Handout 3: metric embeddings

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Given an input graph $G(V, E)$ with an even number of vertices and positive edge costs $c : E \to R^+$, in the bisection problem one needs to select a set $S \subset V$ with $|S| = \frac{|V|}{2}$, so as to minimize $\sum_{u \in S, v \notin S} c(u, v)$. Using an LP relaxation, we showed how the CKR rounding technique [4] leads to a bicreteria approximation: we find a *balanced* cut (namely, S is of size $\frac{n}{3} \leq |S| \leq \frac{2n}{3}$) and the cost of edges cut is within an $O(\log n)$ factor of the minimum bisection.

We discussed finite metric spaces. We saw that every finite metric space (graph with edge lengths) has a probabilistic embedding into Hierarchically Separated Trees (HSTs) such that distances do not contract, and for every pair of vertices, the expected multiplicative stretch for the distance between them is at most $O(\log n)$ (see [3], [6], Chapter 8.5 in [10]).

Convex combinations of tree metrics are ℓ_1 metrics, which in turn are combinations of cut metrics. We used the tree embedding result to derive an $O(\log n)$ approximation for sparsest cut, $\min_{S \subset V} \left[\frac{|E(S, V \setminus S)|}{|S| \cdot |V \setminus S|}\right]$.

We showed that Steiner points can be removed from HSTs (obtaining a tree on the vertices of the original metric space) with only constant multiplicative loss in the distortion.

We showed how tree embeddings can be used in order to derive an $O(\log n)$ approximation for *buy at bulk network design* (see [2], or Chapter 8.6 in [10]). In combination with *spreading metric constraints* [5], tree embeddings can be used to give an $O(\log n)$ approximation for *minimum linear arrangement*.

We have seen the tight relation due to Racke [9, 1] between probabilistic tree embedding that have small stretch (referring to paths and lengths of edges) and those that have small congestion (referring to cuts and capacities of edges). See also Chapters 15.2 and 15.3 in [10], which refer to the latter notion as *cut-tree packings*. We used this to sketch an $O(\log n)$ approximation for min bisection.

We will see Bourgain's embedding of any finite metric space into ℓ_1 in $O(\log^2 n)$ dimensions, and distortion $O(\log n)$. (Chapter 15.1 in [10]). If time permits, we shall see a family of graphs of *treewidth* 2 that embeds into ℓ_1 with distortion $O(1)$, but requires distortion $\Omega(\log n)$ for randomized embeddings into tree metrics [7].

Hand in the homework by July 10, 2019. Though not stated explicitly in the questions, you need to prove that the algorithms that you design achieve what they are claimed to achieve (in terms of approximation ratios and polynomial running time).

1. Question 8.5 in [10]. Given a graph $G(V, E)$ with n vertices and with positive edge costs $c : E \to R^+$, a linear arrangement is a one to one mapping $f : V \to [n]$. The *cut width* of the linear arrangement is $\max_{1 \leq k < n} \sum_{(u,v) \in E | f(u) \leq k < f(v)} c(u, v)$, and the cut width of the graph is the minimum over all linear arrangements of their cut width. Show how the bicriteria $O(\log n)$ approximation for bisection (giving a balanced cut) can be used in order to design an algorithm that finds a linear arrangement whose cut width is within a factor of $O(\log^2 n)$ of the cut width of the graph.

2. Question 8.11 in [10]. Consider a metric (V, d), a vehicle of capacity C and starting point $r \in V$, and k source-destination pairs $\{s_i, t_i\}$ for $1 \leq i \leq k$. The vehicle must deliver from each source s_i an item to its respective destination t_i. The goal in *capacitated dial a ride* problem is to find the shortest possible tour for the vehicle, starting at r, picking each item from its source and delivering it at its destination, and without ever carrying simultaneously more than C items. The vehicle is allowed to visit the same node several times, and to temporarily store items at various nodes.

   (a) Give a factor 2 approximation algorithm when (V, d) is a tree metric (V, T). (Hint appears in question 8.11(a) in [10].)

   (b) Using the answer to 2(a) (even if you have not solved it), give a randomized $O(\log |V|)$ approximation for the capacitated dial a ride problem on arbitrary metrics.

3. Show a polynomial time algorithm that solves min bisection on trees. (Section 15.3 in [10] describes such an algorithm, though you may describe a different algorithm.)

4. Question 15.4 in [10]. Show that the Frechet embedding of Bourgain gives an $O(\log n)$ distortion into ℓ_p for every $p \geq 1$. (In class we considered only ℓ_1.)

5. Question 15.5 in [10]. Recall that in the lecture of May 15 we have seen a factor 2 approximation for multi-cut on trees, and factor $O(\log k)$ (where k is the number of source-sink pairs) for general graphs (the latter result was obtained by CKR rounding of an LP). For general graphs, show a different algorithm that gives an $O(\log n)$ approximation ratio (n is the total number of vertices) by using cut-tree packings. (You may assume that there are polynomial time algorithms for computing cut-tree packings.)

References

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