Complementary balanced words over an arbitrary finite alphabet

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May 23, 2005

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Abstract

A binary word is *balanced* if the numbers of 1-bits in any 2 subwords of equal length differ by at most 1. The structure of $m \geq 3$ complementary balanced binary words χ^j is determined. We prove that they must be characteristic words of Beatty words $\lfloor n\alpha_j + \gamma_j \rfloor$, and that two of the binary words must have identical densities of 1-bits, except possibly in the case where one of the α_j is a proper rational (denominator ≥ 2). For $m \leq 7$, the exceptional case is characterized completely. It is further shown that the above results are equivalent to analogous properties of complementary Beatty words and words on an alphabet of $2m$ letters.

Keywords: complementary words, balanced words, Beatty words, Sturmian words, exact covers

1 Preliminaries

Our aim is to reveal some of the structure of balanced complementary words. Balanced complementary words appear in various areas, such as combinatorics, combinatorial number theory, combinatorial games, dynamical systems, job scheduling and load balancing, optimal routing in queuing networks [2], billiard theory [3]. The paper lies in the interface between combinatorics on words and combinatorial number theory.

Since many authors use different notations/notions/definitions in this area, this paper will be (almost) self-contained. We use the language of *words*, which is isomorphic to that of *sequences*.

Throughout, I_d (d for domain) and I_r (r for range) denote intervals of integers. Usually $I_d = I_r = \mathbb{Z}_{\geq 1}$ is used in the literature, but here we consider the more general case. For α, γ real, $\alpha > 0$, the word $B_n(\alpha, \gamma) = \lfloor n\alpha + \gamma \rfloor$ ($n \in I_d$)

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is called a *Beatty* word with *modulus* α and *shift* γ. Its associated *Sturmian* word is defined by $f_n(\alpha,\gamma) = \lfloor (n+1)\alpha+\gamma \rfloor - \lfloor n\alpha+\gamma \rfloor - \lfloor \alpha \rfloor$ $(n \in I_d)$. Clearly $f_n(\alpha,\gamma) = I_r(f) = \{0,1\}$.

The *characteristic word* (or *indicator word*) $\chi : I_d \to \{0,1\}$ of a Beatty word $\lfloor n\alpha+\gamma \rfloor_{n\in I_d}$ is defined, for all $k \in I_r$ (range of the Beatty word), by

$$\chi_k(\alpha,\gamma) = \begin{cases} 1 & \text{if } \exists\, n \in I_d \text{ such that } \lfloor n\alpha+\gamma \rfloor = k \\ 0 & \text{otherwise.} \end{cases}$$

Let $m \in \mathbb{Z}_{\geq 2}$. Given an interval I_r, denote by $\mathbf{I}$ the word consisting of 1-bits over all of I_r, i.e., $\mathbf{I}_k = 1$ for all $k \in I_r$. We can partition $\mathbf{I}$ into m binary words $\chi^1,\ldots,\chi^m$, i.e., $\chi^i_k\chi^j_k = 0$ for all $i \neq j$ and all $k \in I_r$, and $\cup_{i=1}^m \chi^i = \mathbf{I}$. Such a collection $\{\chi^1,\ldots,\chi^m\}$ constitutes a *partition* of $\mathbf{I}$.

A finite collection of infinite words $\{A^1,\ldots,A^m\}$ of positive integers is said to be *complementary* (over I_r), if $A^i_kA^j_k = 0$ for all $i,j \in \{1,\ldots,m\}$, $i \neq j$ and all $k \in I_d$, and $\cup_{i=1}^m A^i = I_r$. For complementary Beatty words $A^i = \cup_{n\in I_d} \lfloor n\alpha_i+\gamma_i \rfloor$ $(i = 1,\ldots,m)$ we clearly have $\sum_{i=1}^m \alpha_i^{-1} = 1$ if $|I_d| = \infty$.

Let $\chi^1,\ldots,\chi^m$ be binary words. For $j \in \{1,\ldots,m\}$, define a map $V : \{0,1\} \to \{0,\ldots,2m-1\}$ by

$$V(\chi^j_k) = \begin{cases} 2j-2 & \text{if } \chi^j_k = 0 \\ 2j-1 & \text{if } \chi^j_k = 1, \end{cases} \tag{1}$$

for all $k \in I_r$. Denote by R the word all of whose entries are m^2-m+1, i.e., $R_k = m^2-m+1$ for all $k \in I_r$. Incidentally, the sequence $\{m^2-m+1 : m \in \mathbb{Z}_{\geq 0}\}$ appears in many contexts. See sequence A002061 in Sloane's On-Line Encyclopedia [23]. If for every $k \in I_r$ there exists precisely one j satisfying $\chi^j_k = 1$, we define a word F by $F_k = j$.

It turns out that balanced complementary binary words of the form V^j behave the same way as balanced complementary binary words χ^j. One can define other such words with similar behavior over finite alphabets.

For α_j, γ_j real, $\alpha_j > 0$ for all $j \in \{1,\ldots,m\}$, suppose that the Beatty words $B(\alpha_j,\gamma_j)$ partition I_r, i.e., $B(\alpha_i,\gamma_i) \cap B(\alpha_j,\gamma_j) = \emptyset$ for all $i,j \in I_d$, $i \neq j$, and $\cup_{i=1}^m B(\alpha_i,\gamma_i) = I_r$. For every $j \in \{1,\ldots,m\}$, define the words $T^j : j \in \{1,\ldots,m\}$, analogously to the characteristic word, by

$$T^j_k(\alpha_j,\gamma_j) = \begin{cases} k & \text{if } \exists\, n \in I_d \text{ such that } \lfloor n\alpha_j+\gamma_j \rfloor = k \\ 0 & \text{otherwise,} \end{cases}$$

for all $k \in I_r$. Note that the words T^j partition I_r, i.e., $T^i \cap T^j = \emptyset$ for all $i \neq j$, and $\cup_{i=1}^m T^i = I_r$ iff the words $B(\alpha_j,\gamma_j)$ partition I_r.

A (*sub*)*word* u of a word W (notation: $u \in W$) is a concatenation of consecutive letters of W. If $u \in W$, then $|u|$ denotes the number of letters in u, counting repetitions. For example, $2625 \in 872625119$, and $|2625| = 4$. A word u is *finite* if $|u| < \infty$; otherwise it is infinite. If W is a binary word, over $\{0,1\}$, we define the *weight* w of $u \in W$ to be $w(u) = \sum_{a_i \in u} a_i =$ number of 1-bits in u. For an alphabet A and a finite word u over A, we denote by $|u|_p$ the number of occurrences of $p \in A \cap u$.

Definition 1 A word W over a finite alphabet A is *balanced* if for every finite subwords $u, v \in W$ with $|u| = |v|$ we have $||v|_p - |u|_p| \leq 1$ for every $p \in A$.

When $A = \{0, 1\}$, it is convenient to replace the condition $||v|_p - |u|_p| \leq 1$ by $|w(v) - w(u)| \leq 1$. This is clearly equivalent to $||v|_0 - |u|_0| \leq 1$, and so the weight condition is an equivalent condition for balance when $A = \{0, 1\}$.

Definition 2 A number is a *proper* rational, if it has the form p/q, $p, q \in \mathbb{Z}$, $\gcd(p, q) = 1$, with $q \geq 2$.

In the following definition, t is a real number.

Definition 3 Let χ_k, $k \in I_r$ be a binary word, $|I_r| = \infty$. We say that χ has *density* t (notation: $d(\chi) = t$) if for every prefix $I'_r \subseteq I_r$,

$$\lim_{|I'_r| \to \infty} w(\chi \cap I'_r)/|I'_r| = t$$

when the limit exists.

Note. Informally, the density is the percentage of the 1-bits in χ. Similarly, the density of V^j is the percentage of $(2j-1)$-digits in it. If χ is periodic with period p and $|I'_r| = p$, then it is easy to see that $d(\chi) = w(\chi \cap I'_r)/p$.

In the next section we present our main results, together with two examples (Tables 1 and 2). In §3 we prove various auxiliary results which reveal connections between Beatty, Sturmian, balanced, characteristic and almost linear (defined in §3) words. The objective is to show that given any binary balanced word χ, there exists a Beatty word B such that χ is its characteristic word. Almost linear and Sturmian words serve as intermediaries. We can then glean the properties of binary balanced complementary words from those of complementary Beatty words. Instead of binary balanced words χ, we can, equivalently, determine the structure of the balanced words V, or of others, over suitable finite alphabets. In the final §4 we prove the theorems enunciated in §2.

2 Main results

The following are our main results, where in Theorem 1, $j \in \{1, \ldots, m\}$ is intended throughout.

Theorem 1 *Let $\chi^1, \ldots, \chi^m$ be binary words.*

Then the words V^j are balanced and partition R
$\iff$ *the words χ^j are balanced and partition* $\mathbf{I}$
$\iff$ *$\exists$ Beatty words $B(\alpha_j, \gamma_j)$ which partition I_r,*
where χ^j is the characteristic word of $B(\alpha_j, \gamma_j)$
and $d(\chi^j) = d(V^j) = \alpha_j^{-1}$.

Moreover, for any finite subinterval $I'_r \subseteq I_r$, and every j, the computation of the admissible α_j and γ_j in $B(\alpha_j, \gamma_j)$, given the word χ^j, can be done in linear time in $|I'_r|$.

An example illustrating Theorem 1 is depicted in Table 1, where $m = 3$, $\alpha_1 = \sqrt{3}/(\sqrt{3}-1)$, $\alpha_2 = \alpha_3 = 2\sqrt{3}$, $\gamma_1 = \gamma_2 = 0$, $\gamma_3 = -\sqrt{3}$, $d(\chi^2) = d(\chi^3)$, $R_k = \sum_{j=1}^{m} V_k^j = 7$ for all $k \in I_r$, and $F_k = j$ iff $\chi_k^j = 1$.

Table 1: Complementary balanced words for $\alpha_1 = \sqrt{3}/(\sqrt{3}-1)$, $\alpha_2 = \alpha_3 = 2\sqrt{3}$, $\gamma_1 = \gamma_2 = 0$, $\gamma_3 = -\sqrt{3}$. Here B^i denotes $B(\alpha_i, \gamma_i)$.

n	B^1	B^2	B^3	T^1	T^2	T^3	χ^1	χ^2	χ^3	V^1	V^2	V^3	F
1	2	3	1	0	0	1	0	0	1	0	2	5	3
2	4	6	5	2	0	0	1	0	0	1	2	4	1
3	7	10	8	0	3	0	0	1	0	0	3	4	2
4	9	13	12	4	0	0	1	0	0	1	2	4	1
5	11	17	15	0	0	5	0	0	1	0	2	5	3
6	14	20	19	0	6	0	0	1	0	0	3	4	2
7	16	24	22	7	0	0	1	0	0	1	2	4	1
8	18	27	25	0	0	8	0	0	1	0	2	5	3
9	21	31	29	9	0	0	1	0	0	1	2	4	1
10	23	34	32	0	10	0	0	1	0	0	3	4	2
11	26	38	36	11	0	0	1	0	0	1	2	4	1
12	28	41	39	0	0	12	0	0	1	0	2	5	3
13	30	45	43	0	13	0	0	1	0	0	3	4	2
14				14	0	0	1	0	0	1	2	4	1
15				0	0	15	0	0	1	0	2	5	3
16				16	0	0	1	0	0	1	2	4	1
17				0	17	0	0	1	0	0	3	4	2
18				18	0	0	1	0	0	1	2	4	1
19				0	0	19	0	0	1	0	2	5	3
20				0	20	0	0	1	0	0	3	4	2
21				21	0	0	1	0	0	1	2	4	1
22				0	0	22	0	0	1	0	2	5	3
23				23	0	0	1	0	0	1	2	4	1
24				0	24	0	0	1	0	0	3	4	2
25				0	0	25	0	0	1	0	2	5	3
26				26	0	0	1	0	0	1	2	4	1
27				0	27	0	0	1	0	0	3	4	2
28				28	0	0	1	0	0	1	2	4	1
29				0	0	29	0	0	1	0	2	5	3
30				30	0	0	1	0	0	1	2	4	1
31				0	31	0	0	1	0	0	3	4	2

Theorem 2 *Suppose that the binary balanced words* $\{\chi^j : 1 \leq j \leq m\}$, $m \geq 3$, *partition* $\mathbf{I}$. *Then there exist words* χ^i, χ^j, V^i, V^j, $i \neq j$ *with identical density, except when the* χ^j *are characteristic words of Beatty words with moduli*

$$\alpha_j = (2^m - 1)/2^{m-j}, \qquad j \in \{1, \ldots, m\}, \tag{2}$$

and possibly in other cases where one of the χ^h *is the characteristic word of a Beatty word* $B(\alpha_h, \gamma_h)$ *such that* α_h *is a proper rational number. In the exceptional case* (2):
(i) $d(\chi^i)/d(\chi^j) = d(V^i)/d(V^j)$ *is a nonzero power of* 2 *for all* $i, j \in \{1, \ldots, m\}$, $i \neq j$.
(ii) *The word* F *is the palindrome* $F_{n+1} = F_n(n+1)F_n$, *for all* $n \in \mathbb{Z}_{\geq 1}$, *where* $F_1 = 1$.

The case (2) is the *only* exception if m is small:

Theorem 3 *Suppose that the binary balanced words* $\{\chi^j : 1 \leq j \leq m\}$, $3 \leq m \leq 7$, *partition* $\mathbf{I}$. *Then there exist two words* χ^i, χ^j, $i \neq j$ *with identical density, except when the* χ^j *are characteristic words of Beatty words with moduli* (2). *In the exceptional case* (2), (i) *and* (ii) *of Theorem* 2 *hold.*

Table 2 depicts an example for Theorems 2 and 3, where $\alpha_j = (2^m - 1)/2^{m-j}$, $\gamma = -2^{j-1} + 1$, $j \in \{1, \ldots, m\}$, $1 \leq j \leq m$, $m = 3$. Note that the word F is a palindrome.

Table 2: $a_j = (2^m - 1)/2^{m-j}, \gamma_j = -2^{j-1} + 1$, $j = 1, \ldots, m$, $m = 3$.

n	$\lfloor n\frac{7}{4} \rfloor$	$\lfloor n\frac{7}{2} \rfloor - 1$	$n7 - 3$	F(palindr)
1	1	2	4	1
2	3	6	11	2
3	5	9	18	1
4	7	13	25	3
5	8	16	32	1
6	10	20	39	2
7	12	23	46	1

Notes

- The case $m = 3$ of a theorem similar to Theorem 3 was given by Tijdeman [24]. The palindromic structure of the word F appears also in Vuillon [27], Conjecture 1.
- On November 10, 2004, I had a conversation with A. J. Belov and A. L. Chernyatiev. I don't know what their results are, but they described to me complementary non periodic Sturmian words in terms of dynamical

systems, sketching their idea in the form of layers of spinning roulette wheels which are colored with two main colors, and there are various shades of these colors. I told them that the two main colors correspond to two irrational moduli α_1, α_2 satisfying $\alpha_1^{-1}+\alpha_2^{-1} = 1$, and the color shades are the integer multiples of these two irrationals (see also (3) below). A countable set is excluded by them. This turns out to be the set of words generated by integer and rational moduli. In a sense, the case of rational moduli is the most interesting, since much is known about the irrational case, much about the integer case, yet next to nothing about the proper rational case — a rather humbling fact. The nice conversation with Belov and Chernyatiev was the motivation for this paper.

- There is some confusion in the literature about the notions of Sturmian, characteristic and Beatty words. In [24] Sturmian words are words on any finite alphabet. In [1], Ch. 9, they are defined similarly to the definition given above (only for $0 < \alpha < 1$), but only when there is a γ. When $\gamma = 0$ they are called characteristic words. Our characteristic word is also defined there, but no name is given to it. In the authoritative [17], Ch. 2, Sturmian words are binary, and some equivalent definitions of them are given. Also in [27] they are binary. In many papers, some of these notions are given without definition, and the reader has to discern the meaning from the context. Therefore we defined them above, tailored to our use here.

3 Beatty, Sturmian, almost linear, balanced and characteristic words

Definition 4 A word of integers $\{a_n\}_{n\in I_d}$ is said to be *almost linear* if

$$|(a_{n+t} - a_n) - (a_{m+t} - a_m)| \leq 1 \quad (\forall\ m, n, m+t, n+t \in I_d).$$

In [6] it was shown, inter alia,

Theorem I *A word of integers $\{a_n\}_{n\in I_d}$ is almost linear iff it is a Beatty word $B(\alpha, \gamma)$.*

Note that Theorem I is also a sort of balance result. It was shown by Graham et al. [13] (where the terminology "almost linear" was used) for the homogeneous case, but they left the case $\gamma \neq 0$ as an open problem. Given an almost linear word $\{a_n\}_{n\in I_d}$, and any finite subinterval $I'_d \subseteq I_d$, the computation of the admissible numbers α and γ of the Beatty word $B(\alpha, \gamma)$ with values a_i can be done in time linear in $|I'_d|$, by computing the continued fraction expansion of the extreme values of the admissible α [7].

There is a indeed an intimate connection between almost linearity and balance.

Lemma 1 *A Beatty word $B(\alpha,\gamma)=\{a_m\}_{m\in I_d}$ is almost linear iff its associated Sturmian word $F=\{f_n(\alpha,\gamma)\}_{n\in I_d}$ is balanced.*

Proof. For $m,n,t,m+t,n+t\in I_d$ we get, by telescoping terms, $|\sum_{i=n}^{n+t-1} f_i - \sum_{i=m}^{m+t-1} f_i| = |(a_{n+t}-a_n)-(a_{m+t}-a_m)|$. Thus $B(\alpha,\gamma)$ is almost linear iff F is balanced. ■

If $0<\alpha\le 1$, then the characteristic word $\{\chi_n\}_{n\in I_d}$ of the Beatty word with modulus α is trivially balanced, since it is 1 for all $n\in I_d$. Note that if α is irrational, γ real, then there is at most one integer n_0 such that $n_0\alpha+\gamma$ is an integer. (This can also be seen geometrically, by drawing a straight line in the plane with inclination α through the point $(\gamma,0)$.)

There is also a connection between the characteristic word of a Beatty word and the Sturmian word of a related Beatty word. This is shown in Lemmas 2 and 4 for the irrational and rational cases respectively.

Lemma 2 *Let $\alpha>1$ be irrational, γ real, $B(\alpha,\gamma)=\lfloor n\alpha+\gamma\rfloor$, $n\in I_d$. Then $\chi_k(\alpha,\gamma)=f_k(\alpha^{-1},-\gamma\alpha^{-1})$ for all $k\in I_r(B)$, except that if there exists $n_0\in I_d(B)$ such that $n_0\alpha+\gamma=k_0\in I_r$, then $\chi_{k_0-1}(\alpha,\gamma)=0$, $\chi_{k_0}(\alpha,\gamma)=1$, $f_{k_0-1}(\alpha^{-1},-\gamma\alpha^{-1})=1$, $f_{k_0}(\alpha^{-1},-\gamma\alpha^{-1})=0$. Moreover, if I_d is infinite, then $\chi_k(\alpha,\gamma)$ is an aperiodic word.*

Proof. For $k\in I_r(B)$ we have $\chi_k(\alpha,\gamma)=1$

$$\begin{aligned}
&\iff \exists\, n\in I_d \text{ such that } k=\lfloor n\alpha+\gamma\rfloor\\
&\iff \exists\, n\in I_d \text{ such that } k\le n\alpha+\gamma<k+1\\
&\iff \exists\, n\in I_d \text{ such that } k\alpha^{-1}-\gamma\alpha^{-1}\le n<(k+1)\alpha^{-1}-\gamma\alpha^{-1}\\
&\iff \textbf{either } \exists\, n\in I_d \text{ such that } \lfloor k\alpha^{-1}-\gamma\alpha^{-1}\rfloor=n-1,\\
&\qquad \lfloor (k+1)\alpha^{-1}-\gamma\alpha^{-1}\rfloor=n \ \ (\text{if } k<n\alpha+\gamma) \iff f_k(\alpha^{-1},-\gamma\alpha^{-1})=1\\
&\qquad\ \textbf{or } \exists\, n\in I_d \text{ such that } k\alpha^{-1}-\gamma\alpha^{-1}=n<(k+1)\alpha^{-1}-\gamma\alpha^{-1} \iff\\
&\qquad f_{k-1}(\alpha^{-1},-\gamma\alpha^{-1})=1,\ f_k(\alpha^{-1},-\gamma\alpha^{-1})=0 \text{ (since } \alpha^{-1}<1),\\
&\qquad \chi_{k-1}(\alpha,\gamma)=0,\ \chi_k(\alpha,\gamma)=1.
\end{aligned}$$

The last part of the lemma's assertion follows from the facts that $\alpha>1$ and that $\{\{n\alpha+\gamma\}: n\in I_d,\ |I_d|=\infty\}$ is dense in $[0,1)$ [15], Ch. 23, where $\{x\}=x-\lfloor x\rfloor$ denotes the fractional part of x. ■

This result (without the last part) was proved by Allouche and Shallit [1], Ch. 3, for the case $\gamma=0$. They also assumed $I_d=\mathbb{Z}_{\ge 1}$, so the special case $n\alpha+\gamma\in\mathbb{Z}$ didn't arise there.

It almost follows from Lemmas 2 and 1 and Theorem I, that the characteristic word χ of a Beatty word with α irrational is balanced, since F is balanced, but there is the exceptional case mentioned in Lemma 2, in which χ deviates from F in 2 consecutive points. We now show that F remains balanced even when we transpose these 2 values, so χ is balanced over the entire range.

Lemma 3 *For $B = \lfloor n\alpha + \gamma \rfloor$, $\alpha > 1$, suppose that there exists $n_0 \in I_d(B)$ and $k_0 \in I_r(B)$ such that $n_0 = k_0\alpha^{-1} - \gamma\alpha^{-1}$. Put $f'_k(\alpha^{-1}, -\gamma\alpha^{-1}) = f_k(\alpha^{-1}, -\gamma\alpha^{-1})$ for all $k \in I_r$, except that $f'_{k_0-1}(\alpha^{-1}, -\gamma\alpha^{-1}) = 0$, $f'_{k_0}(\alpha^{-1}, -\gamma\alpha^{-1}) = 1$ (a transposition of the values f_{k_0-1} and f_{k_0}). Then $F' = \{f'_k\}_{k \in I_r}$ is also balanced.*

Proof. Let $u, v \in F'$, $|u| = |v|$, $k_{um} = \min\{k \in I_r : \lfloor (k+1)\alpha^{-1} - \gamma\alpha^{-1} \rfloor - \lfloor k\alpha^{-1} - \gamma\alpha^{-1} \rfloor \in u\}$, $k_{uM} = \max\{k \in I_r : \lfloor (k+1)\alpha^{-1} + \gamma\alpha^{-1} \rfloor - \lfloor k\alpha^{-1} + \gamma\alpha^{-1} \rfloor \in u\}$; k_{vm}, k_{vM} are defined analogously. Put $U = [k_{um}, k_{uM}] \cup [k_{vm}, k_{vM}]$. We may assume that $k\alpha^{-1} - \gamma\alpha^{-1} = n_0$ for some $k \in U$, because otherwise $u \cup v \in F$, which is balanced.

Let $\delta = \min\{\{k\alpha^{-1} - \gamma\alpha^{-1}\} : k \in U\}$, $0 < \varepsilon < \delta$. Note that whereas $\lfloor k\alpha^{-1} - \gamma\alpha^{-1} - \varepsilon \rfloor = \lfloor k\alpha^{-1} - \gamma\alpha^{-1} \rfloor$ for all $k \in U$, $k \notin \{k_0, k_0 - 1\}$, we have $\lfloor k_0\alpha^{-1} - \gamma\alpha^{-1} - \varepsilon \rfloor = \lfloor (k_0 - 1)\alpha^{-1} - \gamma\alpha^{-1} - \varepsilon \rfloor = n_0 - 1$ (since $\alpha > 1$). This implies $f_{k_0-1}(\alpha^{-1}, -\gamma\alpha^{-1} - \varepsilon) = 0 = f'_{k_0-1}(\alpha^{-1}, -\gamma\alpha^{-1})$; and also $f_{k_0}(\alpha^{-1}, -\gamma\alpha^{-1} - \varepsilon) = 1 = f'_{k_0}(\alpha^{-1}, -\gamma\alpha^{-1})$.

Now $F(\alpha^{-1}, -\gamma\alpha^{-1} - \varepsilon)$ is balanced by Theorem I and Lemma 1. Hence also F' is balanced. ■

Note that if $\alpha = p/q$ is rational $(p, q \in \mathbb{Z}_{\geq 1},\ \gcd(p, q) = 1)$, then we may assume, without loss of generality, that $\gamma = t/q$, $\quad t \in \mathbb{Z}$.

Lemma 4 *Let $p, q, t \in \mathbb{Z}$, $p > q \geq 1$, $|I_d| \geq p$, and consider the Beatty word $\lfloor np/q + t/q \rfloor$, $n \in I_d$. Then $\chi_k(p/q, t/q) = f_k(q/p, -(t+1)/p)$ for all $k \in I_r$. Moreover, $\chi_k(p/q, t/q)$ is periodic with period p.*

Proof. For $k \in I_r$,

$$\begin{aligned}
\chi_k(p/q, t/q) = 1 \quad &\iff \quad \exists\, n \in I_d \text{ such that } k = \lfloor np/q + t/q \rfloor \\
&\iff \quad \exists\, n \in I_d \text{ such that } k \leq np/q + t/q \leq k + 1 - 1/q \\
&\iff \quad \exists\, n \in I_d \text{ such that } kq/p - t/p \leq n \leq (k+1)q/p - (t+1)/p \\
&\iff \quad \lfloor (k+1)q/p - (t+1)/p \rfloor - \lfloor kq/p - (t+1)/p \rfloor = 1 \\
&\iff \quad f_k(q/p, -(t+1)/p) = 1.
\end{aligned}$$

The last part of the lemma's assertion follows from $\lfloor (n+q)p/q + t/q \rfloor = \lfloor np/q + t/q \rfloor + p$. ■

Corollary 1 *The characteristic word of any Beatty word is balanced.*

Proof. The characteristic word of any Beatty word is identical to the Sturmian word of another Beatty word by Lemmas 2 and 4 with one exception. Every Sturmian word is balanced by Theorem I and Lemma 1. Also in the exceptional case of Lemma 2, the characteristic word is still balanced by Lemma 3. ■

Corollary 2 (i). *Let $G = \{g_n\}_{n \in I_r}$ be a binary aperiodic balanced word. Then there exists a Beatty word $B(\alpha, \gamma)$, α irrational, such that G is its characteristic word χ, except that if there exists $n_0 \in I_d$ such that $n_0\alpha + \gamma = k_0 \in I_r$, then the values $(g_{n_0-1}, g_{n_0}) = (1, 0)$ have to be transposed to $(0, 1)$. The balance of χ is invariant under this transposition.*

(ii). *Let $G = \{g_n\}_{n \in I_r}$ be a binary periodic balanced word. Then there exists a Beatty word $B(\alpha, \gamma)$, α rational, such that G is its characteristic word χ.*

(iii). *Let $G = \{g_n\}_{n \in I_r}$ be a binary balanced word and $B(\alpha, \gamma)$ a Beatty word with characteristic word G. If $|I_r| = \infty$, then G has a density and its value is α^{-1}.*

Proof. (i) Define a word $A = \dots a_i a_{i+1} a_{i+2} \dots$ by $g_n = a_{n+1} - a_n$. For $m, n, t, m+t, n+t \in I_d$ we get, by telescoping terms, $|\sum_{i=n}^{n+t-1} g_i - \sum_{i=m}^{m+t-1} g_i| = |(a_{n+t} - a_n) - (a_{m+t} - a_m)|$. Since G is balanced, A is almost linear. Hence by Theorem I, there exists a modulus and a shift, which we like to denote by α^{-1} and $-\gamma\alpha^{-1}$ respectively, such that $A = \{\lfloor n\alpha^{-1} - \gamma\alpha^{-1} \rfloor : n \in I_d\}$. Its associated Sturmian word is clearly G.

Since G is aperiodic, it contains a 0, say $g_j = 0$. Then $a_{j+1} - a_j = g_j = 0$, which implies that $\alpha > 1$. It then follows from Lemma 2 that $\chi_k(\alpha, \gamma) = g_k(\alpha^{-1}, -\gamma\alpha^{-1})$ for all $k \in I_r(B)$, except that possibly 2 consecutive values of G have to be transposed. The last part of (i) follows from Lemma 3.

(ii) By Theorem I, there exists a modulus and a shift, which we like to denote by q/p and $-(t+1)/p$ respectively, where $p, q, t \in \mathbb{Z}$, such that $A = \{\lfloor nq/p - (t+1)/p \rfloor : n \in I_d\}$. Its associated Sturmian word is clearly G.

If G consists of 0s only, then $\alpha = 0$ will do, and if G consists of 1s only, then $\alpha = 1$. Otherwise, there is a 1 and a 0, and so we conclude, analogously to (i), that $p > q \geq 1$. From Lemma 4 it follows that $\chi_k(p/q, t/q) = g_k(q/p, -(t+1)/p)$ for all $k \in I_r(B)$.

(iii) The existence of $B(\alpha, \gamma)$ follows from (i), (ii). Suppose first that α is irrational. Clearly, $d(\chi_k) = (|I'_r| - \gamma)/\alpha + c$ for $k \in I_r$, where $|c| \leq 2$ and $I'_r \subseteq I_r$. Dividing by $|I'_r|$ gives $(1 - \gamma/|I'_r|)/\alpha + c/|I'_r|$ which tends to α^{-1} as $|I'_r|$ tends to infinity. Now suppose that $\alpha = p/q$ is rational with $\gcd(p, q) = 1$. Then G has period p by Lemma 4. It is easy to see that every interval of length p contains q 1-bits, from which the result follows. ■

The sense of Corollary 2 is that the structure of binary balanced complementary words can be culled from the structure of complementary Beatty words. There is some information about the latter, which will be exploited in the next section.

4 Proof of the main results

Proof of Theorem 1.

The words χ^j partition **I**

$$\begin{aligned}
&\iff \sum_{j=1}^{m} \chi_k^j = 1 \;\; \forall k \in I_r \\
&\iff V^i = 2i-1 \text{ for precisely one } i, \text{ whereas } V^j = 2j-2 \;\; \forall j \neq i \\
&\iff \sum_{j=1}^{m} V_k^j = m^2 - m + 1 \;\; \forall k \in I_r .
\end{aligned}$$

The last line follows from the identity $\sum_{j=1}^{m}(2j-1) = m^2$. In this proof balance was not used. However, it follows directly from Definition 1 and the definition (1) of the words V that the χ^j are balanced iff the V^j are balanced.

Suppose that the words χ^j partition $\mathbf{I}$. For every fixed j, χ^j balanced implies, by Corollary 2(i),(ii), that there exists a Beatty word $B(\alpha_j, \gamma_j)$ such that χ^j is its characteristic word. If $B(\alpha_i, \gamma_i) \cap B(\alpha_j, \gamma_j) = k$ for some $k \in I_r$ and $i, j \in I_d$, $i \neq j$, then $\chi_k^i = \chi_k^j = 1$, contradicting the hypothesis that the χ^j partition $\mathbf{I}$. If $k \notin \cup_{j=1}^{m} B(\alpha_j, \gamma_j)$ for some $k \in I_r$, then $\chi_k^j = 0$ for all $j \in \{1, \ldots, m\}$, contradicting again the hypothesis that the χ^j partition $\mathbf{I}$.

Corollary 2(iii) implies that χ^j has density α_j^{-1}.

Now suppose that the Beatty words $B(\alpha_j, \gamma_j)$ partition I_r. For every fixed j, the characteristic word χ^j of the Beatty word $B(\alpha_j, \gamma_j)$ is balanced by Corollary 1. If $\chi_k^i = \chi_k^j = 1$ for some $k \in I_r$ and $i, j \in I_d$, $i \neq j$, then there exist $n_1, n_2 \in I_d$ such that $B_{n_1}(\alpha_i, \gamma_i) = B_{n_2}(\alpha_j, \gamma_j) = k$, contradicting the complementarity of the words $B(\alpha_j, \gamma_j)$. If there exists $k \in I_r$ such that $\chi_k^j = 0$ for all $j \in \{1, \ldots, m\}$, then $k \notin \cup_{j=1}^{m} B(\alpha_j, \gamma_j)$, again contradicting the complementarity of the words $B(\alpha_j, \gamma_j)$. The density claim follows from Corollary 2.

Note that χ^j can be translated into T^j in time linear in $|I'_r|$, yielding an almost linear word. Applying the $O(|I'_r|)$ algorithm [7] to this almost linear word yields the ranges of the corresponding admissible α and γ. ■

Proof of Theorem 2 By Theorem 1, there exist Beatty words $B(\alpha_j, \gamma_j)$ which partition I_r. Let us examine their moduli. If $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_m$ are all integers, $m \geq 2$, then $\alpha_{m-1} = \alpha_m$. This has been proved using generating functions and complex numbers, see Erdös [8], Znám [28], Newman [19], also known as the Davenport-Rado-Mirsky-Newman result. A first elementary proof was given in [5]. In the integer case we denote the α_j, γ_j by $a_i, b_i \in \mathbb{Z}$ respectively, and the $B(\alpha_j, \gamma_j)$ by $E(a_j, b_j)$, called *exactly covering words* ECW for short, whereas the case of a complementary system with an irrational α^j or a proper rational will be called an *exactly covering family*, or ECF.

By Kronecker's Theorem [15], Ch. 23, if one of the α_j is irrational, then the complementarity of the $B(\alpha_j, \gamma_j)$ implies that all are irrational. The characterization of 2 complementary Beatty words $\{B(\alpha_j, \gamma_j) : 1 \leq j \leq 2\}$ was given in [10], see also O'Bryant [20]. For the irrational case, $\alpha_1^{-1} + \alpha_2^{-1} = 1$ and $\gamma_1/\alpha_1 + \gamma_1 2/\alpha_2 = t \in \mathbb{Z}$ are required, and if $n_0\alpha_1 + \gamma_1 = s \in \mathbb{Z}$, then $s \notin I_d$.

Now if $\{B(\alpha_1, \gamma_1), B(\alpha_2, \gamma_2)\}$ is an ECF and $\{E(a_j, b_j) : 1 \leq j \leq m\}$,

$\{E(a'_j, b'_j) : 1 \leq j \leq m'\}$ are ECWs, then the superposition

$$\{\cup_{j=1}^{m} B(a_j\alpha_1, b_j\alpha_1 + \gamma_1)\} \cup \{\cup_{j=1}^{m'} B(a'_j\alpha_2, b'_j\alpha_2 + \gamma_2)\} \tag{3}$$

is clearly an ECF. Graham [12] showed that conversely, any ECF in which some α_j is irrational, is always the superposition of an ECF with 2 irrational moduli with an ECW, so it is of the form (3). It follows, from the integer case, that if $m + m' \geq 3$, then any ECF always contains two moduli which are the same. Since $d(\chi^j) = d(V^j) = \alpha_j^{-1}$ by Theorem 1, $d(\chi^i) = d(\chi^j) = d(V^i) = d(V^j)$ for some $i \neq j$.

For the proper rational case, however, there is the ECF with distinct moduli $\alpha_j = (2^m - 1)/2^{m-j}$, $j \in \{1, \ldots, m\}$, see [11] (if $I_d = \mathbb{Z}_{\geq 1}$, we can take $\gamma_j = -2^{j-1} + 1$). In this case the densities of all the binary balanced words are distinct, though their ratios are integers or reciprocals of integers. For the proof of (ii) see Tijdeman [26] Theorem. See also Vuillon [27] Conjecture 1. ■

Proof of Theorem 3 Two conjectures were formulated in [11] (see also Erdös and Graham [9]).

Conjecture 1 *If the words $B(\alpha_j, \gamma_j)$, $j \in \{1, \ldots, m\}$, partition I_r and $m \geq 3$, then $\alpha_i/\alpha_j \in \mathbb{Z}$ for some $i, j \in \{1, \ldots, m\}$, $i \neq j$.*

Conjecture 2 *Let $\alpha_1 < \alpha_2 < \ldots < \alpha_m$ be positive real numbers, and let $m \geq 3$. If there are real numbers $\gamma_1, \gamma_2, \ldots, \gamma_m$ such that the words $B(\alpha_j, \gamma_j)$ partition I_r, then $\alpha_j = (2^m - 1)/2^{m-j}$, $j \in \{1, \ldots, m\}$.*

Note that Conjecture 2 implies Conjecture 1. Morikawa dealt with the conjectures in a series of papers during 1982 – 1995. In [16] he classified all rational ECFs with $m = 3$. This classification confirms the truth of Conjecture 2 for $m = 3$. In [18] he did a similar classification for a certain subclass of $m = 4$, verifying Conjecture 2 for this subclass. Altman et al. [2] established it for $m = 4$, and Tijdeman [25], [26] for $m = 5$ and 6. Barát and Varjú [4] extended it to $m = 7$. It follows that Conjecture 2 has been established for $3 \leq m \leq 7$. ■

We also mention that Simpson [21] established the truth of Conjecture 2 if $\min_j \alpha_j \leq 3/2$. See also Simpson [22]. An approach to prove Conjecture 2 by generalizing it to exact multi-coverings was recently taken by Graham and O'Bryant [14]. In [11], Conjecture 1 was proved for $m = 3$ and $m = 4$, as well as in special cases for $m \geq 5$. If Conjecture 2 is true, then of course in Theorem 2 there is no exception beyond its case (2), i.e., Theorem 3 holds for all $m \geq 3$.

In Theorems 2 and 3 we assumed $m \geq 3$. For $m = 2$ the result $d(\chi^1) = d(\chi^2)$, is valid only in the integer case, where it is trivial, i.e., $\alpha_1 = \alpha_2 = 2$.

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