# Handout 4: Linear Programming

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Topics discussed in class on Dec 19: proof of Vizing's theorem (leftover from previous week - does not use LP), 3-approximation of sum of weighted completion times for non-preemptive scheduling on a single machine with release dates.

Topics planned for Dec 26: basic feasible solutions and the Beck Fiala theorem, 4-approximation for metric facility location.

Note: the Beck-Fiala theorem is not discussed in [WS11]. See instead http://www.wisdom.weizmann.ac.il/∼feige/LP15/Lect1LinearAlgebra.pdf (Sections 1.4–1.7).

**Homework** – hand in by January 9. (Grader: Yael Hitron. Recall conventions for homework from Handout 1.)

1. (Problem 4.1 in [WS11].) The following problem arises in telecommunications networks, and is known as the SONET ring loading problem. The network consists of a cycle on $n$ nodes, numbered 0 through $n-1$ clockwise around the cycle. Some set $C$ of calls is given; each call is a pair $(i, j)$ originating at node $i$ and destined to node $j$. The call can be routed either clockwise or counterclockwise around the ring. The objective is to route the calls so as to minimize the total load on the network. The load $L_i$ on link $(i, i+1)$ (modulo $n$) is the number of calls routed through that link, and the total load is $\max_{1 \leq i \leq n} L_i$.

   Give a 2-approximation algorithm for the SONET ring loading problem.

2. Consider the problem of scheduling $n$ jobs on a single machine, where for every job $j$ we are given its processing time $p_j$, and its release date $r_j$. The goal is to find a schedule that minimizes the sum of completion times $\sum C_j$. Show that if preemption is allowed, the problem can be solved in polynomial time.

3. Let $s$ be a slackness parameter which in this homework you can set to be a fixed positive integer of your choice. (A choice of $s = 2$ suffices, but if you find it easier to do the homework with a larger value of $s$, then you may do so.) Let $G$ be an arbitrary graph, let $k$ be an arbitrary positive integer, and let $\alpha_1, \ldots, \alpha_k$ be nonnegative with $\sum_{i=1}^{k} \alpha_i = 1$. Show that one can color the edges of the graph with $k$ colors, such that for every

vertex $v$ and every color class $i$, the number of edges of color $i$ incident with vertex $v$ is between $\lfloor \alpha_i d_v \rfloor - s$ and $\lceil \alpha_i d_v \rceil + s$, where $d_v$ is the degree of vertex $v$.

(Hint: for every edge $e$ and color class $i$ introduce a variable $x_{ei}$ whose intended value is 1 if edge $e$ is colored by color $i$, and 0 otherwise. Then follow the proof technique used in the proof of the Beck-Fiala theorem.)

[Some remarks. The slackness term $s = 2$ can be improved when $k = 2$, and it is an open question whether it can be improved in general. For the special case of bipartite graphs, if $k = 2$ no slackness term is needed at all (namely, $s = 0$), and it is conjectured that this is true for every $k$. For general graphs, if $0 < \alpha_i d_v < 1$ for all $i$ and $v$, then Vizing's theorem implies that no slackness is needed.]