Connectivity of Random High Dimensional Geometric Graphs

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Abstract. We consider graphs obtained by placing n points at random on a unit sphere in $\mathbb{R}^d$, and connecting two points by an edge if they are close to each other (e.g., the angle at the origin that their corresponding unit vectors make is at most $\pi/3$). We refer to these graphs as *geometric graphs*. We also consider a complement family of graphs in which two points are connected by an edge if they are far away from each other (e.g., the angle is at least $2\pi/3$). We refer to these graphs as *anti-geometric graphs*. The families of graphs that we consider come up naturally in the context of semidefinite relaxations of graph optimization problems such as graph coloring.
For both distributions, we show that the largest dimension for which a random graph is likely to be connected is the same (up to an additive constant) as the largest dimension for which a random graph is likely not to have isolated vertices. The phenomenon that connectivity of random graphs is tightly related to nonexistence of isolated vertices is not new, and appeared in earlier work both on nongeometric models and on other geometric models. The fact that in our model the dimension d is allowed to grow as a function of n distinguishes our results from earlier results on connectivity of random geometric graphs.

1 Introduction

For natural numbers n and d and an angle $0 < \theta < 2\pi$, an (n, θ, d)-graph is a collection of n points on the unit sphere in $\mathbb{R}^d$ (equivalently, n unit vectors), with two points (vertices) connected by an edge iff the angle between their corresponding unit vectors is at most θ. For example, when $d = 2$, we have points placed on a circle of radius 1, and two points are neighbors if the angular distance between them is at most θ. Equivalently, each point can be viewed as representing an interval of angular length θ centered at the point, and two intervals are neighbors if they intersect. When $d \geq 3$ the points can be thought of as equal size discs placed on the unit sphere, and two points are neighbors if their discs intersect. In this work we shall be interested in the case when the dimension d is relatively large and scales roughly like $\log n$. We call such graphs high dimensional geometric graphs. Our interest in high dimensional geometric graphs (and their complements that we call *anti-geometric* graphs,

with an edge iff the angle is at least θ) stems from the fact that they come up naturally as solutions for various semidefinite relaxations of combinatorial optimization problems (see [13, 10], for example). The question that we address in the current work is that of connectivity of these graphs. Specifically, we consider the distribution $G_{n,d,\theta}$ in which the locations on the sphere of the n vertices of the (n,θ,d)-graph are chosen uniformly independently at random, and ask for which range of parameters of (n,θ,d) is the graph likely to be connected. This question comes up naturally in the study of algorithms for coloring of random high dimensional anti-geometric graphs [4], and turns out to be more subtle than one might first imagine.

1.1 Definitions and Notation

We use $\mathbb{S}^d$ to denote the unit sphere centered at the origin in $\mathbb{R}^d$, namely, the set of vectors in $\mathbb{R}^d$ of Euclidean norm 1. (Note that in other literature this is sometimes denoted by $\mathbb{S}^{d-1}$, due to the fact that it is a $(d-1)$-dimensional object.) We measure the distance between two points in $\mathbb{S}^d$ by the angle at the origin between the unit vectors that represent these points. Namely, for unit vectors u and v, their *angular distance* is $\arccos(vu)$, where uv denotes their inner product. An (n,θ,d)-graph $G(V,E)$ has as its vertex set V a collection of n points in $\mathbb{S}^d$, and for $u,v \in V$ there is an edge $(u,v) \in E$ iff their angular distance is at most θ, namely, $\arccos(vu) \leq \theta$. We refer to these graphs as *geometric graphs*, and to θ as the *neighborhood radius*. Observe that as we measure angular distance, a geometric graph remains unchanged if we scale the radius of $\mathbb{S}^d$ to be different than 1. We shall also be interested in complements of (n,θ,d)-graphs, in which $(u,v) \in E$ iff $\arccos(vu) \geq \theta$. We refer to these graphs as *anti-geometric graphs*, and to $\pi - \theta$ as the neighborhood radius. (We note that once we fix θ, in our graphs it will not happen that $\arccos(vu) = \theta$, and hence geometric and anti-geometric graphs are indeed complements of each other.)

Related work studied other families of geometric graphs in $\mathbb{R}^d$ (not necessarily on $\mathbb{S}^d$), with geometric distances induced either by the Euclidean norm or by other norms. We shall use the term *neighborhood radius*, typically denoted by r, to denote the geometric distance up to which two points are declared to be neighbors in these models as well.

For given (n,θ,d), we shall be interested in the distribution $G_{n,\theta,d}$ over (n,θ,d)-graphs, in which the n vertices are placed independently uniformly at random in $\mathbb{S}^d$. Likewise, $\bar{G}_{n,\theta,d}$ denotes the distribution over anti-geometric graphs (the complements of (n,θ,d)-graphs) when the n vertices are placed independently uniformly at random in $\mathbb{S}^d$.

Given a graph $G(V,E)$ and two (not necessarily disjoint) sets A and B of vertices, $E_G(A,B)$ denotes the set of edges with one endpoint in A and the other endpoint in B, and $e_G(A,B) = |E_G(A,B)|$ denotes the number of such edges.

Definition 1. *A graph $G(V,E)$ is called an* (n,h)-expander *if $|V| = n$ and for every set of vertices $S \subset V$ with $|S| \leq n/2$ it holds that $e(S, V \setminus S) \geq h|S|$.*

1.2 Our Results

We shall assume that θ is a fixed constant (specifically, $\theta = \pi/3$), and consider increasing values of d.

Given d, scale the sphere $\mathbb{S}^d$ so that its total surface area is 1. Let $\mu(\theta, d)$ denote the area of a spherical cap of angular radius θ. Given a vertex in $\mathbb{S}^d$, its expected number of neighbors in $G_{n,\theta,d}$ is precisely $(n-1)\mu(\theta, d)$. Hence if $n < 1/\mu(\theta, d)$ we expect a constant fraction of the vertices to be isolated (implying that the graph is not connected). It is not difficult to show (see Section 5) that a value of $n_{IV} \simeq \frac{1}{\mu(\theta,d)} \ln(1/\mu(\theta, d))$ serves as a threshold value for isolated vertices: for every $\epsilon > 0$, if $n \geq (1+\epsilon)n_{IV}$ there are unlikely to be isolated vertices, whereas if $n \leq (1-\epsilon)n_{IV}$ there are likely to be isolated vertices. The same applies to $\bar{G}_{n,\theta,d}$ by setting $n_{IV} \simeq \frac{1}{\mu(\pi-\theta,d)} \ln(1/\mu(\pi-\theta, d))$.

Theorem 1. *Let $\theta < \pi/2$ be a fixed constant (e.g., $\theta = \pi/3$) and let d be sufficiently large. There is some universal constant $c \geq 1$ such that if $n \geq cn_{IV}$ then both $G_{n,\theta,d}$ and $\bar{G}_{n,\pi-\theta,d}$ are connected with probability $1 - o(1)$ (where the $o(1)$ term tends to 0 as d grows).*

Our proof of Theorem 1 shows that a value of $c \simeq \pi/\theta$ suffices. One may conjecture that Theorem 1 is true also for some absolute constant c independent of θ, and moreover, that this constant is not much larger than 1.

We are mostly interested in the case that θ is constant, d is a parameter that may grow, and n is exponential in d. Given θ and d, Theorem 1 determines up to a constant multiplicative factor the smallest value of n for which the graph is likely to be connected. Had we fixed θ and n, the same proof would determine up to a constant *additive* term the largest value of d for which the graph is likely to be connected. For example, if $\theta = \pi/3$ and one is given a value of n, our results establish that there is a constant $c_1 \simeq 6.95212$ (see Section 5) and a constant $c_2 > 0$ such that if $d \leq c_1 \log n$ the graph is connected with probability $1 - o(1)$, and if $d \geq c_1 \log n + c_2$ the graph is connected with probability at most $o(1)$. Finally, had we fixed d and n (exponential in d), the proof of Theorem 1 would determine up to an additive term of order $O(1/d)$ the smallest value of θ for which the graph is likely to be connected.

Our proof of Theorem 1 involves two aspects. One is that of establishing various expansion properties of a random (N, θ, d) graph G' when θ is a fixed constant (e.g., $\theta = \pi/3$), d is a parameter that can grow, and N tends to infinity. Establishing these properties involves symmetrization arguments of the type used in [6, 7]. Thereafter, we view the random (n, θ, d) graph G as a random subgraph of G' induced on n random vertices. We wish to show that the expansion properties of G' imply that G is likely to be connected. This is done using the following theorem which is applied in a situation in which the expansion h is of the order of the maximum degree in G, and this maximum degree is so large (e.g., it might be $\frac{N}{\log N}$) so that $\log(N/h)$ is much smaller than $\log N$.

Theorem 2. *Let $G(V, E)$ be an (N, h)-expander of maximum degree Δ. Consider a vertex induced subgraph H that contains qN vertices chosen randomly*

and independently. If $q > \frac{1+o(1)}{h}\ln(\frac{N}{h}) + O\left(\frac{1}{h}\log(\frac{\Delta}{h})\right)$ *then the probability that* H *is connected is at least* $1 - o(1)$*, where the* $o(1)$ *term tends to 0 as* $\frac{N}{h}$ *tends to infinity.*

1.3 Related Work

Connectivity in uniform random graphs (non-geometric) is well understood. Given n vertices which initially have no edges, if one inserts random and independent edges to the graph, with high probability the graph becomes connected exactly at the point where every vertex has degree at least one [5]. Theorem 1 is an approximate version of this tight connection between connectivity and having no isolated vertices. Theorem 1 has several precursors in work on connectivity of various models of geometric graphs [16]. In the most general setting, a "nice" (in particular, connected) domain $D \in \mathbb{R}^d$ is given together with a measure on the domain and a norm. (In our setting the domain is the unit sphere $\mathbb{S}^d$, the measure is uniform, and the norm can be taken to be Euclidean.) One places n points at random in D, and two points are neighbors if the distance between them (according to the given norm) is at most r. The question typically asked is what is the minimum value of r (as a function of n) for which the graph is likely to be connected. We denote the expectation of this value by $R_C(n)$. Equivalently, given the n points one considers a minimum spanning tree (with edge lengths being the distance between the points according to the given norm), and asks what is the length of the longest edge in this spanning tree. It is not hard to see that this is exactly the value $R_C(n)$ that would ensure connectivity. A different question is what is the minimum value of r for which there are not likely to be any isolated vertices. We call the expectation of this value $R_{IV}(n)$. Clearly, $R_C(n) \geq R_{IV}(n)$ for every n. A very general result of Penrose [19] shows that as n tends to infinity, the ratio $\frac{R_C(n)}{R_{IV}(n)}$ tends to 1. There were previous results of this nature in special cases. See [8, 2, 12, 18], for example.

It might appear that the result of Penrose (or other related previous work) implies our Theorem 1 as a special case. However, this overlooks the issue that in previous results that we are aware of, one first fixes the dimension d and then lets n tend to infinity (we refer to this as *asymptotic* n), whereas in our results n is fixed as a function of d (we refer to this as *bounded* n). As a consequence, the statement of previous results only implies that if θ is sufficiently small as a function of d, a theorem such as Theorem 1 holds. In contrast, we prove Theorem 1 when θ is a fixed constant. Once θ is sufficiently small the corresponding graphs acquire geometric properties that are different than those involved in the case when θ is large (when θ is small graph distances approximate well geometric distances on the sphere, whereas when θ is large this is no longer true), and hence proofs of connectivity that apply in one setting might not apply in the other. This aspect is discussed further in the full version of our paper.

Another line of work related to the questions studied in this work is that of *percolation* (see [11] or [17], for example), and specifically, the process referred to as *continuum percolation*. Typically, in the continuum percolation process

one is given a domain D such as a box in $\mathbb{R}^d$ that is symmetric around the origin. One places a point in the origin and additional points at random in the domain, and two points are connected if their distance from each is at most r. The question asked is not that of complete connectivity of the graph, but rather questions such as whether there is a path from the origin to the boundary of the domain, or what is the size of the connected component that contains the origin. A typical situation is that for large enough n there is a given threshold distance R_p such that if $r > R_p$ the origin is likely to be connected to the boundary, whereas if $r < R_p$ the origin is likely to be in a very small component, or even isolated. Obviously, the connectivity threshold is at least as large as the percolation threshold, namely $R_C \geq R_p$. Some of the work on connectivity builds on results from percolation (e.g.,[12] explicitly refers to results in [15], and [19] makes use of proof techniques, a Peierls argument, that is commonly used in percolation). We remark that in percolation theory one typically deals with the regime of asymptotic n rather than bounded n.

The results of [9] may serve to illustrate a difference between our setting of fixed n and the setting that n tends to infinity. Their result is that every monotone graph property (any property that is preserved by adding edges, connectivity being one such example) in random geometric graphs (in their case, the domain is $[0,1]^d$ and the measure is uniform) has a sharp threshold. More specifically, for every n and for every monotone graph property, there is a corresponding threshold distance R (that depends on n and on the property), such that if $r > R+\epsilon$ the property holds almost surely, and if $r < R-\epsilon$, the property almost surely does not hold. Moreover, the value of ϵ tends to 0 as n grows, specifically at a rate $\epsilon \leq O((\frac{\log n}{n})^{1/d})$. In our case of constant θ, the values of n that we consider are only exponential in d, and consequently the results only imply that $\epsilon \leq O(1)$. Moreover, if one inspects the proof technique of [9], it necessarily results in $\epsilon = \Omega(1)$ when θ is constant. As we consider the unit sphere rather than unit cube, this value of ϵ is of the same order of magnitude as the diameter of the whole domain, and hence completely useless. (We do not claim that monotone graph properties do not have sharp thresholds when $n \leq 2^{O(d)}$. We just point out that if they do, establishing this will require proofs that are different than those that apply when n tends to infinity.)

Theorem 2 considers connectivity of random vertex induced subgraphs of expanders. There have been previous studies of connectivity properties of random subgraphs of expanders. However, all previous work that we are aware of either addressed edge induced subgraphs (see for example [1]), or addressed vertex induced subgraphs at a range of parameters that is very different from that of Theorem 2 (see for example [3]).

Low dimensional geometric graphs are sometimes used as models for wireless communication networks (e.g., in [12]), or for physical medium (see for example Chapter 1 in [11]). Random high dimensional geometric graphs such as the ones studied in this paper are not commonly used as a model for physical reality. However, high dimensional geometric graphs come up naturally as solutions to semidefinite programming relaxation to various optimization problems. In par-

ticular, the solution to semidefinite relaxations of graph coloring problems are high dimensional anti-geometric graphs (see [13], though note that the terminology anti-geometric graphs is not used there). Random *anti-geometric* graphs are used in order to construct negative examples showing a large (tight, in some cases) integrality gap for these semidefinite relaxations [6, 7]. The issue of connectivity of these graphs did not come up in these earlier works, but did come up and was left open in more recent study of these negative examples [4].

2 On the Expansion of the Infinite Graph

In this section we define an infinite graph $\mathbb{G}_{\theta,d}$, with the property that an induced graph on a random sample of n vertices from $\mathbb{G}_{\theta,d}$ is distributed as $G \in_R G_{n,\theta,d}$. We will show (under a natural definition of expansion of infinite graphs) that $\mathbb{G}_{\theta,d}$ is an expander.

Given $\boldsymbol{v} \in \mathbb{S}^d$ the set $\left\{\boldsymbol{u} \in \mathbb{S}^d | \arccos(\boldsymbol{v}\boldsymbol{u}) \leq \theta\right\}$ is a *sphere cap* and it is denoted by $C_{\cos(\theta)}(\boldsymbol{v})$. For example, a cap of angular radius $\pi/3$ centered at $\boldsymbol{v}$ is denoted by $C_{\frac{1}{2}}(\boldsymbol{v})$ and a cap of radius $\pi/2$ (a half sphere) is denoted by $C_0(\boldsymbol{v})$. We omit $\boldsymbol{v}$ from the above notation if the location of the center of the cap is not needed. (The subscript denotes $\cos\theta$ rather than simply θ for compatibility with notation in previous work.)

Definition 2. *(The infinite graph $\mathbb{G}_{\theta,d}$). The vertices of $\mathbb{G}_{\theta,d}$ are all the points in $\mathbb{S}^d$ and the edge set of $\mathbb{G}_{\theta,d}$ is all the pairs $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{S}^d$ s.t. $\boldsymbol{u} \in C_{\cos\theta}(\boldsymbol{v})$.*

Every vertex in $\mathbb{G}_{\theta,d}$ has an infinite number of neighbors, therefore the notion of the "degree" of a vertex is represented by the measure of the set of its neighbors. We normalize the uniform measure on the sphere so that the total measure of the (surface area of the) sphere is 1. Given a measurable set S, its measure (the ratio between the surface area of S to the surface area of $\mathbb{S}^d$) is denoted by $|S|$. This measure corresponds to the number of vertices in S. The edge boundary of S (the set of edges with exactly one endpoint in S) is denoted by $\partial(S)$ and its measure is denoted by $|\partial(S)|$. This measure corresponds to the number of edges leaving S. (The measure of $\partial(S)$ is the integral over all points in S of the measures of the sets of their neighbors outside S.)

The following definition for edge expansion is given, for simplicity, specificity for $\mathbb{G}_{\theta,d}$. An equivalent definition to a general infinite graph can be stated in a straightforward manner.

Definition 3. *(Edge expansion for the infinite graph). The edge expansion $h(\mathbb{G}_{\theta,d})$ of a graph $\mathbb{G}_{\theta,d}$ is defined as*

$$h(G) = \inf_{0<|S|\leq\frac{1}{2}} \frac{|\partial(S)|}{|S|}$$

where the minimum is over all measurable sets $S \subset \mathbb{S}^d$ with nonzero measure. For a given set S we call the quantity $h(G,S) := \frac{|\partial(S)|}{|S|}$ the edge expansion of S.

An expander graph is a graph with high edge expansion. To lower bound $h(\mathbb{G}_{\theta,d})$ it suffices to show that the sets with the lowest edge expansion are sphere caps (Lemma 1) and to analyze the expansion of spherical caps (Lemma 2).

Lemma 1. *For all $S \subset \mathbb{S}^d$ s.t. $|S| = m$ $(0 \leq m \leq \frac{1}{2})$ it holds that $|\partial(S)| \geq |\partial(C_a)|$ for the unique a satisfying $|C_a| = m$.*

Proof. The proof is a direct consequence of Theorem 5 in [7]. The theorem shows that the complement of the graph $\mathbb{G}_{\theta,d}$ (u,v are neighbors in the complement graph iff they are not neighbors in the original graph) has the following property: for all $S \subset \mathbb{S}^d$ s.t. $|S| = m$ $(0 \leq m \leq \frac{1}{2})$ it holds that $|\partial(S)| \leq |\partial(C_a)|$ for the unique a satisfying $|C_a| = m$ (here $\partial(S)$ is the edge boundary of the complement graph of $\mathbb{G}_{\theta,d}$). □

In the rest of this section we deal with $\theta = \pi/3$ for geometric graphs and $\theta = 2\pi/3$ for *anti-geometric* graphs, though the results generalize for any θ.

Lemma 2. $h\left(\mathbb{G}_{\pi/3,d}\right) \geq (1/3 - \epsilon)\left|C_{\frac{1}{2}}\right|$ *for ϵ that tends to 0 as d grows.*

Proof. By Lemma 1, the minimum expansion of $\mathbb{G}_{\pi/3,d}$ is attained at a sphere cap. It is not difficult to see that the expansion of sphere caps decreases as their radius increases. Hence among sets of measure at most 1/2, the minimum expansion is attained for the half sphere C_0. Hence $h\left(\mathbb{G}_{\pi/3,d}\right) = h\left(\mathbb{G}_{\pi/3,d}, C_0\right)$. Estimating $h\left(\mathbb{G}_{\pi/3,d}, C_0\right)$ is fairly simple once d is sufficiently large, as we show below.

Fix $\epsilon' > 0$ to be a small constant. For every vertex $\boldsymbol{v}$ of $\mathbb{G}_{\pi/3,d}$ remove those edges to neighbors of $\boldsymbol{v}$ with angle smaller than $\pi/3 - \epsilon'$, thus obtaining a new graph $\mathbb{G}'_{\pi/3,d}$. The ratio $\frac{C_{\cos(\pi/3-\epsilon')}}{C_{\cos(\pi/3)}}$ tends to zero as d grows. (See Theorem 6.) This implies that we have removed only a small fraction (that tends to zero) of the graph edges.

Consider $\boldsymbol{y}$ drawn uniformly from $\mathbb{S}^d$. Any edge e of $\mathbb{G}'_{\pi/3,d}$ has probability $\frac{\pi/3-\epsilon'}{\pi} = 1/3 - \epsilon'/\pi$ to be in $\partial\left(C_0\left(\boldsymbol{y}\right)\right)$ (the analysis is similar to that of the random hyperplane rounding technique of [10]). Hence roughly one third of the edges of the graph are in the edge boundary of C_0. As for the remaining edges, by symmetry half of them are in the half sphere C_0 and half are in its complement. Hence there are essentially as many edges in the edge boundary of C_0 as there are inside C_0, implying that the expansion of C_0 is nearly one third of (the measure of) the degree of its vertices, establishing that that $h\left(\mathbb{G}_{\pi/3,d}, C_0\right) = (1/3 - \epsilon)\left|C_{\frac{1}{2}}\right|$, as desired.

□

2.1 The Expansion of the Infinite Anti-Geometric Graph

The infinite anti-geometric graph $\bar{\mathbb{G}}_{\pi/3,d}$ is defined as follows: its vertices are all the points in $\mathbb{S}^d$ and the edge set of $\bar{\mathbb{G}}_{\pi/3,d}$ is all the pairs $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{S}^d$ s.t. $\boldsymbol{u} \in C_a\left(-\boldsymbol{v}\right)$, (as opposed to $\boldsymbol{u} \in C_a\left(\boldsymbol{v}\right)$ in the case of $\mathbb{G}_{\pi/3,d}$). The graph induced

on a random sample of n vertices from $\bar{\mathbb{G}}_{\pi/3,d}$ is distributed as $G \in_R \bar{G}_{n,\pi/3,d}$. The edge expansion of $\bar{\mathbb{G}}_{\pi/3,d}$, $h\left(\bar{\mathbb{G}}_{\pi/3,d}\right)$, is defined similarly as in Definition 3.

Lemma 3. $h\left(\bar{\mathbb{G}}_{\pi/3,d}\right) \geq h\left(\mathbb{G}_{\pi/3,d}\right)$

Proof. In this proof we shall switch between several graphs. Given a graph H and sets A, B of vertices, the set edges of H with one endpoint in A and the other in B will be denoted by $E(H, A, B)$. Given $A \subset \mathbb{S}^d$ let $\bar{A} := \mathbb{S}^d \setminus A$, i.e the complement set, and let $A^{-1} := \left\{\boldsymbol{x} \in \mathbb{S}^d | -\boldsymbol{x} \in A\right\}$.

To prove the lemma we need to show that

$$\min_{|A|=a} \left|E\left(\bar{\mathbb{G}}_{\pi/3,d}, A, \bar{A}\right)\right| \geq \min_{|A|=a} \left|E\left(\mathbb{G}_{\pi/3,d}, A, \bar{A}\right)\right|$$

for every $0 < a \leq \frac{1}{2}$. Consider an arbitrary $0 < a \leq \frac{1}{2}$.

$$\min_{|A|=a} \left|E\left(\bar{\mathbb{G}}_{\pi/3,d}, A, \bar{A}\right)\right| \geq \min_{\substack{|A|=a \\ |B|=1-a}} \left|E\left(\bar{\mathbb{G}}_{\pi/3,d}, A, B\right)\right|$$

$$= \min_{\substack{|A|=a \\ |B|=1-a}} \left|E\left(\mathbb{G}_{\pi/3,d}, A, B^{-1}\right)\right| = \min_{\substack{|A|=a \\ |B|=1-a}} \left|E\left(\mathbb{G}_{\pi/3,d}, A, B\right)\right|$$

To finish the proof it suffices to show that:

$$\min_{\substack{|A|=a \\ |B|=1-a}} \left|E\left(\mathbb{G}_{\pi/3,d}, A, B\right)\right| = \min_{|A|=a} \left|E\left(\mathbb{G}_{\pi/3,d}, A, \bar{A}\right)\right|$$

We claim that this last equality is a consequence of Theorem 3.5 in [6], which shows the following:

Consider the infinite anti-geometric graph defined on $\mathbb{S}^d$ with parameter θ. Let $0 < a \leq 1$ and let A and B be two (not necessarily disjoint) measurable sets in $\mathbb{S}^d$ of measure $|C_a|$. Let $\boldsymbol{x}$ be an arbitrary vertex of $\mathbb{S}^d$. The minimum of $|E(A, B)|$ is obtained when $A = B = C_a(\boldsymbol{x})$ where C_a is a cap of measure $|C_a|$.

By the above theorem $A = B = C_a(\boldsymbol{x})$ minimizes

$$\min_{\substack{|A|=a \\ |B|=a}} \left|E\left(\bar{\mathbb{G}}_{\pi-\pi/3,d}, A, B\right)\right|.$$

Since $\mathbb{G}_{\pi/3,d}$ and $\bar{\mathbb{G}}_{\pi-\pi/3,d}$ are complement graphs of each other, $A = B = C_a(\boldsymbol{x})$ maximizes

$$\max_{\substack{|A|=a \\ |B|=a}} \left|E\left(\mathbb{G}_{\pi/3,d}, A, B\right)\right| =$$

Equivalently, $A = C_a(\boldsymbol{x})$ and $B = \bar{A}$ maximize

$$\max_{\substack{|A| = a \\ |B| = 1-a}} \left| E\left(\mathbb{G}_{\pi/3,d}, A, \bar{B}\right)\right|.$$

By regularity of the graph $\mathbb{G}_{\pi/3,d}$, it follows that $A = C_a(\boldsymbol{x})$ and $B = \bar{A}$ minimize

$$\min_{\substack{|A| = a \\ |B| = 1-a}} \left| E\left(\mathbb{G}_{\pi/3,d}, A, B\right)\right|$$

proving the claim. □

3 Connectivity of Random Geometric and Anti-geometric Graphs

Having established expansion properties for the infinite graph, we present two proofs of Theorem 1. One proof first "discretizes" the infinite graph, thus obtaining a nearly regular very dense finite graph with expansion properties similar to that of the infinite graph (expansion roughly one third of the degree, for our choice of $\theta = \pi/3$). This dense graph can be thought of as being obtained by taking a finite though extremely large number N of sample points from $\mathbb{G}_{\pi/3,d}$. The formal details of such a discretization are similar to those in [7, 6], and are omitted here. Thereafter, noting the relation $h \geq 0.3\Delta$ between the expansion and maximum degree, one can use Theorem 2, whose proof appears in Section 4.

The other way to prove the main theorem is via a direct proof of sampling from the infinite graph, without performing the discretization first. This may appear in the full version of the paper.

We note that there are alternative approaches that can be used in order to try to prove connectivity of geometric graphs. Specifically, one may try to establish that the graph enjoys a property called *geometric routing*. Essentially, this property means that between every two vertices u and v of the graph there is a path that respects the geometry of the sphere – advancing from u to v along this path decreases the geometric distance to v in every step. However, for our graphs, geometric routing will not work. In fact, the number n of vertices that are required in order to have geometric routing in $G \in_R G_{n,\pi/3,d}$ is such that the average degree of the graph is as high as roughly $n^{0.29}$, rather than only $O(\log n)$ which (as Theorem 1 shows) suffices for connectivity. See more details in the full version of this paper.

4 Connectivity of Subgraphs of Expanders

In this section we prove Theorem 2. We were tempted to try some simpler proof techniques than the ones used in this section, but encountered difficulties in employing them. This issue is discussed further in the full version of this paper.

We shall be concerned with a graph G that is an (n, h)-expander of maximum degree Δ, and a random vertex induced subgraph of G that we denote by H. We use the notation $V, E, n, N(S)$ for set of vertices, set of edges, number of vertices, set of neighbors of S not including S, respectively, all in relation to the graph G.

Our proof of Theorem 2 involves two steps.

1. The first step is similar in nature to percolation. We pick an arbitrarily sampled vertex $v \in V_H$ and then show that with high probability it belongs to a fairly large connected component CC_v in H. The size of the connected component is not measured in terms of the number of vertices that it contains, but rather in terms of the fraction of vertices of the original expander graph G that are neighbors of this connected component. We show that this fraction to be at least half. Namely, $|N_G(CC_v) \bigcup CC_v| \geq n_G/2$.
2. The second step shows that with high probability every vertex $u \in V_H$ has a path consisting only of vertices from H that connects it to CC_v. This step uses the fact that $N_G(CC_v)$ is large, and hence is easy to reach.

In our analysis of Step 1 we shall use the following lemma.

Lemma 4. *For arbitrary $0 < \mu \leq \Delta \leq M$, let $x_i, x_2, \ldots$ be a sequence of nonnegative random variables satisfying $x_i \leq \Delta$ for all i, $E[x_1] \geq \mu$, and $E[x_i|x_1, \ldots, x_{i-1}] \geq \mu$ for $i \geq 2$. Let t be a stopping time, giving the smallest index such that $\sum_{i=1}^{t} x_i \geq M$. Then the following hold:*

1. $E[t] \leq \frac{M+\Delta}{\mu}$.
2. $Pr[t > 2\frac{M}{\mu}] \leq \frac{2\Delta}{M}$.

Proof. Change every random variable x_i to a nonnegative random variable x'_i of expectation exactly μ by reducing its value, if needed. Consider the sequence $y_1, y_2, \ldots$ of random variables in which $y_i = x'_i - \mu$ for all i. The sequence $Y_i = \sum_{j=1}^{i} y_j$ is a Martingale. Let t be a stopping time for the Martingale sequence, giving the smallest index such that $Y_t \geq M - t\mu$. The random variable t has bounded moments, and hence by the optional stopping time theorem for martingales, $E[Y_t] = 0$. Moreover, by the fact that $y_i \leq \Delta - \mu$ and the minimality of t, we have $Y_t \leq Y_{t-1} + (\Delta - \mu) < M - (t-1)\mu + \Delta - \mu = M + \Delta - t\mu$. Hence $E[M + \Delta - t\mu] > 0$ implying $E[t] \leq \frac{M+\Delta}{\mu}$, proving item 1 of the lemma.

Let σ_i^2 be the variance of y_i (conditioned on $y_1, \ldots, y_{i-1}$). Observe that $\sigma_i^2 \leq \frac{\mu}{\Delta}(\Delta - \mu)^2 + \frac{\Delta-\mu}{\Delta}\mu^2 \leq \mu\Delta$. Using the fact that $E[y_i|y_j] = 0$ for $i > j$ we obtain that $var(Y_t) \leq \mu\Delta t$, implying by Chebychev's inequality that $Pr[Y_t < -\mu t/2] \leq \frac{4\Delta}{\mu t}$. For $t = 2\frac{M}{\mu}$ this gives $Pr[Y_t < -M] \leq \frac{2\Delta}{M}$. Observe that $X_{2M/\mu} \geq Y_{2M/\mu} + 2M$, implying item 2 of the lemma. □

The bounds we shall use for Step 1 will be presented in Theorem 4. We first prove Theorem 3 which presents bounds that are incomparable to those of Theorem 4, and whose proof can serve as an introduction to the proof technique of Theorem 4.

Theorem 3. *Let $G(V,E)$ be an n-vertex Δ-regular graph with edge expansion at least h, and let $r \in V$ be an arbitrary vertex. Then with probability at least $1/2$, a random sample U of $\frac{4n}{h}\ln\frac{n}{\Delta}$ vertices contains a subset $S \subset U \cup \{r\}$ with $|N(S) \cup S| \geq n/2$ such that the subgraph induced on S is connected.*

Proof. We expose vertices of U one by one. For $1 \leq i \leq |U|$, let u_i be the ith vertex exposed and let $U_i = \{u_1, \ldots u_i\}$. At every step i of the process we shall maintain a subset $S_i \subset U_i \cup \{r\}$ with $r \in S_i$ such that the subgraph induced on S_i is connected. Specifically, $S_0 = \{r\}$, and $u_i \in S_i$ iff $u_i \in N(S_{i-1})$.

Let us track the growth of $|S_i \cup N(S_i)|$. Initially, $|S_0 \cup N(S_0)| = 1 + \Delta$. For $i \geq 1$ we have that $|S_i \cup N(S_i)| \leq |S_{i-1} \cup N(S_{i-1})| + \Delta$. By the expansion properties of G and averaging arguments, the expected growth in step i satisfies $E[|(S_i \cup N(S_i)) \setminus (S_{i-1} \cup N(S_{i-1}))|] \geq \frac{|S_{i-1} \cup N(S_{i-1})|}{n} h$, as long as $|S \cup N(S)| \leq n/2$.

Partition the growth of S_i into phases, where phase ℓ ends at the smallest value of i for which $|S_i \cup N(S_i)| \geq \ell\Delta$. Then by item 1 of Lemma 4 the expected number of steps that phase $\ell + 1$ takes is at most $\frac{2\Delta n}{h\ell\Delta} = \frac{2n}{h\ell}$. It takes $\frac{n}{2\Delta}$ phases to reach $|S \cup N(S)| \geq n/2$. Hence the expected number of steps required is at most $\sum_{\ell=2}^{\frac{n}{2\Delta}} \frac{2n}{h\ell} < \frac{2n}{h}\ln\frac{n}{\Delta}$. The Theorem follows from Markov's inequality. □

In the statement of Theorem 4 and in its proof, c denotes some sufficiently large constant independent of n, Δ, h.

Theorem 4. *Let $G(V,E)$ be an n-vertex Δ-regular graph with edge expansion at least h, and let $r \in V$ be an arbitrary vertex. Then with probability at least $1/2$, a random sample U of $\frac{cn}{h}\log\frac{\Delta}{h}$ vertices contains a subset $S \subset U \cup \{r\}$ with $|N(S) \cup S| \geq n/2$ such that the subgraph induced on S is connected.*

Proof. Partition U into two parts U' and U" of equal size (namely, $|U'| = |U"| = |U|/2$). A proof similar to that of Theorem 3 (details omitted) implies that with overwhelming probability, U' suffices in order to grow S from its initial size of $|S_0 \cup N(S_0)| = 1 + \Delta$ by a factor of 8, reaching size $|S \cup N(S)| = 8\Delta$. It remains to show that U" can be used in order to grow S further, eventually reaching $|S \cup N(S)| = n/2$.

We expose information about vertices of U" only when needed. The exposure algorithm will proceed in phases. Let $k = \log(4\Delta/h)$. Initially U" is partitioned into $k+1$ sets, $U_0^0, U_0^1, \ldots U_0^k$. Renaming S_0 to be the outcome of the first part, we have $|S_0 \cup N(S_0)| \geq 8\Delta$. We shall have $|U_0^0| = \frac{2n}{h}$, $|U_j^1| = \frac{8n}{h}$ for every $1 \leq j \leq k$. Each phase i is composed of k subphases, where in the jth subphase one scans U_{i-1}^j. In such a scan some vertices are moved to S_{i-1} thus eventually obtaining S_i. The vertices moved to S are replaced in the respective U_{i-1}^j by fresh vertices from U_{i-1}^0. Hence the cardinalities of U^j for $1 \leq j \leq k$ remain unchanged during the exposure algorithm, and $|U^0|$ decreases exactly at the same rate by which $|S|$ increases.

A subphase is considered *successful* if the size of $S \cup N(S)$ increases by a multiplicative factor of at least 2 during the subphase.

Given the current S, a vertex u that is scanned is *good* if $u \in N(S)$ and moreover, $|N(S \cup \{u\})| \geq |N(S)| + \frac{h}{4} + 1$. Only good vertices are added to

S. Observe that the total number of good vertices cannot exceed $\frac{n/2}{h/4}$ (while maintaining $|S \cup N(S)| \leq n/2$), and hence U^0 can indeed compensate for all good vertices.

We now compute the expected contribution of a scanned vertex u, conditioned on all previous subphases being successful. Let S' be the set S at the time u was scanned in the previous phase. Unless u is a fresh vertex from U^0, we need to condition on $u \notin S' \cup N(S')$. Because the previous $k-1$ subphases were successful we have $|N(S) \cup S| \geq \frac{2\Delta}{h}|N(S') \cup S'|$. By the expansion properties of G, the number of edges exiting $N(S) \cup S$ is at least $h(N(S) \cup S)$. At least $\frac{3h}{4}(N(S) \cup S)$ of these edges originate from vertices of $N(S)$ that have at least $h/4$ exiting edges. At most $\Delta\frac{h}{2\Delta}(N(S) \cup S) = \frac{h}{2}(N(S) \cup S)$ of these edges originate from vertices of $S' \cup N(S')$. This leaves at least $\frac{h}{4}(N(S) \cup S)$ edges available for good vertices, implying that the expected contribution of a scanned vertex u is at least $\frac{h}{4n}(N(S) \cup S)$. Hence in expectation not more than $\frac{4n}{h}$ vertices are needed until $N(S) \cup S$ doubles its size. As a subphase contains $\frac{8n}{h}$ vertices, item 2 of Lemma 4 implies that the probability that there is an unsuccessful phase is at most $\sum_{\ell \geq 1} \frac{2\Delta}{4\Delta 2^\ell} \leq 1/2$.

□

Our proof for Theorem 4 did not attempt to optimize the value of the leading constant c.

Each of the bounds in Theorems 3 and 4 may be better than the other, depending on the relative values of n, Δ, h. In our intended applications $\frac{\Delta}{h}$ is much smaller than $\frac{n}{\Delta}$, and hence we shall use Theorem 4.

The requirement that G is regular in Theorems 3 and 4 was made because it simplifies the proofs. This requirement can be removed by slightly adjusting the statement of the theorem.

Theorem 5. *Let $G(V, E)$ be an n-vertex graph with edge expansion at least h, and let Δ denote the degree of the vertex of $\frac{h}{4}$th highest degree (breaking ties arbitrarily). Then with probability at least $1/2$, a random sample U of $\frac{cn}{h}\min[\log\frac{n}{\Delta}, \log\frac{\Delta}{h}]$ vertices contains a subset $S \subset U$ with $|N(S) \cup S| \geq n/2$ whose induced subgraph is connected.*

Proof. Let $H \subset V$ be the set of $h/4$ highest degree vertices in G. We make a preliminary pass over all vertices of U. This pass is successful if $1 \leq |U \cap H| \leq 5c\min[\log\frac{n}{\Delta}, \log\frac{\Delta}{h}]$. This fails with probability $2^{-\Omega(c)}$.

If the preliminary pass achieved its goal, we keep in S only one vertex r chosen arbitrarily from $H \cap S$, and remove from U all other vertices of $H \cap U$. The remaining size of U is at least $(cn/h - 5c)\min[\log\frac{n}{\Delta}, \log\frac{\Delta}{h}]$. This remaining size is still roughly $\frac{cn}{h}\min[\log\frac{n}{\Delta}, \log\frac{\Delta}{h}]$. (This would fail to hold only if h is $\Omega(n)$. However, in that case the proofs of Theorems 3 and 4 apply, requiring only an adjustment of the constants hidden in the c notation.) Observe that for every set $T \subset (V \setminus H)$ of size at most $n/2 - 1$, the set $\{r\} \cup T$ has edge expansion at least $3h/4$ into $V \setminus H$. Replace G by the subgraph G' induced on $V' = (V \setminus H) \cup r$. Observe that in this subgraph the degree of r is at least its

original degree (which was necessarily at least Δ) minus $h/4$. Hence the degree of r in this subgraph is at least $\Delta/2$ (it is not hard to show that $h/4 \leq \Delta/2$), and moreover, no vertex in V' has degree larger than Δ.

It can readily seen that the proofs of Theorems 3 and 4 did not use regularity of G, but rather the following two aspects of Δ: that r has degree at least Δ, and no vertex has degree larger than Δ. For G' the only difference is that the degree of r is at least $\Delta/2$ rather than Δ (and the expansion is also smaller by a constant factor). The proofs of Theorems 3 and 4 apply, requiring only an adjustment of the constants hidden in the c notation. □

We now prove Theorem 2.

Proof. (Theorem 2). Recall that H has $\frac{(1+o(1))n}{h}\ln(\frac{n}{h}) + O\left(\frac{n}{h}\log(\frac{\Delta}{h})\right)$ random vertices. Of them, we use up $O\left(\frac{n}{h}\log(\frac{\Delta}{h})\right)$ random vertices in the proof of Theorem 5, and as a consequence we conclude (with probability that can be made arbitrarily close to 1, by changing the constant in the O notation) that H has a connected component CC_v satisfying $|N(CC_v)\bigcup CC_v| \geq n/2$. This was referred to as Step 1 above. Now we analyze Step 2, which is based on considering the remaining random vertices of H, whose number is at least $\frac{(1+o(1))n}{h}\ln(\frac{n}{h})$.

Let us define a linear order on all vertices of G. The property of this linear order is that for every vertex $u \in V$, either it precedes at least h of its neighbors in this linear order, or $u \in N(CC_v)\bigcup CC_v$. Such a linear order exists by the expansion properties of G. The vertices of $N(CC_v)\bigcup CC_v$ can be placed last in this linear order. As for the set R of remaining vertices, the cardinality of R is at most $n/2$, and hence $e_G(R, V_G \setminus R) \geq h|R|$. Hence at least one vertex in R has h neighbors already placed later than R in the linear order, and this vertex can be placed last among R. This argument can be continued by induction to complete the desired linear order.

Consider now the placement of all vertices of H in this linear order. If for every vertex $u \in H$, either $u \in N(CC_v)\bigcup CC_v$, or there is a vertex $w \in N(u)$ that is also in H and moreover w appears later than u in the linear order, then H is connected (because every vertex has a path to v). In our intended applications $\frac{(1+o(1))n}{h}\ln(\frac{n}{h}) \geq \Omega\left(\frac{n}{h}\log(\frac{\Delta}{h})\right)$, implying that $|H| = O\left(\frac{n}{h}\ln(\frac{n}{h})\right)$. In this case, taking $\frac{(1+o(1))n}{h}\ln(\frac{n}{h})$ random vertices of G in Step 2, a union bound implies that the probability that connectivity fails is at most $|H|e^{-(1+o(1))\ln(n/h)} \leq o(1)$. Note also that even if the condition $\frac{(1+o(1))n}{h}\ln(\frac{n}{h}) \geq \Omega\left(\frac{n}{h}\log(\frac{\Delta}{h})\right)$ does not hold, then in Step 2 we may take $\Omega\left(\frac{n}{h}\log(\frac{\Delta}{h})\right)$ random vertices, and the union bound works as well. □

5 A Note on the Dimension Range

Recall the definition of sphere casps, see Section 2. In this section we use the same bounds as in [7]:

Theorem 6. *(Bounds on the sphere Cap measure).* $\frac{c}{\sqrt{d}}(1-a^2)^{\frac{d-1}{2}} \leq |C_a| \leq \frac{1}{2}(1-a^2)^{\frac{d-1}{2}}$, *where c is some constant independent of d.*

Assume that the dimension d of our graphs $G_{n,\pi/3,d}$ is $c\ln(n)$. It follows that when c gets larger each vertex has fewer neighbors. We would like to determine the values of c for which $G \in_R G_{n,\pi/3,d}$ has isolated vertices with high probability. Let v be a vertex in G it holds that $E[|N(v)|] = (n-1)\left|C_{\frac{1}{2}}\right|$. Theorem 6 implies $\left|C_{\frac{1}{2}}\right|$ can be upper bounded by $\frac{1}{2}\left(1-\frac{1}{2}^2\right)^{\frac{d-1}{2}}$. Therefore the expected number of neighbors of each vertex can be upper bounded:

$$(n-1)\left|C_{\frac{1}{2}}\right| \leq \frac{1}{2}n\left(1-\left(\frac{1}{2}\right)^2\right)^{\frac{d-1}{2}} = \frac{1}{2}ne^{\ln\left(\frac{3}{4}\right)\frac{d-1}{2}}$$

$$= O(1)\,ne^{\ln\left(\frac{3}{4}\right)\frac{d}{2}} = O(1)\,n^{1+\ln\left(\frac{3}{4}\right)\frac{c}{2}}$$

Therefore if $1+\ln\left(\frac{3}{4}\right)\frac{c}{2} < 0 \Rightarrow c > \frac{2}{\ln\left(\frac{4}{3}\right)} = 6.95212$ then (by applying the Markov's inequality) the probability that v is isolated tends to one.

Now we determine the values of c for which $G \in_R G_{n,\pi/3,d}$ has no isolated vertices with high probability. Note that by Fact 6 $\left|C_{\frac{1}{2}}\right|$ can be lower bounded by $\frac{O(1)}{\sqrt{d}}\left(1-\frac{1}{2}^2\right)^{\frac{d-1}{2}}$. Therefore the expected number of neighbors of each vertex can be lower bounded:

$$(n-1)\left|C_{\frac{1}{2}}\right| \geq \frac{O(1)}{\sqrt{d}}n\left(1-\left(\frac{1}{2}\right)^2\right)^{\frac{d-1}{2}} = \frac{O(1)}{\sqrt{d}}n^{1+\ln\left(\frac{3}{4}\right)\frac{c}{2}}$$

Choose c so that the right hand side is somewhat larger than $\ln n$. This requires $1+\ln\left(\frac{3}{4}\right)\frac{c}{2}$ to be slightly larger than 0, which happens for $c \simeq \frac{2}{\ln\left(\frac{4}{3}\right)} = 6.95212$. Then standard large deviation bounds imply each vertex has probability $o(\frac{1}{n})$ of having no neighbors, and then by applying the union bound the probability that there is an isolated vertex is $o(1)$.

6 More on Vertex Percolation in Expander Graphs

We say that a graph G has a *majority component* if G has a connected component containing at least half its vertices. The following corollary is not needed in order to prove Theorem 1, but might be of independent interest.

Corollary 1. *Let $G(V,E)$ be an n-vertex Δ-regular graph with edge expansion at least h. Consider $G[U]$, a subgraph of G induced on a random sample U of $\frac{cn}{h}\log\frac{\Delta}{h}$ vertices. Then with probability at least $1/2$ over the choice of U, the subgraph $G[U]$ contains a majority component.*

Proof. Set the value of c to be large enough so that a simple adaptation of the proof of Theorem 4 implies that every vertex $u \in U$ has probability at least $9/10$ of being in a component S_u with $|S_u \cup N(S_u)| > (n+h)/2$. Observe that if $|S_u \cup N(S_u)| > (n+h)/2$ and $|S_v \cup N(S_v)| > (n+h)/2$ then $|(S_u \cup N(S_u)) \cap (S_v \cup N(S_v))| \geq h$. Moreover, any vertex of U that lies in $(S_u \cup N(S_u)) \cap (S_v \cup N(S_v))$ connects S_u and S_v.

Fix one arbitrary vertex $u \in U$ and analyse $|S_u|$. With probability at least $9/10$ we have $|S_u \cup N(S_u)| > (n+h)/2$. For every $v \neq u$, $v \in U$, we also have probability at least $9/10$ to have $|S_v \cup N(S_v)| > (n+h)/2$. If both events hold, then with overwhelming probability v is in the same component as u (one can reserve a small fraction of the vertices of U specifically for the purpose of checking whether they land in $(S_u \cup N(S_u)) \cap (S_v \cup N(S_v))$). Hence given that $|S_u \cup N(S_u)| > (n+h)/2$, the expected number of vertices not in S_u is at most roughly $|U|/10$, implying that with probability at most roughly $1/5$ it exceeds $|U|/2$. Hence the probability that $|S_u| < |U|/2$ is at most $\frac{1}{5} + \frac{1}{10} + \epsilon < \frac{1}{2}$, where the ϵ term accounts for low probability events ignored n the computation. □

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