The Cyclic Butler University Game

Aviezri Fraenkel

Computer Science and Applied Mathematics

Weizmann Institute of Science

Rehovot, Israel

aviezri.fraenkel@weizmann.ac.il

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Abstract

In a previous book honoring Gardner, a 2-player coin-pushing game on a directed graph (=digraph) without cycles was solved. The coins are placed on selected nodes of the "Butler University map". A move consists of choosing a coin and pushing it to an adjacent node along a directed edge. The player making the last move wins. We consider the same game, but where the digraph may be cyclic. Then there need not be a last move, in which case the outcome is a (dynamic) draw, that is, no player can force a win, but both have always a nonlosing next move. We provide an efficient strategy, consisting of deciding, for every position: (i) who can win, or (ii) whether both can only draw; and (iii) determining the next move that guarantees a win (case (i)) or maintains a draw (case (ii)).

Keywords: Cyclic games, loopy games, Generalized Sprague-Grundy function

In [4], Rebecca Wahl analyzed "The Butler University Game", created by Jerry Farrell. This 2-player game is played on a simplified map of the Butler University campus, see Fig. 1. The labels are the first letters of the various campus buildings. For example, G stands for Gallahue. Wahl explained that the roads between the buildings are directed, because a severe snowstorm led the grounds crews to mark the roads to be one-way only. Thus the map became a *directed graph*, also called, for short, *digraph*.

The game begins by placing coins on some or all of the nodes (buildings). A move consists of selecting a single coin and pushing it to an adjacent node along a directed edge, but only in the direction of the arrow. Multiple occupancy on the nodes is permitted, both in the initial placement and as the result of a move. The players move alternately. The player making the last move wins, and the opponent loses. Note that M is a *leaf*, that is, a node without any outgoing edge. So if all coins are in M, play ends.

It's not immediately clear, however, that every play of the game ends in a finite number of moves. Perhaps a player can cycle through nodes, avoiding to ever enter M. But Fig. 2, which is a redrawing of Fig. 1, shows that the
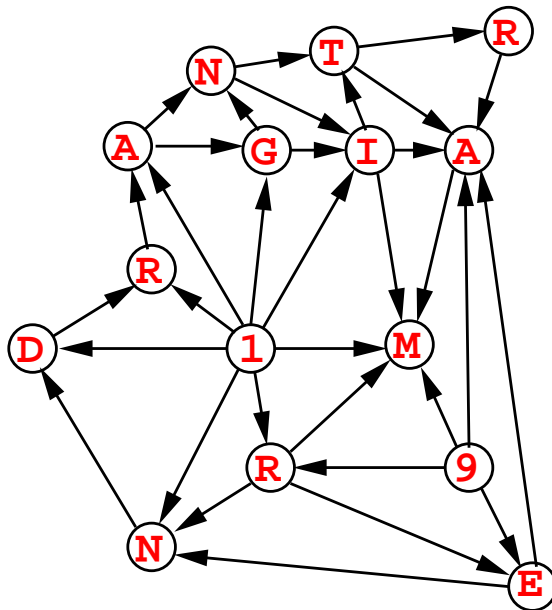
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Figure 1: A simplified map of the Butler University campus.

digraph is *acyclic*, i.e., without cycles. So indeed every play ends with a clear winner and loser. Incidentally, Fig. 2 also shows that the Butler University paid special tribute to Martin Gardner in his 91st year of life. It is to be hoped that in at most another 9 years the University will complete its *Gardner Building*, so the digraph can accommodate the number 100. Using the *Sprague-Grundy function*, the Butler University game can be solved efficiently, i.e., a winning strategy can be computed "in polynomial time", see [1], §2.

The purpose of this note is to extend these results to *cyclic* digraphs — digraphs that contain cycles. Then there may indeed be cases where play of the game doesn't end: no player can force a win, but each has a always a nonlosing next move. We then say that the outcome is a *draw*. Consider for example the case where the snow begins to thaw just a bit, so the road between E and N_2 is back to 2-way traffic, see Fig. 3. If there is a single coin at E or at N_2, the player venturing out of the cycle (E, N_2) — (N_2, E) can be beaten by the opponent. Indeed, if Alice moves $E \rightarrow A_1$, then Bob moves $A_1 \rightarrow M$, winning. If Alice moves $N_2 \rightarrow D$, then Bob moves to R_2, Alice to A_2, and Bob chooses to move to N_1. If now Alice moves to I, then Bob wins immediately by going to M. Otherwise Alice moves to T. Then Bob chooses to move to R_1, Alice to A_1, Bob to M, winning.

Question 1. If coins are placed on some other nodes, in addition to a single coin at E or N_2, does the outcome always remain a draw, or may it become a win for the first player?
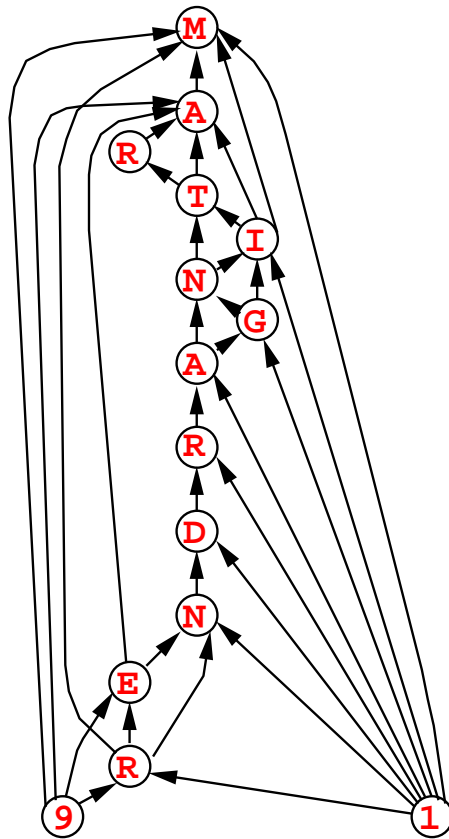
Figure 2: The snowed-in campus map is acyclic.

Every finite cyclic game can be analyzed using the *Generalized Sprague-Grundy function* [3]. We demonstrate on the Butler University digraph how to compute this function.

Label M by 0. Do the same for every unlabeled node that doesn't have an (immediate) *follower* labeled 0, and every unlabeled follower of which has a follower labeled 0. Therefore R_1 is labeled 0. Then N_1 is labeled 0, since the follower I has the follower M labeled 0, and the follower T has the follower R_1 labeled 0. Then R_2 is labeled 0. The next candidate is N_2 since its unlabeled follower D has the follower R_2 already labeled 0. However, it has also the unlabeled follower E, which doesn't have any follower labeled 0. Thus N_2 cannot be labeled 0. Also no other unlabeled node can be labeled 0.

At this stage, every unlabeled node that doesn't have a follower 0 is labeled with a special symbol ∞, larger than every integer. Therefore we dub it *infinity*. This implies that E and N_2 are labeled ∞. But the nodes R_3, 1 and 9 remain unlabeled at this stage, since each has a follower labeled 0.

We now label by 1 every unlabeled node that doesn't have a follower 1, and
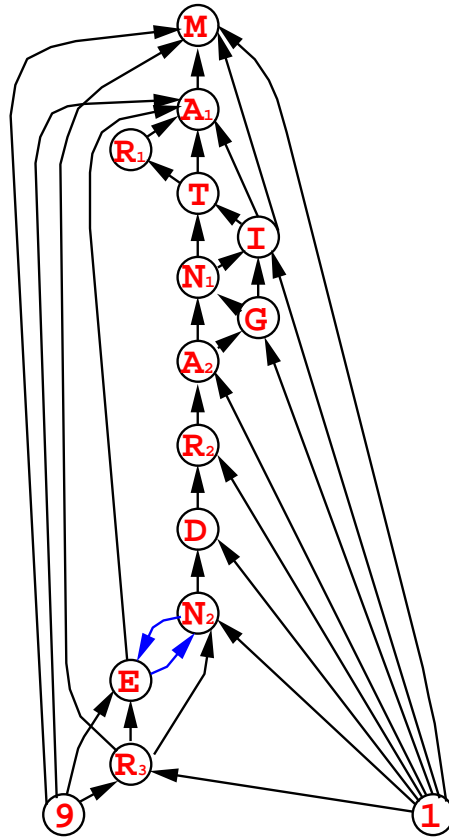
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Figure 3: The snow begins to thaw.

every follower that is either unlabeled or labeled ∞ has a follower labeled 1. Thus A_1 is labeled 1, then G and D are labeled 1, so also R_3 is labeled 1. Every unlabeled node has now a follower 1, so no additional ∞ labels are dispensed.

Repeat this procedure with the label 2. Thus T is labeled 2, then A_2. Now node 9 has the follower ∞ (at E) which doesn't have a follower labeled 2, hence node 9 is labeled ∞. Then I is labeled 3. Node 1 is a candidate for 4, but it has a follower ∞ (at N_2), which doesn't have a follower labeled 4, so node 1 is labeled ∞. Finally, we adjoin to every ∞-label the set of labels of all its finite followers. The labeled digraph is depicted in Fig. 4.

We now show how to use the labels for playing the game optimally. Suppose that one coin is placed on each of the nodes E, R_2 and I, indicated by stars in Fig. 4. We use *Nim-addition* to add the labels of the occupied nodes as follows:

$$\infty(1) \oplus 0 \oplus 3 = \infty(1) \oplus 3 := \infty(1 \oplus 3) = \infty(2).$$

We used $\oplus$ to indicate Nim-addition, which is binary addition without carries. Thus $1 \oplus 3$ (in decimal) $= 01 \oplus 11$ (in binary) $= 10$ (in binary) $= 2$ (in decimal).
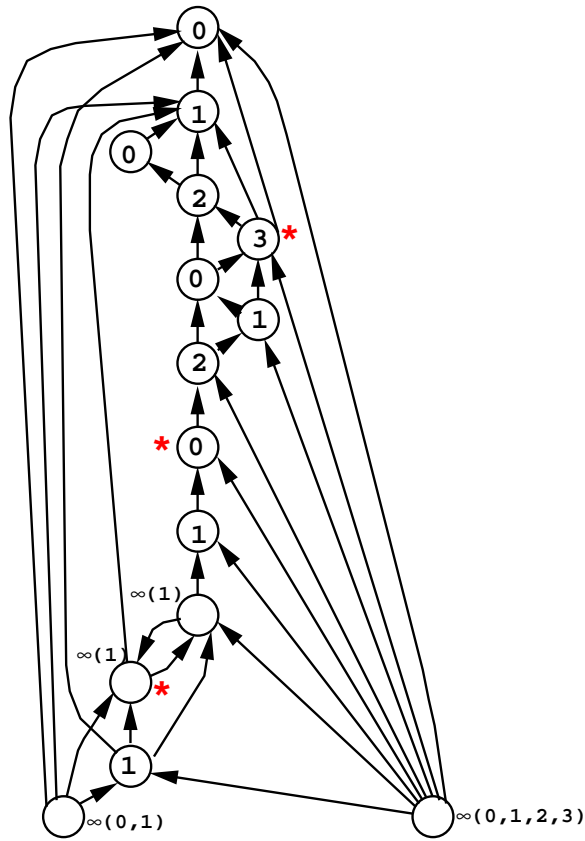
Figure 4: A fully labeled Butler University cyclic digraph.

In fact, there are only 10 types of people in the world: those who understand binary, and those who don't. Since the Nim-sum is $\infty(2)$, which doesn't contain 0, the position is a draw. The way to maintain the draw is by moving in the cycle. If a player moves from a coin on a node labeled ∞ to a node labeled 1, then the opponent can win by moving the coin on the node labeled 3 to a node labeled 1. Then the Nim sum of the resulting position is $0 \oplus 1 \oplus 1 = 1 \oplus 1 = 0$. Every position whose Nim-sum is 0 is a second-player win.

Does this indicate that the answer to Question 1, that placing coins on other nodes in addition to a coin on a node labeled ∞, implies that the outcome necessarily remains a draw?

Well, suppose that the initial placement of coins is as above, except that the coin on the node labeled 0 is shifted one node down, namely, it is placed initially on the node labeled 2. Then the Nim-sum of the initial position is

$$\infty(1) \oplus 2 \oplus 3 = \infty(1) \oplus 1 = \infty(1 \oplus 1) = \infty(0).$$

Since there is a 0 attached to the ∞, the first player can win by moving from

the node labeled ∞ to a node labeled 1. Indeed, the resulting Nim-sum is then $1 \oplus 2 \oplus 3 = 0$, which is a second-player win.

In conclusion, the answer to Question 1 is that the positions of the additional coins determine whether the outcome remains a draw or becomes a win for the first player.

Now suppose that the snow has melted some more, and a loop around building A_2 (Fig. 3) became passable, in addition to the 2-way road opened up before. Note that at A_2 there is now the option of *passing*: a player moving around the loop stays in the same position!

Question 2. Is it true that then every play with a coin on A_2, where there is a loop, is a draw?

Fig. 5 depicts the situation with a coin on A_2 and 2 additional coins, indi-
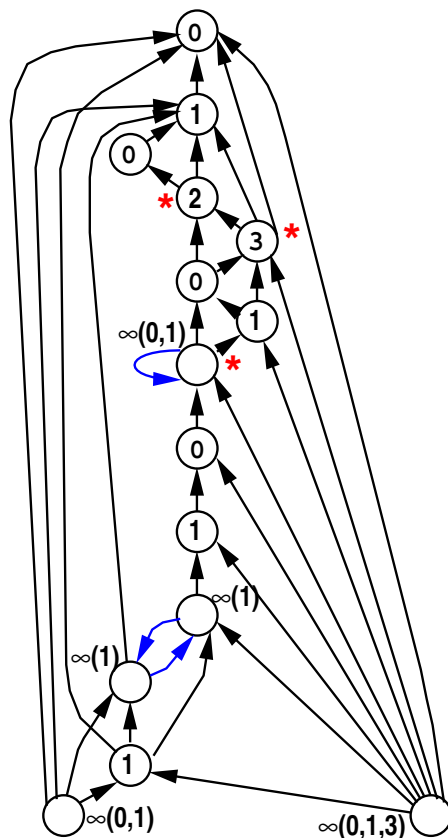

Figure 5: A fully labeled Butler University cyclic digraph, including a loop.

cated by stars, as well as the new labeling induced by the loop. The Nim-sum of the occupied nodes is now $\infty(0, 1) \oplus 2 \oplus 3 = \infty(0, 1) \oplus 1 = \infty(1, 0)$. Thus the first player can win by moving to 1. Indeed, the Nim-sum of this position

is $1 \oplus 2 \oplus 3 = 0$. Note, in fact, that if there is only a single coin at A_2, then the first player can win by moving to the follower labeled 0. However, if the initial position would have only the coin on A_2 and precisely *one* of the other 2 coins depicted by stars on Fig. 5, then the result would be a draw. Thus also the answer to Question 2 is that the outcome depends on the placement of the coins.

The following is a summary of the main steps of the Generalized Sprague-Grundy function labeling algorithm, for any finite digraph.

1. Put $i = 0$.

2. As long as there exists an unlabeled node u such that no follower of u is labeled i and every follower of u that is either unlabeled or labeled ∞ has a follower labeled i, label u by i.

3. Label every unlabeled node that doesn't have a follower labeled i by ∞.

4. If there is still an unlabeled node, put $i \leftarrow i + 1$ (i.e., increase i by 1) and return to 2. Otherwise stop.

A simple short mathematical definition of the Generalized Sprague-Grundy function can be found in [1], §3. The complete labeling algorithm, also in [2], §3.

We end with a homework problem about multiple occupancy. Suppose that finitely many coins are placed on a single node u of a finite digraph, all other nodes being unoccupied. Denote by $\ell(u)$ the label at u. Prove that:

(i) If $\ell(u) = 0$, then no matter how many coins are placed on u, the outcome is a second player win.

(ii) If $0 < \ell(u) < \infty$, then the outcome is a second player win if the number of coins on u is even. Otherwise it's a first player win.

(iii) If $\ell(u) = \infty$, then the outcome is a draw for any number of coins ≥ 2 on u.

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