Nim is Easy, Chess is Hard — But Why??*

Aviezri S. Fraenkel§

January 7, 2007

Department of Computer Science and Applied Mathematics
Weizmann Institute of Science
Rehovot 76100, Israel

Abstract

The game of chess appears to be hard. According to authoritative sources, this is due to the extremely large number of possible chess moves. We refute this argumentation by showing that simple games of moderate size — as an example we consider nim — have a larger number of moves than chess, yet possess a very easy winning strategy. So perhaps chess has also an easy strategy which remains elusive? We argue that this is rather unlikely, in view of several high-complexity aspects of chess, notably the proven Exptime-completeness of $n \times n$ chess.

1 Why is Nim Easy?

In nim, a finite number of piles of finitely many marbles is given. Two players alternate in selecting a pile and removing from it any positive number of marbles, possibly the entire pile. The player first unable to move loses and the opponent wins. Nim is easy. Why? Because it has an easy perfect strategy: write the number of marbles in each pile in binary, and "add" them without carry, an addition also known as XOR (Exclusive Or), denoted by $\oplus$. If the sum is 0, you better be a gentle(wo)man and offer your opponent to move first, because you can win as player II. If the sum is nonzero you can move first and make it 0, thus winning.

Example. Consider 3 piles of sizes 1,2,3. Player I, say, moves $3 \rightarrow 1$, to the position $(1, 2, 1)$. Now player II removes the entire pile of size 2, landing in the position $(1, 1)$. Then player I is forced to take one of the piles of size 1, player II takes the other, winning. In fact, since $1 \oplus 2 \oplus 3 = 0$, player II can win for *every* beginning move of player I.

*Expanded version of a note that appeared in the "Reader's Corner" of *Plus Mag.*, Issue 40 September 2006 http://plus.maths.org/issue40/editorial/index.html#chess

§`fraenkel@wisdom.weizmann.ac.il` `http://www.wisdom.weizmann.ac.il/~fraenkel`

Nim can be played, equivalently, on a board, see Fig. 1. A move consists of selecting a marble and moving it to a neighboring circle along a directed edge. For example, place marbles initially on circles 1,3,4. Player I now chooses to move the marble from circle 4 to circle 2, resulting in marbles on circles 1, 2, 3. Then, say, player II moves $3 \to 1$, to the position $(1, 2, 1)$. The new position consists of 2 marbles on circle 1 and one marble on circle 2, the same as the position reached after the first move of the preceding example. It follows that player I can win. Note that no *marble interaction*, such as *capture* takes place when the 2 marbles meet on circle 1.

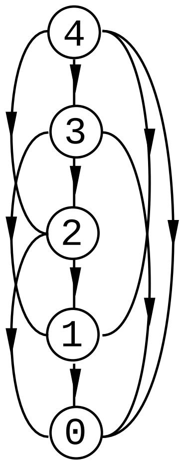

Figure 1: Nim as a board game.

Nim is a game without marble interaction; chess-type games contain capture moves, when one piece removes an opposing piece upon collision. Of course there are many other types of marble interaction moves, such as a jump over a diagonally adjacent piece in checkers, where the piece jumped over is captured, i.e., removed from the board; or without capture, such as in Chinese checkers.

2 Chess' Huge Number of Moves Implies its Computational Intractability...

Finding a perfect strategy for chess seems difficult. Therefore heuristics abound, at least from Shannon [15] to the latest available on the net, such as indicated in Chess Expertise [18] and WolframMathWorld [17].

Why is finding a perfect strategy for chess difficult? Lewis Dartnell [2] wrote, comparing chess with nim: "Chess, however, is almost inconceivably more complex, and the pieces can be arranged on the 64 squares of the board in 10^{44} distinct ways. One mathematician has calculated that there are about $10^{(10^{50})}$ different legal games, which is far more than the number of particles in the entire visible universe. This is effectively an infinite number of permutations, and so in all practical senses it is impossible to play chess perfectly."

A similar reason is given by Marianne Freiberger [5], "In combinatorial games such as chess, the number of possible combinations of moves is astronomical, meaning that a complete analysis is totally unfeasible."

In [6] Steven Goldberg wrote, "There is a way to play chess so that you never lose: the good news is that this has been proved. The bad news is that all the computers in the world will never be able to discover it". The latter claim is based on the statement that "the number of possible moves in chess has been estimated to be approximately 12×10^{81}".

Incidentally, the difference in the estimates of Dartnell and Goldberg is in itself super-astronomical. But the former refers to the number of *games*, whereas the latter to the number of *moves*. Since the main argumentations given above in support of the complexity of chess concerned the length of optimal play, we stick with Goldberg's estimate.

The astronomical number of moves of chess was also given as the reason for its computational intractability by Harel [7], [8], [9]. We quote from the most recent [9], but the text is almost identical in the other two references. "...With chess... the story is quite different. White has 20 possible first moves, and the average number of next moves from an arbitrary chess position is around 35. The number of moves... can easily reach 80 or 100. This means that the number of possibilities to check in a typical game might be something like 35^{100}. ...[This] is many many *many* orders of magnitude larger than the number of protons in the universe... Even if we... assume that each move can be tested in, say, a nanosecond, there is simply no way that computers can explicitly contemplate each and every possibility in any reasonable amount of time. So there is no hope for a perfect chess program. A world champion yes, but a perfect program no."

3 ...Or Does it?

We saw that at least four authors stated, in seven publications, that chess is computationally intractable because of the huge number of its moves. The reasoning has been voted correct unanimously.

It is not hard to see that the number of *positions* of a game of nim with m piles of size at most n, is the number of combinations of n out of $m+n$. For the case $m = n$ this yields on the order of $4^n/\sqrt{n}$ positions. Thus for $n = 140$, i.e., 140 piles, each of size at most 140, the number of *positions* of nim is more than 10 times than Goldberg's number of estimated *moves* of chess. The number of moves of nim is of course considerably larger than the number of its positions. Yet it's trivial to compute the strategy for any such nim game, and a computer can do it in at most a few seconds. A suitable larger nim position will likewise be larger than Harel's estimate, yet can be solved trivially.

The point is that an efficient algorithm may produce more intelligent perfect playing strategies than a dumb search through all possible moves. It enables homing in onto the optimal moves without considering their totality! Conclusion: a large number of moves in a game does *not* imply its computational intractability.

This observation raises the question whether perhaps also chess has an efficient *perfect* strategy, and we simply haven't yet found it. In the rest of this note we will show that this is highly unlikely.

1. Differences between nim and chess that have mathematical ramifications.

• **Cycles**. Nim-like games are finite and "acyclic": no position is assumed twice, but chess-like games are cyclic.

• **Marble interactions**. In nim-like games, marbles coexist peacefully, even on the same circle, whereas they interact in various ways such as jumping, deflecting, capturing, etc., in chess-like games.

• **Partisanship.** A game is *impartial* if the set of (direct) followers of every position is the same for the two players. If this doesn't hold for all positions, the game is *partisan*. Nim-like games are impartial, whereas chess-like games are partisan (the "black" player cannot move a white piece and vice versa).

• **Decomposition**. For solving any large system, we strive to decompose it into a number of smaller tasks, and solve each one individually. This can be done for nim, since the piles are distinct. But chess does not seem to decompose into disjoint parts (except some very simple configurations, such as those consisting of pawns and the two kings only).

• **Termination**. Nim has only one terminating state, when all piles are empty. But in chess, every checkmating configuration is a terminating state, and they abound.

Thus chess appears to be considerably more complex than nim. But the main reason for the apparent complexity of chess is stated now.

2. Chess has been *proved* to be "Exptime-complete" [4].

This statement, roughly, means the following: define chess on a general $n \times n$ board in any "reasonable" way, with one white and one black king.[1] **Then there exists no polynomial time strategy to decide whether an arbitrary position is a player I win or player II win or a draw**. It's *not* just saying that so far no such polynomial strategy has been found for chess, despite much effort and research, so the chance for such a polynomial strategy seems rather small. Rather, it asserts that every attempt to construct a polynomial algorithm that's valid for *every* position in $n \times n$ chess for all n is doomed to failure. In fact, every conceivable or inconceivable algorithm to decide who can win from such a position, must necessarily take time exponential in the size n of the position!

[1] "Reasonable" means that the pieces are subject to the same movement rules as in 8×8 chess, and the number of white and black pawns, rooks, bishops and queens each increases as some fractional power of n. One might think of modifying the movement rule of the knight, since on an 8×8 board its movement is strongly affected by the board boundary. Fortunately, the proof doesn't use any knights.

However, the constructions used in [4] for establishing the *provable* intractability of chess seem to violate the spirit of 8×8 chess, in much the same way as the *conditional* intractabilities of games such as gobang (=GoMoku) [13] and hex [14]. Typical positions in the reduction of [4] do not look like larger versions of typical 8×8 chess endgames. Although there was no attempt there to answer questions of reachability, it seems offhand as though players would have a hard time trying to reach those board positions from any reasonable starting position. (Reachability may not seem quite as unfeasible, perhaps, if we recall the chess rule stating that a pawn reaching the opposite side of the board can become any piece of the same color other than pawn or king [11].)

What can be said, however, is that certain approaches for deciding whether a position in 8×8 chess is a winning position for white may not be very promising, namely, those approaches which work for arbitrary positions and generalize to $n \times n$ boards. Such approaches use time exponential in n, and hence can be useful only if the exponential effect had not yet been felt for $n = 8$.

It's similar to the situation for existential problems, where a polynomial algorithm holds for *all* instances of the problem, and NP-completeness means only that *some* instances are hard — conditional upon NP$\neq$ P. We remark, however, that for all games whose generalized versions have been proved to be hard, no polynomial strategy has been found for their finite commercial-size manifestations. Examples: gobang and hex mentioned above, shogi [1], checkers [3], othello [10], go [12] and more.

An exponential function has the form c^n where $c > 1$ is a constant, such as 2. A polynomial function has the form n^c, with $c > 1$ a constant. Exponential functions grow very fast ("exponentially"), whereas polynomial functions grow at a moderate rate. Thus 2^{40} is bigger than a thousand billion (10^{12}), whereas 40^2 is only $1,600$. The reader may wish to construct a table of the values 2^n and n^2 for n in the range between 30 and 50.

It may be helpful to view the difference between exponential and polynomial functions from a cosmological point of view. Estimates by astrophysicists of the number of particles in the observable universe are currently (2005) on the order of 10^{85} [16]. Now $2^{282} < 10^{85} < 2^{283}$, but 283^2 is only $80,089$.

These type of considerations motivate the following convention used in computational complexity. A problem that admits a polynomial time algorithm is called *tractable*; otherwise it is *intractable.*

4 Epilogue

In conclusion, the provable intractability of chess doesn't exclude the possibility that 8×8 chess has an easy *perfect* strategy, which somehow depends on the number 8 or other small numbers, though this seems unlikely. Moreover, it is likely that the world-wide efforts to improve the heuristic 8×8 computer-chess programs will lead to more Gary Kasparov and other world-chess-champions defeats. It's even conceivable that a perfect polynomial strategy exists that works for *some* positions in $n \times n$ chess for all n. For example, for the very symmetric initial position of chess, it may someday be possible to decide whether white can win or at least draw on $n \times n$ chess for all n. In fact, it's easy to show that the first player in hex has a winning strategy, though the proof doesn't disclose what the strategy is.

Yet no computer program, based on the classical Turing machine model[2], can ever determine with certainty who can win from an arbitrary position in $n \times n$ chess for general n. The reason for this is the inherent complexity of $n \times n$ chess as reflected by its Exptime-completeness — *not* the astronomical number of its moves. It is true that the high complexity of $n \times n$ chess implies that it has a very large number of moves, but not conversely! In particular, the large number of moves of 8×8 chess doesn't imply that it is highly complex, which is the main message of this note.

References

[1] H. Adachi, H. Kamekawa and S. Iwata, Shogi on $n \times n$ board is complete in exponential time, *Trans. IEICE* **J70-D** (1987) 1843–1852 (in Japanese).

[2] L. Dartnell, Practice makes perfect, *Plus Mag.*, Issue 28 January 2004 http://plus.maths.org/issue28/features/dartnell/ .

[3] A. S. Fraenkel, M. R. Garey, D. S. Johnson, T. Schaefer and Y. Yesha, The complexity of checkers on an $n \times n$ board — preliminary report, *Proc. 19th Ann. Symp. Foundations of Computer Science* (Ann Arbor, MI, Oct. 1978), IEEE Computer Soc., Long Beach, CA, 1987, pp. 55–64.

[4] A. S. Fraenkel and D. Lichtenstein, Computing a perfect strategy for $n \times n$ chess requires time exponential in n, *J. Combin. Theory* (Ser. A) **31** (1981) 199–214, MR629595 (83b:68044).

[5] M. Freiberger, Review of "Luck, Logic and White Lies: The Mathematics of Games" By J. Bewersdorff, *Plus Mag.*, Issue 35 May 2005 http://plus.maths.org/issue35/reviews/book3/ .

[6] S. Goldberg, Some very deep mathematical problems that anyone can understand, *J. Recreational Math.*, 19:2, pp. 119-125, 1987.

[7] D. Harel, *The Science of Computing: Exploring the Nature and Power of Algorithms*, Addison-Wesley, 1987; second printing: 1989.

[8] D. Harel, *Algorithmics: The Spirit of Computing*, 2nd ed., Addison-Wesley, 1992; reprinted 1993 (with corrections), 1996, 1997.

[2]Quantum computers or biological computers or other new models might one day do better than Turing machines.

[9] D. Harel, *Computers Ltd. – What They Really Can't Do*, Oxford University Press, 2000.

[10] S. Iwata and T. Kasai, The Othello game on an $n \times n$ board is PSPACE-complete, *Theoret. Comput. Sci.* (Math Games) **123** (1994) 329–340.

[11] T. Just, chief editor, D. B. Burg, editor, *U.S. Chess Federation's Official Rules of Chess*, 5th ed, compiled and sanctioned by the U.S. Chess Federation, Random, House, 2003.

[12] D. Lichtenstein and M. Sipser [1980], Go is Polynomial-Space hard, *J. Assoc. Comput. Mach.* **27** (1980) 393–401. Earlier version appeared in Proc. 19th Ann. Symp. Foundations of Computer Science (Ann Arbor, MI, Oct. 1978), IEEE Computer Soc., Long Beach, CA, 1978, pp. 48–54.

[13] S. Reisch, Gobang ist PSPACE-vollständig, *Acta Informatica* **13** (1980) 59–66.

[14] S. Reisch, Hex ist PSPACE-vollständig, *Acta Informatica* **15** (1981) 167–191.

[15] C. E. Shannon, Programming a computer for playing chess, *Phil. Mag.* **41** (Ser. 7) (1950) 256–275.

[16] http://en.wikipedia.org/wiki/Wikipedia:Size_comparisons

[17] http://mathworld.wolfram.com/Chess.html

[18] http://www.psychology.nottingham.ac.uk/credit/projects/chess_expertise/