Lexicographic Wythoff

David Klein, Aviezri S. Fraenkel

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Abstract

An important aspect of the classic game Wythoff is that its P-positions form a disjoint cover of the positive integers by two sequences. Though generalizations of Wythoff to $K > 2$ piles abound, we believe that the generalization presented here is the first where the P-positions form a disjoint cover of the positive integers by $K > 3$ sequences. To achieve this we add a novel ingredient – we allow pile sizes to increase. This leads, inter alia, to games with infinitely many sub-positions, yet every such game ends with no remaining tokens, due to a lexicographic order $\prec$ imposed on the moves.

1 Introduction

The original definition of Wythoff is [14]:

Definition 1. *The game is played by two persons. Two piles of counters are placed on the table, the number of each pile being arbitrary. The players play alternately and either take from one of the piles an arbitrary number of counters or from both piles an equal number. The player who takes up the last counter or counters, wins.*

A well known property of Wythoff is that the two sequences of its P-positions form a disjoint cover of $\mathbb{N}^+$. We are interested in extending Wythoff to $K > 2$ piles while retaining the property that the K sequences form a disjoint cover of $\mathbb{N}^+$.

There are many generalizations of Wythoff. For example, see most of the 150 bibliographic items in 'Wythoff Visions' [2]. A significant effort has been made to generalize Wythoff to more than two piles, some of the more successful ones are mentioned below.

In [10], the P-positions for a K-Pile game, $K \geq 2$, are constructed using triangular numbers, and the resulting strategy is polynomial-time, whereas most games are either PSPACE-complete or EXPTIME-complete. However, the P-positions do not tile $\mathbb{N}^+$. There is some resemblance between the proofs of [10] and the present paper.

In [4] a generalization of Wythoff to 3 piles is constructed based on the tribonacci word. In this case the P-positions *do* tile $\mathbb{N}^+$. This is further generalized in [3]. It does not look like either game can be extended to $K > 3$ piles.

A quite different generalization is Moore's Nim_k, [11], which is a variation of Nim in which up to k piles can be reduced. Thus Nim_1 is Nim. A tractable strategy can be given by expressing the pile sizes in binary as in Nim, but XOR-ing them to the base $k + 1$. If this sum (without carries) is 0, we have a P-position. Otherwise, it is an N-position, and a move to 0 wins. No polynomial strategy seems to be known for this game. Another generalization appears in Fraenkel, [5].

In [9] it is shown that a natural generalization of Nim to the case of $K > 2$ piles of sizes $[a_1, a_2, ..., a_K]$ is to either remove any positive number of tokens from a single pile, or remove x_i tokens from each pile simultaneously, subject to the conditions: (i) $x_i > 0$ for some i, (ii) $x_i \leq a_i$ for all i, (iii) $x_1 \oplus x_2 \oplus ... \oplus x_K = 0$, where $\oplus$ denotes Nim-sum (also known as addition over GF(2), or XOR). The player making the last move wins and the opponent loses. See also [2] section 4. This game leads to two open conjectures regarding how similar the P-positions are to those of Wythoff. See [12] and [13] for their statement and partial results. Also in this game the P-positions don't tile $\mathbb{N}^+$.

A special case of the game we define in section 5 is analyzed in [8], which itself is a generalization of Wythoff obtained by weakening the constraint of taking equal numbers from both piles.

In a private communication, Professor Shigeki Akiyama shared a generalization similar to ours for the 3 pile case. Though his ruleset is slightly different, we believe that his P-positions may be identical to ours.

The layout of the paper is as follows. In section 2 we introduce our generalization of Wythoff to $K > 2$ piles utilizing lexicographic order. We call the new game Wytlex. In section 3 we present a recursive construction of the P-positions of Wytlex and show that, as in the case of classical Wythoff, they form a disjoint cover of $\mathbb{N}^+$. In section 4 we discuss variable Wytlex with a wider category of moves and prove that again the P-positions form a disjoint cover of $\mathbb{N}^+$. Finally in section 5 we concentrate on 2-Pile variable Wytlex and show that not only do the P-positions always form a disjoint cover of $\mathbb{N}^+$ but for every pair of complementary sequences in a broad class there exists a variable 2-Pile Wytlex with matching P-positions. Most nonhomogeneous Beatty sequences lie within the specified class. The generalization of Wythoff which appears in [8] is a special case of our 2-Pile variable Wytlex.

2 K-Pile Wytlex

A very natural generalization of Wythoff is:

Definition 2. *A two-player game of* Wythoff(K) *is played on* $K \geq 2$ *piles of tokens. A legal move is to choose two of the piles and make on them a legal Wythoff move, and from each of the other piles remove zero or more tokens with no restriction.*

Unfortunately, the 13^{th} and 14^{th} P-positions of Wythoff(4) (in order of increasing total number of tokens) are $[27, 52, 81, 104]$ and $[27, 55, 80, 103]$ and both contain 27. Since we are interested in a generalization of Wythoff where the P-positions form a disjoint cover of $\mathbb{N}^+$, we instead analyze a "Lexified" version of the game as follows.

Recall lexicographic order: Given vectors $\mathbf{A} = [a_1, a_2, ..., a_K]$ and $\mathbf{B} = [b_1, b_2, ..., b_K]$, we say that $\mathbf{B} \prec \mathbf{A}$ if $b_i < a_i$ for the first index i at which they differ. For unordered sets (such as pile sizes in a game) one first arranges the elements into a vector in nondecreasing order, and then applies lexicographic ordering on the corresponding vectors.

Definition 3. *A two-player game of* Wytlex(K) *is played on* $K \geq 2$ *piles of tokens. A legal move is to choose two of the piles and either remove* ***or add*** *any number of tokens from one of them, or remove* ***or add*** *the same number of tokens from both of them. From each of the other piles remove* ***or add*** *zero or more tokens with no restriction. For a move from* $\mathbf{A}$ *to* $\mathbf{B}$ *to be legal, one must have* $\mathbf{B} \prec \mathbf{A}$*.*

A position, $\mathbf{A}$, in Wytlex(K) is specified by giving the sizes of the piles in nondecreasing order $[a_1, a_2, \ldots, a_K]$ with $0 \leq a_1 \leq a_2 \leq \ldots \leq a_K$.

Note: Reducing a pile to size 0 does not change the nature of the game. It is still a K-Pile game, not a (K-1)-Pile game. In such a case we may always move directly from $[0, a_2, a_3, \ldots, a_K]$ to $[0, 0, 0, \ldots, 0]$ by performing a Wythoff move on $[0, a_2]$ to $[0, 0]$ and removing all tokens from the other piles.

2.1 Examples

We give some examples of positions and moves of Wytlex(K) for $K = 3$.

- $[0, 0, 0]$ has no legal moves, since it is first in lexicographical order.
- $[0, 10, 30]$ has a legal move to $[0, 0, 0]$ as follows: Make a Wythoff move on the first two piles to $[0, 0]$ and then remove 30 tokens from the third pile
- $[0, 10, 30]$ has a legal move to $[0, 9, 50]$ as follows: Make a Wythoff move on the first and third piles from $[0, 30]$ to $[0, 50]$ (because in Wytlex we may add tokens) and then remove 1 token from the second pile. Since $[0, 9, 50] \prec [0, 10, 30]$ the move is legal.
- $[10, 10, 20]$ has a legal move to $[5, 12, 12]$ as follows: Make a Wythoff move on the first and second piles adding 2 tokens to each pile. Then remove 15 tokens from the last pile. We thus have 12 tokens in each of the first two piles and 5 in the third. Representing this in nondecreasing order we have $[5, 12, 12]$. Since $[5, 12, 12] \prec [10, 10, 20]$ the move is legal.
- $[1, 2, 3]$ has no legal move to $[0, 0, 0]$ since no two of the piles have a legal Wythoff move to $[0, 0]$

Note that Wytlex(2) is just classical Wythoff. Indeed if tokens are added to one pile, either the other pile remains unchanged or also has tokens added to it. In either case the lexicographic order would increase so such moves are not allowed. That leaves us with the moves of Wythoff(2) which is obviously identical to Wythoff.

One might wonder if the un-natural permission to increase pile size actually "adds" anything to the game. Might these moves be reversible? To show that this is not so, below are the first 13 P-positions of Wytlex(3) versus those of Wythoff(3). Only the last row differs. The position $[23, 39, 58]$

is a P-position in Wythoff(K), but in Wytlex(K) it has a legal move to the previous P-position $[20, 37, 53]$: we make a "Lexified" Wythoff move on the first two piles of $[23, 39, 58]$, adding 14 tokens to each, and remove 38 tokens from the third pile.

Wytlex(3)

n	a_1	a_2	a_3
0	0	0	0
1	1	2	3
2	4	7	10
3	5	9	13
4	6	11	16
5	8	15	22
6	12	21	30
7	14	25	36
8	17	29	41
9	18	31	44
10	19	34	49
11	20	37	53
12	23	42	61

Wythoff(3)

n	a_1	a_2	a_3
0	0	0	0
1	1	2	3
2	4	7	10
3	5	9	13
4	6	11	16
5	8	15	22
6	12	21	30
7	14	25	36
8	17	29	41
9	18	31	44
10	19	34	49
11	20	37	53
12	23	39	58

We note that in Wytlex(K) with $K > 2$ any position T with at least two non-empty piles has an infinite number of sub-positions and there is no upper bound on the number of moves till the game ends. This is true because if the two piles have sizes $0 < a \leq b$ then we can remove 1 token from a and add $n > 0$ tokens to b, leaving the other piles untouched (one of the other piles becomes the untouched pile for the Wythoff move). Since the result is earlier in lexicographic order, the move is legal for all $n > 0$. Even so, due to the constraint on lexicographic order, Wytlex(K) always ends after a finite number of moves.

We would have preferred proving the theorems in this paper for the case of Wythoff(K) instead of Wytlex(K) but we have not been able to do so. In any case, we think that the concept of "Lexifying" a takeaway game has value in its own right and might be usefully applied to other takeaway games.

3 P-positions of Wytlex

Definition 4. *For any set of numbers S and a number x we define the shifted set, $x + S$, to be $\{ x + s \mid s \in S \}$. For two sets S and T we define $S + T = \{ s + t \mid s \in S, t \in T \}$.*

Definition 5. *For a nondecreasing sequence of numbers $\mathbf{A} = [a_1, a_2, \ldots, a_K]$ we define the difference set, $D(\mathbf{A})$, to be $\{ a_j - a_i \mid 1 \leq i < j \leq K \}$. Note that $D(\mathbf{A})$ contains only non-negative numbers.*

Notation:

1. The set of components $a_1, a_2, ..., a_K$ of the vector $\mathbf{A} = [a_1, a_2, \ldots, a_K]$ is denoted *set*($\mathbf{A}$).

2. To enhance readability we use the following notational convention: Subscripts i, j, k, l denote the index of a pile in a position, while superscripts n, m denote different positions. For example, P_i^n may denote the size of the i'th pile of the n'th P-position.

Lemma 1. *If $\mathbf{B} \prec \mathbf{A}$ and $\mathbf{A} \cap \mathbf{B} \neq \varnothing$ then there exists a legal move $\mathbf{A} \to \mathbf{B}$.*

Proof. If $\mathbf{A} \cap \mathbf{B} \neq \varnothing$ there exist i, j such that $a_i = b_j$. We choose pile i as one of the two piles for the Wythoff move, leaving it untouched. We add/remove tokens from the other piles as necessary to move from $\mathbf{A}$ to $\mathbf{B}$. Thus there exists a legal move $\mathbf{A} \to \mathbf{B}$. □

Lemma 2. *If $\mathbf{B} \prec \mathbf{A}$ and $\mathbf{A} \cap \mathbf{B} = \varnothing$ then there exists a legal move $\mathbf{A} \to \mathbf{B}$ if and only if there exist i, j such that $a_j - a_i \in D(\mathbf{B})$.*

Proof. Since $\mathbf{B} \cap \mathbf{A} = \varnothing$ the Wythoff component of the legal move must add/remove the same number T of tokens from both piles. This can happen if and only if there exist indices i, j, k, l such that $a_i - T = b_k$ and $a_j - T = b_l$, which happens if and only if $a_i - b_k = a_j - b_l$ if and only if $a_j - a_i = b_l - b_k$. By switching indices $i \leftrightarrow j$ and $k \leftrightarrow l$ if necessary, we can assume $b_k < b_l$ so $b_l - b_k \in D(\mathbf{B})$ as required. □

Motivation: Due to Lemmas 1 and 2 we want to construct the P-positions recursively so that the n'th P-position has no piles of the same size as in any of the previous P-positions and that the differences in the sizes of its piles don't match any previous differences. This is achieved in the following construction.

Construction 1. *We recursively construct a set of* candidate *P-positions,* $\mathbf{P^n}$ *and the sets* X^n, D^n *as follows:*

$$\mathbf{P^0} = [0, 0, \ldots, 0]$$

and then for all $n > 0$

$$X^n = \bigcup_{0 \le m < n} \mathrm{set}(\mathbf{P^m})$$

$$D^n = \bigcup_{0 \le m < n} D(\mathbf{P^m})$$

and then, for a given n*, define*

$$Q_1^n = X^n$$

and then recursively for each $1 \le i < K$

$$p_i^n = \mathrm{mex}\,\{Q_i^n\}$$

$$Q_{i+1}^n = Q_i^n \cup (p_i^n + D^n) = X^n \bigcup_{j \le i} (p_j^n + D^n).$$

Lemma 3. *For all* $n > 0$ *we have* $p_1^n > p_1^{n-1}$ *and for all* $1 \le i < K$ *we have* $p_{i+1}^n > p_i^n$*. In particular,* $\mathbf{P^n}$ *lists the pile sizes in the correct order.*

Proof. Since $X^n \supset X^{n-1}$ we have $p_1^n \ge p_1^{n-1}$. Since $p_i^{n-1} \in \mathbf{P^{n-1}} \subset X^n$ we have $p_1^n \ne p_1^{n-1}$. So $p_1^n > p_1^{n-1}$.

Since $Q_{i+1}^n \supset Q_i^n$ we have $p_{i+1}^n \ge p_i^n$. Since $0 \in D^1 \subset D^n$ for all $n > 0$, we have $p_i^n = p_i^n + 0 \in Q_{i+1}^n$ and therefore $p_{i+1}^n \ne p_i^n$. So $p_{i+1}^n > p_i^n$. □

Lemma 4. *The sequences* $\{p_1^n\}, \{p_2^n\}, \ldots, \{p_K^n\}$ *for* $n > 0$ *form a disjoint cover of the positive integers.*

Proof. By the definition of $p_1^n = \mathrm{mex}\,\{X^n\}$ it is clear that $\{X^n\}_{n>0}$ is a covering. For all i, $Q_i^n \supset Q_1^n = X^n$ so $p_i^n \notin X^n$. By Lemma 3 $p_{i+1}^n > p_i^n$. So the cover is disjoint. □

Lemma 5. *For all* $n > m$ *there is no legal move from* $\mathbf{P^n}$ *to* $\mathbf{P^m}$*.*

Proof. By Lemma 4 no pile in $\mathbf{P^n}$ has the same number of tokens as a pile in $\mathbf{P^m}$. So by Lemma 2 a legal move exists only if there exist i, j with $p_i^n - p_j^n \in D(\mathbf{P^m})$. But then from Lemma 3 we have $i > j$. So $p_i^n \in (p_j^n + D^n) \subset Q_i^n$ contradicting $p_i^n = \mathrm{mex}\,\{Q_i^n\}$. □

We denote by P the set of candidate P-positions, $\{\mathbf{P^n}\}_{n\geq 0}$. We denote the complement of P by N, namely the set of candidate N-positions.

Lemma 6. *For every position in N there exists a legal move to some position in P.*

Proof. Let $\mathbf{V} \in N$. By Lemma 4 $v_1 = p_k^n$ for some n, k. If $k > 1$ then by Lemma 3 $v_1 = p_k^n > p_1^n$ so $\mathbf{P^n} \prec \mathbf{V}$. So the move $\mathbf{V} \to \mathbf{P^n}$ leaving v_1 unchanged is legal.

If $k = 1$ then $v_1 = p_1^n$ and there exists a first i such that $v_j = p_j^n$ for all $j < i$ and $v_i \neq p_i^n$ (since $\mathbf{V} \neq \mathbf{P^n}$). If $p_i^n < v_i$ then again the move $\mathbf{V} \to \mathbf{P^n}$ is legal.

If $v_i < p_i^n$ then, since $p_i^n = \operatorname{mex}\{Q_i^n\}$,

$$v_i \in Q_i^n = X^n \bigcup_{j<i} (p_j^n + D^n) = X^n \bigcup_{j<i} (v_j + D^n).$$

If $v_i \in X^n$ then v_i equals some p_l^m and since $v_1 = p_1^n > p_1^m$, by Lemma 3 the move $\mathbf{V} \to \mathbf{P^m}$ leaving v_i untouched is legal.

Finally, if

$$v_i \in \bigcup_{j<i} (v_j + D^n),$$

then $v_i - v_j \in D(\mathbf{P^m})$ for some $m < n$. Since $v_1 = p_1^n > p_1^m$, by lemma (2) the move $\mathbf{V} \to \mathbf{P^m}$ is legal. $\square$

Theorem 1. *The P-positions other than $\mathbf{0}$ of Wytlex(K) are given recursively by construction (1). The corresponding sequences $\{p_1^n\}, \{p_2^n\}, \ldots, \{p_K^n\}$ for $n > 0$ form a disjoint cover of the positive integers.*

Proof. By Lemmas 5 and 6 P and N are the P-positions and N-positions of the game. By Lemma 4 the P-positions other than $\mathbf{0}$ form a disjoint cover of the positive integers. $\square$

The complexity of the above recursive algorithm depends on the size of the sets Q_i^n, which in turn depends on the sizes of X^n and D^n. Obviously $|X^n| \leq nK$ and $|D^n| \leq nK^2$. So $|Q_i^n| \leq |Q_K^n| \leq nK^3$. The recursive construction of P^n thus requires at most n^2K^3 steps. For fixed K this is $O(n^2)$. We have tried to find a much more efficient arithmetic or algebraic representation of the P-positions a' la those in [8] but have had no success. Using the method of [1] we have shown that the P-positions of Wytlex(3) can't be described by any Beatty sequence, homogeneous or not. The same remains true far into the sequence, even after ignoring P-positions with total pile size less than 2 million.

4 Variable K-Pile Wytlex

We now extend the above definitions and theorems to a wider class of games which we call *Variable* Wytlex(K). As motivation for the extension we first reformulate the definition of a legal move in Wytlex(K).

Definition 6. *A move from* $\mathbf{A}$ *to* $\mathbf{B}$ *in Wytlex(*K*) is legal if* $\mathbf{B} \prec \mathbf{A}$ *and either* $\mathbf{A} \cap \mathbf{B} \neq \varnothing$ *or* $D(\mathbf{A}) \cap D(\mathbf{B}) \neq \varnothing$

By Lemmas 1 and 2 the new definition is equivalent to the old one given in Definition 3.

Before proceeding to the definition of a variable Wytlex(K) we will also need the following

Definition 7. *We define the open interval* (i, j) *as the set of integers* $\{ k \mid i < k < j \}$*. We define similarly the closed and half open intervals* $[i, j], [i, j), (i, j]$*.*

And finally,

Definition 8. *A game of* Variable Wytlex(K) *has the same positions as those of Wytlex(*K*). In addition there is given a function* $f : \mathbb{N}^{\mathbb{K}} \to \mathbb{N}^{+}$*. A move from* $\mathbf{A}$ *to* $\mathbf{B}$ *is legal if* $\mathbf{B} \prec \mathbf{A}$ *and either* $\mathbf{A} \cap \mathbf{B} \neq \varnothing$ *or* $D(\mathbf{A}) \cap \Big(D(\mathbf{B}) + \big[0, f(\mathbf{B})\big) \Big) \neq \varnothing$*.*

We denote the variable Wytlex(K) defined by f as Wytlex(K, f).

An example Wytlex(K, f) for $K = 3$ is given by $f(\mathbf{B}) \equiv f([B_1, B_2, B_3]) \equiv 1 + B_3 - B_2$. We list the first few positions $\mathbf{B}$ and the values of $D(\mathbf{B})$, $f(\mathbf{B})$ and $D(\mathbf{B}) + \big[0, f(\mathbf{B})\big)$. The *Legality* column specifies whether a move from $\mathbf{A} = [2, 4, 6]$ to $\mathbf{B}$ is legal and, if so, at least one reason why. Note that $D(\mathbf{A}) = \{2\}$. Since the number of possible sub-positions is infinite, the below table obviously doesn't contain *all* legal moves from $\mathbf{A}$.

Variable 3-Pile Wytlex						
B_1	B_2	B_3	$D(\mathbf{B})$	$f(\mathbf{B}) = 1 + B_3 - B_2$	$D(\mathbf{B}) + \big[0, f(\mathbf{B})\big)$	Legality
0	0	0	$\{0\}$	1	$\{0\}$	Illegal
0	0	1	$\{0,1\}$	2	$\{0,1,2\}$	D's intersect
0	0	2	$\{0,2\}$	3	$\{0,1,2,3,4\}$	D's intersect
0	1	1	$\{0,1\}$	1	$\{0,1\}$	Illegal
0	1	2	$\{1\}$	2	$\{1,2\}$	D's intersect
0	2	2	$\{0,2\}$	1	$\{0,2\}$	D's intersect
1	1	1	$\{0\}$	1	$\{0\}$	Illegal
1	1	2	$\{0,1\}$	2	$\{0,1,2\}$	D's intersect
1	2	2	$\{0,1\}$	1	$\{0,1\}$	A and B intersect
2	2	2	$\{0\}$	1	$\{0\}$	A and B intersect

Lemma 7. *If* $\mathbf{B} \prec \mathbf{A}$ *and* $\mathbf{A} \cap \mathbf{B} = \varnothing$ *then there exists a legal move* $\mathbf{A} \to \mathbf{B}$ *if and only if there exist* $i < j$ *such that* $a_j - a_i \in D(\mathbf{B}) + \big[0, f(B)\big)$.

Proof. Immediate from the definition of a legal move. □

Construction 2. *We duplicate construction (1) with a single change. We replace the definition of* D^n *with*

$$D^n = \bigcup_{0 \le m < n} D(\mathbf{P^m}) + \big[0, f(\mathbf{P^m})\big).$$

Theorem 2. *The P-positions other than* $\mathbf{0}$ *of Wytlex(*K, f*) are given recursively by construction (2) and form a disjoint cover of the positive integers.*

Proof. The corresponding proofs of Lemmas 3-6 and Theorem 1 remain unchanged after replacing Lemma 2 with Lemma 7. □

5 Variable 2-Pile Wytlex and Complementary Sequences

Theorem 3. *The P-positions of Wytlex(*$2, f$*) are given by* $p_1^n = \operatorname{mex}\{X^n\}$ *and* $p_2^n = p_1^n + \sum_{m=0}^{n-1} f(\mathbf{P^m})$.

Proof. $p_1^n = \operatorname{mex}\{X^n\}$ follows directly from construction 2 and Theorem 2. We prove $p_2^n = p_1^n + \sum_{m=0}^{n-1} f(\mathbf{P^m})$ by induction. Since, by convention, the empty sum is zero, we have $p_2^0 = p_1^0 + 0 = 0$ so $\mathbf{P^0} = [0, 0]$ as required. For $n > 0$ we have

$$\begin{aligned}
D^n &= \bigcup_{0 \le m < n} D(\mathbf{P^m}) + \big[0, f(\mathbf{P^m})\big) \\
&= \bigcup_{0 \le m < n} \sum_{s=0}^{m-1} f(\mathbf{P^s}) + \big[0, f(\mathbf{P^m})\big) \\
&= \bigcup_{0 \le m < n} \left[\sum_{s=0}^{m-1} f(\mathbf{P^s}), \sum_{s=0}^{m-1} f(\mathbf{P^s}) + f(\mathbf{P^m})\right) \\
&= \bigcup_{0 \le m < n} \left[\sum_{s=0}^{m-1} f(\mathbf{P^s}), \sum_{s=0}^{m} f(\mathbf{P^s})\right) \\
&= \left[0, \sum_{s=0}^{n-1} f(\mathbf{P^s})\right).
\end{aligned}$$

from Theorem 2, looking back at construction (1), we have

$$\begin{aligned}
p_2^n &= \operatorname{mex}\{Q_2^n\} = \operatorname{mex}\{X^n \cup (p_1^n + D^n)\} \\
&= \operatorname{mex}\left\{X^n \cup p_1^n + \left[0, \sum_{s=0}^{n-1} f(\mathbf{P^s})\right)\right\} \\
&= \operatorname{mex}\left\{X^n \cup \left[p_1^n \, , \, p_1^n + \sum_{s=0}^{n-1} f(\mathbf{P^s})\right)\right\}.
\end{aligned}$$

But X^n contains all the integers less than p_1^n and $\big[p_1^n \, , \, p_1^n + \sum_{s=0}^{n-1} f(\mathbf{P^s})\big)$ contains the rest of the integers up to $p_1^n + \sum_{s=0}^{n-1} f(\mathbf{P^s})$, so

$$p_2^n = p_1^n + \sum_{s=0}^{n-1} f(\mathbf{P^s})$$

(the last equality is true because, by induction, all elements of X^n are less than $p_1^n + \sum_{s=0}^{n-1} f(\mathbf{P^s})$). $\square$

Note: If we choose f to be a constant function, $f(\mathbf{X}) = a$ for all $\mathbf{X}$ with $a > 0$, then the P-positions of Wytlex$(2, f)$ are

$$p_1^n = \operatorname{mex}\left\{X^N\right\}, \; p_2^n = p_1^n + na$$

which are the same as the P-positions of the generalization of Wythoff introduced in [8].

Definition 9. *An ordered pair of sequences,* $(\{y^n\}_{n>0}, \{z^n\}_{n>0})$ *which form a disjoint cover of* $\mathbb{N}^+$ *is* monotonic *if* $y^1 < z^1$ *and for all* $n > 0$, $z^{n+1} - y^{n+1} > z^n - y^n$.

It is obvious that $y^1 = 1$ and $y^n < z^n$ for all n and therefore that $y^n = \text{mex}\{\{y^i\}_{i<n}, \{z^i\}_{i<n}\}$.

From Theorem 3 we know that the P-positions, $(\{p_1^n\}_{n>0}, \{p_2^n\}_{n>0})$, other than $[0,0]$ of Wytlex$(2, f)$, form a monotonic disjoint cover of $\mathbb{N}^+$. The converse is also true:

Theorem 4. *For every monotonic disjoint cover of* $\mathbb{N}^+$, $(\{y^n\}_{n>0}, \{z^n\}_{n>0})$ *there exists a function* f *such that the P-positions other than* $[0,0]$ *of Wytlex(*$2, f$*) are* $(y^n, z^n)_{n>0}$.

Proof. Define $f(0,0) = z^1 - y^1 = z^1 - 1 > 0$. For all $n \in \mathbb{N}^+$ either $n = y^i$ or $n = z^i$. If $n = y^i$ define $f(y^i, z^i) = (z^{i+1} - y^{i+1}) - (z^i - y^i)$. For all other (n, m) the value of $f(n, m)$ will turn out to be irrelevant, so define $f(n, m)$ to be an arbitrary positive integer (for example, 1). Then, since $(\{y^n\}, \{z^n\})$ is monotonic, f defines a function from $\mathbb{N}^2 \to \mathbb{N}^+$ and therefore defines a game Wytlex$(2, f)$. If $[a^n, b^n]$ are the P-positions other than $[0,0]$ of Wytlex$(2, f)$ then we prove by induction that for all $n > 0$, $a^n = y^n$ and $b^n = z^n$.

For the case $n = 1$ we have $a^1 = 1 = y^1$ and $b^1 = a^1 + f(0) = a^1 + z^1 - y^1 = z^1$. Assume by induction that $a^m = y^m$ and $b^m = z^m$ for all $m < n$. Then

$$\begin{aligned} a^n &= \text{mex}\{\{a^m\}_{m<n}, \{b^m\}_{m<n}\} \\ &= \text{mex}\{\{y^m\}_{m<n}, \{z^m\}_{m<n}\} = y^n, \end{aligned}$$

and

$$\begin{aligned} b^n &= a^n + \sum_{m=0}^{n-1} f(a^m, b^m) \\ &= a^n - a^{n-1} + a^{n-1} + \sum_{m=0}^{n-2} f(a^m, b^m) + f(a^{n-1}, b^{n-1}) \\ &= a^n - a^{n-1} + b^{n-1} + f(a^{n-1}, b^{n-1}) \\ &= y^n - y^{n-1} + z^{n-1} + f(a^{n-1}, B^{n-1}) \\ &= y^n - y^{n-1} + z^{n-1} + ((z^n - y^n) - (z^{n-1} - y^{n-1})) \\ &= z^n. \end{aligned}$$

$\square$

Corollary 1. *In particular, if two Beatty sequences* $\lfloor np+\beta_p \rfloor_{n>0}$, $\lfloor nq+\beta_q \rfloor_{n>0}$ *form a disjoint cover of* $\mathbb{N}^+$ *with* $q \geq 3$ *then they are the P-positions of some Wytlex(*$2, f$*).*

Proof. First we note that $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x+y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1$. Since $1/p + 1/q = 1$ we have $p < 2$. But then

$$\begin{aligned}\lfloor (n+1)q+\beta_q \rfloor - \lfloor (n+1)p+\beta_p \rfloor &= \lfloor nq+\beta_q+q \rfloor - \lfloor np+\beta_p+p \rfloor \\ &\geq \lfloor nq+\beta_q \rfloor + \lfloor q \rfloor - (\lfloor np+\beta_p \rfloor + \lfloor p \rfloor + 1) \\ &= \lfloor nq+\beta_q \rfloor + \lfloor np+\beta_p \rfloor + \lfloor q \rfloor - \lfloor p \rfloor - 1 \\ &\geq \lfloor nq+\beta_q \rfloor + \lfloor np+\beta_p \rfloor + 1.\end{aligned}$$

So the Beatty sequences form a monotonic disjoint cover of $\mathbb{N}^+$and the result follows immediately from Theorem 4. $\square$

6 Further Work

It would seem that many other token taking games that have been discussed in the literature would also be amenable to “Lexification” which might lead to interesting games in their own right. For classic Nim, where each move is restricted to taking tokens from a single pile, “Lexification” would add nothing. Similarly, we noticed that “Lexification” has no effect on classic 2-Pile Wythoff. But there are many games which allow taking from more than two piles. For example in [9] it is shown that a natural generalization of Nim to the case of $K > 2$ piles of sizes $[a_1, a_2, ..., a_K]$ is to either remove any positive number of tokens from a single pile, or remove x_i tokens from each pile simultaneously, subject to the conditions: (i) $x_i > 0$ for some i, (ii) $x_i \leq a_i$ for all i, (iii) $x_1 \oplus x_2 \oplus ... \oplus x_K = 0$, where $\oplus$ denotes Nim-sum. This game has some interesting open conjectures regarding its P-positions. We can “Lexify” this game by allowing some of the x_i to be negative (thus adding tokens) and requiring that moves be to positions which are earlier in lexicographic order.

Two additional directions of research specific to Wytlex would be: To investigate Wytlex$(2, f)$ for f linear, or, more generally, Wytlex(K, f) for suitable functions f that produce 'interesting' games; To reduce the time complexity of calculating the P-positions, for example by an algebraic expression or using an enumeration system.

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