On optimal strategies for a hat game on graphs

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Abstract

The following problem was introduced by Marcin Krzywkowski as a generalization of a problem of Todd Ebert. After initially coordinating a strategy, n players each occupy a different vertex of a graph. Either blue or red hats are placed randomly and independently on their heads. Each player sees the colors of the hats of players in neighboring vertices and no other hats (and hence, in particular, the player does not see the color of his own hat). Simultaneously, each player either tries to guess the color of his own hat or passes. The players win if at least one player guesses correctly and no player guesses wrong. The value of the game is the winning probability of the strategy that maximizes this probability. Previously, the value of such games was derived for certain families of graphs, including complete graphs of carefully chosen sizes, trees, and the 4-cycle.

In this manuscript we conjecture that on every graph there is an optimal strategy in which all players who do not belong to the maximum clique always pass. We provide several results that support this conjecture, and determine among other things the value of the hat game for any bipartite graph and any planar graph that contains a triangle.

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1 Introduction

The following hat problem was formulated by Todd Ebert [4], and has since been studied in several papers, as well as becoming a popular mathematics puzzle question. There are n players who may coordinate a strategy before the game begins. Each player gets a hat whose color is selected randomly and independently to be blue with probability 1/2 and red otherwise. Each player can see the colors of all other hats but not of his own. Simultaneously, each player may guess a color or pass. The players win if at least one player guesses correctly the color of his own hat, and no player guesses wrong. The goal is to find a strategy that maximizes the probability of winning. This maximum probability is called the value of the game.

If the game was different and only one pre-specified player would need to guess, then the value of the game would be 1/2. In fact, even if all players need to guess the colors of

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their hat simultaneously, they still win with probability 1/2. One strategy achieving this is for each player to guess blue if he sees an even number of blue hats, and red otherwise. The players all guess correctly if and only if the number of blue hats is odd. The twist offered by Ebert's game is that players are allowed to pass. If $n = 2$, this does not increase their winning probability. (This can be verified by an easy case analysis.) However, whenever $n \geq 3$ the right to pass makes a big difference. For example, when $n = 3$ the following strategy (played by all players) wins with probability 3/4. If the two hats that the player sees have the same color, the player announces the other color. Otherwise he passes. The players lose only if all three hats are of the same color. It can be shown that as n increases the value of Ebert's game approaches 1.

Marcin Krzywkowski [7] introduced the following variant of the hat problem. The players are placed on vertices of a graph, and a player can only see the colors of hats of his neighbors. The requirement for winning remains the same. (Variations of a similar nature to other hat games are studied in [2].) If the graph is a complete graph, this is exactly Ebert's original problem. In [7] it is shown that if the graph is a tree, the value of the corresponding game is 1/2. In [8] the same result is shown when the graph is C_4 (a cycle on four vertices).

1.1 Our results

Given a graph G, the value of the hat game on graph G will be denoted by $h(G)$.

For a graph G, let $\omega(G)$ denote the size of the maximum clique (complete graph) in G. Let K_n denote the complete graph on n vertices. We conjecture that the value of the hat game on a graph G is completely determined by the size of the largest clique in G.

Conjecture 1 *For every graph, $h(G) = h(K_{\omega(G)})$.*

The inequality $h(G) \geq h(K_{\omega(G)})$ is straightforward. The following strategy for G has success probability at least $h(K_{\omega(G)})$. Let K be a maximum clique in G. Players not in K always pass. Those players in K ignore those players not in K, and simply play the optimal strategy for K.

Hence the real content of Conjecture 1 is in the inequality $h(G) \leq h(K_{\omega(G)})$. This is consistent with the results obtained by Krzywkowski on trees and C_4. In both cases $\omega(G) = 2$, $h(K_2) = 1/2$, and the value for the corresponding hat games are indeed 1/2. In this manuscript we provide additional results consistent with our conjecture.

For certain values of $\omega(G)$, we are able to prove the conjecture.

Theorem 2 *Conjecture 1 is true whenever $\omega(G) + 1$ is a power of 2.*

Our attempt to prove the conjecture for other values of $\omega(G)$ has not succeeded so far. However, we have identified certain families of graphs for which the conjecture is true. Let $\chi(G)$ denote the chromatic number of a graph G (the number of colors that suffice to color its vertices so that the endpoints of every edge receive different colors). Observe that $\chi(G) \geq \omega(G)$ for every graph.

Theorem 3 *Conjecture 1 is true whenever $\chi(G) = \omega(G)$. More generally, Conjecture 1 is true whenever $h(K_{\chi(G)}) = h(K_{\omega(G)})$.*

Among other things, Theorem 3 implies that for all bipartite graphs the value of the hat game is 1/2. This both generalizes and provides an alternative proof to the results of Krzywkowski (since both trees and C_4 are bipartite). The second part of Theorem 3 implies (among other things) that for all planar graphs that contain a triangle, the value of the hat game is 3/4.

1.2 Notation

We refer to the hat colors as 0 or 1 rather than blue or red. A hat *configuration* is a placement of hats of random colors on the heads of players. When there are n players, a configuration is naturally represented as a string in $\{0,1\}^n$. Every player can observe only that part of the configuration that corresponds to the colors of the hats of his neighbors. A strategy for a single player is a function from the vector of colors observable to the player to $\{0, 1, \text{pass}\}$. A strategy for the players is a collection of strategies, one for each player.

1.3 Organization of the paper

In Section 2 we review the known results regarding the hat game on complete graphs. In Section 3 we develop techniques that prove Theorem 2. The same techniques can be used in order to show that given a graph G, computing the value of $h(G)$ is NP-hard. This is shown in Section 4. In Section 5 we prove Theorem 3. In Section 6 we show that the value of the hat game on a union of two disjoint graphs is no higher than the value on one of the graphs. This is a result one would expect to have if Conjecture 1 is true. Section 7 provides a discussion of some of the implications of Conjecture 1. Not knowing whether Conjecture 1 is true, these implications are presented as conjectures, and may serve as intermediate steps towards proving (or refuting) Conjecture 1.

2 Complete graphs

Here we review known results (see e.g. [9]) regarding the hat game on complete graphs. Our terminology might be somewhat different than that used in previous work.

Let K_n denote the complete graph (clique) on n vertices. Let H_n denote the hypercube graph of dimension n (with 2^n vertices labelled by binary strings on length n, with two vertices connected by an edge if the Hamming distance of their corresponding labels is 1). The space of all hat configurations for K_n is naturally represented by the hypercube H_n. An edge in H_n along dimension i connects two configurations that differ only in the color of the hat of player i. Hence if player i guesses in one of the configurations, he does so also in the other configuration, and his guess is correct in exactly one of the two configurations.

A *dominating set* in a graph is a set D of vertices such that every vertex not in D has a neighbor in D. Let $\gamma(G)$ denote the size of the minimum size dominating set in G.

Proposition 4 *The value of the hat game in a complete graph K_n is exactly*

$$h(K_n) = 1 - \frac{\gamma(H_n)}{2^n}$$

Proof. Fix an arbitrary strategy S for the players in K_n. Given a configuration in which the players win, there must be a neighboring configuration in H_n on which some player guesses wrong. Hence the union of configurations on which some player guesses wrong and those on which no player guesses must form a dominating set in H_n. If follows that $h(K_n) \leq 1 - \frac{\gamma(H_n)}{2^n}$.

Given an arbitrary dominating set D in H_n, the following strategy wins on all configurations not in the dominating set. Each player, upon seeing the hats of the other players, checks whether he himself having a hat of color 0 would make the configuration a member of D. If so, he guesses 1. Else, he checks whether he himself having a hat of color 1 would make the configuration a member of D. If so, he guesses 0. If none of these two cases hold, the player passes. It can easily be seen that this strategy fails only if the configuration is in D. It follows that $h(K_n) \geq 1 - \frac{\gamma(H_n)}{2^n}$. ■

It is well known that if (and only if) $n+1$ is a power of 2, then $\gamma(H_n) = 2^n/(n+1)$. The codewords of the Hamming code serve as a corresponding dominating set (and divisibility requirements prove the only if direction). Hence we have the following corollary.

Corollary 5 *If $n+1$ is a power of 2, then $h(K_n) = n/(n+1)$. In particular, $h(K_1) = 1/2$, $h(K_3) = 3/4$, and $H(K_7) = 7/8$.*

For general values of n, the exact value of $\gamma(H_n)$ appears not to be known. The asymptotic behavior of $\gamma(H_n)$ was studied in [6] where it was shown to behave like $(1 + o(1))2^n/n$. For small values of n the value of $\gamma(H_n)$ can be deduced from a combination of degree constraints and integrality constraints. Clearly, if a graph G of n vertices has maximum degree d, then $\gamma(G) \geq n/(d+1)$. It follows that $\gamma(H_n) \geq 2^n/(n+1)$. This together with integrality constraints implies that $\gamma(H_2) \geq \lceil 4/3 \rceil = 2$, and that $\gamma(H_4) \geq \lceil 16/5 \rceil = 4$. Using also $h(K_2) \geq h(K_1)$ and $h(K_4) \geq h(K_3)$ we see that $h(K_2) = 1/2$ and $h(K_4) = 3/4$. Moreover, the fact that $2^n/(n+1)$ is integer only when n is a power of 2 implies that Corollary 5 captures all values of n for which $h(K_n) = n/(n+1)$. When $n+1$ is not a power of 2, then necessarily $h(K_n) < n/(n+1)$.

3 An upper bound

Here we present a simple upper bound on $h(G)$.

Given a graph G with n vertices, consider the set $C(G)$ of all 2^n hat configurations. A strategy S for the players partitions $C(G)$ into three sets, the set $W_S(G)$ on which S leads the players to win, the set $L_S(G)$ on which S leads the players to lose because some player guesses wrong, and the set $N_S(G)$ on which no player using S makes a guess. We now compare the relative sizes of $W_S(G)$ and $L_S(G)$. For this, we construct a bipartite graph B with $W_S(G)$ as the left hand side vertices and $L_S(G)$ as the right hand side vertices. We have an edge (u, v) between a vertex $u \in W_S(G)$ and a vertex $v \in L_S(G)$ if and only if the configurations of u and v differ only in the color of one hat (say, the hat of player i), and the respective player (player i) guesses a color in both configurations. Note that to player i both configurations look identical as he does not see the color of his own hat. Hence if he guesses in one of the configurations he makes the same guess in the other. His guess is correct in exactly one of the configurations.

Proposition 6 *In the bipartite graph B, every vertex of $W_S(G)$ has degree at least 1.*

Proof. For every vertex $u \in W_S(G)$, there is some player (say player i) that guesses correctly. Let v be the configuration that differs from u only in the color of hat i. Then i guesses also in v and guesses wrong, and so $v \in L_S(G)$ and the edge (u, v) exists. ■

Proposition 7 *In the bipartite graph B, every vertex of $L_S(G)$ has degree at most $\omega(G)$.*

Proof. Let v be a vertex in $L_S(G)$. Let T be the set of players that guess wrong in the hat configuration that corresponds to v. (We consider only players who guessed wrong in v, because a configuration u that differs from v only in the color of the hat of a player who guessed correctly in v cannot belong to $W_S(G)$.) Now let $T' \subset T$ be the set of players i in T such that the configuration u that differs from v only in the color of the hat of player i belongs to $W_S(G)$. We claim that the vertices of G that correspond to the players of T' must form a clique in G. Suppose otherwise. Then there are two players in T', say i and j, that do not see each other in G. Hence if the color of the hat of i is flipped, player j is not aware of this and continues to guess wrong as in configuration v. ■

Let m denote the number of edges in B. Counting edges once from the $W_S(G)$ side and once from the $L_S(G)$ side, the two previous propositions imply that $|W_S(G)| \le m \le \omega(G)|L_S(G)|$. Hence we have proved the following theorem.

Theorem 8 *For every graph G,*

$$h(G) \le \frac{\omega(G)}{1 + \omega(G)}$$

Proof. Follows from the following chain of inequalities:

$$|C(G)| \ge |W_S(G)| + |L_S(G)| \ge |W_S(G)| + \frac{|W_S(G)|}{\omega(G)} = \frac{(1 + \omega(G))|W_S(G)|}{\omega(G)}$$

■

Theorem 8 implies Theorem 2 which we restate here as a corollary.

Corollary 9 *If $\omega(G) + 1$ is a power of 2, then*

$$h(G) = \frac{\omega(G)}{1 + \omega(G)}$$

Proof. When $\omega(G) + 1$ is a power of 2, the inequality $h(G) \ge \omega(G)/(1 + \omega(G))$ follows from Corollary 5 (using also Proposition 12). The inequality $h(G) \le \omega(G)/(1 + \omega(G))$ is Theorem 8. ■

4 Remarks on computational complexity

Given a graph, how difficult is it to determine $h(G)$? If G does not contain a triangle then we have seen that $1/2 \leq h(G) < 2/3$, implying a 3/4 approximation. If G does contain a triangle then we have seen that $3/4 \leq h(G) < 1$, again implying a 3/4 approximation. Since checking whether G contains a triangle can be done in polynomial time it follows that one can approximate $h(G)$ within a ratio of 3/4. Presumably, also better approximation ratios are possible.

We now discuss the difficulty of approximating $1 - h(G)$, which also implies that computing $h(G)$ exactly is NP-hard. Let $\omega_2(G)$ denote the integer t such that $t + 1$ is a power of 2 and $t \leq \omega(G) \leq 2t$. Then we have the following approximate characterization of $h(G)$.

Proposition 10 *For every graph,*

$$\frac{1}{1+\omega(G)} \leq 1 - h(G) \leq \frac{1}{1+\omega_2(G)}$$

Proof. Since G contains a clique of size $t = \omega_2(G)$ we have that $h(G) \geq h(K_t)$. But since $t + 1$ is a power of 2, we have by Corollary 5 that $h(K_t) = 1 - 1/(t + 1)$. It follows that $1 - h(G) \leq 1/(t + 1) = 1/(1 + \omega_2(G))$.

By Theorem 8 we have that $h(G) \leq 1 - 1/(1+\omega(G))$, implying $1 - h(G) \geq 1/(1+\omega(G))$. ■

Proposition 10 implies that the value $h(G)$ (and hence $1 - h(G)$) determines the value of $\omega(G)$ up to a factor of 2. However, it is known that it is NP-hard to approximate the maximum clique size with ratios of $n^{1-\epsilon}$ [5, 11]. Hence we can easily deduce the following corollary.

Corollary 11 *For every $\epsilon > 0$, given a graph G on n vertices as input, it is NP-hard to approximate $1 - h(G)$ within a ratio of $n^{1-\epsilon}$.*

5 Graph transformations

In this section we prove Theorem 3, among other things.

We say that a graph G is a subgraph of G' if it can be obtained from G' by using only the operations of removing edges and removing vertices. The following proposition, which was already used in previous parts of our paper and in earlier work [7], is presented for completeness.

Proposition 12 *If G is a subgraph of G', then $h(G) \leq h(G')$.*

To prove Proposition 12, it suffices to show that any strategy for G can also be implemented in G' with exactly the same success probability. Showing this is straightforward (players corresponding to vertices not in G always pass, information over edges not in G is ignored), and we spare the reader of a formal proof.

Sometimes we shall use the contra-positive of Proposition 12. Namely, if G' is a supergraph of G, then $h(G') \geq h(G)$.

We now reach a key observation. Given a graph G and a vertex v, let $N(v)$ denote the set of neighbors of v. We say that a vertex v is neighborhood-dominated in G if there is some other vertex u in G with $N(v) \subseteq N(u)$. Let us denote by $G - v$ the subgraph of G obtained by removing v and all edges incident with v.

Lemma 13 *Let v be a neighborhood-dominated vertex in G. Then $h(G - v) = h(G)$.*

Proof. The inequality $h(G - v) \leq h(G)$ follows from Proposition 12. Hence it remains to prove $h(G - v) \geq h(v)$.

Let S denote an optimal strategy for G, achieving $h(G)$. Let u be a vertex that neighborhood-dominates v. Partition all possible hat configurations into two classes: the class $C_=$ in which the hats of u and v have the same color, and the class $C_{\neq}$ in which the hats of u and v have different colors. Note that $h(G)$ is the average of the success probability obtained by S over these two classes, and hence on one of these classes S succeeds with probability at least $h(G)$. We present the proof for the case this happens for the class $C_=$. (The class $C_{\neq}$ is treated in a similar way, after renaming blue as red and red as blue when one considers the color of the hat of vertex v. Moreover, the proof will imply that in both classes the success probability of S must be exactly $h(G)$, as otherwise we will get $h(G - v) > h(G)$, contradicting Proposition 12.)

Observe that in the class $C_=$, the color of the hat of u uniquely determines the color of the hat of v. Hence to specify an input for $C_=$, it suffices to specify the colors of all hats except for that of v. We show a strategy S' for the hat game on $G - v$ that succeeds exactly on those inputs from the class $C_=$ that the strategy S for G succeeds on. This is done as follows. Any vertex of $G - v$ that is neither u nor an original neighbor of v uses in S' exactly the same strategy as in S. Any vertex w which was a neighbor of v in G does the following. Being also a neighbor of u, vertex w observes the color of the hat of u and deduces that had the graph been G and the input been taken from $C_=$, the color of the hat of v would have been the same. Now w plays according to S under this assumption. Finally, it remains to describe how u plays. Since $N(v) \subset N(u)$ in G, vertex u knows what v would have seen had the game been played on G. Hence in addition to its own output, u can produce v's output. So u produces two outputs, one for v and one for u. If both of them are to pass, then u passes. If one of them is to pass, and the other is a guess, then u outputs the guess. It does not matter whether this guess was intended to be a guess by u or by v, because in $C_=$ both colors are the same, and hence the guess is correct if and only if it matches the color of u's hat. Finally, if both outputs are guesses, u can output his own guess. (Once u guesses, an additional guess by v cannot improve the success probability.)
■

Lemma 13 easily implies the results of [7] for trees. Let T be an arbitrary tree with at least three vertices. Then any leaf in this tree, say v, is neighborhood-dominated by some other vertex of the tree. Hence $h(T) = h(T - v)$, by Lemma 13. Continuing in this fashion one is eventually left with a tree on two vertices, proving that $h(T) = h(K_2) = 1/2$.

It does not take much work to extend the proof to arbitrary bipartite graphs. Let B be a bipartite graph, with bipartization (U, V). Add all the missing edges between U and V to obtain a complete bipartite graph G. By Proposition 12 we have that $h(B) \leq h(G)$. Now in G, every two vertices in the same side of the bipartization have the same set of neighbors, hence one of the vertices can be removed by Lemma 13. Continuing in this way we again are left with a single edge, showing that $h(B) \le h(G) = h(K_2) = 1/2$. As $h(B) \ge H(K_1) = 1/2$, it follows that for every bipartite graph B (and trees are special cases of bipartite graphs) $h(B) = 1/2$.

The following theorem implies Theorem 3.

Theorem 14 *For every graph G, the inequality $h(G) \le h(K_{\chi(G)})$ holds. In particular, if $\omega(G) = \chi(G)$, then $h(G) = h(K_{\omega(G)})$.*

Proof. Given a graph G, partition it into $\chi(G)$ color classes (with no edges within a color class). Add all missing edges between the color classes, thus obtaining a graph G'. Proposition 12 implies that $h(G) \le h(G')$. Now all vertices within the same color class have the same set of neighbors. Hence Lemma 13 can be used to remove all but one vertex from each color class, remaining with $K_{\omega(G)}$, and maintaining $h(G') = h(K_{\omega(G)})$. ■

A well known class of graphs for which $\omega(G) = \chi(G)$ is that of perfect graphs (where the equality $\omega(G) = \chi(G)$ holds not only for the graph but also for all its subgraphs). Hence Theorem 14 characterizes the value of the hat game for all perfect graphs. Bipartite graphs are a special case of perfect graphs. In fact, by the strong perfect graph theorem [3], every graph for which neither the graph nor its complement contains an induced odd cycle of length at least 5 (an *odd hole*) is perfect.

Using the fact that $h(K_i)$ is not strictly increasing in i, Theorem 14 can be used to characterize the value of the hat game for additional classes of graphs.

Corollary 15 *For every planar graph P that contains a triangle (in particular, for every triangulated planar graph), $h(P) = 3/4$.*

Proof. Since P contains a triangle, $h(P) \ge h(K_3) = 3/4$.

By the famous four color theorem, $\chi(P) \le 4$. Recall that in Section 2 we have seen that $h(K_4) = 3/4$. Hence $h(P) \le h(K_4) = 3/4$.

It follows that $h(P) = 3/4$. ■

In Section A in the appendix we provide an alternative approach for proving Theorem 14.

6 Unions of disjoint graphs

Given two disjoint graphs G_1 and G_2, let $G_1 + G_2$ denote the union of these two graphs (the disconnected graph that results by placing the two graphs side by side, on disjoint sets of vertices). It is natural to ask whether $h(G_1 + G_2) = \max\{h(G_1), h(G_2)\}$. Conjecture 1 if true would imply such a statement. Here we provide a direct proof of this statement.

Theorem 16 *For every two graphs G_1 and G_2, their disjoint union satisfies:*

$$h(G_1 + G_2) = \max\{h(G_1), h(G_2)\}$$

Proof. Clearly $h(G_1 + G_2) \geq \max\{h(G_1), h(G_2)\}$. It remains to show that $h(G_1 + G_2) \leq \max\{h(G_1), h(G_2)\}$.

Without loss of generality, assume that $h(G_1) \geq h(G_2)$, and let k denote the size of the maximum clique in $G_1 + G_2$. An argument similar to Proposition 10 implies that $h(G_1) \geq k/(k+2)$.

Now let us assume that there is some strategies S_1 for G_1 and S_2 for G_2 that on G_1+G_2 succeed with higher probability than S_1 succeeds on G_1 alone. We shall show that in this case the success probability on G_1+G_2 is smaller than $k/(k+2)$. This means that ignoring G_2 and replacing S_1 by an optimal strategy for G_1 is always at least as good as combining S_1 with S_2. Namely, adding G_2 to G_1 does not increase the value of the hat game.

Let w_i, ℓ_i, p_i be the probabilities of winning, losing when somebody guesses wrong, and everybody passing, when using strategy S_i in graph G_i. Assume without loss of generality that $w_1 \geq w_2$. (The case $w_2 > w_1$ is treated in a similar way.) The probability of winning in $G_1 + G_2$ is $w_1(1 - \ell_2) + p_1 w_2$. Observe that Section 3 implies that $w_i \leq \ell_i k$ (where k is the maximum clique size). Hence replacing ℓ_2 by w_2/k and rearranging, the probability of winning is at most $w_1 + w_2(p_1 - w_1/k)$. Observe that $p_1 = 1 - w_1 - \ell_1 \leq 1 - w_1 - w_1/k$. So the probability of winning is at most $w_1 + w_2(1 - w_1(k+2)/k)$. For G_2 to contribute anything, it must be the case that $w_1 < k/(k+2)$. Hence we make this assumption. This then implies that also $w_2 < k/(k+2)$, because $w_2 \leq w_1$. Hence the derivative of $w_1 + w_2(1 - w_1(k+2)/k)$ with respect to w_1 is positive, implying that the expression attains its maximum value when w_1 does. But setting w_1 to the maximum value of $k/(k+2)$ the winning probability in $G_1 + G_2$ is also $k/(k+2)$, which is no improvement over what can be attained for G_1 alone. ■

Another notion of a union of two graphs is when both graphs have the same set of vertices, and the union is only in terms of the edges. Let us denote this union by $G(V, E_1 \cup E_2)$, where V is the set of vertices and E_1 and E_2 are the two sets of edges. To analyze the range of possible values for $h(G(V, E_1 \cup E_2))$ as a function of $h(G(V, E_1))$ and $h(G(V, E_2))$, we appeal to some well known results in Ramsey theory (see [10] for example). Proposition 10 implies that $h(G)$ is approximately equal to $1 - 1/\omega(G)$. If $|V| = n$, E_1 is a random set of edges and E_2 is its complement, then with high probability both $\omega(G(V, E_1)) \simeq 2\log_2 n$ and $\omega(G(V, E_2)) \simeq 2\log_2 n$. On the other hand, $G(V, E_1 \cup E_2)$ is the complete graph and hence $\omega(G(V, E_1 \cup E_2)) = n$. So this shows that taking the union of two sets of edges may cause an exponential drop in $1 - h(G)$ (from $1/2\log_2 n$ to $1/n$). The drop cannot be more than exponential by standard results in Ramsey theory: if $G(V, E_1 \cup E_2)$ has a clique of size k, then either $G(V, E_1)$ or $G(V, E_2)$ must have a clique of size at least $(\log_2 k)/2$.

We remark that the effect of taking unions of graphs was previously studied in other contexts (see [1] for example), and influenced the choice of questions addressed in the current section.

7 Conjectures

In this section we present some implications of Conjecture 1. These implications are stated here as conjectures.

It is well known (and follows from Proposition 4) that for complete graphs, there is an optimal strategy for the players in which in every configuration, at least one player does not pass. Conjecture 1 if true would imply that the same holds for every graph.

Conjecture 17 *For every graph G, there is an optimal strategy for the players in which in every configuration, at least one player does not pass.*

Theorem 2 proves Conjecture 1 whenever $\omega(G) + 1$ is a power of 2. Hence the first value of $\omega(G)$ for which the conjecture is open is $\omega(G) = 2$. We state this special case as a separate conjecture.

Conjecture 18 *The value of the hat game on a graph is larger than 1/2 if and only if the graph has a triangle.*

Let us discuss Conjecture 18 briefly. We have seen that it holds for bipartite graphs. It is pretty easy to extend it to some non-bipartite graphs. Consider for example the hat game on an odd cycle of length $3q$, where q is odd and sufficiently large. If some player always passes then one may fix the color of its hat to the color that maximizes the probability that the remaining players win, and then remove this vertex from the cycle. This does not decrease the winning probability. The remaining players may simply pretend that the removed player is still in the cycle with a hat of the color to which it was fixed. (This argument appears in previous work. See for example Theorem 4 in [8].) Thereafter the graph becomes bipartite, and the winning probability is at most $1/2$. It remains to deal with the case in which each player has nonzero probability of guessing. But then each player guesses with probability at least $1/4$, and makes a wrong guess with probability at least $1/8$. Consider now q players each at distance at least 3 from each other. Since their neighborhoods are disjoint, it follows that the events of them guessing wrong are independent. Hence the probability of none of them guessing wrong is $(7/8)^q$, which is smaller than $1/2$ for large enough q.

As an intermediate step towards proving Conjecture 18, one may try to provide a short proof (that hopefully can be generalized) that for the 5-cycle the value of the hat game is $1/2$. Determining the value of the hat game on the Petersen graph may serve as an indication of whether Conjecture 18 is true.

Observe that Conjecture 18 is equivalent to the statement that any minimal graph for which the hat game has value greater than $1/2$ cannot have a vertex whose neighbors form an independent set. Perhaps a statement like this can be proved using techniques similar to the proof of Lemma 13.

Let us note that Conjecture 18, if true, implies that there is no graph for which the hat game has value v for $1/2 < v < 3/4$.

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A An alternative proof for chromatic number upper bound

For completeness, we present here another approach for proving Theorem 14. We shall present it only for the case of bipartite graphs, but the generalization for r-partite graphs (where r is the chromatic number) is straightforward.

Theorem 19 *The value of the hat game in bipartite graphs is 1/2.*

Proof. The value of the hat game on any graph is at least 1/2. Hence it remains to show that the value is not more than 1/2.

Consider an arbitrary bipartite graph with sides L and R. For concreteness and without loss of generality, let $|L| = |R| = k$. Hence altogether there are 2^{2k} possible hat configurations. The colors of hats in L can be thought of as a random 0/1 string of length k. We say that two strings are *twins* if one is the bitwise complement of the other. For example, the strings 001101 and 110010 are twins. Clearly every string has exactly one twin. There are 2^k possible strings for L. They form 2^{k-1} twins. The same holds for R. Now consider all possible pairs of twins, one from L and one from R. There are $2^{2(k-1)}$ such pairs, and each such pair induces four possible hat configurations (there are four possible ways of selecting one string from each twin). These pairs form a complete partition of the hat configuration space (each configuration appears in exactly one pair). We show that for every such pair, conditioned on the input configuration being one that is induced by the pair, the players win with probability at most $1/2$ (even if they are told which pair induces the configuration). This of course implies that the value the game on bipartite graphs is at most $1/2$ as well.

Consider an arbitrary pair of twins. Let $(s, \bar{s})$ denote the twins for L and $(t, \bar{t})$ the twins for R. Assume for the sake of contradiction that the players win in three of the induced configurations, and without loss of generality let these configurations be $[s, t]$, $[s, \bar{t}]$ and $[\bar{s}, t]$. Then at least one player p guessed in $[s, t]$ and assume without loss of generality that $p \in L$. Then p also guesses in the configuration $[\bar{s}, t]$, because his view is identical in those two configurations. But p's guess is correct in exactly one of $[s, t]$ and $[\bar{s}, t]$ because he has different colored hats in these two configurations. This contradicts the assumption that the players win on both these configurations. ■

B Some examples

It is not true (and not claimed by Conjecture 1) that in every optimal strategy, all guessing players form a clique. Consider for example two K_4 that share a vertex v. The value of the hat game is $3/4$ (since the graph contains a triangle and is 4-colorable). Removing v, the graph decomposes into two triangles. Each triangle has a strategy with success probability $3/4$. We may let one triangle play if the color of v's hat is 0, and the other triangle play if the color of v's hat is 1. Note that in this strategy v never guesses.

Call a strategy a *no waste* strategy if it enjoys the following three properties.

1. There are no wasted configurations: in every configuration, at least one player makes a guess.
2. There are no wasted correct guesses: in every configuration in which more than one player guesses, all guesses are wrong.
3. There are no wasted players: every player guesses in some configuration.

We show here an example of a no waste strategy in which the graph is not a clique.

The graph is the complete graph on four vertices, a, b, c, d, except for the edge (a, b) that is missing. In the no waste strategy, whenever a player guesses he guesses 0.

Player a guesses when the hat colors for c, d are $1, 0$.

Player b guesses when the hat colors for c, d are $1, 1$.

Player c guesses when the hat colors for a, b are $1, 1$.

Player d guesses when the hat colors for a, b are not $1, 1$, and in addition the hat color for c is 0.

The four players contribute $1/8 + 1/8 + 1/8 + 3/16 = 9/16$ to the success probability. There are two configurations in which two players guess wrong simultaneously: $(1, 1, 1, 0)$ (a and c guess wrong) and $(1, 1, 1, 1)$ (b and c guess wrong).

The value of the hat game on the above graph is $3/4$, because the graph contains a triangle and is 3-colorable.