Treewidth and graph minors (homework)

We shall touch upon the theory of *Graph Minors* by Robertson and Seymour. As a motivating example, we show that maximum weight independent set on trees can be solved in polynomial time, using dynamic programming. (Note however that bandwidth is NP-hard on trees.) We then introduce the notion of a tree decomposition of a graph.

Definition: A *tree decomposition* of a graph $G(V,E)$ is a tree T, where each node i of T is labeled by a subset (*bag*) $B_i \subset V$ of vertices of G, each edge of G is in a subgraph induced by at least one of the B_i, and the nodes of T labeled by any vertex $v \in V$ are connected in T. The *treewidth* of G is the minimum integer p such that there exists a tree decomposition G with all subsets of cardinality at most $p+1$.

Theorem: For every graph $G(V,E)$ of treewidth p and $X \subset V$, there is a vertex separator S, $|S| \leq p+1$, that partitions X into two subsets X_1 and X_2 of size at most $2|X|/3$, with all paths between X_1 and X_2 going through S.

We shall show that a tree has treewidth 1, a series-parallel graph has treewidth 2, a k-clique has treewidth $k-1$, and an n by n grid has treewidth $\Theta(n)$.

Definition: H is a *minor* of G if it can be obtained from G by a sequence of operations of taking subgraphs and edge *contractions* (merging endpoints together).

Kuratowski showed that non-planar graphs must contain either K_5 or $K_{3,3}$ as minors.

Theorem (planar minor): Let H be a planar graph. If G has no H-minor, then the treewidth of G is bounded by some function of H, independent of G. (Not proved in class.)

Theorem: For graphs that have bounded treewidth, a corresponding tree decomposition can be found in polynomial (in fact, linear) time.

We shall use this and dynamic programming to design a polynomial time algorithm for k-coloring graphs of bounded treewidth.

Robertson and Seymour show that every family of graphs that is closed w.r.t. taking minors has a finite *obstruction set* of minors (such as K_5 or $K_{3,3}$ for planarity). They further show that for every fixed H, testing whether G contains H as a minor can be done in time $O(n^3)$. A consequence of their theory is that any property of graphs that is inherited by minors (such as being embeddable in 3-dimensional space without linked cycles) can be *decided* in polynomial time. (Their theory does not necessarily produce an explicit algorithm, and if it does, the hidden constants in the running time are often huge.)

Homework.

1. Recall that graphs of treewidth at most 1 are those without a K_3 minor (trees) and graphs of treewidth at most 2 are those without a K_4 minor (series parallel graphs). Prove that for every p there is a finite list of forbidden minors (that depends on p)

such that graphs of treewidth at most p are those without any of these subgraphs as a minor.

2. Prove the following approximate min-max relation between treewidth and grid minors – for every graph, its treewidth (which is a minimization problem) is "approximately" related to the size of its largest grid minor.

   (a) If a graph has a k by k grid as a minor, then it has treewidth at least $\Omega(k)$.

   (b) If a graph has treewidth more than p then it has an $f(p)$ by $f(p)$ grid as a minor, for some nonnegative function $f(p)$ that tends to infinity as p grows.

3. Show that for every planar graph H there is a large enough grid for which H is a minor. (As a consequence, proving the Theorem "planar minor" in the special case that H is required to be a grid implies the theorem for all planar H.)

If needed, see footnote[1] for **Hints**.

[1] Hints.

1. Show that if H is a minor of G then necessarily the treewidth of H is no larger than the treewidth of G. Thereafter refer to the theory of Robertson and Seymour.
2. (For 2(b)). Use Theorem "planar minor". You need not specify f explicitly, and in fact it is still not known what the best f can be in this relation.
3. Consider a planar (with no crossing edges) drawing of H, where edges are lines with some nonzero thickness and vertices are discs with nonzero radius (similar to the object that one would get by using paper and pencil to draw a planar graph).