k+1) = Md+k$ curvilinear algebraic edges and at least k vertices. By the Euler formula one can place an upper bound on the number of faces of the joint graph, $F = 2-k+E \leqslant 2+M(2d+1)$.

Thus D gets subdivided into a known number of connected components, each of them belonging to exactly one triangle T of the triangulation of $\mathbb{R}^2 \smallsetminus \Sigma$. This gives the required upper bound on the total number of triangles covering D. □

Having proved this proposition, we can freely pass from one simply connected domain to another, provided that they remain, say, bounded by a known number of line segments or circular arcs.

2.6. Relative quasialgebraicity. In order to carry out the induction in the number of singular points when proving Theorem 3, we need a relative analog of quasialgebraicity of a Picard–Vessiot field in a given domain U, eventually containing singularities. This technical definition means a possibility of explicitly count zeros and poles not "everywhere in $\mathbb{C}$" , but rather "in U" (with the standard provision for multivalued functions to be restricted on triangles). Unlike the global case, in the relative case the bound is allowed to depend on the distance between the boundary of U and the singular locus, yet this dependence must be computable.

This distance should take into account the eventual singular point at infinity, so for any point set $S \subset \mathbb{C}$ we define

$$\operatorname{dist}(S,\, \Sigma \cup \infty) = \inf_{t \in S,\ t_j \in \Sigma} \{|t-t_j|,\ |t^{-1}|\}. \tag{2.5}$$

If S is compact and disjoint with Σ, then this distance is strictly positive.

Let $U \subset \mathbb{C}$ be a polygonal domain (for our purposes it would be sufficient to consider only rectangles and their complements). Denote by $\lceil x \rceil$ the integer part of a real number $x > 0$ plus 1.

Definition 6. A Picard–Vessiot field $\mathbb{C}(X)$ built from solutions of a system (1.1) of Fuchsian or special type, is said to be *quasialgebraic in* U, if:

1. the boundary ∂U of the domain U is bounded and contains no singularities of the system (1.1), so that $\operatorname{dist}(\partial U,\, \Sigma \cup \infty) = \rho > 0$,
2. the number of isolated zeros and poles of any function $f \in \mathbb{C}(X)$ in any triangle $T \Subset U \smallsetminus \Sigma$ free from singular points, is bounded by a primitive recursive function $\mathfrak{N}''(d,\, n,\, m,\, r,\, s)$ of the degree $d = \deg f$, the parameters $n,\, m,\, r$ of the field and the inverse distance $s = \lceil 1/\rho \rceil \in \mathbb{N}$.

If the Riemann sphere $\mathbb{C}P^1$ is tiled by finitely many domains $U_1, \ldots, U_k$ and the field $\mathbb{C}(X)$ is quasialgebraic in each U_i, then it is globally quasialgebraic provided that the boundaries of the domains never pass too close to the singularities.

3. Alignment of singular points

3.1. Transformations of Picard–Vessiot fields. In this section we show how Theorem 3 implies Theorems 1 and 2.

This reduction consists in construction of an algebraic (ramified, multivalued) change of the independent variable $z = \varphi(t)$ using some rational function $\varphi \in \mathbb{C}(t)$, that "transforms" an arbitrary Fuchsian system (1.1)–(1.2) to another Fuchsian system with respect to the new variable z, having all singular points z_j on $[-1, 1]$ and only real residue matrices. Speaking loosely, one has to "fold" the Riemann sphere $\mathbb{C}P^1$ until all singular points (both the initial singularities and the singularities created when folding) fall on the unit segment. Meanwhile the norms of residues of the obtained system should remain explicitly bounded. The general idea of using such transformation belongs to A. Khovanskii [Kho95].

The word "transforms" appears in the quotation marks since the ramified algebraic changes of the independent variable do not preserve rationality of the coefficients. Indeed, if $X(t)$ is a (germ of) fundamental matrix solution to a linear system (1.1) with rational matrix of coefficients $A(t)$ and $z = \varphi(t)$ is a rational map of the t-sphere onto the z-sphere, the transform[1] $X'(z) = X(\varphi^{-1}(z))$ of $X(t)$ by the *algebraic* (*non*-rational) change of the independent variable $\psi = \varphi^{-1}$ does not in general satisfy a system of equations with rational coefficients. Performing the change of variables, we obtain

$$\frac{dX'(z)}{dz} = \left(1 \Big/ \frac{d\varphi}{dz}(\varphi^{-1}(z))\right) A(\varphi^{-1}(z))X'(z),$$

and see that the new matrix of coefficients has in general only algebraic entries. This means that the field $\mathbb{C}(\psi^* X)$ generated by entries of the pullback $\psi^* X = X \circ \psi = X \circ \varphi^{-1}$, is not a differential extension of $\mathbb{C}(z)$: it has to be completed.

Remark 4. For any multivalued function $f \in \mathbb{C}(X)$ ramified over Σ, its pullback (transform) $\psi^* f(z) = f(\varphi^{-1}(z))$ by $\psi = \varphi^{-1}$ is a multivalued function ramified over the union $\Sigma' = \varphi(\Sigma) \cup \operatorname{crit} \varphi$ consisting of the *direct* φ-image of Σ and the critical locus $\operatorname{crit} \varphi$ over which φ^{-1} is ramified.

Lemma 2 (Alignment of singularities of Fuchsian systems)**.** *For any Picard–Vessiot field $\mathbb{C}(X)$ of Fuchsian type, there can be constructed an algebraic transformation $\psi = \varphi^{-1}$ inverse to a rational map $\varphi \in \mathbb{C}(t)$, and a Picard–Vessiot field $\mathbb{C}(Y)$, also of Fuchsian type, such that*:

(1) $\psi^* \mathbb{C}(X) = \mathbb{C}(\psi^* X) \subseteq \mathbb{C}(Y)$,
(2) *for any $f \in \mathbb{C}(X)$ of degree d, the pullback $\psi^* f \in \mathbb{C}(Y)$ has degree $\leqslant d$ relative to $\mathbb{C}(Y)$,*
(3) *the ramification locus of $\mathbb{C}(Y)$ consists of only finite real points, and the corresponding residue matrices are all real,*

[1]The prime $'$ never in this article is used to denote the derivative.

(4) *the degree of the map* φ *and the parameters of the field* $\mathbb{C}(Y)$*, in particular its height, are bounded by computable functions of the parameters of the field* $\mathbb{C}(X)$*.*

In addition, if the spectral condition (1.4) *held for the initial Fuchsian system generating the field* $\mathbb{C}(X)$*, then it will also hold for the Fuchsian system generating the field* $\mathbb{C}(Y)$*.*

3.2. Special vs. Fuchsian systems: derivation of Theorem 2 from Theorem 3. Obviously, quasialgebraicity of $\mathbb{C}(Y)$, when proved, implies that its subfield $\mathbb{C}(\psi^* X)$ is also quasialgebraic. By Proposition 1, this means also quasialgebraicity of $\mathbb{C}(X)$. Indeed, φ-preimages and φ-images of triangles are semialgebraic domains of explicitly bounded complexity, provided the degree of the map φ is known.

As soon as all singular points of a Fuchsian system are finite and aligned along the real axis $\mathbb{R}$, by a suitable affine transformation of the independent variable they can be brought to the segment $[-1, 1]$. This transformation does not affect the residues of the Fuchsian system, therefore all additional conditions from the definition of the special class can be satisfied: in this case $A'_k = 0$ and the growth exponent at infinity is also zero.

The folding construction shows that the question on quasialgebraicity of an arbitrary Picard–Vessiot field $\mathbb{C}(X)$ of Fuchsian type can be reduced using Lemma 2 to the question on quasialgebraicity of a certain auxiliary field of special type, with explicitly bounded parameters.

The proof of Lemma 2 is completely independent of the rest of the paper and is moved to Appendix A. From this moment on we concentrate exclusively on the proof of Theorem 3.

4. Computability of index

4.1. Index. For a multivalued analytic function $f(t) \not\equiv 0$ ramified over a finite locus Σ, and an arbitrary piecewise-smooth oriented path $\gamma \subset \mathbb{C} \smallsetminus \Sigma$ avoiding this locus, denote by $V_\gamma(f)$ the index of f along γ.

More precisely, if f has no zeros on γ, then $\operatorname{Arg} f$ admits selection of a continuous branch along γ and the real number $V_\gamma(f)$ is defined as $\operatorname{Arg} f(\text{end}) - \operatorname{Arg} f(\text{start})$ (this number is an integer multiple of 2π if γ is closed). If f has zeros on γ, then $V_\gamma(f)$ is defined as

$$V_\gamma(f) = \limsup_{c \to 0} V_\gamma(f - c),$$

where the upper limit is taken over values $c \notin f(\gamma)$. By definition, index of the identically zero function is set equal to 0 along any path.

4.2. Index of segments distant from the critical locus. Consider a linear system (1.1), (2.1) from the special class $\mathcal{S}(n, m, r)$ with the singular locus Σ and the corresponding Picard–Vessiot field $\mathbb{C}(X)$.

If γ is a bounded rectilinear segment not passing through singular points, then the distance $\rho = \operatorname{dist}(\gamma, \Sigma \cup \infty)$ from γ to $\Sigma \cup \infty$, defined as in (2.5), is positive.

Lemma 3. *For any Picard–Vessiot field* $\mathbb{C}(X)$ *of special type the absolute value of index of any function* $f \in \mathbb{C}(X)$ *along any rectilinear segment* $\gamma \subset \mathbb{C} \smallsetminus \Sigma$ *is bounded* *by a computable function of $d = \deg f$, the parameters of the field and the inverse distance $s = \lceil 1/\rho \rceil$, $\rho = \operatorname{dist}(\gamma,\ \Sigma \cup \infty)$.*

This result is a manifestation of the general *bounded meandering principle*, see [NY97], [NY99b]. It is derived in Appendix B from theorems appearing in these references.

4.3. Index of small arcs. As the segment γ approaches the singular locus, the bound for the index along γ given by Lemma 3 explodes. However, for Picard–Vessiot fields of Fuchsian and special types, the index along small circular arcs around all finite singularities remains computable.

Lemma 4. *Assume that the Picard–Vessiot field $\mathbb{C}(X)$ is of Fuchsian or special type and $t_j \in \Sigma$ is a finite ramification point.*

Then there exist arbitrarily small circles around t_j (not necessarily centered at t_j but containing it strictly inside) with the following property. The absolute value of the index of any function $f \in \mathbb{C}(X)$ of degree d along any part of any such circle is bounded by a computable function of d and the parameters of the field.

In other words, index along some (not all) small circular arcs around every finite singularity is bounded by a computable function of the available data. The proof also appears in Appendix B.

4.4. Corollaries on quasialgebraicity. The above results on computability of the index immediately imply some results on relative quasialgebraicity.

Corollary 1. *Any Picard–Vessiot fields of Fuchsian or special type is relatively quasialgebraic in any domain U free from ramification points.*

Proof. Indeed, any "polynomial" $f \in \mathbb{C}[X]$ has an explicitly bounded index along any triangle $T \subset U$ by Lemma 3, since all sides of T are at least ρ-distant from Σ, where $\rho = \operatorname{dist}(\partial U,\ \Sigma \cup \infty)$. By the argument principle, this index bounds the number of zeros. The answer is given in terms of $\lceil \rho^{-1} \rceil$ and the parameters of the field. The assertion on zeros and poles of rational functions from $\mathbb{C}(X)$ follows from the remark in Section 2.5.1. □

If U has singular points of the system, still some of the solutions may be analytic or meromorphic (but not ramified) at these points. In this case Lemma 3 allows to prove quasialgebraicity of the appropriate *subfield*. Assume that U is a polygon (say, triangle) and $H = \{h_1, \dots, h_k\} \subset \mathbb{C}(X|U) \cap \mathcal{M}(U)$ is a collection (list) of functions from $\mathbb{C}(X)$ that after restriction on U are single-valued (i. e., meromorphic) there. Without loss of generality we may assume that h_i are in fact holomorphic in $U \setminus \Sigma$, multiplying them if necessary by the appropriate polynomials from $\mathbb{C}[t] \subset \mathbb{C}(X)$.

Corollary 2. *If $\mathbb{C}(X)$ is a Picard–Vessiot field of Fuchsian or special type and $H \subset \mathbb{C}(X) \cap \mathcal{O}(U)$ is a collection of branches holomorphic in a triangular domain U, then the subfield $\mathbb{C}(H) \subseteq \mathbb{C}(X)$ is quasialgebraic in U.*

Proof. The argument principle can applied to polynomials $f \in \mathbb{C}[H]$ and the boundary ∂U. Lemma 3 gives a computable upper bound on the number of zeros f in U hence in any triangle $T \subset U$. The case of rational functions from $\mathbb{C}(H)$ is treated as above. □

Remark 5. The domain U may be not a triangle but a polygon with any apriori bounded number of sides. As usual, if the bound for functions $f \in \mathbb{C}(H)$ is to be expressed in terms of the degree of f with respect to the initial generators, then the degree of H (the maximum of degrees of all h_i) must be explicitly specified, as explained in Section 2.5.2.

5. Isomonodromic reduction

The two corollaries from Section 4.4 suggest that an obstruction to the (relative) quasialgebraicity lies in the non-trivial monodromy of the Picard–Vessiot field. The results of this section show that this is the *only* obstruction: at least for Picard–Vessiot fields of special type, the monodromy group is the only factor determining whether the field is quasialgebraic or not.

5.1. Isomonodromic systems, isomonodromic fields. Let $U \subseteq \mathbb{C}$ be a (polygonal) bounded domain and

$$\dot X = A(t)X, \qquad \dot Y = B(t)Y, \qquad A,\, B \in \operatorname{Mat}_n(\mathbb{C}(t)), \tag{5.1}$$

two linear systems of the same size with rational coefficients. Denote by $\varSigma_A$ and $\varSigma_B$ their respective singular loci.

Definition 7. The two systems are said to be isomonodromic in the domain U, if:

(1) they have the same singular points inside U, $\varSigma_A \cap U = \varSigma_B \cap U = \varSigma$, and no singular points on the boundary ∂U, and
(2) all monodromy operators corresponding to loops *entirely belonging to* U, are simultaneously conjugated for these two systems.

Two Picard–Vessiot fields are said to be isomonodromic in U, if the respective linear systems are isomonodromic there.

Choosing appropriate fundamental solutions (e. g., their germs at a nonsingular point $t_* \in U \smallsetminus \varSigma$), one can assume without loss of generality that the monodromy factors are all equal:

$$\forall \gamma \subset U, \quad X^{-1} \cdot \Delta_\gamma X = Y^{-1} \cdot \Delta_\gamma Y.$$

The matrix ratio $H = XY^{-1}$ of *these* solutions is single-valued in U (note that this may not be the case for another pair of fundamental solutions). Note also that after continuing H along a path leaving U, we arrive at a different branch of H which may well be ramified even in U.

Let $U \subset \mathbb{C}$ be a rectangle with sides parallel to the real and imaginary axes and having no singularities on the boundary.

Lemma 5 (Isomonodromic reduction principle)**.** *Two Picard–Vessiot fields $\mathbb{C}(X)$ and $\mathbb{C}(Y)$ of the special type, isomonodromic in a rectangle U, are either both quasialgebraic, or both not quasialgebraic in U.*

More precisely, if one of two isomonodromic fields admits a computable upper bound $\mathfrak{N}_1(d, n, m, r)$ for the number of zeros (poles) in U, then the other also admits the similar bound $\mathfrak{N}_2(d, n, m, r, s)$, additionally depending (also in a computable way) on the inverse distance $s = \lceil \operatorname{dist}^{-1}(\partial U, \varSigma_A \cup \varSigma_B \cup \infty)\rceil$.

The proof of the lemma occupies the rest of this section. The strategy is, assuming that $\mathbb{C}(X)$ is quasialgebraic, to prove quasialgebraicity of the *compositum* $\mathbb{C}(X, Y)$, the Picard–Vessiot field generated by entries of both fundamental solutions. Clearly, this is sufficient since $\mathbb{C}(Y) \subseteq \mathbb{C}(X, Y)$ will in this case also be quasialgebraic. To count zeros of a function from $\mathbb{C}(X, Y)$ we use a modification of the popular derivation-division algorithm [Kho91], [IY91], [Rou98] based on the Rolle lemma for real functions. The role of the derivation in this modified algorithm is played by the operator $\operatorname{Im}(\cdot)$ of taking the imaginary part.

5.2. Real closeness. First we observe that the Picard–Vessiot field of special type is closed by taking real or imaginary parts. We say that the system (1.1) is *real*, if the coefficients matrix $A(t)$ of this system is real on $\mathbb{R}$.

Proposition 2. *Let $\sigma \subset \mathbb{R} \smallsetminus \Sigma$ be a real segment free from singularities of a real system* (1.1)*. Then for any function $f \in \mathbb{C}(X)$ there exist two functions from $\mathbb{C}(X)$, denoted respectively by $\operatorname{Re}_\sigma f$ and $\operatorname{Im}_\sigma f$, such that*

$$\operatorname{Re}(f|_\sigma) = (\operatorname{Re}_\sigma f)|_\sigma, \qquad \operatorname{Im}(f|_\sigma) = (\operatorname{Im}_\sigma f)|_\sigma.$$

The degree of both functions does not exceed $\deg f$.

In other words, the real/imaginary part of any branch of any function f on any *real* segment σ extends again on the whole set $\mathbb{C} \smallsetminus \Sigma$ as an analytic function from the same field $\mathbb{C}(X)$ and the degree is not increased.

Proof. Since the assertion concerns only the field, it does not depend on the choice of the fundamental solution $X(t)$. Choose this solution subject to an initial condition $X(t_*) = E \in \mathrm{GL}(n, \mathbb{R})$, where $t_* \in \sigma$ is arbitrary point on the segment. Then the solution remains real-valued on σ, $X(t) \in \mathrm{GL}(n, \mathbb{R})$ for any $t \in \sigma$, since the system (1.1) is real. The "obvious" operators Re_σ, Im_σ, defined on the generator set X of $\mathbb{C}(X)$ as

$$\operatorname{Re}_\sigma X = X, \quad \operatorname{Im}_\sigma X = 0; \qquad \operatorname{Re}_\sigma c = \operatorname{Re} c, \quad \operatorname{Im}_\sigma c = \operatorname{Im} c \quad \forall c \in \mathbb{C},$$

can be naturally and uniquely extended on the field $\mathbb{C}(X)$ of rational combinations with constant complex coefficients. They possess all the required properties. □

Remark 6. Note that, unlike single-valued functions, the result in general depends on the choice of σ (this explains the notation). Yet if the restriction $h \in \mathbb{C}(X|U)$ is a single-valued branch in a domain U intersecting the real axis, then $\operatorname{Re}_\sigma h$ and $\operatorname{Im}_\sigma h$ do not depend for the choice of a segment $\sigma \subset U \cap \mathbb{R}$. This immediately follows from the uniqueness theorem for meromorphic functions.

5.3. Index and real part: the Petrov lemma. Let $f\colon \sigma \to \mathbb{C}$ be a complex-valued function without zeros on a line segment $\sigma \subset \mathbb{C}$, and $g\colon \sigma \to \mathbb{R}$ is the imaginary part of f, a real-valued function on σ. We consider the case when g is real analytic on σ and hence has only isolated zeros. The following elementary but quite powerful trick was conceived by G. Petrov [Pet90].

Proposition 3. *If $g = \operatorname{Im} f \not\equiv 0$ has k isolated zeros on σ, then the index of f on σ is bounded in terms of k:*

$$|V_\sigma(f)| \leqslant \pi(k+1).$$

Proof. This is an "intermediate value theorem" for the map of the interval σ to the circle, sending $t \in \sigma$ to $f(t)/|f(t)|$. If $V_\sigma(f) \geqslant \pi$, then the image of f should cover two antipodal points on the circle, and hence contain one of the two arcs connecting them on the sphere. But any such arc intersects both real and imaginary axis.

The general case is obtained by subdividing σ on parts along which variation of argument is equal to π. □

5.4. Compositum of Picard–Vessiot fields. If $\mathbb{C}(X)$ and $\mathbb{C}(Y)$ are two Picard–Vessiot fields generated by fundamental solutions of the two systems (5.1), their compositum $\mathbb{C}(X, Y)$, the field generated by entries of both fundamental solutions, is again a Picard–Vessiot field. This is obvious: the block diagonal matrix $\operatorname{diag}\{X, Y\}$ satisfies a linear system with the block diagonal coefficients matrix $\operatorname{diag}\{A, B\}$. Notice that this system will be of Fuchsian (resp., special) class if both systems (5.1) were from this class. The dimension, number of singularities and the height of the system generating the compositum field, are computable in terms of the respective parameters of the original fields.

As a consequence, for any function $f \in \mathbb{C}(X, Y)$ its index along segments or arcs described in Lemmas 3 and 4, is explicitly computable in terms of the degree of f and the parameters of the two fields.

5.5. Demonstration of Lemma 5. As was already mentioned, the goal would be achieved if we prove that the compositum $\mathbb{C}(X, Y)$ in U is quasialgebraic provided that $\mathbb{C}(X)$ is quasialgebraic there. The proof begins with a series of preliminary simplifications.

1. We assume that the fundamental solutions X, Y are chosen with coinciding monodromy factors, so that their matrix ratio $H = YX^{-1}$ is single-valued hence meromorphic in U. Its degree in $\mathbb{C}(X, Y)$ is $n+1$ (Example 2). Thus to prove the lemma, it is sufficient to prove quasialgebraicity of the field $\mathbb{C}(X, H)$:

$$\mathbb{C}(Y) \subset \mathbb{C}(X, Y) = \mathbb{C}(X, YX^{-1}) = \mathbb{C}(X, H).$$

2. By Remark 6, $\operatorname{Re} H$ and $\operatorname{Im} H$ are unambiguously defined matrix functions, holomorphic in $U \smallsetminus \Sigma$ and real on $U \cap \mathbb{R}$. Replacing H by $\operatorname{diag}\{\operatorname{Re} H, \operatorname{Im} H\}$ if necessary, we may assume that H itself is real on $\mathbb{R} \cap U$.

3. Since H has no poles in $U \smallsetminus \Sigma$, to prove quasialgebraicity it is sufficient to consider only "polynomials in H", i. e., the ring $\mathbb{C}(X)[H] \subset \mathbb{C}(X)(H) = \mathbb{C}(X, H)$, cf. with Section 2.5.1. By the assumed quasialgebraicity of $\mathbb{C}(X)$, the number of poles of each such function, represented as $\sum x_\alpha(t) H^\alpha$, $x_\alpha \in \mathbb{C}(X)$, is computable in terms of $\deg x_\alpha \leqslant \deg f$, the parameters of the field $\mathbb{C}(X)$ and the relative geometry of U and $\Sigma_A \cup \Sigma_B$. Therefore it remains only to majorize the number of zeros of functions from the ring $\mathbb{C}(X)[H]$.

4. Any function from $f = \mathbb{C}(X)[H]$ can be represented as the finite sum

$$f = \sum_{i=1}^{\nu} x_i h_i, \qquad x_i \in \mathbb{C}(X), \;\; h_i \in \mathbb{C}[H], \;\; \operatorname{Im} h_i|_{\mathbb{R}} = 0. \tag{5.2}$$

The number ν and the degrees $\deg_X x_i$, $\deg_H h_i$ are all explicitly computable in terms of $\deg_{X,Y} f$ and n.

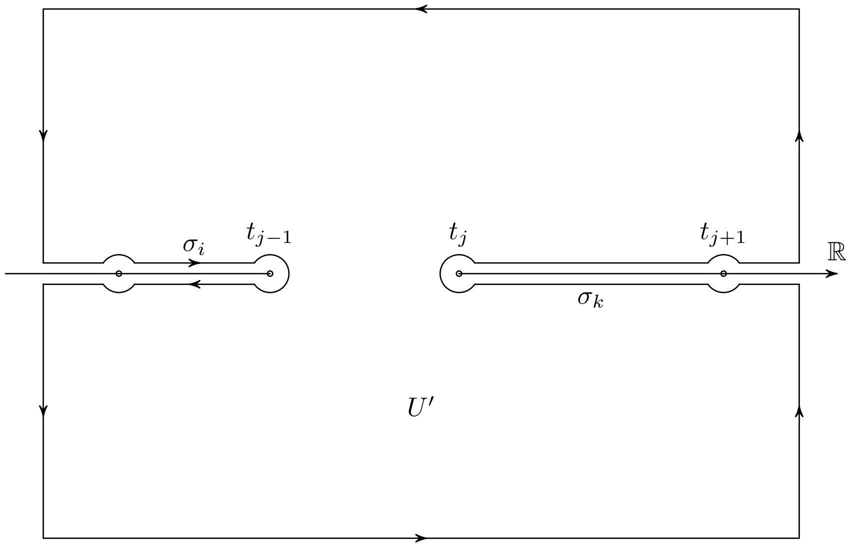

FIGURE 1. Slit rectangle U

After all this preparatory work we can prove by induction in ν, the number of terms in (5.2), that the number of isolated zeros of any function of such form in any triangle $T \subset U \smallsetminus \varSigma$, is computable in terms of the parameters of the fields $\mathbb{C}(X)$ and $\mathbb{C}(Y)$, the degrees $\deg_X x_i$ and $\deg_H h_i$ and the distance from U to $\varSigma_A \cup \varSigma_B \cup \infty$. The proof is very much similar to the popular "differentiation–division algorithm" except that the role of derivation play the operators $\operatorname{Im}_{\sigma_i}$ for which functions h_j behave like constants.

5. For $\nu = 1$ the assertion is immediate. Zeros of $x_1 h_1$ in any triangle $T \subset U \smallsetminus \varSigma$ are either zeros of $x_1 \in \mathbb{C}(X)$ or zeros of h_1. The number of zeros of the first kind in any triangle is bounded by virtue of quasialgebraicity of $\mathbb{C}(X)$. The number of zeros of second kind is bounded by Corollary 2, since h_1 is holomorphic with computable index along the boundary. Thus the base of induction is established.

6. For a sum (5.2) involving $\nu > 1$ terms, we first divide f by x_ν. This division may result in a loss or acquisition of a computable number of zeros or poles (occurring as poles and zeros of x_ν respectively). The degree of x_j, $j \neq 1$, as well as the degree of f may increase at most by $\deg x_i \leqslant d$. After such division we may without loss of generality assume that $x_\nu \equiv 1$, that is,

$$f = h_\nu + \sum_{i=1}^{\nu-1} x_i h_i, \qquad h_1, \ldots, h_\nu \in \mathbb{C}[H], \quad x_1, \ldots, x_{\nu-1} \in \mathbb{C}(X).$$

Being convex, an arbitrary triangle $T \subset U \smallsetminus \varSigma$ may intersect $\mathbb{R}$ by at most one connected segment: $T \cap \mathbb{R} \subseteq \sigma \subset \mathbb{R} \smallsetminus \varSigma$, where σ is one of the intervals between consecutive real singularities. Consider the domain obtained by slitting U along the two rays $\mathbb{R} \smallsetminus \sigma$ and in addition delete small disks around singular points as explained in Lemma 4. The result will be a simply connected domain U', still containing T, whose boundary consists of three types of arcs/segments (see Fig. 1):

(1) the "distant part", the union of 6 segments belonging to the original boundary of ∂U,
(2) "small arcs" around singular points $t_j \in \varSigma$ (semi-circular or circular),
(3) at most $2m$ real segments $\sigma_j \subset \mathbb{R} \smallsetminus \varSigma$ (upper and lower "shores" of the slits).

As before, the number of poles of f in U' is bounded by a computable function (all of them must be poles of the functions $x_i \in \mathbb{C}(X)$). Thus to majorize the number of zeros of f in T, it is sufficient to majorize the index of the boundary $\partial U'$. The contribution from the first two parts (the part sufficiently distant from the singular locus and small arcs around Fuchsian singularities) is computable by Lemmas 3 and 4. What remains is to estimate the contribution of the index of f along different real segments σ_i with endpoints at the singular locus.

7. The index $V_\sigma(f)$ along each such segment $\sigma = \sigma_j$ by Proposition 3 is bounded in terms of the number of zeros of the imaginary part $(\operatorname{Im} f)|_\sigma$ which coincides with $\operatorname{Im}_\sigma f$ on σ. Computation of $\operatorname{Im}_\sigma f$ yields (recall that $x_\nu = 1$)

$$\operatorname{Im}_\sigma f = \operatorname{Im}_\sigma h_\nu + \sum_{i=1}^{\nu-1} \operatorname{Im}_\sigma(x_i h_i) = 0 + \sum_{i=1}^{\nu-1} h_i \operatorname{Im}_\sigma x_i = \sum_{i=1}^{\nu-1} h_i \widetilde{x}_i, \tag{5.3}$$

where the functions $\widetilde{x}_i = \operatorname{Im}_\sigma x_i$ again belong to $\mathbb{C}(X)$ by Proposition 2.

8. This computation allows to conclude that the contribution of each segment σ_j to the total index of the boundary is majorized by the number of zeros on $\sigma_j \subset U \smallsetminus \varSigma$ of an auxiliary function $f_j \in \mathbb{C}(X)[H]$ which already appears represented by (5.3) as a sum of $\leqslant \nu - 1$ terms in the pattern (5.2). By the inductive assumption, this number is bounded by a computable function of the degree, the parameters and the distance between U and $\varSigma$.

Thus the total number of zeros of f in U' is shown to be bounded by a primitive recursive function of admissible parameters (the algorithm for computing this function is summarized in Section 5.7 below). The proof of Lemma 5 is complete. □

5.6. Quasialgebraicity near infinity. The singular point at $t = \infty$ is non-Fuchsian for systems of the special class, and hence it deserves a separate treatment.

Let U be a *complement to the rectangle*,

$$U = \mathbb{C} \smallsetminus \{|\operatorname{Re} t| < 2, \ |\operatorname{Im} t| < 1\}$$

and $\mathbb{C}(Y)$ a Picard–Vessiot field of special type with the given parameters n, m, r (we retain the notation compatible with that used earlier in the proof of Lemma 5). Denote by γ the (interior) boundary of U. By definition of the special class, all finite ramification points of $\mathbb{C}(Y)$ are outside U.

Lemma 6. *The field $\mathbb{C}(Y)$ is quasialgebraic in U provided that the spectral condition (1.4) holds for the "large loop" γ going around infinity.*

Proof. Despite its different appearance, this is again an isomonodromic reduction principle: the role of the second system is played by an appropriate Euler system.

1. If the monodromy operator M for the large loop has all eigenvalues on the unit circle, then there exists an Euler system

$$\dot{X} = t^{-1}AX, \qquad A \in \operatorname{Mat}_n(\mathbb{C}),$$

isomonodromic with the initial system in U, whose matrix residue A has only real eigenvalues in the interval $[0, 1)$. Since conjugacy (gauge transformation) by constant nondegenerate matrices does not affect the isomonodromy, one can assume that the norm of A is bounded, say, by n.

2. The Picard–Vessiot field $\mathbb{C}(X)$ is quasialgebraic by Lemma 1 globally, hence in U as well.

3. To prove quasialgebraicity of $\mathbb{C}(Y) \subseteq \mathbb{C}(X, H)$, where $H = YX^{-1}$, we follow the same strategy. More precisely, in this case choose U' to be the domain U slit along the negative real semiaxis and with a "small neighborhood of infinity" (i. e., complement to a disk of large radius) deleted.

As before, we have to majorize the number of roots of functions from $\mathbb{C}(X)[H]$ via the index of the boundary. This bound will serve all triangles $T \subset U$ not intersecting the negative semiaxis (if T intersects the negative semiaxis, the slit must be along the positive semiaxis).

4. The boundary of U' also consists of three types of segments/arcs:

(1) the "distant part", the union of 5 segments belonging to the original boundary of $\gamma = \partial U$,
(2) one circular "small arc" γ_∞ around infinity,
(3) 2 real segments $\sigma_j \subset \mathbb{R} \smallsetminus \Sigma$ (upper and lower "shores" of the slit).

The only difference from the previous case is that the index of γ_∞ is not guaranteed to be computable, since Lemma 4 does not apply to the infinite singular point. Yet the following simple argument shows that for functions from $\mathbb{C}(X)[H]$ the index is bounded *from above*. In case the monodromy at infinity is trivial, this would correspond to bounding the order of pole; note that the lower bound for the above index (corresponding to the multiplicity of a root at the singularity $t = \infty$) would be much harder to majorize (cf. with [Bol02]).

Proposition 4. *If the orientation on γ_∞ is chosen counterclockwise (as a part of $\partial U'$), then for any function $f \in \mathbb{C}(X)[H] \subset \mathbb{C}(X, Y)$ the index $V_{\gamma_\infty}(f)$ is bounded from above by a computable function of $d = \deg_{X,H} f$ and the parameter r of the special class only.*

Proof of the proposition. We claim that any function $f \in \mathbb{C}(X)[H] \subseteq \mathbb{C}(X, Y)$ admits an asymptotic representation $f(t) = t^\lambda \ln^k t(c + o(1))$ with a real *growth exponent* λ, natural $k < n$ and a nonzero complex constant $c \in \mathbb{C}$ near infinity (more precisely, in any slit neighborhood of infinity). Moreover, we claim that the growth exponent λ is bounded from above in terms of $\deg_{X,H} f$ and the parameters of the field $\mathbb{C}(X, Y)$.

Indeed, for functions $\mathbb{C}[X]$ this is obvious, since $\mathbb{C}[X]$ as a ring is generated over $\mathbb{C}$ by monomials $t^\lambda \ln_k t$ with $\lambda \in [0, 1)$ and $k \leqslant n$. For the field $\mathbb{C}(X)$ it follows from the fact that a ratio of two functions with the specified asymptotics is again a function with the same asymptotical representation and the exponent being the difference of the growth exponents. Thus the growth exponent for $f \in \mathbb{C}(X)$ with $\deg f = d$ is between $-d$ and d.

By definition of the special class, all entries of Y and Y^{-1} grow no faster than t^r as $t \to \infty$ along any ray. This means that H, H^{-1} may have a pole of order $\leqslant r+1$ at infinity, hence the growth exponent of any function $f \in \mathbb{C}[H]$ of degree d can be at most $(r+1)d$.

Since the growth exponent of a sum is the maximum of the growth exponents of the terms entering with nonzero coefficients, the growth exponent of any $f \in \mathbb{C}(X)[H]$ is bounded in terms of $\deg f$ and r as asserted.

It remains only to note that the index of any function $f = t^\lambda \ln^k t(c+o(1)$ along a circular arc with sufficiently large radius and the angular length less than 2π, is no greater than $2\pi\lambda + 1$. This proves the proposition. □

5. The proof of Lemma 6 is completed exactly as the proof of Lemma 5. Namely, to majorize inductively the index of any sum (5.2) along the boundary $\partial U'$, we estimate the contribution of the "distant part" by Lemma 3, the contribution of the arc γ by Proposition 4. The contribution of two remaining real intervals is majorized by the number of zeros of two auxiliary functions (the imaginary parts of f/x_ν) having the same structure (5.2) but with fewer terms, $\nu - 1$. □

5.7. Algorithm. The above proof of Lemma 5 is algorithmic (for Lemma 6 this is even more transparent). Let $f \in \mathbb{C}(Y)$ be an arbitrary function of degree d. The procedure aimed to majorize the number of its zeros in U (i. e., in any triangle $T \subset U \smallsetminus \Sigma$) looks as follows.

1°. Using the identity $Y = H^{-1}X$, we can express f as an element of $\mathbb{C}(X, H)$ and then as a ratio of two "polynomials", $f = f_1/f_2$, $f_1, f_2 \in \mathbb{C}(X)[H]$. The degree d' of f_i in X, H is immediately computable, as well as the number of terms $\nu \leqslant d'$ in the representation (5.2).

2°. We start from the list of two functions $\mathcal{L} = \{f_1, f_2\}$, both written in the form of a sum (5.2). Then we transform this list, repeatedly using the following two rules:

(1) if $f = \sum_{i=1}^k x_i h_i$ is in $\mathcal{L}$ and $x_k \not\equiv 1$, then f is replaced by 2 functions, x_k and $f/x_k = (\sum_{i=1}^{k-1} x_i/x_k) + h_k$;
(2) if $x_k \equiv 1$, then $f \in \mathcal{L}$ is replaced by $2m-2$ functions $\operatorname{Im}_{\sigma_j} f$, the imaginary parts of f on each of the real segments σ_j, $j = 1, \dots, 2(m-1)$.

These steps are repeated no more than $2d'$ times, after which $\mathcal{L}$ contains no more than $(2m)^{d'}$ functions, all of them belonging to $\mathbb{C}(X)$.

3°. After each cycle, the maximal degree $\max\deg x_i$ of all coefficients of all functions in $\mathcal{L}$ is at most doubled (the imaginary part does not change it, division by the last term increases at most by the factor of 2). Thus all functions obtained after the loop is finished, will have explicitly bounded degrees not exceeding $d'' = d' \cdot 2^{d'}$.

4°. Since the field $\mathbb{C}(X)$ is quasialgebraic, the number of isolated zeros of all functions is bounded in terms of their degrees and the parameters of the field $\mathbb{C}(X)$.

It remains to add the computed bounds together, adding the computable contribution coming from the index of all "distant" and "small" arcs occurring in the process. This contribution is bounded by a computable function of d'', the maximal degree ever to occur in the construction, the inverse distance from ∂U to the combined singular locus, and the parameters of the compositum field $\mathbb{C}(X, Y)$. The algorithm terminates. It will later be inserted as a "subroutine" into one more simple "loop algorithm" to obtain the computable bound in Theorem 3.

6. Isomonodromic surgery

Existence of a system isomonodromic to a given one in a given domain may be problematic in the Fuchsian class, but such system can be always found in the special class of systems. Moreover, the parameters of this new system can be explicitly majorized in terms of the parameters of the initial system.

6.1. Matrix factorization problem. Let $R = \{a < |t| < b\} \subset \mathbb{C}$ be a circular annulus centered at the origin on the complex plain and $H(t)$ a matrix function, holomorphic and holomorphically invertible in the annulus. It is known, see [Bir13], [GK60], [FM98], [Bol00], that H can be factorized as follows,

$$H(t) = F(t)\, t^D G(t), \qquad D = \operatorname{diag}\{d_1, \dots, d_n\},$$

where the matrix functions F and G are holomorphic and holomorphically invertible in the sets $U = \{|t| < b\} \subset \mathbb{C}$ and $V = \{|t| > a\} \cup \{\infty\} \subset \mathbb{C}P^1$ respectively (their intersection is the annulus R), and D is the diagonal matrix with the integer entries uniquely defined by H.

We need a quantitative version of this result. Unfortunately, it is impossible to estimate the sup-norms of the matrix factors F and G and their inverses in terms of the matrix norm of H, as we would like to have. Still a weaker assertion in this spirit can be proved.

Assume that a given matrix function H, holomorphic and holomorphically invertible in the annulus R, admits factorization by two matrix functions F, G which satisfy the following properties,

$$\begin{aligned} &H(t) = F(t)G(t), \\ &F,\, F^{-1} \quad \text{holomorphic and invertible in } U = \{|t| < b\}, \\ &G,\, G^{-1} \quad \text{holomorphic and invertible in } V = \{a < |t| < +\infty\} \\ &\qquad\qquad \text{and have a finite order pole at } t = \infty. \end{aligned} \tag{6.1}$$

A matrix function $G(t)$, having a unique pole of finite order $\leqslant \nu$ at infinity, can be represented as follows,

$$G(t) = \widetilde{G}(t) + G_1 t + G_2 t^2 + \cdots + G_\nu t^\nu, \qquad G_i \in \operatorname{Mat}_n(\mathbb{C}), \tag{6.2}$$

with $\widetilde{G}(t)$ holomorphic and *bounded* in $\{|t| > a\}$. Similarly, if the inverse matrix $G^{-1}(t)$ has a unique pole at infinity, then

$$G^{-1}(t) = \widetilde{G}'(t) + G_1' t + G_2' t^2 + \cdots + G_\nu' t^\nu, \qquad G_i' \in \operatorname{Mat}_n(\mathbb{C}), \tag{6.3}$$

with a holomorphic bounded term $\widetilde{G}'(t)$.

We will say that the decomposition (6.1), (6.2)–(6.3) is *constrained* in a smaller annulus $R' = \{a' < |t| < b'\} \Subset R$, $a < a' < b' < b$, by a finite positive constant C, if (as usual, we minimize the number of constraining parameters)

$$\begin{aligned} \|F(t)\| + \|F^{-1}(t)\| &\leqslant C, & |t| &< b, \\ \|\widetilde{G}(t)\| + \|\widetilde{G}'(t)\| &\leqslant C, & |t| &> a, \\ \|G_1\| + \cdots + \|G_\nu\| + \|G_1'\| + \cdots + \|G_\nu'\| + \nu &\leqslant C. \end{aligned} \tag{6.4}$$

Let $R' = \{a' < |t| < b'\} = U' \cap V' \Subset R$ be the annulus, $a' = a + \frac{1}{8}(b-a) > a$, $b' = b - \frac{1}{8}(b-a) < b$, and $U' = \{|t| < b'\}$, $V' = \{|t| > a'\}$.

Lemma 7 (Constrained matrix factorization)**.** *Assume that the conformal width $b/a-1$ and the exterior diameter of the annulus R are constrained by the inequalities*

$$b < q, \qquad b/a > 1 + 1/q, \qquad q \in \mathbb{N}. \tag{6.5}$$

Assume that the matrix function H is bounded together with its inverse in R,

$$\|H(t)\| + \|H^{-1}(t)\| \leqslant q', \qquad a < |t| < b, \quad q' \in \mathbb{N}. \tag{6.6}$$

Then there exist two matrix functions F and G, holomorphic and holomorphically invertible in smaller circular domains U' and V' respectively, such that the decomposition (6.1) *is constrained in R' by a bound C given by a computable* (*primitive recursive*) *function $\mathfrak{C}(q, q')$ of the natural parameters q, q'.*

If H is real on intersection with the real axis $R \cap \mathbb{R}$, then F and G can be also found real on $\mathbb{R}$.

The proof is given in Appendix C.

6.2. Isomonodromic surgery. Consider a linear system (1.1), (2.1) from the special class $\mathcal{S}(n, m, r)$ and assume that its singular locus Σ consists of two subsets Σ_1, Σ_2 sufficiently well apart. We prove that the system is isomonodromic to a "simpler" system (having fewer singular points), in appropriate simply neighborhoods of each subset Σ_i. The proof is constructive and we show that the new systems can be also chosen from the special classes with the respective parameters (n, m_i, r_i), $i = 1, 2$, explicitly bounded in terms of the parameters of the initial system and the width of the gap between Σ_1 and Σ_2.

Lemma 8. *Assume that the singular locus Σ of a system* (1.1), (2.1) *from the special class is disjoint with the annulus $R = \{a < |t| < b\}$ constrained by a natural parameter $q \in \mathbb{N}$ as in* (6.5).

Then there exists a system from the special class, isomonodromic with the initial system (1.1) *in the disk $\{|t| < b\}$ and having no other finite singularities.*

The parameters of the new system are bounded by a computable function of the "width parameter" $q \in \mathbb{N}$ of the annulus and the parameters of the initial system.

The monodromy group of the new system satisfies the spectral condition (1.4) *if this condition was satisfied by the initial system* (1.1).

Proof. The construction is fairly standard, see [Bol00], [AI88]. We have only to verify the computability of all the relevant constraints.

Let γ be the loop running counterclockwise along the middle circle $\{|t| = \frac{1}{2}(a+b)\}$ of the annulus and $X(t)$ a solution defined by the condition $X(t_*) = E$ at some positive real point t_* inside the annulus.

By construction the loop γ cannot be very close to $\varSigma$ and hence the matrix $A(t)$ is explicitly bounded on γ. By the Gronwall inequality the monodromy factor M of this solution and its inverse M^{-1} both have the norm explicitly bounded in terms of the available data, hence the spectrum of M belongs to some annulus $R^* \subset \mathbb{C}$ on the λ-plane (actually we will need only the case when this spectrum belongs to the unit circle).

The annulus R^* can be slit along a meridian $\{\operatorname{Arg} t = \text{const}\}$ so that the boundary γ^* of the resulting simply connected domain will be sufficiently distant from the spectrum of M. Consider the matrix logarithm defined by the matrix Cauchy integral [Lan69],

$$A_0 = \frac{1}{2\pi i} \oint_{\gamma^*} (\lambda E - M)^{-1} \ln \lambda \, d\lambda \in \operatorname{Mat}_{n\times n}(\mathbb{C}).$$

By construction, $\|A_0\|$ is bounded by a computable function of the admissible parameters, and the multivalued function $X'(t) = t^{A_0}$ has the same monodromy along the loop γ as the initial system. Therefore the matrix ratio $H(t) = X(t)(X'(t))^{-1}$ is single-valued, and holomorphic. By the Gronwall inequality, H, H^{-1} are bounded in the smaller sub-annulus $R'' = \{a'' < |t| < b''\}$, $a'' = a + \frac{1}{4}(b-a)$, $b'' = b - \frac{1}{4}(b-a)$, together with the inverse H^{-1} by a computable function.

Consider the factorization (6.1) of H, described in Lemma 7 (with R replaced by R''). Then the two multivalued functions $Y(t) = F^{-1}(t)X(t)$ and $Z(t) = G(t)X'(t)$, the first defined in U' and the other in V', coincide on the intersection $R' = U' \cap V'$, so that the meromorphic matrix-valued 1-forms $dY \cdot Y^{-1}$ and $dZ \cdot Z^{-1}$, defined in the domains $\{|t| < b\}$ and $\{|t| > a\}$, in fact coincide on the intersection R'. Hence they are restrictions of some globally defined meromorphic matrix 1-form $B(t)\,dt$ with a rational matrix function $B(t)$.

We claim that $B(t)$ has a form (2.1) and compute (i. e., majorize) its number of singularities m and height r. Indeed, by construction $B(t)$ is real on $R \cap \mathbb{R}_+$ hence everywhere on $\mathbb{R}$, and

$$B(t) = \begin{cases} -F^{-1}(t) \cdot \dot F(t) + F^{-1} A(t) F(t), & |t| < b, \\ \dot G(t) G^{-1}(t) + t^{-1} G(t) A_0 G^{-1}(t), & |t| > a. \end{cases}$$

The terms $F^{-1}(t) \cdot \dot F(t)$ and $\dot G(t) \cdot G^{-1}(t)$ are holomorphic at all finite points of their respective domains of definition. Thus the matrix function $B(t)$ has finite poles only at those poles of $A(t)$, which are inside the disk $\{|t| < a\}$. The residue of $B(t)$ at a singular point t_j is $F^{-1}(t_j) A_j F(t_j)$, where A_j is the residue of $A(t)$ at this point. Since residues of the initial system were bounded by the parameters of the special class while the decomposition (6.1) was explicitly constrained, all residues of $B(t)$ at all finite points are real and explicitly bounded.

It is easy to verify that the order of pole of $B(t)$ at $t = \infty$ and its Laurent coefficients are bounded: this follows from computability of the order of pole of G and G^{-1} at infinity and the explicit bounds (6.4) on the matrix Laurent coefficients G_i, G_i'.

It remains to verify the regularity of the singular point at infinity and compute the growth exponent of the solution $V(t)$. This is also obvious, since the growth exponents of G, G^{-1} are explicitly bounded whereas the growth exponents of $X'(t)$ and its inverse are bounded by $\|A_0\|$.

The last assertion of the lemma (on the monodromy group) is obvious: any simple loop avoiding the part of the spectrum inside the annulus, can be deformed to remain also inside the annulus, hence avoiding also singularities outside. The gauge transformation replacing X by $F^{-1}X$, does not affect the monodromy factors for the loops on which the transformation is defined. Hence the spectrum of all such loops remains the same for the initial and the constructed system. □

7. Demonstration of the main Theorem 3

In this section we prove our main result, Theorem 3, by induction in the number of finite singular points. The idea is to break the singular locus Σ into two (both nonempty) subsets sufficiently apart from each other, cover them by two disjoint rectangles U_1, U_2 whose boundaries are distant from Σ, and find for each rectangle a system from the special class (with fewer singularities), isomonodromic to the given one. Application of the isomonodromic reduction principle would yield then a computable global upper bound for the number of zeros of solutions.

7.1. Stretch and break. Given a system (1.1) from the special class (2.1) with m finite singular points, we cover the complex plane $\mathbb{C}$ by three polygonal domains U_1, U_2, U_3 (two rectangles and a complement to a rectangle, all with sides parallel to the imaginary and real axes) so that in each of these domains the problem of counting zeros and poles of functions from $\mathbb{C}(X)$ is reduced to the same problem restated for the three auxiliary fields $\mathbb{C}(Y_i)$, $i = 1, 2, 3$, each having no more than $m-1$ ramification points.

The first step is to stretch the independent variable t so that the singular locus Σ is not collapsing. By definition of the standard class, $\Sigma \subset [-1, 1]$, and one can always make an affine transformation (stretch) $t = at' + b$, $a, b \in \mathbb{R}$, $0 < a \leqslant 1$, $|b| \leqslant 1$, which would send the extremal points $t_{\min} < t_{\max}$ of Σ to ± 1.

Such transformation preserves the form (2.1) of the coefficients matrix, eventually affecting only the magnitude of the Laurent coefficients A'_k at infinity. However, because of the inequalities $|a| < 1$, $|b| < 1$ the impact will be limited. Indeed, the linear transform $t = at'$ with $|a| < 1$ can only decrease the norms $\|A'_i\|$, while the bounded translation $t = t' + b$, $|b| < 1$, may increase them by a factor at most 2^p.

Thus without loss of generality one can assume that the endpoints of Σ coincide with ± 1. Since there are m singular points, one of the intervals, say, (t_j, t_{j+1}) should be at least $2/(m-1)$-long. Therefore the straight line $\ell = \{\operatorname{Re} t = \frac{1}{2}(t_j + t_{j+1})\}$ subdivides the singular locus into two parts, each of them at least $1/(m-1)$-distant from ℓ.

Let $U = \{|\operatorname{Re} t| \leqslant 2,\ |\operatorname{Im} t| < 1\}$ be the rectangle containing inside it the singular locus. Denote by U_1 and U_2 two parts, on which U is subdivided by ℓ, and let $U_3 = \mathbb{C} \smallsetminus U$ be the complement.

For each $\Sigma_i = \Sigma \cap U_i$, $i = 1, 2$, one can construct the annulus satisfying the assumptions of Lemma 8 with $q \leqslant m$ (eventually, translated by no more than 1 along the real axis). Thus there exist two systems, isomonodromic with the initial system (1.1), in the domains U_1, U_2, whose boundaries are at least $1/(m-1)$-distant from all singular points. In U_3 the initial system has only one singularity (at infinity) and hence is isomonodromic with an appropriate Euler system generating the corresponding Picard–Vessiot field $\mathbb{C}(Y_3)$.

Notice that all three auxiliary systems have fewer (at most $m-1$) finite singular points, and all belong to the appropriate special classes whose parameters are expressed by computable functions of the parameters of the initial system.

7.2. Algorithm for counting zeros. Given a Picard–Vessiot field $\mathbb{C}(X)$ of special type described by the given set of parameters n, m, r and a function $f \in \mathbb{C}(X)$ of degree d from this field, we proceed with the following process.

1°. Stretch the singular locus Σ and subdivide $\mathbb{C}$ into three domains U_1, U_2, U_3 as explained in Section 7.1, and for each subdomain construct the auxiliary field $\mathbb{C}(Y_i)$, $i = 1, \ldots, 3$, isomonodromic to $\mathbb{C}(X)$ in U_i. Each is again a special Picard–Vessiot field with no more than $m-1$ points and all other parameters explicitly bounded.

2°. By the isomonodromic reduction principle (Lemma 5 for U_1, U_2 and Lemma 6 for U_3) the question on the number of zeros of f in U_i can be reduced to that for finitely many functions $f_{i,\alpha}$ from $\mathbb{C}(Y_i)$, each of them having the degree (relative to the corresponding field) no more than 2^d. This step of the algorithm is realized by the "subroutine" described in Section 5.7 which itself is an iterated loop. Yet the number of iterations is no greater than $O(2^d)$.

3°. The steps 1° and 2° have to be repeated for each of the functions $f_{i,\alpha}$ of known degrees in the respective Picard–Vessiot fields of special type with explicitly known parameters (actually, only the height has to be controlled).

4°. After m loops of the process explained before, all functions would necessarily belong to the Euler fields. Their number, their degrees and the heights of the corresponding Euler fields will be bounded by a computable function of the parameters of the process. Since the spectral condition (1.4) for each of these fields will be satisfied, Lemma 1 gives an upper bound for the number of isolated zeros of each of these functions in any triangle free from singularities.

5°. Assembling together all these inequalities for the number of zeros together with the computable contribution from the index of "distant" boundary segments and "small" arcs, gives an upper bound $\mathfrak{N}'(d, n, m, r)$ for the number of zeros of the initial function f of degree d.

6°. The running time of this algorithm is obviously bounded by an explicit expression depending only on d, n, m, r. As is well-known [MR67], [Mac72], the bound itself must be therefore a primitive recursive function of its arguments. This completes the proof of Theorem 3.

7.3. Fields with pairwise distant ramification points. The arguments described above prove also the following result described earlier in Section 1.4.

Assume that the natural number $s \in \mathbb{N}$ is so large that the singular points $t_1, \ldots, t_m$ of the Fuchsian system (1.1), (1.2) satisfy the inequalities

$$|t_i - t_j| \geqslant 1/s, \quad |t_i| \leqslant s, \qquad s \in \mathbb{N}. \tag{7.1}$$

Theorem 4. *A Picard–Vessiot field $\mathbb{C}(X)$ of Fuchsian type is quasialgebraic, provided that its singular points $t_1, \dots, t_m$ satisfy the inequality (7.1) and all residues $A_1, \dots, A_m, A_\infty = -\sum_1^m A_j$ have only real eigenvalues. Besides the parameters n, m, r, the bound for the number of zeros in this case depends on the additional parameter s.*

Proof. To prove Theorem 4, it is sufficient to subdivide $\mathbb{C}$ into several rectangles U_i, $i = 1, \dots, m$ each containing only one singularity, with boundaries sufficiently distant from the singular locus (the parameter s controls this distance from below). The spectral condition for the small loops γ_j around each singular point t_j, finite or nor, is automatically verified: the eigenvalues $\mu_{k,j}$ of the corresponding monodromy operators $M_j \in \mathrm{GL}(n, \mathbb{C})$ are exponentials of the eigenvalues $\lambda_{k,j}$ of the residues A_j, $\mu_{k,j} = \exp 2\pi i \lambda_{k,j}$ [Bol00], [AI88].

Application of the isomonodromic reduction principle allows to derive quasialgebraicity of $\mathbb{C}(X)$ from that for each Euler field. Note that in this case there no induction in the number of singular points is required, the isomonodromic surgery is a trivial exercise of constructing a matrix logarithm and the spectral condition becomes an algebraic assumption on the residues. □

Since any three points on the sphere $\mathbb{C}P^1$ can always be placed by an appropriate conformal isomorphism to $0, 1, \infty$ so that (7.1) is satisfied with $s = 1$, we have the following corollary.

Corollary 3. *Any Picard–Vessiot field with 3 ramification points, is quasialgebraic, provided that all three residue matrices have only real eigenvalues.*

Appendix A. Folding of Fuchsian systems

We first show how to complete the field $\mathbb{C}(\psi^* X)$ to a Picard–Vessiot field $\mathbb{C}(Y)$ in the case when $\psi(z) = \sqrt{z}$ is the algebraic map inverse to the *standard fold* $\varphi_0 \colon t \mapsto t^2$. Our main concern is how to place an upper bound on the height of the "folded" system.

Then we construct the rational map φ as an alternating composition of simple folds and conformal isomorphisms of $\mathbb{C}P^1$ so that it would align all singularities of any given Fuchsian system (1.1)–(1.2) on the real axis, while increasing in a controllable way the height r and the dimension n. On the final step we symmetrize the system to achieve the condition that all residues are real.

A.1. Simple fold. Consider the Fuchsian system (1.1)–(1.2), assuming that both $t = 0$ and $t = \infty$ (the critical values of the standard fold φ_0) are nonsingular for it:

$$t_j \neq 0, \quad j = 1, \dots, m, \qquad \sum_{j=1}^m A_j = 0. \tag{A.1}$$

We claim that the collection of $2n$ functions $(x_1(t), \dots, x_n(t), tx_1(t), \dots, tx_n(t))$, after the change of the independent variable $t = \sqrt{z}$ satisfies a Fuchsian system of $2n$ linear ordinary differential equations which can be obtained by separating the even and odd parts $A_\pm(\cdot)$ of the matrix $A(t) = A_+(t^2) + t\,A_-(t^2)$.

More precisely, the (column) vector function $y(z) = (x(\sqrt{z}),\ \sqrt{z}\,x(\sqrt{z}))$ satisfies the system of linear ordinary differential equations with a rational matrix of coefficients $B(z)$,

$$\begin{aligned} &\frac{dy}{dz} = B(z)y, \qquad B(z) = \frac{1}{2}\begin{pmatrix} A_-(z) & z^{-1}A_+(z) \\ A_+(z) & A_-(z) + z^{-1}E \end{pmatrix}, \\ &A_+(z) = \sum_{j=1}^{m} \frac{t_j A_j}{z - z_j}, \quad A_-(z) = \sum_{j=1}^{m} \frac{A_j}{z - z_j}, \qquad z_j = t_j^2. \end{aligned} \tag{A.2}$$

This can be verified by direct computation. The explicit formulas (A.2) allow to compute residues of B, showing that it is indeed Fuchsian under the assumptions (A.1). Note that even if there are two points $t_j = -t_i$ with $z_i = z_j$, the folded system nevertheless has only simple poles.

The points $z_j = t_j^2$ are finite poles for $B(z)$ with the corresponding residues B_j given by the formulas

$$B_j = \frac{1}{2}\begin{pmatrix} A_j & z_j^{-1}A_j \\ z_j A_j & A_j \end{pmatrix}, \qquad j = 1, \ldots, m. \tag{A.3}$$

In addition to these singular points (obviously, simple poles), the matrix function $B(z)$ has two additional singular points $z = 0$ and $z = \infty$. The residues at these points are

$$B_0 = \frac{1}{2}\begin{pmatrix} 0 & -\sum z_j^{-1}A_j \\ 0 & E \end{pmatrix}, \qquad B_\infty = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ -\sum z_j A_j & -E \end{pmatrix}. \tag{A.4}$$

Inspection of these formulas immediately proves the following result.

Proposition 5. *The transform $\mathbb{C}(\psi^* X)$ of the Picard–Vessiot field of Fuchsian type satisfying the condition* (A.1) *is a subfield of the Picard–Vessiot field $\mathbb{C}(Y)$ built from solutions of the system* (A.2).

This system has the double dimension, its singular locus on $\mathbb{C}P^1$ consists of the squares $z_j = t_j^2$, $j = 1, \ldots, m$, and the points 0 and ∞.

The height r' of the folded system (A.2) *is bounded by a computable function of the height r of the initial system and the inverse distance $s = \lceil \operatorname{dist}^{-1}(\Sigma, \{0, \infty\}) \rceil = \lceil \max_{j=1,\ldots,m}(|t_j|,\ |t_j^{-1}|) \rceil$ between the polar locus of* (1.1) *and the critical locus of the fold.*

Of course, this "computable function" can be immediately written down, but our policy is to avoid explicit formulas.

A.2. Monodromy of the folded system. We show now that the spectral condition (1.4) is preserved by the standard fold. This is not completely trivial.

The proof begins by constructing a fundamental matrix solution $Y(z)$ to the system (A.2). Let $z = b$ be a nonsingular point for the latter, in particular, $b \neq 0, \infty$, and denote by $t_+(z)$, $t_-(z)$, two branches of the root $\sqrt{z}$ for z near b, with $t_+(b) = -t_-(b)$ denoted by a. Let $X_\pm(t)$, be two germs of two fundamental matrix solutions for the system (1.1) defined near the points $\pm a$ (they must not necessarily be analytic continuations of each other). Then the formal computations of the previous sections show that (the germ at a of) the analytic matrix function

$$Y(z) = \begin{pmatrix} X_+(t_+(z)) & X_-(t_-(z)) \\ t_+(z)\, X_+(t_+(z)) & t_-(z)\, X_-(t_-(z)) \end{pmatrix}, \qquad t_\pm \colon (\mathbb{C}, b) \to (\mathbb{C}, \pm a), \tag{A.5}$$

satisfies the system (A.2). In order to show that it is nondegenerate, subtract from its $(n+k)$th row the kth row multiplied by $t_+(z)$. The result will be a block upper triangular matrix with two $n \times n$-blocks $X_+(t_+(z))$ and $(t_-(z) - t_+(z))\, X_-(t_-(z))$ on the diagonal, each of which is nondegenerate near $z = b$ since $b \neq 0, \infty$.

Proposition 6. *If for the system* (1.1) *all monodromy operators associated with simple loops have eigenvalues only on the unit circle, then this also holds true for the folded system* (A.2)*.*

Proof. Consider any simple loop γ on the z-plane, starting at the point a and avoiding the folded locus $\Sigma' = \{0, z_1, \dots, z_m, \infty\}$. Denote by $\gamma_\pm$ its two preimages on the t-plane, obtained by analytic continuation of the branches $t_\pm(z)$ along γ. Two cases are to be distinguished.

If γ was not separating 0 and ∞, then both branches are single-valued on γ and hence both $\gamma_\pm$ will be closed loops on the t-plane. Denote by $M_\pm$ the respective monodromy matrices for $X_\pm$ along $\gamma_\pm$. Clearly, the monodromy of the whole matrix $Y(z)$ along such a loop will be block-diagonal: $\Delta_{\gamma_\pm} X_\pm = X_\pm M_\pm$, $\Delta_{\gamma_\pm} t_\pm = t_\pm$, and hence

$$\Delta_\gamma \begin{pmatrix} X_+ & X_- \\ t_+X_+ & t_-X_- \end{pmatrix} = \begin{pmatrix} X_+ & X_- \\ t_+X_+ & t_-X_- \end{pmatrix} \cdot \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix}.$$

The spectrum of such matrix is the union of spectra of $M_\pm$, hence the assertion of the lemma trivially holds.

In the other case when γ separates 0 and ∞, the branches of $t_\pm$ will change sign after going around it, hence the images $\gamma_\pm = t_\pm(\gamma)$ will be non-closed arcs connecting two symmetric points $\pm a$. However, making the second turn around γ will restore the closedness: both $\gamma'_+ = \gamma_- \circ \gamma_+$ and $\gamma'_- = \gamma_+ \circ \gamma_-$ will be *simple* closed loops issued from the points $t_+(a)$ and $t_-(a)$ respectively. Denote by $M'_\pm$ the monodromy matrices of the solutions $X_\pm$ along the corresponding loops $\gamma'_\pm$. Then, since $\Delta_{\gamma'_\pm} t_\pm = (-1)^2 t_\pm = t_\pm$, the *square* $\Delta_\gamma^2 Y$ of the monodromy factor for the solution Y will be block-diagonal, $\operatorname{diag}(M'_+, M'_-)$ and hence its spectrum is on the unit circle.

But if the eigenvalues of the square of a matrix are on the unit circle, then the eigenvalues of the matrix itself are also of unit modulus, and the proof of the lemma is complete. □

A.3. Placement of poles by a conformal isomorphism. Applying the simple fold that preserves the real axis can decrease the number of non-real poles, if some of them were on the imaginary axis. Thus combining simple folds with real shifts $t \mapsto t + c$, $c \in \mathbb{R}$ sufficiently many times, one may hope to place all poles of the resulting system on the real axis.

However, it can happen that after a shift one of the poles would appear at or arbitrarily closely to $t = 0$. This will result in an arbitrarily large height of the folded system (A.2). Besides, the point $t = \infty$ becomes a pole after the first fold.

In order to be able to iterate folding, instead of shifts (affine translations) between the folds we will make conformal isomorphisms that will keep all singularities away from the critical locus $\{0, \infty\}$ of the standard fold. Such isomorphisms can be always found.

Let $\rho = \rho_m = \pi/(2m-2)$ be the constant depending only on the number m of the points.

Proposition 7. *Let $t_1, \ldots, t_m$ be any m points on $\mathbb{C}P^1$ (real or not), and $t_1 \notin \mathbb{R}P^1$. Then there exists a conformal isomorphism of the sphere $\mathbb{C}P^1$ which takes t_1 into $i = +\sqrt{-1}$, preserves the real line and such that the images of the points $t_2, \ldots, t_m$ are all in the annulus $\rho_m \leqslant |z| \leqslant \rho_m^{-1}$.*

Proof. After an appropriate affine transformation preserving $\mathbb{R}$, we may already assume that $t_1 = +i$ while other points t_j, $j = 2, \ldots, m$ can be arbitrary.

Using the (inverse) stereographic projection, we can identify $\mathbb{C}P^1$ with the Euclidean sphere $S^2 \subset \mathbb{R}^3$ so that the real line becomes its equator E, whereas the pair of points $\pm i$ is mapped into the north and south poles N, S respectively. The points 0 and ∞ become a pair of antipodal points on E.

The conformal isomorphism we are looking for, will be constructed as a rigid rotation of the sphere around the axis NS so that some two opposite points of the equator, sufficiently distant from all other points $t_2, \ldots, t_m$, will occupy the positions for 0 and ∞ respectively. Such a pair of opposite points always exists for the following simple reason.

Consider $2m-2$ points, $t_2, \ldots, t_m$ and their antipodes on the sphere, and a spherical cap of geodesic radius less than $\rho = \rho_m = \pi/(2m-2)$ around each point. The union of these caps cannot completely cover the equator E whose length is 2π. Therefore there is a point b on the equator, at least ρ-distant from all a_j and their antipodes. Hence both b and anti-b are ρ-distant from all $t_2, \ldots, t_m$.

It remains to observe that after the rotation sending b to 0 and anti-b to ∞ and returning to the initial affine chart on $\mathbb{C}P^1$ by the (direct) stereographic projection, the distance from 0 to any of t_j, equal to $|t_j|$, will be no smaller than its spherical counterpart, and the same for the distance $|1/t_j|$ from t_j to infinity. Since the stereographic projection of the sphere on the plane is conformal (and obviously so is the rigid rotation of the sphere), the result will be a conformal isomorphism as required. □

A.4. Proof of Lemma 2. We construct an alternate sequence of conformal isomorphisms and simple folds as follows. Assuming that $k \geqslant 0$ singularities already belong to the real axis, choose one that is not in $\mathbb{R}$. Using the conformal transformation described in Proposition 7, place the chosen point to $+\sqrt{-1}$ while preserving the real line and keeping all other singularities away from $\{0, \infty\}$. Then after the standard fold the number of poles of the system off the real line will be by one less than before and the construction continues by induction in k.

After at most $m-3$ folds all singular points of the initial system will be aligned along the real axis (without loss of generality the first three points can be assumed already real).

The chain of folds and conformal isomorphisms determines the chain of Fuchsian systems (defined on different copies of $\mathbb{C}P^1$): we start from the initial system and then either make a conformal change of the independent variable (the procedure that does not change neither the residues nor the monodromy group) or replace the system by the folded system (A.2) of the double dimension. If the intermediate conformal isomorphisms are chosen as in Proposition 7, then norms of the residues of the suspended system after the subsequent simple folds could exceed those of the initial system at most by a constant factor depending only on m. All this follows from Proposition 5.

The spectral condition (1.4) is also preserved in this suspension (Proposition 6). Thus the system obtained at the end satisfies all assertions of the Lemma 2 except that its residue matrices are possibly not real yet. We show that this can be corrected by one more doubling of dimension, this time involving reflection in the real axis rather than reflection in the origin.

Let $A(t)$ be a rational matrix function not necessarily real for $t \in \mathbb{R}$ but with all singular points only in $\mathbb{R} \cup \infty$. Denote by $A^*(t) = \overline{A(\overline{t})}$ another, also rational matrix function. Together with the system $\dot{x} = A(t)x$ consider its "mirror image" $\dot{x}^* = A^*(t)x^*$ obtained by reflection in the real axis, where x^* is the new dependent complex vector variable.

Notice that $A^*(t)$ and $A(t)$ take conjugate values for all $t \in \mathbb{R}$. The variables $u = \frac{1}{2}(x + x^*)$ and $v = \frac{1}{2i}(x - x^*)$, $u, v \in \mathbb{C}^n$, together satisfy the system of equations

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} A + A^* & i(A - A^*) \\ -i(A - A^*) & A + A^* \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

and the matrix of this system is real on $\mathbb{R}$.

Applying this last dimension doubling to the system obtained on the preceding step, we construct a new system with the matrix of coefficients real on $\mathbb{R}$. This completes the proof of Lemma 2. ◻

Appendix B. Zeros of functions defined by systems of polynomial ordinary differential equations

In this section we derive Lemmas 3 and 4 from the general theorems on zeros of functions defined by systems of polynomial differential equations [NY99b], [Yak99].

B.1. A complex analog of the Vallée Poussin theorem. Consider a linear nth order differential equation with analytic coefficients $a_k(t)$ bounded in a domain $U \subset \mathbb{C}$,

$$\begin{aligned} &y^{(n)} + a_1(t)\, y^{(n-1)} + \cdots + a_{n-1}(t)\dot{y} + a_n(t)y = 0, \\ &|a_k(t)| \leqslant C \qquad \forall t \in U \subset \mathbb{C}, \qquad k = 1, \dots, n. \end{aligned} \tag{B.1}$$

The following result can be considered as a complex generalization of a known result by C. de la Vallée Poussin on non-oscillation of solutions of linear equations in the real domain [dlVP29].

Theorem I [Yak99]**.** *If $y = f(t) \not\equiv 0$ is a nonzero solution of the equation* (B.1) *and $\gamma = [t_0, t_1] \subset U$ a rectilinear segment of length $|\gamma|$, then the index $V_\gamma(f)$ of f along γ is explicitly bounded from two sides,*

$$|V_\gamma(f)| \leqslant \pi(n+1)(1+3|\gamma|C). \quad \square \tag{B.2}$$

B.2. Index of small circular arcs around a Fuchsian singularity. If (B.1) has a singularity (pole of the coefficients) in U, the coefficients become unbounded and Theorem I is not applicable. Yet if the singularity is Fuchsian (i. e., if the coefficient a_k in (B.1) has a pole of order $\leqslant k$ at this point, see [AI88], [Inc44]), then the index can be majorized along small circular arcs centered at the singular point.

Let $\mathcal{E} = t\frac{d}{dt}$ be the Euler differential operator and $\widetilde{a}_1(t), \dots, \widetilde{a}_n(t)$ holomorphic germs at $(\mathbb{C}, 0)$. Consider the differential equation

$$\mathcal{E}^n y + \widetilde{a}_1(t)\mathcal{E}^{n-1}y + \cdots + \widetilde{a}_{n-1}\mathcal{E}y + \widetilde{a}_n(t)y = 0, \qquad \mathcal{E} = t\frac{d}{dt}, \;\; t \in (\mathbb{C}, 0). \tag{B.3}$$

After expanding the powers $\mathcal{E}^k$ and division by t^n the equation (B.3) becomes an equation of the form (B.1) with coefficients $a_k(t)$ having a pole of order $\leqslant k$ (i. e., Fuchsian) at the origin $t = 0$. Conversely, any linear equation (B.1) having a Fuchsian singular point at the origin, can be reduced to the form (B.3) with appropriate analytic coefficients $\widetilde{a}_k(t)$ obtained as linear combinations of holomorphic germs $t^k\, a_k(t)$ with bounded constant coefficients.

In the "logarithmic time" $\tau = \ln t$ the Euler operator $\mathcal{E}$ becomes the usual derivative $\frac{d}{d\tau}$ and hence the whole equation (B.3) takes the form (B.1) with the coefficients $a_i(\tau) = \widetilde{a}_i(e^\tau)$ defined, analytic and $2\pi i$-periodic, hence bounded, in a shifted left half-plane $\{\operatorname{Re}\tau < -B\}$ for some $B \in \mathbb{R}$. Circular arcs of angular length φ, $0 < \varphi < 2\pi$, with center at the origin, become rectilinear segments of (Euclidean) length $\varphi \leqslant 2\pi$ in the chart τ. Application of Theorem I to this equation proves the following result.

Corollary 4. *Assume that the coefficients of the Fuchsian equation* (B.3) *are explicitly bounded at the origin, $|\widetilde{a}_i(0)| \leqslant C$, $i = 1, \dots, n$.*

Then index $V_\gamma(f)$ of any solution $f \not\equiv 0$ along any sufficiently small circular arc $\gamma \subset \{|t| = \varepsilon\}$ of angular length less than 2π is explicitly bounded from two sides,

$$|V_\gamma(f)| \leqslant \pi(n+1)(1+6\pi C). \quad \square \tag{B.4}$$

A globalization of this result looks as follows. Let $\Sigma = \{t_1, \dots, t_m\} \subset \mathbb{C}$ be a finite point set, $t_i \neq t_j$, δ a monic polynomial with simple roots on Σ and $\mathcal{D}$ the differential operator of the following form,

$$\mathcal{D} = \delta(t)\frac{d}{dt}, \qquad \delta(t) = (t-t_1)\cdots(t-t_m) \in \mathbb{C}[t]. \tag{B.5}$$

Consider the linear differential equation with Fuchsian singularities on Σ,

$$\mathcal{D}^n y + a_1(t)\,\mathcal{D}^{n-1}y + \cdots + a_{n-1}(t)\,\mathcal{D}y + a_n(t)y = 0, \tag{B.6}$$

where $a_k \in \mathbb{C}[t]$ are polynomials in t. We will refer to the sum of absolute values of (complex) coefficients of a polynomial $p \in \mathbb{C}[t]$ as its *height*.

Proposition 8. *If the degrees and heights of $\delta \in \mathbb{C}[t]$ and all polynomial coefficients $a_k \in \mathbb{C}[t]$ of the equation (B.6) do not exceed $r \in \mathbb{N}$, then for any nonzero solution f of this equation*:

(1) *if γ is a line segment such that $\operatorname{dist}(\gamma, \Sigma \cup \infty) \geqslant 1/s$ for some $s \in \mathbb{N}$, then $|V_\gamma(f)|$ is bounded by a computable function of n, m, r and s;*

(2) *if t_j is sufficiently distant from other singularities, i. e., $\operatorname{dist}(t_j, \Sigma \smallsetminus \{t_j\}) \geqslant 1/s$ for some $s \in \mathbb{N}$, and $\gamma_\varepsilon \subset \{|t - t_j| = \varepsilon\}$ is a simple circular arc around t_j, then $\limsup_{\varepsilon \to 0} |V_{\gamma_\varepsilon}(f)|$ is bounded by a computable function of n, m, r and s.*

Proof. Both assertions follow from Theorem I and Corollary 4.

To prove the first assertion, one has to pass from the representation (B.6) to the expanded form

$$y^{(n)} + \frac{\widetilde{a}_1(t)}{\delta(t)}\, y^{(n-1)} + \frac{\widetilde{a}_2(t)}{\delta^2(t)}\, y^{(n-2)} + \cdots + \frac{\widetilde{a}_{n-1}(t)}{\delta^{n-1}(t)}\, \dot y + \frac{\widetilde{a}_n(t)}{\delta^n(t)}\, y = 0 \tag{B.7}$$

and use the explicit lower bound for $|\delta(t)|$ on γ. To prove the second assertion assuming for simplicity that $t_j = 0$, one has to substitute $\mathcal{D} = \widetilde{\delta}(t) \cdot \mathcal{E}$, where $\widetilde{\delta} = \delta/t$, and reduce (B.6) to the form (B.3). The lower bound on $\widetilde{\delta}(0)$ coming from the fact that all other singular points are at least $1/s$-distant from $t = 0$, implies then an upper bound for $\widetilde{a}_j$. $\square$

B.3. From systems to high order equations. Consider a system (1.1) of linear ordinary differential equations with rational coefficients and represent its matrix $A(t)$ in the following form,

$$A(t) = \frac{1}{\delta(t)}\, P(t), \qquad P = \sum_{k=0}^{d} P_k t^k, \quad \delta(t) = \prod_{j=1}^{m} (t - t_j), \tag{B.8}$$

where $P(t)$ is a matrix polynomial. The height of P is defined as the total norm of its matrix coefficients $\sum_k \|P_k\|$. Note that the denominator $\delta(t)$ is a monic polynomial.

The system (1.1), (B.8) can be reduced to one high order linear equation (B.6) or (B.7) in a variety of ways. Yet only a few of them allow for an explicit control of the height of coefficients of (B.6) in terms of the height of P and δ in (B.8). One of such reductions is outlined below (another approach, recently suggested in [Gri01], is based on completely different ideas and may eventually lead to better bounds).

Let $p_0(t, x) = c_1 x_1 + \cdots + c_n x_n$ be an arbitrary linear form on $\mathbb{C}^n$ with constant coefficients $c_1, \ldots, c_n \in \mathbb{C}$. Consider the sequence of polynomials $p_i(t, x) \in \mathbb{C}[t, x_1, \ldots, x_n]$, $i = 1, 2, \ldots$, defined by the recurrent formula

$$p_{i+1}(t, x) = \delta(t) \frac{\partial p_i(t, x)}{\partial t} + \sum_{j,k=1}^{n} \frac{\partial p_i(t, x)}{\partial x_j} P_{jk}(t) x_k \tag{B.9}$$

where $P_{jk}(t) \in \mathbb{C}[t]$ are entries of the polynomial matrix function $P(t)$. In other words, p_{i+1} is the Lie derivative of the polynomial $p_i \in \mathbb{C}[t, x]$ along the polynomial vector field in $\mathbb{C}^{n+1}$,

$$\delta(t)\frac{\partial}{\partial t} + \sum_{j,k=1} P_{jk}(t)x_k\frac{\partial}{\partial x_j} \tag{B.10}$$

orbitally equivalent to the system (1.1), (B.8). Obviously, all p_i are linear forms in the variables $x_1, \ldots, x_n$.

The chain of polynomial ideals

$$(p_0) \subseteq (p_0, p_1) \subseteq (p_0, p_1, p_2) \subseteq \cdots \subseteq (p_0, \ldots, p_k) \subseteq \cdots \tag{B.11}$$

in the Noetherian ring $\mathbb{C}[t, x]$ eventually stabilizes. Hence for some $\ell < +\infty$ one must necessarily have an inclusion

$$p_\ell \in (p_0, \ldots, p_{\ell-1}),$$

meaning that for an appropriate choice of polynomials $b_1, \ldots, b_\ell \in \mathbb{C}[t, x]$, the identity $p_\ell = \sum_{i=1}^{\ell} b_i p_{\ell-i}$ must hold. Since all p_i are linear forms in x_i, truncation of this identity to keep only linear terms implies that

$$\begin{gathered} p_\ell(t, x) + a_1(t)p_{\ell-1}(t, x) + \cdots + a_\ell(t)p_0(t, x) = 0, \\ a_i(t) = -b_i(t, 0) \in \mathbb{C}[t], \end{gathered} \tag{B.12}$$

with *univariate* polynomial coefficients $a_i(t) \in \mathbb{C}[t]$. Restricting this identity on the solution $x(t)$ of the linear system (1.1), (B.8), we immediately see that its kth derivative $\frac{d^k}{dt^k}y$ is the restriction $p_k(t, x(t))$ on the same solution. The conclusion is that the function $y = p_0(t, x(t))$ satisfies the equation (B.6) with the polynomial coefficients $a_i(t) \in \mathbb{C}[t]$, having Fuchsian singularities only at the roots of δ.

The polynomial identity (B.12) is by no means unique (this concerns both the length ℓ and the choice of the coefficients a_i). However, one can *construct* the identity (B.12) of *computable* length ℓ and with coefficients $a_i \in \mathbb{C}[t]$ of *explicitly bounded heights*.

Assume that a rational matrix function $A = P/\delta$ of the form (B.8) has explicitly bounded degree and height:

$$\deg P,\, \deg\delta \leqslant m, \qquad \text{height of } P,\, \delta \leqslant r. \tag{B.13}$$

Theorem II [NY99b, especially Appendix B]**.** *There exist explicit elementary functions, $\mathfrak{l}(n, m)$ and $\mathfrak{r}(n, m, r)$ with the following properties.*

For any linear system (1.1)–(B.8) constrained by (B.13), and any linear form $p_0(x)$ with constant coefficients, one can construct polynomial identity between the iterated Lie derivatives (B.12) so that its length and the degrees of the coefficients are bounded by $\mathfrak{l}$ and their heights by $\mathfrak{r}$ respectively. □

This is a particular case of a more general theorem proved in [NY99b] for Lie derivatives along an arbitrary polynomial (not necessarily linear) vector field in any dimension. The functions $\mathfrak{l}$ and $\mathfrak{r}$ can be explicitly computed and their growth rates for large values of n, m, r estimated. These growth rates are very large:

$$\begin{gathered} \mathfrak{l}(n, m) \leqslant n^{m^{O(m^2)}}, \qquad \mathfrak{r}(n, m, r) \leqslant (2+r)^{\mathfrak{m}(n,m)}, \\ \mathfrak{m}(n, m, m) \leqslant \exp\exp\exp\exp(4n\ln m + O(1)) \qquad \text{as } n, m \to \infty. \end{gathered} \tag{B.14}$$

The enormity of these bounds is another reason why we never attempted to write explicitly the bounds whose computability is asserted by our main results (Theorems 1, 2 and 3). However, the simpler assertion of Theorem 4 can eventually be made explicit with not-too-excessive bounds based on the mentioned results of Grigoriev [Gri01].

B.4. Proof of Lemma 3. Assertion of this lemma immediately follows from Theorem II and Proposition 8 in view of the construction from Section B.3. Indeed, a linear system from the special class $\mathcal{S}(n, m, r)$ has the coefficients matrix $A(t)$ which, after reducing to the common denominator, can be written as a ratio $A = P/\delta$ with explicitly bounded height and degree of the polynomial matrix numerator P and scalar monic denominator δ.

Any linear combination f (a function of degree 1 in the respective ring $\mathbb{C}[X]$) satisfies then the linear equation (B.6) and the first assertion of Proposition 8 together with the bounds provided by Theorem II, yields an upper bound for the index of f along any segment $\gamma \subset \mathbb{C} \smallsetminus \Sigma$ sufficiently distant from the singular locus to admit a lower bound for $|\delta(t)|$ on it.

To treat polynomials of higher degrees, one may either verify by inspection that the proof of Theorem II can be modified to cover also chains starting from a polynomial $p_0 \in \mathbb{C}[t, x]$ of any degree d, the bounds depending on d in an explicit and computable way. An alternative is to notice that if $x = (x_1, \dots, x_n)$ is solution of the linear system $\dot x = Ax$, $A = \{a_{ij}(t)\}_{i,j=1}^n$, then the pairwise products $x_i x_j$ satisfy the linear system

$$\tfrac{d}{dt} x_{ij} = \sum_{k=1}^{n} (a_{ik}\, x_{kj} + a_{jk}\, x_{ik}), \qquad i, j = 1, \dots, n,$$

of dimension $n(n+1)/2$, whose coefficients have explicitly bounded degree and height. A similar system appears for monomials $x_{ijk} = x_i x_j x_k$ of degree 3 and in general for any finite degree. Arranging all these systems until the degree d in one system $\dot Y_d = A_d(t) Y_d$ with the block diagonal matrix, we obtain a rational linear system such that the functions of degree 1 in $\mathbb{C}[Y_d]$ exhaust all polynomials of degree $\leqslant d$ from $\mathbb{C}[X]$. The entries of the large matrix A_d are selected among the entries of $A = A_1$, hence all other parameters (except for the dimension) will remain the same, as well and the number and location of the singularities. This completely reduces the polynomial case to the linear one treated first.

The index of a rational function $f \in \mathbb{C}(X)$ of degree d does not exceed the sum of indices of its numerator and denominator. This observation completes the proof of the lemma. □

B.5. Proof of Lemma 4. The same arguments as used above (with the reference to the second rather than the first assertion of Proposition 8), prove also the assertion of Lemma 4 in the particular case when the system is from the special class and the singular point t_j is finite and at least $1/s$-distant from all other finite singularities. The bound for the index in this case depends on the additional natural parameter $s \in \mathbb{N}$, not allowed in the formulation of the lemma.

If we were dealing with a Fuchsian system, this would pose no problem: by a suitable Möbius transformation of the sphere $\mathbb{C}P^1$ one could move any given point away from the rest of Σ. Since conformal changes of t preserve the Fuchsian form and do not affect the matrix residues, the new system will belong to the same Fuchsian class $\mathcal{F}(n, m, r)$.

However, the special class is not invariant by Möbius transformations (and even affine changes of t may well affect the condition $\Sigma \subset [-1, 1]$). To prove Lemma 4 one has to use arguments necessary involving "non-special" systems.

Consider the system (1.1) with the coefficient matrix $A(t)$ of the form (2.1), having a simple pole at the origin $t_1 = 0$ and make a non-affine conformal map

$$z = \frac{t}{t-c} \iff t = c\frac{z}{z-1}, \qquad 0 \neq c \in \mathbb{R}, \tag{B.15}$$

which preserves the origin and takes infinity to the point $z = 1$.

The new coefficient matrix $A(z) = A(t)\,dt/dz$ after such transformation takes the form

$$A(z) = \frac{A_\infty}{z-1} + \sum_{j=1}^{m} \frac{A_j}{z - z_j} + \sum_{k=2}^{r+2} \frac{B_k}{(z-1)^k}. \tag{B.16}$$

Assume that $|c| < 1$. Then the total norm $\|B_2\| + \cdots + \|B_{r+2}\|$ of the Laurent coefficients B_k at the point $t = 1$ is explicitly bounded in terms of r (the common upper bound for the height of the initial system (2.1) and the order of its pole at infinity).

Indeed, the (scalar) 1-form $t^k\,dt$ is transformed by the change of variables (B.15) into the rational 1-form $-c^{k+1}\bigl(1 + \frac{1}{z-1}\bigr)^k \frac{dz}{(z-1)^2} = c^{k+1}\omega_k$, where ω_k is a rational 1-form whose Laurent coefficients are depending only on k and bounded from above. This means that each matrix-valued 1-form $t^k A_k'(t)\,dt$ is transformed into a rational matrix 1-form with arbitrarily small matrix residues, provided that $|c| > 0$ is sufficiently small (depending on k). In other words, the residues B_k in (B.16) can be made arbitrarily small by a suitable choice of c small enough.

After the transformation (B.15) all finite nonzero singular points t_j will occur at $z_j = t_j/(t_j - c)$. Choosing $|c|$ sufficiently small, one can arrange them arbitrarily close to the point $z = 1$, in particular, inside the annulus $\{\frac12 < |z| \leqslant 2\}$. Then the height of the monic denominator $\delta(z) = (z-1)^{r+2}\prod_1^m (z - z_j)$ will be then explicitly bounded.

Finally, the residues A_j at all Fuchsian points remain the same, thus for all sufficiently small values of $|c|$, the transformation (B.15) brings the matrix $A(z)$ of the system into the form (B.16) with $\sum \|A_j\| \leqslant r$, $\sum_k \|B_k\| \leqslant 1$, $z_1 = 0$ and $\frac12 \leqslant |z_j| \leqslant 2$ for $j = 2, \ldots, m$. After reducing it to the common denominator form (B.8), we obtain a system with explicitly bounded height of P and δ, which has a singular point $z = 0$ at least 1/2-distant from all other points which are still on the segment $[-2, 2]$.

Now one can safely reduce this system to the linear equation (B.6) as described in Theorem II and apply the second assertion of Proposition 8 to obtain computable two-sided bounds on the index of small circular arcs on the z-plane around the origin. The images of these arcs on the t-plane will be arbitrarily small circular arcs around t_j, though not necessarily centered at this point. The proof of Lemma 4 is complete. $\square$

Appendix C. Quantitative factorization of matrix functions in an annulus

As often happened before, the proof with quantitative bounds is essentially obtained by inspection of existing "qualitative" proofs, supplying quantitative arguments when necessary.

The standard proof of matrix factorization theorem that we "quantify", is the one from [Bol00]. Namely, for the matrix function $H(t)$ *sufficiently close* to the (constant) unity matrix E, one can prove existence of *holomorphic and holomorphically invertible* decomposition $H = FG$, including the point at infinity (i. e., with both G and G^{-1} bounded in the outer domain). Then, using approximation by rational functions, we prove that there exist *meromorphic and meromorphically invertible* decomposition with explicit bounds on the number of zeros and poles of the determinants $\det F$, $\det G$. Finally, we show that all zeros and poles can be forced to "migrate" to the single point at infinity, while retaining control over all magnitudes.

Throughout this section R is the annulus $\{a < |t| < b\}$ with a, b constrained by the natural parameter q as in (6.5), and U, V denote respectively the interior and exterior disks, $U = \{|t| < b\}$, $V = \{|t| > a\} \subset \mathbb{C}$, so that $R = U \cap V$.

C.1. Holomorphic factorization of near-identity matrix functions.

Lemma 9. *There exists a computable function $\mathfrak{N}(q)$ of one integer argument q with the following property.*

For a given annulus R whose dimensions are determined by the integer parameter $q \in \mathbb{N}$ as in (6.5), and any holomorphic invertible matrix function H in the annulus, close to identity enough to satisfy the condition

$$\|H(t) - E\| \leqslant 1/\mathfrak{N}(q), \qquad t \in R, \tag{C.1}$$

there exists the decomposition $H = FG$ with F and G holomorphic and holomorphically invertible in the respective circular domains U and V. The matrix functions F, G satisfy the constraints

$$\|F(t)\| + \|F^{-1}(t)\| \leqslant \mathfrak{N}(q), \qquad \|G(t)\| + \|G^{-1}(t)\| \leqslant \mathfrak{N}(q) \tag{C.2}$$

in their domains.

Scheme of the proof. The identity $H = FG$ can be considered as a functional equation to be solved with respect to F and G for the given matrix function H. Representing the three functions as $H = E + C$, $F = E + A$, $G = E + B$, this identity can be rewritten as

$$C = A + B + AB. \tag{C.3}$$

This equation is nonlinear with respect to the pair of matrix functions (A, B). Its "linearization" is obtained by keeping only "first order" terms,

$$C = A + B. \tag{C.4}$$

Solution of this linearized equation can be immediately given by the Cauchy integral. The boundary γ of the annulus R consists of two circular arcs, the interior arc γ_a and the exterior arc γ_b (properly oriented). Writing

$$C(t) = \frac{1}{2\pi i}\oint_{\gamma} \frac{C(z)\,dz}{z-t} = \frac{1}{2\pi i}\oint_{\gamma_a} \frac{C(z)\,dz}{z-t} + \oint_{\gamma_b} \frac{C(z)\,dz}{z-t}.$$

The first term which we denote by $B(t)$, is holomorphic in $V = \{|t| > a\}$. The second term is respectively holomorphic in $U = \{|t| < b\}$ and after being denoted by $A(t)$, constitutes together with B a holomorphic solution of the linearized problem (C.4). This solution considered as an integral operator $C(\cdot) \mapsto (A(\cdot),\, B(\cdot))$ is of explicitly bounded norm. Indeed, if $\|C(t)\|$ is bounded on γ by 1, then for each $t \in R$ at least one of the distances $\operatorname{dist}(t,\, \gamma_a)$ or $\operatorname{dist}(t,\, \gamma_b)$ will be no smaller than $1/2q$, where q is the width parameter of the annulus as described in (6.5). This means that the norm of one of the integrals is explicitly bounded in terms of q, but the second integral is then also bounded by virtue of (C.4). Note that this boundedness is specific for the annulus having two disjoint parts γ_a, γ_b of the boundary γ.

The nonlinear equation (C.3) can be now solved by the Newton method. If $H = E + C$ and $\|C\| \leqslant \varepsilon$, then factorization of H can be reduced to factorization of $(E + A)^{-1}H(E + B)^{-1}$: if the latter is factorized as $F'G'$ as required, the former can also be factorized as $(E + A)F' \cdot G'(E + B)$. The new difference $C' = (E + A)^{-1}(E + C)(E + B)^{-1} - E$ is of *quadratic* order of magnitude, $\|C'\| \leqslant L\varepsilon^2$, where $L = L(q)$ is the norm of the above described integral operator solving the linearized equation. The process of linearization and solving the linearized equation can be continued.

The super-exponential convergence of Newton-type iterations guarantees that if the original difference $\max_{t\in R} \|H(t) - E\|$ was smaller than, say, $1/2L(q)$, then the process converges and the ultimate result, constructed as the infinite product of matrices fast converging to the identity E in the respective domains U, V, will be of matrix norms explicitly bounded in terms of q only. The accurate estimates can be extracted from [Bol00] if necessary.

If C is real on $R \cap \mathbb{R}$, then clearly the solution of the linearized equation (C.4) is also real on the real line, and this condition is preserved in the iterations of the Newton method. This completes the proof of the lemma. □

C.2. Meromorphic solutions of the factorization problem. The following construction is also standard.

Proposition 9. *Any matrix function H satisfying the assumptions of Lemma 7, can be approximated by a rational matrix function*

$$M(t) = \sum_{k=-d}^{d} M_k t^k, \qquad M_k \in \operatorname{Mat}_n(\mathbb{C}), \tag{C.5}$$

so that the product $M(t)H^{-1}(t)$ satisfies the condition (C.1):

$$\|M(t)H^{-1}(t) - E\| \leqslant 1/\mathfrak{N}(q)$$

in the smaller annulus $R' = \{a' < |t| < b'\} \Subset R$, $a' = a + \frac{1}{4}(b-a)$, $b' = b - \frac{1}{4}(b-a)$.

The degree d and norms of the matrix coefficients M_k will be bounded by primitive recursive functions of q and q'. If H is real on $R \cap \mathbb{R}$, then M will also be real on $\mathbb{R}$.

Proof. Consider the Laurent expansion for H, converging uniformly on $R' \Subset R$ since H is holomorphic in the annulus R. The rate of this convergence can be explicitly estimated in terms of q' and q, see (6.5) and (6.6). Thus one can immediately estimate the number of terms of the Laurent expansion that would be sufficient to keep to satisfy the required accuracy of the approximation so that

$$\|H(t) - M(t)\| \leqslant 1/q'\mathfrak{N}(q).$$

Then $H^{-1}M - E$ and $MH^{-1} - E$ will be as small as asserted on R'. The rest is obvious. □

This proposition together with Lemma 9 applied to $H^{-1}M$, guarantees for an arbitrary holomorphically invertible matrix function H existence of a factorization of the form

$$\begin{gathered} H = FG, \qquad G = G'M, \\ F,\, G' \text{ holomorphic in } U',\, V' \text{ respectively, } M \text{ rational,} \end{gathered} \tag{C.6}$$

where F, G' are holomorphic, holomorphically bounded and invertible in $U' = \{|t| < b'\} \Subset U$, $V' = \{|t| > a'\} \Subset V$, and are bounded together with their inverses there by a computable function of q, q' only.

C.3. Expulsion of poles of G^{-1}. Proof of Lemma 7. The term F in the decomposition (C.6) already meets the requirements imposed in (6.4).

To achieve the conditions required from the second term G, we multiply it from the left by several rational matrix functions, at the same time multiplying F from the right by their inverses. The construction, due to G. Birkhoff [Bir13], is utterly classical. Our task is only to verify that neither these functions nor their inverses are not uncontrollably large.

The matrix function G has a unique pole of order $\leqslant d$ at infinity, but may have a number of "*zeros*" $z_,\, \dots,\, z_k$ (poles of the inverse matrix G^{-1}). Their number k and position are determined by zeros of $\det G = \det G' \det M$, that is, by zeros of the rational function $\det M(t)$ of known degree, since the term G' is holomorphically invertible; k is bounded by a computable function of q, q'. We are interested only in those poles that are in V'.

Consider an arbitrary root $t = z$ of the determinant $\det M(t)$. The matrix $G = G(z)$ has nontrivial left null space, since $\det G(z) = 0$. Let $C = C_z$ be the constant matrix whose first row is a (left) null vector $v = v_z \in \mathbb{C}^n$ of $G(z)$ of unit Hermitian norm, while the other rows are chosen among the rows of the identity matrix E in such a way that both $\|C\|$ and $\|C^{-1}\|$ are bounded by n. This choice is always possible: it is sufficient to delete from the collection of all coordinate vectors in $\mathbb{C}^n$ the vector which has the largest Hermitian scalar product with v.

By construction, the product $C_z G(t)$ at the point $t = z$ has the zero first row. Let $r_z(t) = |z|/(t - z)$ be the rational function with the simple pole at $t = z$. Then the matrix product $\operatorname{diag}\{r_z(t),\, 1,\, \dots,\, 1\} \cdot CG(t)$ continues to be holomorphic at the point z. The correction term $Q_z(t) = \operatorname{diag}\{r_z(t),\, 1,\, \dots,\, 1\} \cdot C$ is bounded together with its inverse in the thinner annulus $R'' = \{a'' < |t| < b''\} \Subset R'$, the "middle belt" of R', $a'' = a' + \frac{1}{4}(b' - a')$, $b'' = b' - \frac{1}{4}(b' - a')$. Note that this bound is *uniform* as z varies in $V' = \{b' < |z|\}$. Besides, $Q_z(t)$ is bounded together with its inverse also in U'. All these bounds follow from two-sided bounds on $r_z(t)$ in the respective domains and are expressed by computable functions of q, q'. Replacing the decomposition $H = FG$ by $H = FQ_z^{-1}Q_zG$, we obtain a new factorization of H that has fewer poles of the right term in V' (both terms remain holomorphic and the left is holomorphically invertible in U').

This expulsion of a pole of G^{-1} from $t = z$ to $t = \infty$ has to be repeated for each pole $z_j \in V'$ of the initial matrix $G = G'M$ (the order is inessential). As a result, we replace the decomposition (C.6) by

$$H = FQ^{-1} \cdot QG, \quad Q = Q(t) \text{ rational without poles or zeros in } R'. \tag{C.7}$$

By construction, both QG and $(QG)^{-1}$ have only one (multiple) pole at $t = \infty$, and all four matrices FQ^{-1}, QG and their inverses, are explicitly bounded in R'' in the sense of the norm by a computable function of q, q'.

The order of pole of QG at infinity is no greater than the initial order d of pole of M. The order of pole of $(QG)^{-1}$ does not exceed k, the degree of $\det M$ (the number of singular points that were expelled). Finally, the norms of the Laurent coefficients of $(QG)^{\pm 1}$ at infinity can be majorized using the Cauchy estimates (differentiating the Cauchy integral along the inner boundary of the annulus R''). These bounds together prove the assertion of Lemma 7.

The only remaining problem is to choose Q being real on $\mathbb{R}$ provided that G is real there. In this case non-real "zeros" of G come in conjugate pairs z, $\bar{z}$ and the corresponding left null vectors (rows) v, $\bar{v} \in \mathbb{C}^n$ are complex conjugate. Both such zeros can be expelled simultaneously if the corresponding rational matrix factor $Q_{z,\bar{z}}(t)$ has the first row of the form

$$|z| \left(\frac{v}{t - z} + \frac{\bar{v}}{t - \bar{z}} \right) \in \mathbb{R}^n, \qquad t \in \mathbb{R}.$$

This vector function has the norm bounded from above away for all pairs of points $\{z, \bar{z}\}$ uniformly over all $z \in V'$ in exactly the same way as the function $r_z(t)$ above, and as before the inverse Q^{-1} is of bounded norm provided that the other rows of Q were appropriately chosen among the rows of the identity matrix E exactly as before.

This remark concludes the proof of Lemma 7. □

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