Linear Programming – Handout 1

March 15 and 22, 2015

Parts of the course will overlap the book: Jiri Matousek, Bernd Gartner: Understanding and Using Linear Programming (Springer 2007), which is available freely on the web.

http://link.springer.com/book/10.1007%2F978-3-540-30717-4.

Topics covered in first lectures: formulating problems as a linear program, transformations among various forms, basic feasible solutions, the Beck-Fiala theorem

Homework. Hand in by April 12.

1. Suppose that you are given a system of linear equations $Ax = b$, and are asked to find a solution that minimizes a linear objective function $c^t x$. (This is similar to an LP in standard form, but without the nonnegativity constraints.) What are sufficient and necessary conditions for this problem to have a bounded optimal solution?

2. Consider the problem of minimizing the ratio

$$\frac{c^t x}{f^t x}$$

of two linear functions, subject to all the following constraints:

   - $Ax \geq b$,
   - $f^t x \geq 1$,
   - $c^t x \geq -8$,
   - $c^t x \leq 8$.

   Show how linear programming can be used as a subroutine so as to find the optimal solution within any degree of accuracy (the running time may depend on the degree of accuracy required). (Hint: consider the problem of deciding whether the objective function is at most a given value.)

3. Consider an LP in general form, and suppose that among the constraints of the LP there are n constraints that are linearly independent. (Here we do not distinguish between main constraints and nonnegativity constraints.) A basic feasible solution (bfs) is one that satisfies n linearly independent constraints with equality. Show that if the LP is feasible and the value of the optimal solution is bounded then the LP has a bfs that is optimal.

4. Let s be a slackness parameter which in this homework you can set to be a fixed positive integer of your choice. (A choice of $s = 2$ suffices, but if you find it easier to do the homework with a larger value of s, then you may do so.) Let G be an arbitrary graph and let $\alpha_1, \ldots, \alpha_k$ be nonnegative with $\sum_{i=1}^{k} \alpha_i = 1$. Show that one can color the edges of the graph such that for every vertex v and every color class i, the number of edges of color i incident with vertex v is between $\lfloor \alpha_i d_v \rfloor - s$ and $\lceil \alpha_i d_v \rceil + s$, where d_v is the degree of vertex v.

   (Hint: for every edge e and color class i introduce a variable x_{ei} whose intended value is 1 if edge e is colored by color i, and 0 otherwise. Then follow the proof technique used in the proof of the Beck-Fiala theorem.)

   [Some remarks. The slackness term $s = 2$ can be improved when $k = 2$, and it is an open question whether it can be improved in general. For the special case of bipartite graphs, if $k = 2$ no slackness term is needed at all (namely, $s = 0$), and it is conjectured that this is true for every k.]