Summary: Linear Programming and Combinatorial Optimization

February 8, 2023

Basic concepts to be familiar with. (See definitions in lecture notes.)

- Linear programs: feasible solution, optimal solution, standard form, canonical form, slack variables, Fourier-Motzkin elimination, basic feasible solution, degeneracy.
- Math background in geometry and linear algebra: convex set, linear and affine subspace, linear and affine independence, affine transformation, polyhedron, bounded set, polytope, simplex, hyperplane, half space, supporting hyperplanes and faces of polytopes (vertices, edges, facets), extreme points, eigenvalues and eigenvectors, Cramer's rule.
- Terminology related to simplex algorithm: pivoting rules, reduced costs, cycling, Klee-Minty cube, tableau method.
- LP Duality: primal and dual LPs (and how to transform one into the other), Farkas lemma, weak duality, strong duality, complementary slackness.
- Terminology related to Ellipsoid algorithm: ellipsoid and its mathematical definition using positive definite matrices, bounding ball, bounded ball, separation oracles, strongly polynomial time algorithms.
- Terminology related to interior point methods (no lecture notes): log barrier, analytic center, central path.
- Generalization beyond linear programming: semidefinite programming, the Lovasz theta function.
- LP and SDP relaxations of integer programs: integrality gaps, total unimodularity, half-integral solution, rounding, randomized rounding, random hyperplane, multi-dimensional normal random variable.
- Related concepts encountered during the course: Shadow prices, Helly's theorem, Minimax theorem, Dilworth's theorem, Perfect graphs.

Topics covered in course.

- Beck-Fiala Theorem.
- The simplex algorithm, the Klee-Minty cube, upper bounds on the diameter of the graph of a polytope (Kalai).
- Vertex cover in bipartite graphs, $\frac{1}{2}$-approximation in general graphs, max flow min cut theorem, integrality of min cost flow.
- Ellipsoid algorithm.
- Low distortion embeddings into Euclidean space.
- Finding a maximum independent set and a minimum coloring in perfect graphs.
- Coloring 3-colorable graphs with a relatively small number of colors (Karger-Motwani-Sudan).

Topics that we have not covered but are worth knowing.

- Interior point methods (was only given as a reading assignment).
- Multiplicative Weights method and solving LPs approximately.
- Duality for general convex programs such as SDPs.
- Polynomial time algorithms for solving integer linear programs with a small number of variables (Lenstra).
- More examples of using LP and SDP relaxations as approximation algorithms, such as approximating max-SAT (Goemans and Williamson [SIDMA 1994]), approximating max-cut (Goemans and Williamson [JACM 1995]).

A taste of well known open questions.

1. The simplex algorithm.
   - Hirsch's conjecture: does the skeleton graph of polytopes in n dimension always have diameter polynomial in n? Linear in n? Current best known upper bound is $n^{O(\log n)}$.
   - Is there a polynomial time computable pivoting rule that ensures polynomial convergence of the simplex algorithm?
   - Is there a polynomial time computable randomized pivoting rule that ensures polynomial convergence of the simplex algorithm? Subexponential convergence is known.
   - Are there strongly polynomial time (possibly randomized) algorithms for solving linear programs?

2. Beck-Fiala theorem. Improve the bounds on the discrepancy of 2-coloring hypergraphs of maximum degree d. Get below $O(d)$, hopefully to $O(\sqrt{d})$.

3. Is there a polynomial time algorithm that approximates minimum vertex cover within a ratio better than $2 - \epsilon$, for some fixed $\epsilon > 0$? (If so, the *unique games conjecture* is false.)

4. Is there a polynomial time algorithm that colors 3-colorable graphs of maximum degree Δ with less than $\Delta^{1/3}$ colors?

There are many other less difficult research projects in these areas.