A New Heap Game

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Abstract

Given $k \geq 3$ heaps of tokens. The moves of the 2-player game introduced here are to either take a positive number of tokens from at most $k-1$ heaps, or to remove the *same* positive number of tokens from all the k heaps. We analyse this extension of Wythoff's game and provide a polynomial-time strategy for it.

Keywords: multi-heap games, efficient strategy, Wythoff game

1 Introduction

We propose the following two-player game on k heaps with finitely many tokens, where $k \geq 3$. There are two types of moves: (i) remove a positive number of tokens from up to $k-1$ heaps, possibly $k-1$ entire heaps, or, (ii) remove the *same* positive number of tokens from all the k heaps. The player making the last move wins.

Any position in this game can be described in the following standard form: $(m_0, \ldots, m_{k-1})$ with $0 \leq m_0 \leq \ldots \leq m_{k-1}$, where m_i is the number of tokens in the i-th heap. Given any game Γ, we say informally that a *P-position* is any position u of Γ from which the *P*revious player can force a win, that is, the opponent of the player moving from u. An *N-position* is any position v of Γ from which the *N*ext player can force a win, that is, the player who moves from v. The set of all P-positions of Γ is denoted by $\mathcal{P}$, and the set of all N-positions by $\mathcal{N}$. Denote by $F(u)$ all the followers of u, i.e., the set of all positions that can be reached in one move from the position u. It is then easy to see that:

$$\text{For every position } u \text{ of } \Gamma \text{ we have } u \in \mathcal{P} \text{ if and only if } F(u) \subseteq \mathcal{N}\ ;$$
$$\text{and } u \in \mathcal{N} \text{ if and only if } F(u) \cap \mathcal{P} \neq \emptyset\ . \quad (1)$$

For $n \in \mathbb{Z}^0$, denote the n-th *triangular number* by $T_n = \frac{1}{2}n(n+1)$. We prove,

Theorem 1. *Every P-position of the game can be written in the form $(T_n, m_1, \ldots, m_{k-1})$, where the $(k-1)$-tuples $(m_1, \ldots, m_{k-1})$ range over all the (unordered) partitions of $(k-1)T_n + n$ with parts of size $\geq T_n$. In other words, $\mathcal{P} = \bigcup_{n=0}^{\infty} P_n$, where*

$$\begin{aligned} P_n \quad = \quad & \Big\{(T_n, m_1, \ldots, m_{k-1}) : \sum_{i=1}^{k-1} m_i = (k-1)T_n + n \ , \\ & T_n \leq m_1 \leq \ldots \leq m_{k-1},\ n \in \mathbb{Z}^0\Big\} \ . \end{aligned} \tag{2}$$

Example. For $k = 4$,

$$P_n = \{(T_n, m_1, m_2, m_3) : m_1 + m_2 + m_3 = 3T_n + n,\ n \in \mathbb{Z}^0\} \ .$$

The first few P-positions are:

$P_0 = \{(0, 0, 0, 0)\}$
$P_1 = \{(1, 1, 1, 2)\}$
$P_2 = \{(3, 3, 3, 5),\ (3, 3, 4, 4)\}$
$P_3 = \{(6, 6, 6, 9),\ (6, 6, 7, 8),\ (6, 7, 7, 7)\}$
$P_4 = \{(10, 10, 10, 14),\ (10, 10, 11, 13),\ (10, 10, 12, 12),\ (10, 11, 11, 12)\}$
$P_5 = \{(15, 15, 15, 20),\ (15, 15, 16, 19),\ (15, 15, 17, 18),$
$(15, 16, 16, 18),\ (15, 16, 17, 17)\}.$

2 The Proof

Throughout, as in (2), every k-tuple $(T_n, m_1, \ldots, m_{k-1})$, $(m_0, \ldots, m_{k-1})$ or $(k-1)$-tuple $(m_1, \ldots, m_{k-1})$ is arranged in nondecreasing order. Any of the first two tuples is also called a position (of the game) or partition (of $kT_n + n$); and the third is also a partition (of $(k-1)T_n + n$). The terms m_i are called components (of the tuple) or parts (of the partition).

Lemma 1. *Given any partition $(m_1, \ldots, m_{k-1})$ of $(k-1)T_n + n$, where each part has size $\geq T_n$. Then each part has size $< T_{n+1}$.*

Proof. We have,

$$(k-1)T_n + n - m_{k-1} = \sum_{i=1}^{k-2} m_i \geq (k-2)T_n \ .$$

Hence for all $i \in \{1, \ldots, k-1\}$, $m_i \leq m_{k-1} \leq T_n + n = T_{n+1} - 1$. ■

Lemma 2. *Let $k \geq 3$ and $n \in \mathbb{Z}^0$. Every integer in the semi-closed interval $t \in [T_n, T_{n+1})$ appears as a component in some position of P_n. It appears in P_m for no $m \neq n$.*

Proof. The smallest component in P_n is T_n, and by Lemma 1, the largest part cannot exceed $T_n + n = T_{n+1} - 1$. Hence $t \in [T_n, T_{n+1})$ appears as a component in P_m for no $m \neq n$. Let $t \in [T_n, T_{n+1})$, say $t = T_n + j$, $0 \leq j \leq n$. Then for $k \geq 3$, T_n+j appears in the partition $\{m_1, \ldots, m_{k-1}\} = \{T_n^{k-3}, T_n+n-j, T_n+j\}$ of $(k-1)T_n + n$, where T_n^{k-3} denotes $k-3$ copies of T_n, and so $T_n + j$ appears in some position of P_n. ■

Proof of Theorem 1. It follows from (1) that it suffices to show two things: (I) A player moving from any position in P_n lands in a position which is in P_m for no m. (II) From any position which is in P_m for no m, there is a move to some P_n, $n \in \mathbb{Z}^0$. The fact that (I) and (II) suffice in general for characterizing $\mathcal{P}$ and $\mathcal{N}$, is shown in [9] for the case of games without cycles, based on a formal definition of the P- and N-positions, and a proof of (1). (It is not true for cyclic games: given a digraph consisting of two vertices u and v, and an edge from u to v, and an edge from v to u. Place a token on u. The two players alternate in pushing the token to a follower. The outcome is clearly a *draw*, since there is no last move. However, putting $\mathcal{P} = \{u\}$, $\mathcal{N} = \{v\}$, satisfies (1).)

(I) Let P_n be any k-tuple of the form (2). Removing tokens from up to $k-1$ heaps, including the first heap, results in a position Q such that the first element is in P_j for some $j < n$, yet there is a heap whose size is a component in P_n. Thus $Q \in P_m$ for no m by Lemma 2. Removing tokens from up to $k-1$ heaps, excluding the first heap, results in a position Q whose last $k-1$ components sum to a number $< (k-1)T_n + n$. Since, however, the first component is in T_n, Q is not of the form (2). Hence $Q \in P_m$ for no m.

So consider the move from P_n which results in $Q = (T_n - t, m_1 - t, \ldots, m_{k-1} - t)$ for some $t \in \mathbb{Z}^+$. If $Q \in P_m$ for some $m < n$, then $T_n - t = T_m$. Then $(T_n - t) + (m_1 - t) + \ldots + (m_{k-1} - t) = kT_n + n - kt = kT_m + m$. Thus, $0 = k(T_n - T_m - t) = m - n < 0$, a contradiction. Hence $Q \in P_m$ for no m.

(II) Let $(m_0, \ldots, m_{k-1})$ be any position which is in P_m for no m. Since $\bigcup_{n=0}^{\infty} [T_n, T_{n+1})$ is a partition of $\mathbb{Z}^0$, we have $m_0 \in [T_n, T_{n+1})$ for precisely one $n \in \mathbb{Z}^0$. Put $L = \sum_{i=1}^{k-1} m_i$.

Case (i). $m_0 = T_n$. If $L > (k-1)T_n + n$, then removing $L - (k-1)T_n - n$ from a suitable subset of $\{m_1, \ldots, m_{k-1}\}$, results in a position in P_n. So suppose that $L < (k-1)T_n + n$. Then $L = (k-1)T_n + j$ for some $j \in \{0, \ldots, n-1\}$. Subtracting $T_n - T_j$ from all components then leads to a position in P_j. Indeed, $m_0 - (T_n - T_j) = T_j$, and $\sum_{i=1}^{k-1} (m_i - (T_n - T_j)) = (k-1)T_j + j$.

Case (ii). $T_n < m_0 < T_{n+1}$, say $m_0 = T_n + j$, $j \in \{1, \ldots, n\}$. Suppose first that $L \geq (k-1)T_n + n + j$. If $m_1 < T_{n+1}$, subtract j from m_0 to get to T_n. By the first part of Lemma 2, m_1 is a part in some partition of $(k-1)T_n + n$. Then reduce, if necessary, a subset of the m_i for $i > 1$, so that $m_1 + \sum_{i=2}^{k-1} m_i' = (k-1)T_n + n$. Here and below, m_i' denotes m_i after a suitable positive integer may have been subtracted from it. If $m_1 \geq T_{n+1}$, then decrease m_1 to T_n. Then $T_n + \sum_{i \neq 1} m_i \geq T_n + j + T_n + (k-2)T_{n+1} \geq kT_n + (k-2)(n+1) + 1 \geq kT_n + n + 2 > kT_n + n$, since $k \geq 3$. Again by Lemma 2, m_0 is a part in some partition of $(k-1)T_n + n$. So reducing, if necessary, a subset of the m_i for $i \geq 2$, we get $m_0 + \sum_{i=2}^{k-1} m_i' = (k-1)T_n + n$.

So consider the case $L \leq (k-1)T_n + n + j$. We claim that subtracting $m_0 - T_m$ from all components of $(m_0, \ldots, m_{k-1})$ leads to a position in T_m, where $m = L - (k-1)m_0$. First note that $m = \sum_{i=1}^{k-1} m_i - (k-1)m_0 \geq 0$, and $m = L - (k-1)m_0 \leq (k-1)T_n + n + j - (k-1)m_0 = n - (k-2)j \leq n - j < n$ (since $k \geq 3$), so $0 \leq m < n$, as required. Secondly, $m_0 - (m_0 - T_m) = T_m$, and $\sum_{i=1}^{k-1} (m_i - (m_0 - T_m)) = L - (k-1)(m_0 - T_m) = (k-1)T_m + m$. (Note that for $L = (k-1)T_n + n + j$ we provided two winning moves. The second leads to a win faster than the first.)

In conclusion, we see that $\bigcup_{i=0}^{\infty} P_i = \mathcal{P}$. ■

3 Aspects of the Strategy

We observe that the *statement* of Theorem 1 tells a player whether or not it is possible to win by moving from any given position. The *proof* of the theorem shows how to compute a winning move, if it exists. Together they form a *strategy* for the game.

The strategy can, in fact, be computed in polynomial time. Given any position $Q = (m_0, \ldots, m_{k-1})$ of the game. Its input size is $\Theta\left(\sum_{i=0}^{k-1} (\log m_i)\right)$. Solving $m_0 = n(n+1)/2$ leads to $n = \lfloor (\sqrt{1+8m_0} - 1)/2 \rfloor$. By Theorem 1, $Q \in \mathcal{P}$ if and only if $m_0 = T_n$, where $n = (\sqrt{1+8m_0} - 1)/2$ is an integer, and $\sum_{i=1}^{k-1} m_i = (k-1)T_n + n$. Otherwise $Q \in \mathcal{N}$, and the proof of Theorem 1 indicates how to compute a winning move to a P_n-position. All of this can be done in time which is polynomial in the input size. The point is that there are no operations that require m_i steps for any i; the computation of n involves only $O(\log m_0)$ operations, since any integer N is represented succinctly by $\Theta(\log N)$ digits.

It is also of interest to estimate the density of the P-positions in the set of all game positions. Subtracting $T_n - 1$ from each m_i in the sum of (2), we get partitions of the form

$$x_1 + \ldots + x_{k-1} = n + k - 1 \ , \quad 1 \leq x_1 \leq \ldots \leq x_{k-1} \leq n + 1 \ ,$$

where $x_i = m_i - (T_n - 1)$. The number $p_{k-1}(n+k-1)$ of partitions of $n+k-1$ into $k-1$ positive integer parts is estimated in [12, Ch. 4]. It is a polynomial of degree $k-1$ in $n+k-1$, whose leading term is $(n+k-1)^{k-2}/(k-2)!$. Thus the number of positions P_n for $n \leq N$ is estimated by $\pi(N) = \sum_{n=0}^{N} (n+k-1)^{k-2}/(k-2)!$. It is easy to see that

$$\int_{-1}^{N} (x+k-1)^{k-2}/(k-2)! \, dx \leq \pi(N) \leq \int_{0}^{N+1} (x+k-1)^{k-2}/(k-2)! \, dx \ ,$$

leading to

$$\frac{(N+k-1)^{k-1} - (k-2)^{k-1}}{(k-1)!} \leq \pi(N) \leq \frac{(N+k)^{k-1} - (k-1)^{k-1}}{(k-1)!} \ .$$

The total number of positions up to P_N is the number of partitions of the form $m_0+\ldots+m_{k-1}=n$, $0 \le m_0 \le \ldots \le m_{k-1}$, where n ranges from 0 to kT_N+N. Adding 1 to all the parts, we get partitions of the form $x_0+\ldots+x_{k-1}=n+k$, $1 \le x_0 \le \ldots \le x_{k-1} \le n+k$, whose number is $p_k(n+k)$. As above, the total number of positions is thus estimated by $\nu(N)=\sum_{n=0}^{kT_N+N}(n+k)^{k-1}/(k-1)!$. Using integration as above, we get

$$\frac{(kT_N+N+k)^k-(k-1)^k}{k!} \le \nu(N) \le \frac{(kT_N+N+k+1)^k-k^k}{k!} \ .$$

For large N, the ratio is thus about

$$\frac{\pi(N)}{\nu(N)} \approx \frac{k}{kT_N+N+k}\left(\frac{N+k}{kT_N+N+k}\right)^{k-1} \ .$$

Dividing the numerator and denominator of the second fraction by N^{k-1} results in $\pi(N)/\nu(N)=O(1/N^{k+1})$. We see that the P-positions are rather rare, so our game sticks to the majority of games in the sense of [14] and [15]. The rareness of P-positions in general, is, in fact, consistent with the intuition suggested by (1): a position is in $\mathcal{P}$ if and only if *all* of its followers are in $\mathcal{N}$, whereas for a position to be in $\mathcal{N}$ it suffices that one of its followers is in $\mathcal{P}$. The scarcity of the P-positions is the reason why game strategies are usually specified in terms of their P-positions, rather than in terms of their N-positions.

4 Epilogue

In the heap games known to us, such as those discussed in [1], the moves are restricted to a *single* heap (which might, in special cases, be split into several subheaps). We know of three exceptions. One is Moore's Nim_k, [13], where up to k heaps can be reduced in a single move (so Nim_1 is ordinary Nim). Another one is superficially similar to the present game, in that the moves are also to take from all k heaps or from $\le k-1$ heaps, with some restrictions. But there a heap may also be split into new heaps [8]. The third is Wythoff's game, Wyt, [16], [3], [4], [17], where a move may affect up to two heaps. The motivation for the present note was to extend Wythoff's game to more than two heaps.

Wyt is played on two heaps. The moves are to either remove any positive number of tokens from a single heap, or to remove the *same* positive number of tokens from both heaps. Denoting by (x,y) the positions of Wyt, where x and y denote the number of tokens in the two heaps with $x \le y$, the first eleven P-positions are listed in Table 1. The reader may wish to guess the next few entries of the table before reading on.

For any finite subset $S \subset \mathbb{Z}^0$, define the *M*inimum *EX*cluded value of S as follows: $\operatorname{mex} S = \min \mathbb{Z}^0 \setminus S$ = least nonnegative integer not in S [1]. Note that if $S=\emptyset$, then $\operatorname{mex} S = 0$. The general structure of Table 1 is given by:

$$A_n = \operatorname{mex}\{A_i, B_i : 0 \le i < n\},\ B_n = A_n + n \quad (n \in \mathbb{Z}^0) \ .$$

Table 1: The first few P-positions of Wyt.

n	A_n	B_n
0	0	0
1	1	2
2	3	5
3	4	7
4	6	10
5	8	13
6	9	15
7	11	18
8	12	20
9	14	23
10	16	26

Since the input size of Wyt is *succinct*, namely $\Theta(\log(x+y))$, one can see that the above characterization of the P-positions implies a strategy which is exponential. A polynomial strategy for Wyt can be based on the observation that $A_n = \lfloor n\alpha \rfloor$, $B_n = \lfloor n\beta \rfloor$, where $\alpha = (1+\sqrt{5})/2$ is the golden section, $\beta = (3+\sqrt{5})/2$. Another polynomial strategy depends on a special numeration system whose basis elements are the numerators of the simple continued fraction expansion of α. These three strategies can be generalized to Wyt_a, proposed and analysed in [5], where $a \in \mathbb{Z}^+$ is a parameter of the game. The moves are as in Wyt, except that the second type of move is to remove say $k > 0$ and $l > 0$ from the two heaps subject to $|k-l| < a$. Clearly Wyt_1 is Wyt.

The generalization of Wyt to more than two heaps was a long sought-after problem. In [6] it is shown that the natural generalization to the case of $k \geq 2$ heaps is to either remove any positive number of tokens from a single heap, or say $l_1, \ldots, l_k$ from all of them simultaneously, where the l_i are nonnegative integers with $\sum_{i=1}^{k} l_i > 0$ and $l_1 \oplus \ldots \oplus l_k = 0$, and where $\oplus$ denotes Nim-sum (also known as addition over GF(2), or XOR). In particular, the case $k = 2$ is Wyt. But the actual computation of the P-values seems to be difficult.

The heap-game considered here is a generalization of the *moves* of Wyt, but not of its strategy. In fact, it doesn't specialize to the case $k = 2$; we used the fact that $k \geq 3$ in several places of the proof. However, the P-positions of the present game have a compact form, the exhibition of which was the purpose of this note.

The game proposed here has the rather exceptional property that although it is succinct, it has a *simple* polynomial strategy. Normally, an extra effort is required for showing that succinct games have a polynomial strategy. Different families of succinct games seem to require different methods for recovering a polynomial winning strategy, when it exists.

For example, in *octal* games, invented by Guy and Smith [11], a linearly ordered string of beads may be split and or reduced according to rules encoded in octal. See also [1, Ch. 4], [2, Ch. 11]. The standard method for showing that an octal game is polynomial, is to demonstrate that its *Sprague-Grundy* function (the 0s of which constitute the set of P-positions) is periodic. Periodicity has been established for a number of octal games. Some of the periods and or preperiods may be very large; see [10]. Another way to establish polynomiality is to show that the Sprague-Grundy function values obey some other simple rule, such as forming an arithmetic sequence, as for Nim.

Sometimes polynomiality is established by a nonstandard method. An arithmetic procedure, based on the Zeckendorf numeration system [18], is used for recovering polynomiality for Wythoff's game. In [7] and [8], games were proposed and analysed, and suitable numeration systems were used to establish polynomiality. For Wythoff's game, polynomiality can be proved also using the integer value function; a special numeration system isn't essential. In [7], [8] it was shown that the integer value function cannot be used to establish polynomiality for the games defined there. But the question remains whether there is some *polynomial* algorithm not based on numeration systems for some of these new games.

We remark finally that the Sprague-Grundy function g of a game provides a strategy for the *sum* of several games. The computation of g for Nim_k, $k \geq 2$, and Wyt_a, $a \geq 1$ seems to be difficult. It would be of interest to compute the g-function for the present game. Perhaps this is also difficult.

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