Linear Programming Duality – Handout3

May 3, 2015

LP-duality asserts that if the optimal solution of the primal linear program below exists and is bounded, then the same holds for the dual linear program, and both optimal solutions have the same value. (Here A_i denotes row i and A^j denotes column j.)

Primal: *minimize* $c^T x$ subject to
$A_i x \geq b_i$, $1 \leq i \leq h$
$A_i x = b_i$, $h < i \leq m$
$x_j \geq 0$, $1 \leq j \leq \ell$.

Dual: *maximize* $b^T y$ subject to
$(A^j)^T y \leq c_j$, $1 \leq j \leq \ell$
$(A^j)^T y = c_j$, $\ell < j \leq n$
$y_i \geq 0$, $1 \leq i \leq h$.

It may happen that neither the primal LP nor its dual are feasible, or that one is unbounded and the other is not feasible.

Whenever one studies a linear program, it is informative to look at its dual.

Let x and y be feasible solutions to the primal and dual problem. Then they are optimal iff they satisfy the *complementary slackness* conditions. Namely $y^T(Ax - b) = 0$ and $(c^T - y^T A)x = 0$.

Duality theory places linear programming in NP $\bigcap$ coNP, and implies that finding a feasible solution to a linear program is essentially as difficult as finding an optimal solution. It can be used in order to prove results that predated LP duality, such as von-Neumann's minimax theorem for 0-sum two person games, and Farkas' lemma (either $Ax = b$ has a solution with $x \geq 0$, or there is a vector y such that $y^T A \geq 0$ but $y^T b < 0$). We will comment on the relevance of the minimax theorem to proving lower bounds for randomized algorithms (Yao's minimax principle).

If all coefficients of a linear program are integer and the constraint matrix A is *totally unimodular*, then the linear program (if feasible) has an integer optimal solution.

Vertex cover (the complement of independent set) can be formulated as:
minimize $\sum_{i=1}^{n} v_i$ *subject to*:
$v_i + v_j \geq 1$, for each edge (v_i, v_j),
$v_i \in \{0, 1\}$, for each vertex v_i.
NP-hardness of vertex cover implies that integer programming is NP-hard.

We relax the integer program to a linear program by replacing the $v_i \in \{0, 1\}$ constraints by nonnegativity constraints $v_i \geq 0$. The gap between the value of the optimal solution for integer program and its relaxed fractional program is known as the *integrality gap*. For the case of vertex cover, it is a multiplicative factor of at most 2.

By studying the dual of this linear program we will derive a polynomial time algorithm for vertex cover in bipartite graphs.

We shall also use duality and total unimodularity to prove the min-cut/max-flow theorem.

Homework. Hand in by May 17.

1. Helly's theorem says that for any collection of convex bodies in R^n, if every $n+1$ of them intersect, then there is a point lying in the intersection of all of them. Prove Helly's theorem for the special case that the convex bodies are polytopes. Your proof should be based on linear programming duality. Try to keep it short (but without harming correctness).

   Guidance. Recall that a polytope is the intersection of half spaces. Hence the intersection of polytopes is also the intersection of half spaces. Equivalently, it is the solution to a system of inequalities $Ax \geq b$. Hence Helly's theorem for polytopes is equivalent to proving that if a system of inequalities $Ax \geq b$ does not have a solution, then we can select $n+1$ of the inequalities such that the resulting system does not have a solution. This system of inequalities can be cast as the constraints of an LP (you would need to also choose some objective function for the LP – make the choice that would make the resulting proof as simple as possible). Going to the dual, one would like it to be feasible but unbounded (which you will need to prove). Thereafter, by a variation on the notion of basic feasible solutions one would need to show that $n+1$ nonzero dual variables suffice for an unbounded solution. Removing the remaining variables and taking the dual of the resulting LP would give an infeasible version of the primal, but with less constraints. Filling the gaps in the above sketch and putting all parts together, you should be able to deduce Helly's theorem.

2. We saw in class that the LP that we gave for maximum matching (the dual of the LP for vertex cover) has a *half integral* (all variables receive values from $\{0, \frac{1}{2}, 1\}$) optimal solution. Prove that the same holds for the LP for vertex cover. Moreover, prove that all vertices of the polytope defined by the vertex cover LP are half integral (a theorem due to Nemhauser and Trotter). Equivalently, for every linear objective function (costs on the vertices, positive, negative or 0) that one wishes to minimize, there is an optimal solution that is half integral. (There are several known proofs for this theorem. One proof, following directly from the definition of a vertex, is by a shifting argument, showing that every solution that is not half integral can be expressed as an average of two other solutions. Another proof is by a clever reduction to the bipartite case.)

3. Prove that the vertex-arc incidence matrix for directed graphs is totally unimodular.