Linear Programming – Handout 2

December 7, 2022

The **simplex** algorithm was designed by G. Danzig in 1947. It works remarkably fast in practice. The algorithm is composed of two phases, which in essence are:

Phase 1: Find a vertex of the polyhedron.

Phase 2: Reduce the value of the objective function by walking from vertex to vertex along the edges of the polyhedron, until a minimum vertex is found.

We show how this geometric interpretation of the algorithm can be carried out algebraically. We show the tableau implementation in which the number of arithmetic operations performed per pivot operation is $O(mn)$. We briefly discuss issues such as handling degeneracies and pivot selection rules.

Degeneracies are recognized by the appearance of a zero on the right column of the tableau. Bland's rule can be used to prevent cycling. Choose the lowest possible column to enter the basis, and the lowest possible column to leave it. (Columns are not rearranged between pivot steps).

Though the number of pivot operations is potentially exponential, in practice it is small (typically $O(m)$).

Klee and Minty (1972) were the first to demonstrate that the simplex algorithm can be tricked to visit exponentially many vertices. The Klee-Minty cube (or *squashed cube*) is an LP in canonical form with n variables, $2n$ constraints and 2^n vertices in which there is a monotone path traversing all vertices (and hence the simplex algorithm might make exponentially many pivot steps). It can be represented as the following LP (in general form): *minimize* x_n subject to $0 \leq x_1 \leq 1$, and $\epsilon x_{i-1} \leq x_i \leq 1 - \epsilon x_{i-1}$. The simplex algorithm starting at $(0, 0, \ldots, 1)$ visits all 2^n vertices, if Bland's rule is used.

A variation on the LP of Klee and Minty is given by Chvatal (the case $n = 2$ is given as homework).

Maximize $\sum_{j=1}^{n} k^{n-j} x_j$

subject to

$2 \sum_{j=1}^{i-1} k^{i-j} x_j + x_i \leq k^{2(i-1)}$ for $1 \leq i \leq n$

$x_i \geq 0$ for all i

In the LP, k needs to be a sufficiently large constant ($k > 2$).

For pivot rules that are used in practice there are known (artificial) examples that demonstrate exponential behavior. It is an open question whether there is any pivot rule that ensures termination in a polynomial number of steps. Moreover, it is not known whether in general there is a path of polynomial length leading to the minimum vertex. Hirsch conjectured that the minimum vertex can always be reached by a (monotone) path of length at most $m - n$. (Here m is the total number of halfspaces defining the polytope, which for an LP includes the non-negativity constraints.) Though the conjecture in this strong form is by now known to be false, it may not be far from the truth.

We shall prove the following theorem of Kalai: From any vertex on any polyhedron with n dimensions and m facets there is a monotonically decreasing path of length $O((n + \log m)^{\log m})$ to the minimum vertex.

Homework. Hand in by December 21.

1. At a degenerate vertex, a variable with negative reduced cost might enter the basis without improving the value of the objective function. A. Hoffman designed an example showing that cycling (a sequence of changes of bases that eventually returns to the original basis) can occur in the simplex algorithm. Can the following most simple form of cycling occur at a degenerate vertex: first x_1 enters the basis and x_2 leaves it; then x_2 enters the basis and x_1 leaves it? If yes, provide an example where this happens. If no, explain why not.

2. Consider the following LP, with $k > 2$ (corresponding to the special case of $n = 2$ for Chvatal's LP).

   Maximize $kx_1 + x_2$ subject to

   $x_1 \leq 1$

   $2kx_1 + x_2 \leq k^2$

   $x_1, x_2 \geq 0$

   Convert it to an LP in standard form by adding slack variables s_1 for the first constraint and s_2 for the second constraint. Now use the tableau method for the simplex algorithm, with columns of the tableau ordered as x_1, x_2, s_1, s_2 and a last column for b, with an additional row for the reduced costs. Starting with the basis (s_1, s_2), present the sequence of tableaus that arises when the pivot rule is that of taking the entering variable to be that of most negative reduced cost (when minimizing the negative of the objective function). Explain in each step which variable enters the basis, which variable exists the basis, and how they are found from inspecting the tableau. Where is the assumption that $k > 2$ used?