Maximin fair allocations with two item values

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February 22, 2022

Abstract

We consider allocation of indivisible items to agents with additive valuations and equal entitlements. We show that if there are only two possible item values, then there is an allocation that gives every agent a bundle of value at least her maximin share.

1 Introduction

We consider allocation of a set M of m indivisible items to a set N of n agents with equal entitlement. Throughout this short note we only consider the setting in which every agent $i \in N$ has an additive valuation function v_i. An allocation $A = (A_i, \ldots, A_n)$ is a partition of M into n disjoint bundles, where agent i receives bundle A_i. An MMS_i partition is a partition of M into n disjoint bundles $B_1^i, \ldots, B_n^i$ that maximizes $\min_j v_i(B_j^i)$. This last value is referred to as the *maximin share* of agent i, and is denoted by MMS_i. An MMS allocation is an allocation that gives every agent i a bundle of value at least MMS_i. There are allocation instances with additive valuations for which no MMS allocation exists [5, 4].

Let S be the support for item values, in the sense that for every agent i and item e_j it holds that $v_i(e_j) \in S$. We seek sufficient conditions on S under which an MMS allocation always exists. One such sufficient condition is when $S = \{0, 1, 2\}$ [2]. Here we present an additional sufficient condition.

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Theorem 1 *For every two values $a < b$, if valuation functions are additive and all item values are either a or b, then an MMS allocation exists, and can be found in polynomial time.*

We present two examples for instances addressed by Theorem 1.

Example 2 *Consider two agents with the same valuation function, and five items of values 3,3,2,2,2. An MMS allocation needs to give both items of value 3 to the same agent, and the remaining items to the other agent. In contrast, an allocation that balances the number of items of value b (in this example, $b = 3$) that different agents receive is not an MMS allocation.*

Example 3 *Consider two agents and six items, of values 3,3,3,3,3,3 for agent 1 ($MMS_1 = 9$), and values 5,5,3,3,3,3 for agent 2 ($MMS_2 = 11$). An MMS allocation needs to give at least three items to agent 2. In contrast, an allocation that maximizes Nash Social Welfare (the product of values received by agents) is not an MMS allocation (it will give only two items to agent 2, because $12 \cdot 10 \geq 13 \cdot 9$).*

2 The proof

Throughout, we shall assume that the instance has the *identical ordering* (IDO) property, namely, for every agent i and two items e_j and e_k, if $j < k$ then $v_i(e_j) \geq v_i(e_k)$. By results of [3], IDO can be assumed without loss of generality. Namely, for additive valuations, any allocation algorithm that produces MMS allocations for IDO instances can be transformed into an algorithm that produces MMS allocations for arbitrary (not necessarily IDO) instances. Moreover, if the former algorithm runs in polynomial time, so does the latter algorithm.

We now prove Theorem 1.

Proof. Some details of the proof depend on whether $a \geq 0$ (in which case valuations are non-negative and items are *goods*) or $a < 0$ (in which case either all items are *chores*, or some are goods and some are chores, a setting referred to as *mixed manna*).

The proof of the theorem is by induction on n. For $n = 1$, the theorem trivially holds. For the inductive step, $n \geq 2$, and we may assume that the theorem holds for all instances in which the number of agents is $n - 1$.

Given an instance I with $n \geq 2$ agents, let $i \in N$ be the agent for which $v_i(M)$ is largest (breaking ties arbitrarily). As there are only two item values, this also implies that MMS_i is largest. We may assume that $MMS_i \neq 0$, as otherwise giving agent i all the items is an MMS allocation (agent i gets non-negative value, whereas other agents get 0 value, which is at least as high as their MMS). In the ordered sequence of items, i has the longest prefix of items of value b. Call this prefix $p_b(i)$. Let $B_1, \ldots, B_n$ be a partition of M into bundles, such that $v_i(B_j) \geq MMS_i$ for every $j \in [n]$.

- If $a \geq 0$, then without loss of generality, we assume that B_1 is the bundle containing the smallest number of items (breaking ties arbitrarily). Hence $|B_1| \leq \frac{m}{n}$. Note that $|B_1| \geq 1$, as MMS_i in this case is strictly positive.

- If $a < 0$, then without loss of generality, we assume that B_1 is the bundle containing the largest number of items (breaking ties arbitrarily). Hence $|B_1| \geq \frac{m}{n}$.

Let n_b (n_a, respectively) denote the number of items in B_1 that have value b (value a, respectively) according to v_i. That is: $n_b = |\{e \in B_1 \mid v_i(e) = b\}|$ and $n_a = |\{e \in B_1 \mid v_i(e) = a\}|$. Necessarily $n_a, n_b \geq 0$ and $n_a + n_b \geq 1$.

Allocate to agent i the last n_b items in $p_b(i)$ (each such item has v_i value b), and the last n_a items (each such item has v_i value a). Hence i gets value at least MMS_i, as she gets a bundle of value equal to $v_i(B_1)$. In fact, for simplicity of terminology and without loss of generality, we assume that indeed B_1 was composed of precisely those items that we give to i.

Consider the instance I' that remains (without agent i and the items of B_1). It has only $n-1$ agents. We claim that for every agent (except i that already got her MMS), her MMS in I' (partitioning M' to $n-1$ bundles) is at least as high as her MMS in I. Given the claim, the theorem follows by applying the inductive hypothesis.

We now prove the claim. Consider an arbitrary agent j, and let $p_b(j)$ denote the prefix of items in I of value b with respect to v_j. Recall that $p_b(i) \geq p_b(j)$. There are two cases to consider.

Some item of $p_b(j)$ is allocated to i. In this case i and j agree on the values of all items, except possibly for those in B_1. Hence $v_i(B_k) = v_j(B_k)$ for each of the remaining bundles B_k, with $2 \leq k \leq n$. As $MMS_j \leq MMS_i$ (implied by $p_b(j) \subseteq p_b(i)$), it follows that $B_2, \ldots, B_n$ form a partition of the items of I' in which the value of each bundle to agent j is at least her original MMS_j. Hence MMS_j in I' is not smaller than MMS_j in I.

No item of $p_b(j)$ is allocated to i. In this case, according to v_j, all items of B_1 have value a.

- If $a \geq 0$ then $|B_1| \leq \frac{m}{n}$. The MMS_j partition of M contains a bundle B with at least $\frac{m}{n}$ items. For such a bundle B we have that:

  - $v_j(e) \geq v_j(e')$ for every $e \in B$ and $e' \in B_1$.
  - $|B| \geq |B_1|$.
  - B contains only items of non-negative value.

  The above three conditions imply that for entitlement $\frac{1}{n-1}$, the MSS_j value for the set $M \setminus B_1$ is at least as large as the MMS_j value for the set $M \setminus B$. As the MMS_j in the latter case is no smaller than MMS_j for the original input instance I (because B is a bundle in the corresponding MMS_j partition), the same holds for the former case.

- If $a < 0$ then $|B_1| \geq \frac{m}{n}$. In this case we take B to be a bundle (from agent j's MMS partition) with at most $\frac{m}{n}$ items. We have that:

  - $v_j(e) \geq v_j(e')$ for every $e \in B$ and $e' \in B_1$.
  - $|B| \leq |B_1|$.
  - B_1 contains only items of negative value.

  The proof now proceeds as in the case $a \geq 0$.

Finally, we note that the proof of the theorem provides a polynomial time algorithm for computing the MMS allocation. For the agent i with highest $v_i(M)$, compute an MMS_i partition (this can be done in polynomial time because there are only two item values). If $MMS_i = 0$, give all items to agent i. If $MMS_i \neq 0$, then of all bundles in the MMS_i partition, if $a \geq 0$, give i the bundle with the smallest number of items, and if $a < 0$, give i the bundle with the largest number of items. Continue in an inductive manner with the instance that remains (with fewer items, and one less agent). ■

Acknowledgements

This work benefited from discussions with Alexey Norkin. The authors of [1] provided encouragement to make this short note public.

References

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