Theoretical Computer Science 29 (1984) 49–73
North-Holland

WYTHOFF GAMES, CONTINUED FRACTIONS, CEDAR TREES AND FIBONACCI SEARCHES

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Communicated by M. Nivat
Received February 1983
Revised November 1983

Abstract. Recursive, algebraic and arithmetic strategies for winning generalized Wythoff games in misère play are given. The notion of cedar trees, a subset of binary trees, is introduced and used for consolidating these and the normal play strategies. A connection to generalized Fibonacci searches is indicated.

1. Introduction

Let a be a positive integer. Given two piles of tokens, two players move alternately in a generalized Wythoff game. The moves are of two types: a player may remove any positive number of tokens from a *single* pile, or he may take from both piles, say k (>0) from one and l (>0) from the other, provided that $|k-l|<a$. Note that passing is not allowed: each player at his turn has to remove at least one token. In *normal* play, the player first unable to move is the loser, his opponent the winner. In *misère* play, the outcome is reversed: the player first unable to move is the winner, his opponent the loser.

In this paper we show how to beat our adversary recursively, algebraically and arithmetically in misère play, analogously to the three strategies given in [3] for normal play. In addition we introduce the notion of *cedar trees* and use it to consolidate the strategies of normal play and of misère play. This permits us to beat our adversary in both normal and misère play from the top of a single cedar tree. A connection between cedar trees and generalized Fibonacci searches is also indicated.

The classical Wythoff game (see, e.g., Wythoff [9] or Yaglom and Yaglom [10]) is the normal play version for the parameter choice $a=1$, that is a player taking from both piles has to take the *same* number from both. Denote by S_1 and S_2 the previous-player-winning positions of normal and misère play respectively. Our results imply, in particular, the interesting fact that S_1 is identical to S_2 except for

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the first two positions when $a=1$ (for which case the game is *tame* in the sense of Berlekamp, Conway and Guy [1]), whereas $S_1 \cap S_2 = \emptyset$ for every $a>1$.

The recursive and algebraic characterizations of the previous-player-winning positions are presented in Sections 2 and 3 respectively. Some prerequisite results on continued fractions and systems of numeration are briefly presented in Section 4. These results are used for giving the arithmetic characterization of the previous-player-winning positions in Section 5. In Section 6 the notion of cedar trees is introduced, and in the final section 7 it is used for consolidating normal and misère play strategies.

Notation

Unless otherwise specified, we assume misère play. Game positions are denoted by (x, y) with $x \le y$, where x denotes the number of tokens in one pile and y the number of tokens in the other pile. Positions from which the Previous player can win whatever move his opponent will make are called *P-positions*, and those from which the Next player can win whatever his opponent will make are called *N-positions*. Thus $(0, 1)$ is a P-position for every a, because the Next player has to move to $(0, 0)$ and so Previous wins; $(1, b)$, $b > 1$ is an N-position for every a: the Next player moves to $(0, 1)$ and wins. For $a=2$, the position $(2, 5)$ is a P-position: if Next moves to $(0, 2)$, $(0, 3)$, $(0, 4)$, $(0, 5)$, $(1, 2)$, $(1, 3)$, $(1, 4)$ or $(1, 5)$, then Previous, using a move of the first type, moves to $(0, 1)$ and wins. If Next moves to $(2, 2)$, $(2, 3)$ or $(2, 4)$, then Previous, using a move of the second type, can again move to $(0, 1)$.

The set of all P-positions is denoted by P, and the set of all N-positions by N.

2. Recursive characterization of the *P*-positions

A list of the first few P-positions (E_n, H_n) for the cases $a=1$ and $a=3$ is displayed in Tables 1 and 2. The tables have an interesting structure. First note that (at least for $n \le 11$), $H_n - E_n = n$ in Table 1, and $= 3n+1$ in Table 2. It is a bit harder to notice that $E_n = \text{mex}\{E_i, H_i : 0 \le i < n\}$ for both, where, for any set S, if $\bar{S}$ denotes the complement of S with respect to the nonnegative integers, then mex $S = \min \bar{S} =$ least nonnegative integer not in S (mex stands for *m*inimum *ex*cluded value). Thus mex $\emptyset = 0$. If we define (E_n, H_n) in the indicated manner for all n, then $(E_{12}, H_{12}) = (19, 31)$ for $a=1$ and $(16, 53)$ for $a=3$.

We now prove that the pairs (E_n, H_n) constitute the set P of P-positions for every $n \ge 0$.

Theorem 2.1. *The P-positions for misère Wythoff games are the following:*

(i) *For* $a=1$: $(E_0, H_0) = (2, 2)$.

$$E_n = \text{mex}\{E_i, H_i : 0 \le i < n\}, \quad H_n = E_n + n \qquad (n \ge 1).$$

Table 1
The first few P-positions of the misère Wythoff game for $a = 1$.

n	E_n	H_n
0	2	2
1	0	1
2	3	5
3	4	7
4	6	10
5	8	13
6	9	15
7	11	18
8	12	20
9	14	23
10	16	26
11	17	28

Table 2
The first few P-positions for the misère Wythoff game for $a = 3$.

n	E_n	H_n
0	0	1
1	2	6
2	3	10
3	4	14
4	5	18
5	7	23
6	8	27
7	9	31
8	11	36
9	12	40
10	13	44
11	15	49

(ii) *For* $a > 1$:

$$E_n = \text{mex}\{E_i, H_i \colon 0 \leq i < n\}, \quad H_n = E_n + an + 1 \qquad (n \geq 0).$$

Proof. From the definition of E_n and H_n as given in the theorem it follows that if $E = \bigcup_{n=0}^{\infty} E_n$ and $H = \bigcup_{n=0}^{\infty} H_n$, then, for every $a > 1$, E and H are *complementary* sets of numbers, that is, $E \cup H = Z^0$ (the set of nonnegative integers), and $E \cap H = \emptyset$. The last equality is true since if $E_n = H_m$, then $n > m$ implies that E_n is the mex of a set containing $H_m = E_n$, a contradiction; and $n \leq m$ is impossible since $H_m = E_m + am + 1 \geq E_n + an + 1 > E_n$. For $a = 1$, E and H are *covering* sets, that is, $E \cup H = Z^0$. (In fact, $E \cap H = \{2\}$ is easily proved as above.) Thus E and H are covering for every $a \geq 1$.

In order to prove the theorem it evidently suffices to show two things: (i) A player moving from some (E_n, H_n) lands in a position not of the form (E_i, H_i). (ii) Given any position $(x, y) \neq (E_i, H_i)$ (except for $(x, y) = (0, 0)$), there is a move to some (E_n, H_n). (It is useful to note that these two conditions are also necessary: the definition of P and N implies that *all* positions reachable in one move from a P-position are N-positions; whereas at least one P-position is reachable in one move from an N-position.)

(i) A move of the first type from (E_n, H_n) clearly produces a position not of the form (E_i, H_i). Suppose that a move of the second type from (E_n, H_n) produces a position (E_i, H_i). Then $i \neq n$. A move of the second type satisfies

$$|(H_n - H_i) - (E_n - E_i)| = |(H_n - E_n) - (H_i - E_i)| = |(n - i)a| < a,$$

which implies $i = n$, a contradiction.

(ii) Let (x, y) with $x \leq y$ be a position not of the form (E_i, H_i) $(i \geq 0)$. If $(x, y) = (0, 0)$, then Next wins without doing anything. So we may assume $(x, y) \neq (0, 0)$.

Since E and H are covering, every nonnegative integer appears in one of $\{E_n\}$ or $\{H_n\}$. Therefore either $x = H_n$ or $x = E_n$ for some $n \geq 0$.

Case 1. $x = H_n$. Then move $y \to E_n$. (This move always exists since $y \geq x = H_n \geq E_n$, and at least one inequality is strict.)

Case 2. $x = E_n$. If $y > H_n$, then move $y \to H_n$. If $y = E_n$, then move to (E_0, H_0). We may thus assume that $E_n < y < H_n$. In particular, $n > 1$ for $a = 1$. Let $d = y - x - \varepsilon$, $m = \lfloor d/a \rfloor$, where

$$\varepsilon = \varepsilon(a) = \begin{cases} 0 & \text{if } a = 1, \\ 1 & \text{if } a > 1. \end{cases}$$

Then move $(x, y) \to (E_m, H_m)$. This is a legal move, since

(a) $m \geq 0$,

(b) $d = y - E_n - \varepsilon < H_n - E_n - \varepsilon = an$, hence $m = \lfloor d/a \rfloor \leq d/a < n$,

(c) $y = E_n + d + \varepsilon > E_m + d + \varepsilon \geq E_m + am + \varepsilon = H_m$,

(d) $|(y - H_m) - (x - E_m)| = |(y - x) - (H_m - E_m)| = |d - am| < a$. □

Note that whereas the *statement* of Theorem 2.1 characterizes the P-positions, its *proof* indicates explicitly how to win, starting from an N-position. The characterization and move-specification constitute together a *strategy* for the game. Thus Theorem 2.1 and its proof provide a strategy for misère Wythoff games in which each P-position can be computed from the previous ones.

For computing a strategy, consider a position (x, y) with $0 \leq x \leq y$ $((x, y) \neq (0, 0))$. We may assume, here and in the sequel, that $y \leq x + ax + 1$, since for $y > x + ax + 1$ we have $(x, y) \in N$; and (x, y) and $(x, x + ax + 1)$ have then the same winning strategy. At most $O(x)$ computation steps are needed for computing the table of P-positions. Once the table is given, only $O(\log x)$ steps are required to locate x in it by binary search. Since also the next move can be computed in $O(\log x)$ steps, the total number of steps for computing the strategy is only $O(\log x)$, which is linear in the input size $O(\log x)$. However, a given table permits to compute the strategy for piles of bounded size only, and the table itself has exponential size. In the next section we give a closed form for the nth P-position, which enables us to beat our adversary using an explicit rather than only an implicit recursive strategy, which is always polynomial (in time and space).

3. An algebraic characterization of the *P*-positions

Let

$$\alpha = \alpha(a) = \tfrac{1}{2}(2 - a + \sqrt{a^2 + 4}), \qquad \beta = \beta(a) = \alpha + a.$$

α is the positive root of the quadratic equation $\xi^{-1}+(\xi+a)^{-1}=1$. Thus α and β are irrational for every positive integer a, and satisfy $\alpha^{-1}+\beta^{-1}=1$. Let $\gamma=\gamma(a)=\alpha^{-1}$, $\delta=\delta(a)=\gamma+1$. Then

$$\frac{\gamma}{\alpha}+\frac{\delta}{\beta}=\gamma\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)+\frac{1}{\beta}=\frac{1}{\alpha}+\frac{1}{\beta}=1.$$

It thus follows immediately from [2, Theorem II] that the sets

$$E'=\{E'_n : n=0,1,2,\ldots\},\qquad H'=\{H'_n : n=0,1,2,\ldots\}$$

are complementary, where $E'_n=\lfloor n\alpha+\gamma\rfloor$, $H'_n=\lfloor n\beta+\delta\rfloor$.

Let $a>1$. Note that $E'_0=0=E_0$, $H'_0=1=H_0$, and $H'_n=E'_n+an+1$. Moreover, $\text{mex}\{E'_i, H'_i : 0\le i<n\}=E'_n$ $(n\ge 0)$, since $\{E'_n\}$ and $\{H'_n\}$ are increasing sequences and E' and H' are complementary: if the mex were not E'_n, then E'_n would never be obtained! This shows that $E'_n=E_n$, $H'_n=H_n$ $(n\ge 0)$. We have proved the second part of the following theorem.

Theorem 3.1. *The P-positions of misère Wythoff games are the following:*

(i) *For* $a=1$: $(E_0, H_0)=(2,2)$ $(E_1, H_1)=(0,1)$,

$$(E_n, H_n)=(\lfloor n\alpha\rfloor, \lfloor n\beta\rfloor)\quad (n\ge 2),$$

where $\alpha=\alpha(1)$, $\beta=\beta(1)$.

(ii) *For* $a>1$,

$$(E_n, H_n)=(\lfloor n\alpha+\gamma\rfloor, \lfloor n\beta+\delta\rfloor)\quad (n\ge 0),$$

where $\alpha=\alpha(a)$, $\beta=\beta(a)$, $\gamma=\gamma(a)$, $\delta=\delta(a)$.

Proof. The first part of the theorem is proved in essentially the same way as the second part. □

A strategy based on this observation can be realized as follows for every $a>1$. It is easy to see that $n\alpha+\gamma$ is irrational for every n. Since $\alpha>1$,

$$x=\lfloor n\alpha+\gamma\rfloor \Leftrightarrow x<n\alpha+\gamma<x+1$$

$$\Leftrightarrow \frac{x-\gamma}{\alpha}<n<\frac{x-\gamma+1}{\alpha}\Leftrightarrow\left\lfloor\frac{x-\gamma+1}{\alpha}\right\rfloor=\left\lfloor\frac{x-\gamma}{\alpha}\right\rfloor+1,$$

where (x,y) with $x\le y$ is a game position. Therefore either $x=\lfloor n\alpha+\gamma\rfloor=E_n$ where $n=\lfloor (x-\gamma+1)/\alpha\rfloor$, or else, by complementarity, $x=\lfloor n\beta+\delta\rfloor=H_n$ where $n=\lfloor (x-\delta+1)/\beta\rfloor$. We have thus reduced the situation to that considered in cases (ii) and (i) in the proof of Theorem 2.1, and hence the move selection made there can be followed. The strategy is similar for $a=1$, hence the details are omitted. For implementing this strategy, α, β, γ and δ have to be computed and stored to a precision of $O(\log x)$ digits, such that $(x-\gamma+1)\alpha^{-1}$ and $(x-\delta+1)\beta^{-1}$ have still at least one significant digit to the right of the (decimal, say) point.

In order to give yet another, unexpected, way for beating our opponent, we resort to the theory of continued fractions.

4. Continued fractions and systems of numeration

Let α be an irrational number satisfying $1<\alpha<2$. Denote its *simple continued fraction* expansion by

$$\alpha = 1+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cdots}}} = [1, a_1, a_2, a_3, \ldots],$$

where the a_i are positive integers. Its *convergents* $p_n/q_n = [1, a_1, \ldots, a_n]$ satisfy the recursion

$$p_{-1}=1,\ p_0=1,\ p_n=a_np_{n-1}+p_{n-2} \quad (n\geq 1),$$

$$q_{-1}=0,\ q_0=1,\ q_n=a_nq_{n-1}+q_{n-2} \quad (n\geq 1).$$

For the basic facts of the theory of continued fractions, see, for example, Hardy and Wright [4], Olds [7] or Perron [8].

In the next theorem we present two systems of numeration, one based on the numerators p_i and one on the denominators q_i of the convergents of α. The two systems are called *p-system* and *q-system* in the sequel.

Theorem 4.1. *Every positive integer can be written uniquely in the form*

$$N=\sum_{i=0}^{m} s_ip_i,\quad 0\leq s_i\leq a_{i+1},\ s_{i+1}=a_{i+2}\Rightarrow s_i=0\ \ (i\geq 0),$$

and also in the form

$$N=\sum_{i=0}^{n} t_iq_i,\quad 0\leq t_0<a_1,\ 0\leq t_i\leq a_{i+1},\ t_i=a_{i+1}\Rightarrow t_{i-1}=0\ \ (i\geq 1).$$

Table 3 displays the representation of the first few nonnegative integers in the p and q-systems for the case $a_i=3$ $(i\geq 1)$.

For the proof of Theorem 4.1 and Lemma 4.2 below (which is needed later), see [3].

Lemma 4.2. *Let*

$$G_{i+1}=a_{i+1}p_i+a_{i-1}p_{i-2}+\cdots+a_{k+1}p_k,$$

Table 3
The representation of the first few nonnegative integers in the p and q-systems for the case $a_i = 3$ $(i \geq 1)$.

n	p_2 13	p_1 4	p_0 1	q_2 10	q_1 3	q_0 1
0	0	0	0	0	0	0
1			1			1
2			2			2
3			3		1	0
4		1	0		1	1
5		1	1		1	2
6		1	2		2	0
7		1	3		2	1
8		2	0		2	2
9		2	1		3	0
10		2	2	1	0	0
11		2	3	1	0	1
12		3	0	1	0	2
13	1	0	0	1	1	0
14	1	0	1	1	1	1
15	1	0	2	1	1	2
16	1	0	3	1	2	0
17	1	1	0	1	2	1
18	1	1	1	1	2	2
19	1	1	2	1	3	0
20	1	1	3	2	0	0

where $k = 0$ *if* i *is even,* $k = 1$ *if* i *is odd. Then* $G_{i+1} = p_{i+1} - 1$.

(Informally, G_{i+1} is the equivalent in the p-system of $99\ldots9$ in the decimal system.)

We close this short section with three definitions which will be useful in the next sections.

Definition 4.3 (Representations and their interpretations). Relative to a simple continued fraction $\alpha = [1, a_1, a_2, \ldots]$, define a *representation* R to be an $(m+1)$-tuple

$$R = (d_m, d_{m-1}, \ldots, d_1, d_0),$$

where

$$0 \leq d_i \leq a_{i+1} \quad \text{and} \quad d_{i+1} = a_{i+2} \Rightarrow d_i = 0 \qquad (i \geq 0).$$

If it is known that $d_{i-1} = d_{i-2} = \cdots = d_0 = 0$, we also write $R = (d_m, \ldots, d_i)$ instead of $(d_m, \ldots, d_i, 0, \ldots, 0)$. The *p-interpretation* I_p of a representation $R = (d_m, \ldots, d_0)$ is the number $I_p = \sum_{i=0}^{m} d_i p_i$. The *q-interpretation* of R is the number $I_q = \sum_{i=0}^{m} d_i q_i$, provided that $d_0 < a_1$; otherwise R has no q-interpretation. Given any positive integer k, we say that its *p-representation* $R_p(k)$ (or *q-representation* $R_q(k)$) is $(d_m, \ldots, d_0)$ if

$$k = \sum_{i=0}^{m} d_i p_i \quad \left(\text{or } k = \sum_{i=0}^{m} d_i q_i,\ d_0 < a_1\right).$$

We shall later be interested in p-interpretations of q-representations! Thus for $\alpha = [1, \dot{3}]$ where $\dot{x}$ denotes the infinite concatenation of x with itself, the decimal number 15 has q-representation 112 (see Table 3), whose p-interpretation is 19. Thus $I_p(R_q(15)) = I_p(112) = 19$.

Definition 4.4 (Left and right shifts of representations). If $R = (d_m, \ldots, d_0)$ is any representation (which might be $R_p(k)$ or $R_q(k)$ for some positive integer k), then the representation $R' = (d_m, \ldots, d_0, 0)$ is called a *left shift* of R. In other words, R' is obtained from R by shifting each digit d_i of R left by one place and inserting a zero at the right. If $R = (d_m, \ldots, d_1, d_0)$ is any representation, then the representation $R'' = (d_m, \ldots, d_1)$ is called a *right shift* of R.

Definition 4.5 (Lexicographic ordering of representations). Given two representations $R_1 = (d_m, \ldots, d_0)$ and $R_2 = (c_m, \ldots, c_0)$, we say that R_1 is *larger* than R_2 or R_2 is smaller than R_1 ($R_1 > R_2$ or $R_2 < R_1$) if there is some $j \in [0, m]$ such that $d_j > c_j$ and $d_i = c_i$ $(i > j)$.

Note that $R_1 > R_2$ if and only if $I_p(R_1) > I_p(R_2)$.

5. An arithmetic characterization of the *P*-positions

We use the numeration systems introduced in the previous section to give a quite different characterization of the P-positions. Comparing Tables 2 and 3 we notice three interesting patterns. To make them more conspicuous, we unite Tables 2 and 3 in the form of Table 4. Below we prove that these patterns do indeed hold for every $\alpha = [1, \dot{a}]$, $a > 1$, in the form of the following three properties.

Whenever we say that a representation R ends in a certain string, we mean that this string constitutes the right-hand end of R.

Property 1. *The set* $\{E_n : n \geq 0\}$ *is identical to the set of numbers with p-representations ending in one of* (i) 3, 4, ..., a, (ii) 01 ... 1even, 01 ... 12even, or (iii) c1 ... 1odd, c1 ... 12odd, *where c denotes any digit in the range* $1 < c < a$, *and* even (odd) *at the end of a string means that the number of consecutive trailing* 1*'s is* even (odd), *followed by the digit* 2 *where indicated. The set* $\{H_n : n \geq 0\}$ *is identical to the set of numbers with p-representations ending in one of* (iv) 01 ... 1odd, 01 ... 12odd, *or* (v) c1 ... 1even ≥ 2, c1 ... 12even.

Table 4
The representation of the first few P-positions (E_n, H_n) and n in the p and q-systems for $\alpha = [1, \dot{3}]$ ($a = 3$, misère play).

n	E_n	H_n	$R_p(E_n)$			$R_p(H_n)$				$R_q(n)$		
			p_2 13	p_1 4	p_0 1	p_3 43	p_2 13	p_1 4	p_0 1	q_2 10	q_1 3	q_0 1
0	0	1	0	0	0				1	0	0	0
1	2	6			2			1	2			1
2	3	10			3			2	2			2
3	4	14		1	0		1	0	1		1	0
4	5	18		1	1		1	1	1		1	1
5	7	23		1	3		1	2	2		1	2
6	8	27		2	0		2	0	1		2	0
7	9	31		2	1		2	1	1		2	1
8	11	36		2	3		2	2	2		2	2
9	12	40		3	0		3	0	1		3	0
10	13	44	1	0	0	1	0	0	1	1	0	0
11	15	49	1	0	2	1	0	1	2	1	0	1
12	16	53	1	0	3	1	0	2	2	1	0	2
13	17	57	1	1	0	1	1	0	1	1	1	0
14	19	62	1	1	2	1	1	1	2	1	1	1
15	20	66	1	1	3	1	1	2	2	1	1	2
16	21	70	1	2	0	1	2	0	1	1	2	0
17	22	74	1	2	1	1	2	1	1	1	2	1
18	24	79	1	2	3	1	2	2	2	1	2	2
19	25	83	1	3	0	1	3	0	1	1	3	0
20	26	87	2	0	0	2	0	0	1	2	0	0
21	28	92	2	0	2	2	0	1	2	2	0	1
22	29	96	2	0	3	2	0	2	2	2	0	2
23	30	100	2	1	0	2	1	0	1	2	1	0
24	32	105	2	1	2	2	1	1	2	2	1	1
25	33	109	2	1	3	2	1	2	2	2	1	2
26	34	113	2	2	0	2	2	0	1	2	2	0
27	35	117	2	2	1	2	2	1	1	2	2	1
28	37	122	2	2	3	2	2	2	2	2	2	2
29	38	126	2	3	0	2	3	0	1	2	3	0
30	39	130	3	0	0	3	0	0	1	3	0	0
31	41	135	3	0	2	3	0	1	2	3	0	1
32	42	139	3	0	3	3	0	2	2	3	0	2

Property 2. *Denote the least significant digit of $R_p(E_n)$ by t. Then $R_p(H_n)$ is the left shift $R'_p(E_n)$ of $R_p(E_n)$ with the last digit (zero) replaced by* 1 *(if $t = 0$ or* 1*) or by* 2 *and t replaced by $t-1$ (if $1 < t \leq a$) ($n \geq 0$).*

Property 3. *Let n be any nonnegative integer. If $R_q(n)$ ends in* 01 . . . 1even *or in* $c1 \ldots 1$odd ($1 < c < a$)*, then $E_n = I_p(R_q(n))$. If $R_q(n)$ ends in* 01 . . . 1odd *or in* $c1 \ldots 1$even, *then $E_n = I_p(R_q(n)) + 1$ ($n \geq 0$).*

For proving these properties we need two further auxiliary results. Let $\alpha = [b, a_1, a_2, \ldots]$ with convergents $\{p_i/q_i\}$, where b is any integer. Let $D_i = \alpha q_i - p_i$ $(i \geq -1)$. From the theory of continued fractions it is known that

$$-1 = D_{-1} < D_1 < D_3 < \cdots < 0 < \cdots < D_4 < D_2 < D_0 = \alpha - b,$$
$$|D_{i-1}| > |D_i| \quad (i \geq 0).$$

Lemma 5.1

$$D_j + \sum_{i=1}^{m} a_{j+2i} D_{j+2i-1} = D_{j+2m} \quad (j \geq -1).$$

Proof. For a proof, see [3].

Lemma 5.2. *Let b be any integer, a any positive integer and $\alpha = [b, \dot{a}]$. Then*

$$\sum_{i=0}^{m} D_i = a^{-1}(D_m + D_{m+1} + b + 1 - \alpha) \quad (m \geq 0).$$

Proof. True for $m = 0$. If it is true for m, then

$$\sum_{i=0}^{m+1} D_i = a^{-1}(D_m + D_{m+1} + b + 1 - \alpha) + D_{m+1}$$
$$= a^{-1}(aD_{m+1} + D_m + D_{m+1} + b + 1 - \alpha)$$
$$= a^{-1}(D_{m+1} + D_{m+2} + b + 1 - \alpha) \quad \text{since } aD_{m+1} + D_m = D_{m+2}. \qquad \square$$

For proving Property 3 it evidently suffices to show that the following four relations hold for every $j \geq 0$:

$$\text{(i)} \quad n = \sum_{i=0}^{2j-1} q_i + \sum_{i=2j+1}^{k} d_i q_i \Rightarrow \lfloor n\alpha + \alpha^{-1} \rfloor = \sum_{i=0}^{2j-1} p_i + \sum_{i=2j+1}^{k} d_i p_i \quad (k \geq 0)$$

$$\text{(ii)} \quad n = \sum_{i=0}^{2j} q_i + \sum_{i=2j+1}^{k} d_i q_i, \quad d_{2j+1} > 1$$
$$\Rightarrow \lfloor n\alpha + \alpha^{-1} \rfloor = \sum_{i=0}^{2j} p_i + \sum_{i=2j+1}^{k} d_i p_i \quad (k \geq 1),$$

$$\text{(iii)} \quad n = \sum_{i=0}^{2j} q_i + \sum_{i=2j+2}^{k} d_i q_i \Rightarrow \lfloor n\alpha + \alpha^{-1} \rfloor = 1 + \sum_{i=0}^{2j} p_i + \sum_{i=2j+2}^{k} d_i p_i \quad (k \geq 0),$$

$$\text{(iv)} \quad n = \sum_{i=0}^{2j-1} q_i + \sum_{i=2j}^{k} d_i q_i, \quad d_{2j} > 1$$
$$\Rightarrow \lfloor n\alpha + \alpha^{-1} \rfloor = 1 + \sum_{i=0}^{2j-1} p_i + \sum_{i=2j}^{k} d_i p_i \quad (k \geq 0).$$

Relation (i) is evidently equivalent to

$$0 \leq n\alpha + \alpha^{-1} - \sum_{i=0}^{2j-1} p_i - \sum_{i=2j+1}^{k} d_i p_i < 1$$

for n as given in (i). This is equivalent to:

$$\text{(v)}\quad 0 \leq \sum_{i=0}^{2j-1} D_i + \sum_{i=2j+1}^{k} d_i D_i + \alpha^{-1} < 1 \quad (k \geq 0).$$

Similarly, (ii), (iii) and (iv) are equivalent, respectively, to

$$\text{(vi)}\quad 0 \leq \sum_{i=0}^{2j} D_i + \sum_{i=2j+1}^{k} d_i D_i + \alpha^{-1} < 1 \quad (d_{2j+1} > 1,\ k \geq 1),$$

$$\text{(vii)}\quad 1 \leq \sum_{i=0}^{2j} D_i + \sum_{i=2j+2}^{k} d_i D_i + \alpha^{-1} < 2 \quad (k \geq 0),$$

$$\text{(viii)}\quad 1 \leq \sum_{i=0}^{2j-1} D_i + \sum_{i=2j}^{k} d_i D_i + \alpha^{-1} < 2 \quad (d_{2j} > 1,\ k \geq 0).$$

Proof of (v). We proceed to prove (v). By Lemma 5.1,

$$\sum_{i=2j+1}^{k} d_i D_i \leq \sum_{i=1}^{k} aD_{(2j+1)+(2i-1)} = D_{2j+2k+1} - D_{2j+1} < -D_{2j+1},$$

$$\sum_{i=2j+1}^{k} d_i D_i \geq \sum_{i=1}^{k} aD_{2j+2i-1} = D_{2j+2k} - D_{2j} > -D_{2j}.$$

Hence by Lemma 5.2 (with $b = 1$ here and below),

$$\sum_{i=0}^{2j-1} D_i + \sum_{i=2j+1}^{k} d_i D_i < a^{-1}(D_{2j-1} + D_{2j} + 2 - \alpha) - D_{2j+1}$$

$$= a^{-1}(2 - \alpha - (a-1)D_{2j-1} - (a^2 - 1)D_{2j}).$$

Now

$$D_{2j+1} = aD_{2j} + D_{2j-1} = (a+1)D_{2j} + D_{2j-1} - D_{2j}.$$

Since $D_{2j} > -D_{2j+1}$, we thus get

$$-D_{2j-1} < D_{2j+1} + D_{2j} - D_{2j-1} = (a+1)D_{2j}.$$

Thus $-(a-1)D_{2j-1} \leq (a^2 - 1)D_{2j}$, hence

$$\sum_{i=0}^{2j-1} D_i + \sum_{i=2j+1}^{k} d_i D_i < a^{-1}(2 - \alpha).$$

Let α_2 be the negative root of $\xi^2 + (a-2)\xi - a = 0$ (α is the positive root). Then $\alpha\alpha_2 = -a$, $\alpha + \alpha_2 = 2 - a$. Therefore

$$a^{-1}(2 - \alpha) = a^{-1}(a + \alpha_2) = a^{-1}(a - a\alpha^{-1}) = 1 - \alpha^{-1},$$

proving the right-hand side of (v). For the proof of the left-hand side, write

$$\sum_{i=1}^{2j-1} D_i + \sum_{i=2j+1}^{k} d_i D_i \geq a^{-1}(D_{2j-1} + D_{2j} + 2 - \alpha) - D_{2j}$$

$$= a^{-1}(D_{2j-1} - (a-1)D_{2j} + 2 - \alpha)$$

$$\geq a^{-1}(D_{-1} - (a-1)D_0 + 2 - \alpha)$$

$$= 1 - \alpha = a + \alpha_2 - 1 = a(1 - \alpha^{-1}) - 1 \geq -\alpha^{-1}. \quad \square$$

Proof of (vi). For proving (vi) we use $d_{2j+1} > 1$ and Lemma 5.1 to get

$$\sum_{i=2j+1}^{k} d_i D_i < 2D_{2j+1} + \sum_{i=1}^{k} aD_{(2j+1)+(2i-1)} = D_{2j+1} + D_{2j+2k+1} < D_{2j+1},$$

and, as above, $\sum_{i=2j+1}^{k} d_i D_i > -D_{2j}$. Hence by Lemma 5.2,

$$\sum_{i=1}^{2j} D_i + \sum_{i=2j+1}^{k} d_i D_i < a^{-1}(D_{2j} + D_{2j+1} + 2 - \alpha) + D_{2j+1}$$

$$= a^{-1}(D_{2j+1} + D_{2j+2} + 2 - \alpha) = \sum_{i=0}^{2j+1} D_i \leq 1 - \alpha^{-1},$$

where the last inequality follows as in the proof of (v). On the other hand,

$$\sum_{i=0}^{2j} D_i + \sum_{i=2j+1}^{k} d_i D_i > a^{-1}(D_{2j} + D_{2j+1} + 2 - \alpha) - D_{2j}$$

$$= a^{-1}(2 - \alpha - (a-1)D_{2j} + D_{2j+1})$$

$$\geq a^{-1}(2 - \alpha - (a-1)D_0 + D_1) = 0 > -\alpha^{-1}. \quad \square$$

Proof of (vii). For proving (vii) we again start with Lemma 5.1:

$$\sum_{i=2j+2}^{k} d_i D_i \leq \sum_{i=1}^{k} aD_{(2j+1)+(2i-1)} = D_{2j+2k+1} - D_{2j+1} < -D_{2j+1},$$

$$\sum_{i=2j+2}^{k} d_i D_i \geq \sum_{i=1}^{k} aD_{(2j+2)+(2i-1)} = D_{2j+2k+2} - D_{2j+2} > -D_{2j+2}.$$

Thus

$$\sum_{i=0}^{2j} D_i + \sum_{i=2j+2}^{k} d_i D_i < a^{-1}(D_{2j} + D_{2j+1} + 2 - \alpha) - D_{2j+1}$$

$$= a^{-1}(2 - \alpha + D_{2j} - (a-1)D_{2j+1})$$

$$\leq a^{-1}(2 - \alpha + D_0 - (a-1)D_1)$$

$$= \alpha + a - a\alpha = 1 - (\alpha - 1)(a - 1).$$

Since $\alpha > 1$, we have $-(\alpha-1)(a-1) \leq 0 < 1-\alpha^{-1}$. Thus

$$1-(\alpha-1)(a-1) < 2-\alpha^{-1},$$

proving the right-hand side of (vii). In the other direction,

$$\sum_{i=0}^{2j} D_i + \sum_{i=2j+2}^{k} d_i D_i > a^{-1}(D_{2j}+D_{2j+1}+2-\alpha) - D_{2j+2}$$
$$= a^{-1}(2-\alpha-(a-1)D_{2j}-(a^2-1)D_{2j+1}).$$

We proceed in a way similar to the proof of (v):

$$D_{2j+2} = (a+1)D_{2j+1} + D_{2j} - D_{2j+1},$$

hence

$$-D_{2j} > D_{2j+2} + D_{2j+1} - D_{2j} = (a+1)D_{2j+1},$$

so

$$-(a-1)D_{2j} - (a^2-1)D_{2j+1} \geq 0.$$

Hence

$$\sum_{i=0}^{2j} D_i + \sum_{i=2j+2}^{k} d_i D_i > a^{-1}(2-\alpha) = 1-\alpha^{-1}. \qquad \square$$

Proof of (viii). Finally we prove (viii) by writing

$$\sum_{i=2j}^{k} d_i D_i \leq \sum_{i=1}^{k} a D_{(2j-1)+(2i-1)} = D_{2j+2k-1} - D_{2j-1} < -D_{2j-1}.$$

Since $d_{2j} > 1$,

$$\sum_{i=2j}^{k} d_i D_i > 2D_{2j} + \sum_{i=1}^{k} a D_{2j+2i-1} = D_{2j} + D_{2j+2k} > D_{2j}.$$

Thus

$$\sum_{i=0}^{2j-1} D_i + \sum_{i=2j}^{k} d_i D_i < a^{-1}(D_{2j-1}+D_{2j}+2-\alpha) - D_{2j-1}$$
$$= a^{-1}(2-\alpha-(a-1)D_{2j-1}+D_{2j})$$
$$\leq a^{-1}(2-\alpha-(a-1)D_{-1}+D_0)$$
$$= 1 < 2-\alpha^{-1},$$

since $\alpha > 1$. In the other direction,

$$\sum_{i=0}^{2j-1} D_i + \sum_{i=2j}^{k} d_i D_i > a^{-1}(D_{2j-1}+D_{2j}+2-\alpha) + D_{2j}$$
$$= a^{-1}(D_{2j}+D_{2j+1}+2-\alpha) = \sum_{i=0}^{2j} D_i > 1-\alpha^{-1},$$

as in (vii). $\square$

Proof of Property 1. The first part of Property 3 implies that if $R_q(n)$ ends in $01\ldots1$even or in $c1\ldots1$odd, then also $R_p(E_n)=R_q(n)$ ends in the same strings. The second part of Property 3 implies that if $R_q(n)$ ends in $01\ldots1$odd, then $R_p(E_n)$ ends in $01\ldots12$even; and if $R_q(n)$ ends in $c1\ldots1$even, then $R_p(E_n)$ ends in $3,4,\ldots,a$ or in $c1\ldots12$odd. Since the sets $\{E_n: n\geq 0\}$ and $\{H_n: n\geq 0\}$ are complementary, the latter set of numbers has representations which are the complement of the representations of the former set. This proves Property 1. □

Proof of Property 2. For proving Property 2, note that the transformation of f defined in its statement is a bijection since it has an inverse f^{-1}: Shift $f(R_p(E_n))$ right; if $d_0=2$, then put $d_1\leftarrow d_1+1$ (d_1d_0 is the right trailing end of $f(R_p(E_n))$). This evidently produces E_n. Moreover, by Property 1, the sets $\{R_p(E_n): n\geq 0\}$ and $\{f(R_p(E_n)): n\geq 0\}$ are complementary.

We now proceed by induction. The assertion is true for $n=0$. If it is true for all $n<m$, then $f(R_p(E_m))\neq R_p(H_n)$, $n<m$. In fact, $I_p(f(R_p(E_m)))$ is the smallest number of $\{H_n\}$ not yet obtained for $n<m$. If $H_m\neq I_p(f(R_p(E_m)))$, then $I_p(f(R_p(E_m)))$ can never be obtained for $n>m$, contradicting the complementarity of $\{R_p(E_n)\}$ and $\{f(R_p(E_n)\}$. □

Now suppose we are given a position (x, y) with $0\leq x\leq y$ ($a>1$). We may assume $(x, y)\neq(0, 0)$. To obtain a strategy based on Properties 1, 2 and 3, compute $R_p(x)$. If it ends in one of the strings (iv) or (v) of Property 1, then $x=H_k$ for some $k\geq 0$, and a winning move is $(x, y)\rightarrow(I_p(f^{-1}(R_p(x))), x)\in P$. If $R_p(x)$ ends in one of the strings (i), (ii) or (iii) of Property 1, then $x=E_k$ for some $k\geq 0$. If $y>I_p(f(R_p(x)))$, then the move $(x, y)\rightarrow(x, I_p(f(R_p(x))))\in P$ is a winning move. If $y=I_p(f(R_p(x)))$, then $(x, y)\in P$, so we cannot win when starting from the given position (x, y). If $x=y$, then the move $(x, y)\rightarrow(0, 1)\in P$ is winning. Finally, if $x<y<I_p(f(R_p(x)))$, then let $m=\lfloor(y-x-1)/a\rfloor$. If $R_q(m)$ ends in $01\ldots1$even or $c1\ldots1$odd ($1<c<a$), then $E_m=I_p(R_q(m))$ by Property 3. Otherwise $E_m=I_p(R_q(m))+1$. In either case a winning move is $(x, y)\rightarrow(E_m, E_m+am+1)\in P$. The strategy for $a=1$ is similar and is therefore omitted.

The complexity analysis of this algorithm is very similar to that performed at the end of [3], and is based on the fact that $p_n\sim Kg^{n+1}$, where $K=\alpha/\sqrt{a^2+4}$, $g=\frac{1}{2}(a+\sqrt{a^2+4})$. Since $n\sim\log_g(x/K)$ and g increases with a, this strategy implementation, which requires $n=O(\log x)$ steps, is more efficient than even the algebraic one of Section 3 when a is large.

In order to consolidate the above strategies of misère Wythoff games as well as those of normal Wythoff games, we now proceed to introduce the notion of a cedar tree.

6. Cedar trees

Relative to a simple continued fraction $\alpha=[1, a_1, a_2, \ldots]$ we define the following subclass of binary trees. The subclass consists of *c*ontinu*ed* fr*a*ction *r*epresentation trees (*cedar* trees). A cedar tree $C_{m,d} = C_{m,d}(\alpha)$ of order m is defined as follows:

(i) For $m<0$ or $d=d_m$ outside the range $(1, 2, \ldots, a_{m+1})$, $C_{m,d}$ is empty.

(ii) For $1 \leq d = d_m \leq a_{m+1}$ and $m \geq 0$, the root of $C_{m,d}$ is any representation of the form $(d_n, \ldots, d_m)$.

Note that the order m of the tree is the index m of the least significant nonzero digit d_m of the representation of the tree's root. The root of the left subtree is the representation $(d_n, \ldots, d_{m+1}, d_m - 1, 1)$ of order $m-1$ $(m \geq 1)$. If $d = d_m < a_{m+1}$, then the root of the right subtree is the representation $(d_n, \ldots, d_m + 1)$ of order m. If $d_m = a_{m+1}$, then the root of the right subtree is the representation $(d_n, \ldots, d_m, 0, 1)$ of order $m-2$ $(m \geq 2)$. This inductive definition is illustrated schematically in Fig. 1 for the case $m=2$ and $a_3=4$. If the root of $C_{m,d}$ is the representation (d_m), we denote the tree by $C^0_{m,d}$. Fig. 2 illustrates the unique cedar tree $C^0_{2,1}([1, \dot{3}])$ in which the numbers above and below the nodes should be ignored for the moment.

We now derive two subfamilies of trees from the family of cedar trees. A *p-tree* $T_{m,d}$ derived from a cedar tree $C_{m,d}$ is the p-interpretation of $C_{m,d}$, that is, every node of $C_{m,d}$ is replaced by its p-interpretation to make up $T_{m,d}$. Similarly, a *q-tree* $\tau_{m,d}$ derived from $C_{m,d}$ is the q-interpretation of $C_{m,d}$, but with one proviso: since $t_0 < a_1$ (see Theorem 4.1 above) and the nodes of $C_{m,d}$ ending in a_1 are precisely the leaves of $C_{m,d}$, the form of a $\tau_{m,d}$-tree is that of $T_{m,d}$, but without the leaves of the latter. Thus a $T_{m,d}$-tree turns into a $\tau_{m,d}$-tree in the fall, after having shed all its leaves, and the process is reversed in the spring! The notation $C^0_{m,d}$ carries over to $T^0_{m,d}$ and $\tau^0_{m,d}$ in an obvious manner.

For example, the numbers above the nodes in Fig. 2 are the p-interpretations of the nodes. Replacing the nodes of $C^0_{2,1}([1, \dot{3}])$ by these p-interpretations gives $T^0_{2,1}([1, \dot{3}])$. The numbers under the nodes of $C^0_{2,1}([1, \dot{3}])$ are the q-interpretations. So pruning the leaves of $C^0_{2,1}([1, \dot{3}])$ and replacing its remaining nodes by their q-interpretations gives $\tau^0_{2,1}([1, \dot{3}])$.

We need the following definitions. The *rightmost* (*leftmost*) *descendant* v of a node u is the descendant at the end of a chain of right (left) descendants of u which has no further right (left) son. (Note that v is not necessarily a leaf.) A *label* of an edge (u, v) of a cedar tree is a bit 0 or 1 attached to the edge according to whether v is a left or right son of u. If w is any vertex of a cedar tree and $(e_1, \ldots, e_m)$ is the path of edges between the root and w, then $m+1$ is the *length* of the path and $(c_1, \ldots, c_m) \in \{0, 1\}^m$ is the *trace* $\mathrm{tr}(w)$ of w, where c_i is the label of edge e_i

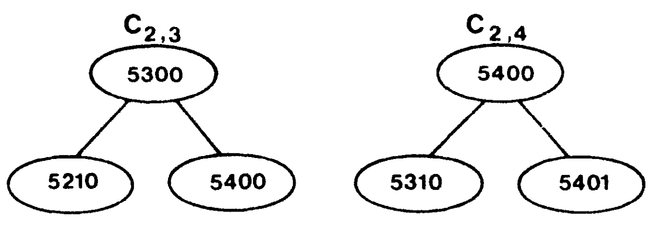

Fig. 1. The inductive construction of a cedar tree for $m=2$, $a_3=4$.

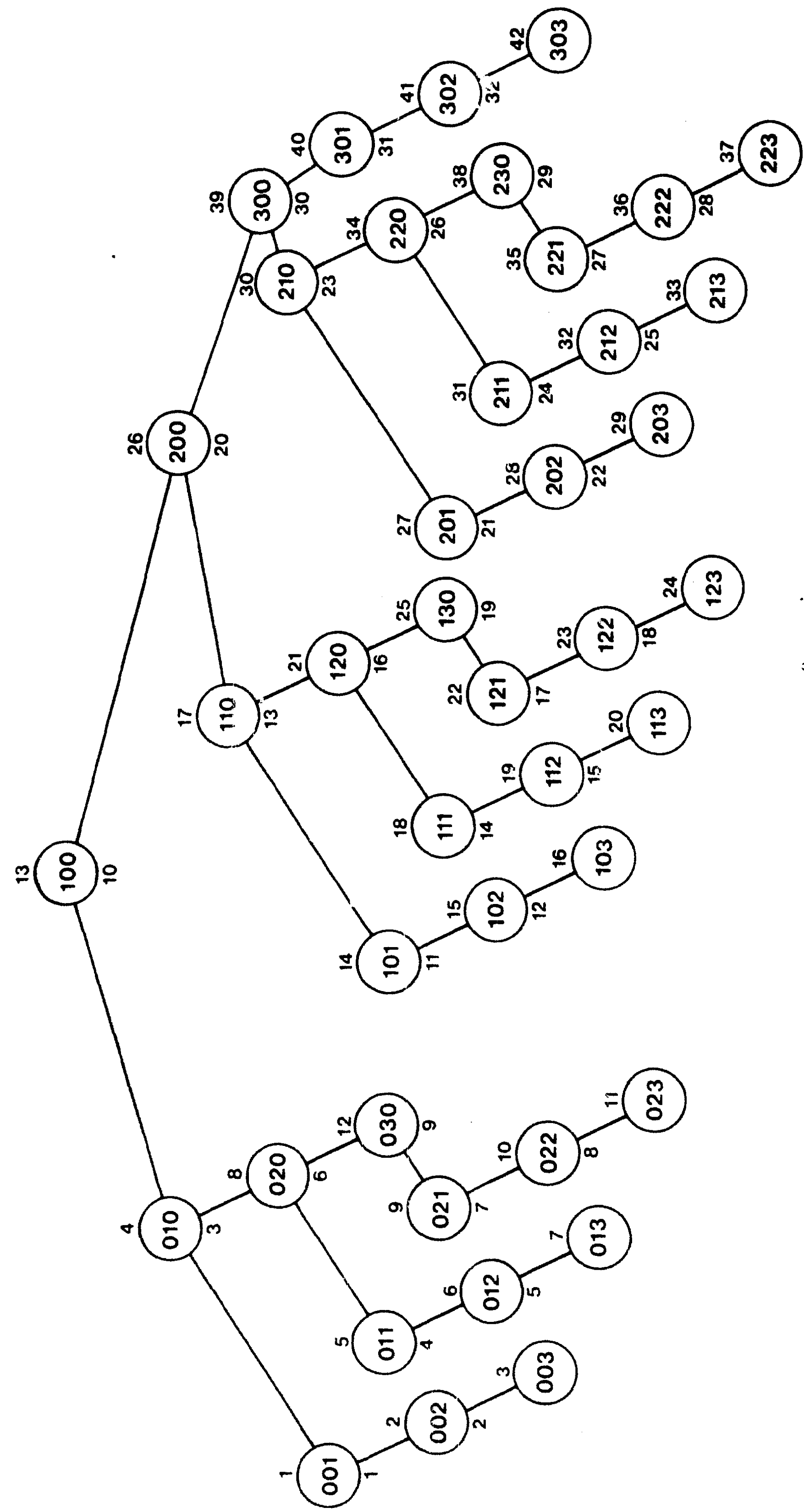

Fig. 2. Illustration of the cedar tree $C^{0}_{2,1}([1,3])$.

$(1 \leq i \leq m)$. The order of the sequence is such that c_1 is the label of the edge e_1 from the root to its son, and c_m is the label of the edge e_m leading into w. Note that the trace of the root of a cedar tree is the empty sequence. If $\mathrm{tr}(u) = (0, c_2, \ldots, c_k)$, then $\mathrm{tr}(v) = \mathrm{tr}'(u) = (c_2, \ldots, c_k)$ is a *left shift* of $\mathrm{tr}(u)$. Also $\mathrm{tr}(u) = \mathrm{tr}''(v) = (0, c_2, \ldots, c_k)$ is a *right shift* of $\mathrm{tr}(v)$.

The basic properties of cedar trees and some other properties needed for the applications are enunciated in the 'cedar tree ten commandments' Theorem 6.1 below.

The reader may find it useful to verify the properties on the cedar tree of Fig. 2 while reading the theorem.

Theorem 6.1. *Let $\alpha = [1, a_1, a_2, \ldots]$ be an irrational number and $C_{m,d} = C_{m,d}(\alpha)$ a cedar tree ($d = d_m$, $m \geq 0$). For assertions* (ix) *and* (x) *we assume: $a_i = a$ $(i \geq 1)$ where a is any positive integer (since the proof rests on Lemma 5.2), $\gamma = \alpha^{-1}$, and we restrict attention to a cedar tree $C^0_{m,1}(\alpha)$.*

(i) *Let u be any node in $C_{m,d}$ or in the $T_{m,d}$ or $\tau_{m,d}$-tree derived from $C_{m,d}$. Then every node in the left subtree of u is smaller than u and every node in the right subtree of u is larger than u.*

(ii) *The trees $C_{m,d}$ and $T_{m,d}$ have $p_{m+1} - (d-1)P_m - 1$ nodes each and $\tau_{m,d}$ has $q_{m+1} - (d-1)q_m - 1$ nodes.*

(iii) *Every number from among $\{1, 2, \ldots, P_{m+1} - 1\}$ appears exactly once in $T^0_{m,1}$ and every number from among $\{1, 2, \ldots, q_{m+1} - 1\}$ appears exactly once in $\tau^0_{m,1}$.*

(iv) *Pruning the leaves of $C_{m,d}$ and replacing the nodes by their q-interpretations gives $\tau_{m,d}$. Moreover, the number of leaves of $c_{m,1}$ (or $T_{m,1}$) is $p_{m+1} - q_{m+1}$; this number is q_m if $a_i = a$ $(i \geq 1$, a any positive integer).*

(v) *Let $R_1 = (d_n, \ldots, d_{k+1}, d_k)$ be a node of $C_{m,d}$ with $d_k \neq 0$, $k > 0$. Then the leftmost descendant of the right subtree of R_1, if any, is $R_2 = (d_n, \ldots, d_{k+1}, d_k, 0, \ldots, 0, 1)$ ($k-1$ intervening 0's), and $I_p(R_2) = I_p(R_1) + 1$. The rightmost descendant of the left subtree of R_1 is*

$$R_3 = (d_n, \ldots, d_{k+1}, d_k - 1, a_k, 0, \ldots, 0, a_1) \quad (k \text{ odd})$$

or

$$(d_n, \ldots, d_{k+1}, d_k - 1, a_k, 0, \ldots, a_2, 0) \quad (k \text{ even}).$$

In either case, $I_p(R_3) = I_p(R_1) - 1$.

(vi) *The longest path from the root of $C_{m,1}$ to a leaf has length $L_m = \sum_{i=1}^{m+1} a_i$; the shortest path has length $l_m = a_1 + m$ (if $a_i > 1$ for $i \geq 1$) or $\lfloor \frac{1}{2}(m+3) \rfloor$ (if $a_i = 1$ for $i \geq 1$).*

(vii) *If $\mathrm{tr}(u) = (0, c_2, \ldots, c_k)$ for u in $C^0_{m,1}$, then there exists a node v in $C^0_{m,1}$ with $\mathrm{tr}(v) = \mathrm{tr}'(u) = (c_2, \ldots, c_k)$, and v is a left shift of u. Conversely, if v is a node in $C^0_{m,1}$ ending with 0 and $\mathrm{tr}(v) = (c_2, \ldots, c_k)$, then the right shift u of v is in $C^0_{m,1}$ and $\mathrm{tr}(u) = (0, c_2, \ldots, c_k)$.*

(viii) *If $R_q(n)$ ends in an even number of zeros in $C^0_{m,1}$, then* $\text{tr}(R_p\lfloor n\alpha\rfloor) = \text{tr}(R_q(n))$. *Otherwise* $\text{tr}(R_p\lfloor n\alpha\rfloor)$ *is* $\text{tr}(R_q(n))$ *followed by* 0 *and as many* 1*'s as possible until a rightmost descendant is reached.*

(ix) *Let u be a node in $C^0_{m,1}$ ending in a digit t with* $\text{tr}(u) = (0, c_2, \ldots, c_k)$. *If $t < a$, then there exists a node v in $C^0_{m,1}$ with* $\text{tr}(v) = (c_2, \ldots, c_k, 1, 0, \ldots, 0)$ *(maximal number of trailing* 0*'s until a leftmost descendant is reached) and $v = u'$, except that the last digit* 0 *of v is replaced by* 1. *Suppose $a > 1$. If $t > 0$, then there exists a node v in $C^0_{m,1}$ with* $\text{tr}(v) = (c_2, \ldots, c_k, 0, 1)$ *and $v = u'$, except that the last two digits $(t, 0)$ of v are replaced by $(t-1, 2)$. Conversely, if v and* $\text{tr}(v)$ *have the specified forms, then $(0, c_2, \ldots, c_k)$ is the trace of a node in $C^0_{m,1}$ which is the right shift v'' of v with the last digit t of v'' replaced by $t+1$, if the last digit of v is* 2.

(x) *If $R_q(n)$ ends in* 01 … 1even *or in* $c1 \ldots 1$odd $(1 < c < a)$, *then* $\text{tr}(R_p\lfloor n\alpha + \gamma\rfloor) = \text{tr}(R_q(n))$. *Otherwise* $\text{tr}(R_p\lfloor n\alpha + \gamma\rfloor)$ *is* $\text{tr}(R_q(n))$ *followed by* 1 $(n > 0)$.

Proof. (i) We prove the result for $T_{m,d}$. The proof for $\tau_{m,d}$ and $C_{m,d}$ is the same. If u in $T_{m,d}$ has the form $K_{m+1} + dp_m$ for some $1 \le d \le a_{m+1}$ where $R_p(K_{m+1})$ has the form $(d_n, \ldots, d_{m+1})$, then it follows from the definition of cedar trees that every node in the right subtree of u contains the summand $K_{m+1} + dp_m$, in addition to other summands. The left subtree T' of u has root $v = K_{m+1} + (d-1)p_m + p_{m+1}$. Assuming the result inductively for T', the largest node of T' is a rightmost descendant of v, whose value is

$$K_{m+1} + (d-1)p_m + a_m p_{m-1} + a_{m-2}p_{m-3} + \cdots = K_{m+1} + dp_m - 1 < u$$

by Lemma 4.2.

(ii) Again we prove the assertion for $T_{m,d}$ only. Since obviously $T_{m,d}$ and $T^0_{m,d}$ have the same number of nodes, it suffices to restrict attention to $T^0_{m,d}$. We proceed by induction on m for any d. A tree $T^0_{0,d}$ has obviously $a_1 - d + 1 = p_1 - (d-1)p_0 - 1$ vertices. Given $T^0_{m,d}$ $(m \ge 1)$. Each of the nodes $dp_m, (d+1)p_m, \ldots, a_{m+1}p_m$ in the branch emanating from the root on the right has a left subtree of the form $T_{m-1,1}$ (see Fig. 3). The number of nodes in the branch and in the $(a_{m+1} - d + 1)$ left subtrees is $(a_{m+1} - d + 1)p_m$ by the induction hypothesis. In addition, the node $a_{m+1}p_m$ has a right subtree of the form $T_{m-2,1}$, which has $p_{m-1} - 1$ nodes. Hence the total number of nodes of $T_{m,d}$ is

$$(a_{m+1} - d + 1)p_m + p_{m-1} - 1 = p_{m+1} - (d-1)p_m - 1.$$

(iii) Once again we prove the result for $T^0_{m,1}$ only. From (i) it follows that the smallest element in $T^0_{m,1}$ is the leftmost descendant which is evidently 1. The largest element is the rightmost descendant, and it is $a_{m+1}p_m + a_{m-1}p_{m-2} + \cdots = p_{m+1} - 1$ by Lemma 4.2. It also follows from (i) that all values in $T_{m,d}$ are distinct. Since the number of nodes is $p_{m+1} - 1$ by (ii) and all are in the range $[1, p_{m+1} - 1]$, every integer in this range must appear precisely once in $T^0_{m,1}$. (Induction on m without using (i) and Lemma 4.2 could have been used as an alternative proof.)

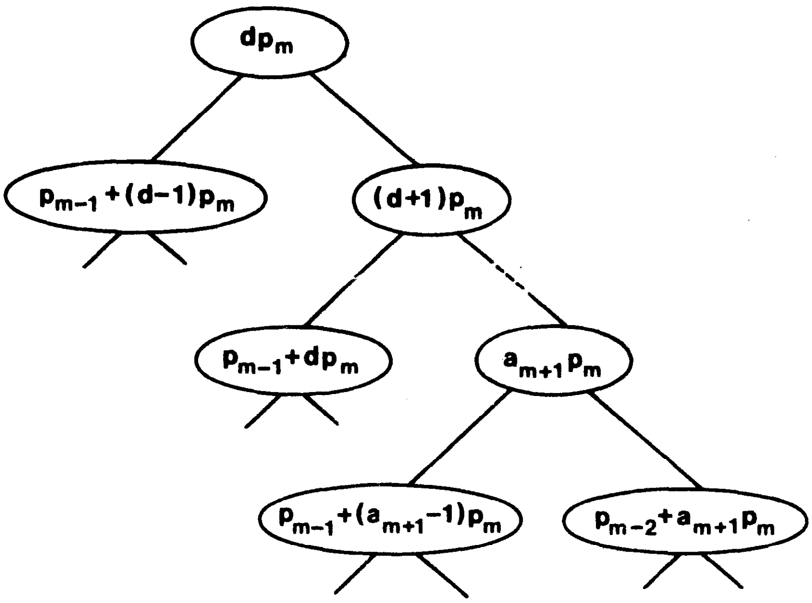

Fig. 3. The first few branches of $T^0_{m,d}$.

(iv) The first part was already proved. By (ii), the number of leaves of $T_{m,1}$ is $(p_{m+1}-1)-(q_{m+1}-1)=p_{m+1}-q_{m+1}$. For $\alpha=[1,\dot{a}]$ we have $p_i=q_i+q_{i-1}$ $(i\geq 0)$. This is indeed the case for $i=0$ and 1 by inspection. Assume true for $i<n$ $(n\geq 2)$. Then

$$p_n=ap_{n-1}+p_{n-2}=a(q_{n-1}+q_{n-2})+(q_{n-2}+q_{n-3})=q_n+q_{n-1}.$$

(v) The results directly follow from the definition of cedar trees and from Lemma 4.2.

(vi) It suffices to compute the longest and shortest path from the root of $T^0_{m,1}$—rather than $C_{m,1}$—to a leaf. By inspection, $L_0=l_0=a_1$, $L_1=a_2+a_1$, $l_1=l_0+1$. Since $L_{m-1}\geq L_{m-2}$ we have (see Fig. 3 in which we now put $d=1$),

$$L_m=\max\{1+L_{m-1},\, a_{m+1}+L_{m-1}\}=a_{m+1}+L_{m-1},$$

and the result $L_m=\sum_{i=1}^{m+1} a_i$ follows by induction on m. Since also $l_{m-1}\geq l_{m-2}$ we have,

$$l_m=\min\{1+l_{m-1},\, a_{m+1}+l_{m-2}\}.$$

If $a_i>1$ $(i\geq 1)$, then, assuming the result inductively, we get that $l_m=\min\{a_1+m,\, a_{m+1}+a_1+m-2\}=a_1+m+\min\{0,\, a_{m+1}-2\}=a_1+m$. If $a_i=1$ $(i\geq 1)$, then

$$l_m=1+l_{m-2}=1+\lfloor\tfrac{1}{2}(m+1)\rfloor=\lfloor\tfrac{1}{2}(m+3)\rfloor$$

by the induction hypothesis.

(vii) For traces of length 1, $\mathrm{tr}(u)=(c_1)=(0)$. The desired vertex v in this case is the root with the empty trace. Moreover, $u=(0,1,0,\ldots,0)$ $(m-1$ trailing zeros) and $v=(1,0,0,\ldots,0)$ (m trailing zeros), and so v is a left shift of u. Given u with

$\mathrm{tr}(u) = (0, c_2, \ldots, c_k)$ $(k \geq 2)$. By the induction hypothesis on the trace length, for the father u_1 of u with $\mathrm{tr}(u_1) = (0, c_2, \ldots, c_{k-1})$, there exists a node v_1 with $\mathrm{tr}(v_1) = (c_2, \ldots, c_{k-1})$, and v_1 is a left shift of u_1. Thus if $u_1 = (d_n, \ldots, d_l, 0, \ldots, 0)$, $d_l \neq 0$ (l trailing zeros), then $v_1 = (d_n, \ldots, d_l, 0, \ldots, 0)$ ($l+1$ trailing zeros). If u is a left son of u_1, then $l > 0$ and so a fortiori v_1 has a left son v. The trace of u is then $(0, c_2, \ldots, c_k)$ $(c_k = 0)$ and that of v is $(c_2, \ldots, c_k)$. Moreover, $u = (d_n, \ldots, d_l - 1, 1, 0, \ldots, 0)$ ($l-1$ trailing zeros) and v has the same form but with l trailing zeros, and so v is a left shift of u. A similar argument holds if u and v are right sons of u_1 and v_1 respectively.

Now let v be a node ending in zero and let u be the right shift of v. Since $I_p(u) < I_p(v)$, (iii) implies that $I_p(u)$ appears in $T^0_{m,1}$, the p-tree derived from $C^0_{m,1}$. Hence u appears in $C^0_{m,1}$. Since $I_p(v) < p_{m+1}$, we have $I_p(u) < p_m$. Hence if $\mathrm{tr}(v) = (c_2, \ldots, c_k)$ and $\mathrm{tr}(u) = (b_1, b_2, \ldots, b_k)$, then $b_1 = 0$. By the first part of (vii), we know that the left shift v of u has trace $(b_2, \ldots, b_k) \equiv (c_2, \ldots, c_k)$. Hence u has trace $(0, c_2, \ldots, c_k)$.

(viii) The result follows from [3, Theorem 4]. The first part of that theorem implies that if $R_q(n)$ ends in an even number of zeros, then $R_p\lfloor n\alpha \rfloor = R_q(n)$ is represented by a single node of $C^0_{m,1}$. The second part of the theorem and (v) imply that if $R_q(n)$ ends in an odd number of zeros, then $R_p\lfloor n\alpha \rfloor$ is a rightmost descendant of the left subtree of $R_q(n)$, which implies the result.

(ix) From (vii) we know that the left shift $w = u'$ is in $C^0_{m,1}$ and $\mathrm{tr}(w) = (c_2, \ldots, c_k)$. If $t < a$, then w has a right son x, hence a leftmost descendant v of x (which may be x itself). Clearly $\mathrm{tr}(v) = (c_2, \ldots, c_k, 1, 0, \ldots, 0)$; and $I_p(v) = I_p(w) + 1$ by (v). This implies that replacing the last digit of w by 1 gives v, completing the proof of the first part. For proving the second part, note that since w ends in 0, w has a left son y. If $t > 0$, then y ends in $t-1$, 1. Since $a > 1$, y has a right son z which ends in $t-1$, 2, and $\mathrm{tr}(z) = (c_2, \ldots, c_k, 0, 1)$. The converse of the first two parts is clear.

(x) The result directly follows from Property 3. □

7. Some uses of cedar trees

7.1 Search decision trees

A tree $T^0_{m,1}$ ([1]) is the decision tree of a so-called *Fibonacci search* (see, e.g., Knuth [6, Section 6.2.1]). More generally, a tree $T^0_{m,1}$ ([1, $\bar{a}$]) ($a \geq 1$ any integer) can be considered as the decision tree of a generalized Fibonacci search algorithm which, given a table of n numbers in increasing order, starts by comparing the argument searched for with the (n/g)th number (where $g = (p_{m+1} - 1)/p_m$), and iterates this procedure on the smaller blocks (see [6, Section 6.2.1, Exercise 20]). A generalized search in this sense is in fact defined by $T^0_{m,1}(\alpha)$ and by $\tau^0_{m,1}(\alpha)$ for every real number α. A seemingly first application of such searches is indicated below.

7.2. Wythoff games

In order to show how cedar trees can be used to consolidate Wythoff game strategies, it is useful to take stock of the main results obtained so far.

Normal play

We have the following three characterizations of the P-positions (A_n, B_n):

(I) $A_n = \operatorname{mex}\{A_i, B_i: 0 \leq i < n\}$, $B_n = A_n + an$ $(n \geq 0)$.

(II) $A_n = \lfloor n\alpha \rfloor$, $B_n = \lfloor n\beta \rfloor$, $\alpha = (2-a+\sqrt{a^2+4})/2$, $\beta = \alpha + a$ $(n \geq 0)$.

(III) (a) $\{A_n\}$ ($\{B_n\}$) is the set of all numbers whose p-representation relative to $\alpha = [1, \dot{a}]$ ends in an (1) even ((2) odd) number of zeros.

(b) $R_p(B_n) = R'_p(A_n)$.

(c) If $R_q(n)$ ends in (1) an even number of zeros, then $A_n = I_p(R_q(n))$; if it ends in (2) an odd number of zeros, then $A_n = I_p(R_q(n)) - 1$ $(n \geq 0)$.

Misère play

We have the following three characterizations of the P-positions (E_n, H_n):

(IV) (i) For $a = 1$, $(E_0, H_0) = (2, 2)$,

$$E_n = \operatorname{mex}\{E_i, H_i: 0 \leq i < n\}, \quad H_n = E_n + n \quad (n \geq 1).$$

(ii) For $a > 1$,

$$E_n = \operatorname{mex}\{E_i, H_i: 0 \leq i < n\}, \quad H_n = E_n + an + 1 \quad (n \geq 0).$$

(V) (i) For $a = 1$, $(E_0, H_0) = (2, 2)$, $(E_1, H_1) = (0, 1)$,

$$E_n \lfloor \tfrac{1}{2} n(1+\sqrt{5}) \rfloor, \quad H_n = \lfloor \tfrac{1}{2} n(3+\sqrt{5}) \rfloor \qquad (n \geq 2).$$

(ii) For $a > 1$,

$$E_n = \lfloor n\alpha + \gamma \rfloor, \quad H_n = \lfloor n\beta + \delta \rfloor \qquad (n \geq 0),$$

where $\alpha = \tfrac{1}{2}(2 - a + \sqrt{a^2+4})$,

$\beta = \alpha + a$, $\gamma = \alpha^{-1}$, $\delta = \gamma + 1$.

(VI) (i) For $a = 1$, E_n, H_n are the same as A_n, B_n in (III) above $(n \geq 2)$, except that $(E_0, H_0) = (2, 2)$, $(E_1, H_1) = (0, 1)$.

(ii) For $a > 1$, (a) E_n is the set of all numbers with p-representation ending in one of (1) $3, 4, \ldots, a$, $01\ldots1$even, $01\ldots12$even, $c1\ldots1$odd or $c1\ldots12$odd $(1 < c < a)$, and H_n is the set of all numbers with p-representation ending in one of (2) $01\ldots1$odd, $01\ldots12$odd, $c1\ldots1$even≥ 2 or $c1\ldots12$even $(n \geq 0)$. (b) $R_p(H_n)$ is $R'_p(E_n)$ with the last digit (zero) replaced by 1 (if the last digit t of $R_p(E_n)$ is 0 or 1) or by 2 and t replaced by $t-1$ (if $1 < t \leq a$) $(n \geq 0)$. (c) If $R_q(n)$ ends in (1) $01\ldots1$even or in $c1\ldots1$odd, then $E_n = I_p(R_q(n))$. If $R_q(n)$ ends in (2) $01\ldots1$odd or in $c1\ldots1$even, then $E_n = I_p(R_q(n)) + 1$ $(n \geq 0)$.

Note. Consider the following alternative definition of Wythoff games. The rules are as defined in Section 1, with the additional requirement that no player is ever permitted to move to a position of the form (x, x) $(x \geq 0)$. The player first unable to move is the loser, his opponent the winner. Then obviously the last position of the game is $(0, 1)$. Moreover, it is easy to see that this class S_2 of normal Wythoff games is equivalent to the class S_1 of misère Wythoff games according to the definition of Section 1 for every $a > 1$. But for $a = 1$ we get a different game, whose P-positions are consistent with those for $a > 1$ rather than different from them. For the alternative definition, the above summary for misère play can thus be simplified by omitting IV(i), V(i) and VI(i), and omitting $a > 1$ in IV(ii), V(ii) and VI(ii). (Note that the inequalities (v)–(viii) in Section 5 hold also for $a = 1$; the proofs did not use $a > 1$.)

As an illustration for the alternative definition we present the first few P-positions for $a = 1$ of the alternatively defined game in Table 5. But in the remaining part of this section we shall resort back to our original definition of misère Wythoff games given in Section 1.

Table 5
The representation of the first few P-positions (E_n, H_n) and n in the p and q-systems for $\alpha = [\bar{1}]$ in the alternatively defined Wythoff game ($a = 1$, normal play).

n	E_n	H_n	$R_p(E_n)$						$R_p(H_n)$							$R_q(n)$					
			p_5 13	p_4 8	p_3 5	p_2 3	p_1 2	p_0 1	p_6 21	p_5 13	p_4 8	p_3 5	p_2 3	p_1 2	p_0 1	q_5 8	q_4 5	q_3 3	q_2 2	q_1 1	q_0 1
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
1	2	4					1	0					1	0	1					1	0
2	3	6				1	0	0				1	0	0	1				1	0	0
3	5	9			1	0	0	0			1	0	0	0	1			1	0	0	0
4	7	12			1	0	1	0			1	0	1	0	1			1	0	1	0
5	8	14		1	0	0	0	0		1	0	0	0	0	1		1	0	0	0	0
6	10	17		1	0	0	1	0		1	0	0	1	0	1		1	0	0	1	0
7	11	19		1	0	1	0	0		1	0	1	0	0	1		1	0	1	0	0
8	13	22	1	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0
9	15	25	1	0	0	0	1	0	1	0	0	0	1	0	1	1	0	0	0	1	0
10	16	27	1	0	0	1	0	0	1	0	0	1	0	0	1	1	0	0	1	0	0
11	18	30	1	0	1	0	0	0	1	0	1	0	0	0	1	1	0	1	0	0	0

We shall now apply cedar trees to consolidate Wythoff game strategies. A connection between cedar trees and the first, second and third characterizations of the P-positions is given by the following algorithm:

While traversing a $C^{0}_{m,1}([1, \bar{a}])$ cedar tree $(a \geq 1)$ in *inorder* (see, e.g., Horowitz and Sahni [5, Section 6.1.1]), put $I_p(n)$ of every node u visited into a list A', B', E' or H' according to whether u ends in one of the strings of the form (III)(a)(1), (III)(a)(2), (VI)(ii)(a)(1) or (VI)(ii)(a)(2). Then A', B', E' and H' are the beginning segment of an $A_n = \lfloor n\alpha \rfloor$-sequence, a $B_n = \lfloor n\beta \rfloor$-sequence, an $E_n =$

$\lfloor n\alpha+\gamma\rfloor$-sequence and an $H_n=\lfloor n\beta+\delta\rfloor$-sequence. There are some 'boundary conditions': The A_n, B_n sequences have to be preceded by $A_0=0$, $B_0=0$; the E_n sequence by $E_0=0$. For $a=1$, the E_n and H_n sequences should be replaced by the A_n and B_n sequences ($n\geq 2$) and preceded by $(E_0, E_1)=(2,0)$ and $(H_0, H_1)=(2,1)$. In practice, this procedure may be simulated without actually constructing $C^0_{m,1}([1,\dot{a}])$: Start with a vector $(0,\ldots,0)$ ($m+1$ 0's) and, by 'adding 1', cycle through all representation vectors up to $(a0a0\ldots b)$, where $b=a$ (m even), $b=0$ (m odd).

The correctness of this algorithm is a direct consequence of the strategies summarized above and the fact that an inorder traversal of a cedar tree with property (i) of Theorem 6.1 sorts the entries in increasing lexicographic order.

A more intimate connection between cedar trees and the third characterizations of the P-positions will now be presented. We assume $a>1$ for misère play, since for $a=1$ the strategy is the same as for normal play except that the first two P-positions are different.

Given a position (x, y) in a normal or misère Wythoff game. We may assume $0<x<y$, since for $x=0$ and for $x=y$, the situation is quite clear. Let m be the smallest positive integer satisfying $p_m>x$, and let $u=R_p(x)$. We can compute u and $\mathrm{tr}(u)$ simultaneously by 'searching for u' in an (imaginary) p-tree with root p_m, proceeding in binary search tree fashion: 'Turn left' ('right') whenever x is smaller (larger) than the current node z. This simply means that if $z=\sum_{i=k}^{m} d_i p_i$ ($d_k\neq 0$), then 'turning left' amounts to replacing d_k by d_k-1 and adding p_{k-1} ($k\geq 1$); and 'turning right' means replacing d_k by d_k+1 (if $d_k<a_{k+1}$) or adding p_{k-2} (if $d_k=a_{k+1}$, $k\geq 2$).

The following algorithm is based on representations. Other variants, based on traces or a mixture of traces and representations can easily be formulated. An algorithm based on traces instead of representations has the advantage that binary sequences rather than representations are being handled and compared.

Statements in parentheses and in brackets refer to normal and misère play respectively. Comments appear in curly brackets.

Algorithm

Step 1. {Compute u}. Using the above search method, compute $u=R_p(x)$.

Step 2. {$x=(B_n)[H_n]$}. If u ends in a string of the form ((III)(a)(2)) [(VI)(ii)(a)(2)], make the move $(x,y)\to(I_p(w),x)\in P$, where $w=u''$ [with the rightmost digit t of u'' replaced by $t+1$ if the rightmost digit of u is 2]. **End.**

Step 3. {$x=(A_n)[E_n]$}. Denote by (s,t) the two rightmost digits of u. If u ends in a string of the form ((III)(a)(1)) [(VI)(ii)(a)(1)], let $v=u$ if ($t<a$) [$t=0$ or 1] ($v=u$ except that $(s+1,0)\leftarrow(s,t)$ if $t=a$) [$v=u$ except that $t-1\leftarrow t$ if $t>1$]. Compute $I_q(v)=n$. Then $(B_n)[H_n]=x+an$ [+1]. If $y=(B_n)[H_n]$, then $(x,y)\in P$. **End.**

Step 4. {$y>(B_n)[H_n]$}. If $y>(B_n)[H_n]$, make the move $(x,y)\to(x,(B_n)[H_n])\in P$. **End.**

Step 5. $\{(A_n<y<B_n)[E_n<y<H_n]\}$. Compute $d=\lfloor(y-x\ [-1])/a\rfloor$. Using the above search method, compute $w=R_q(d)$. If w ends in a string of the form ((III)(c)(1)) [(VI)(c)(1)], make the move $(x,y)\to(I_p(w), I_p(w)+ad\ [+1])\in P$. **End.** If w ends in a string of the form ((III)(c)(2)) [(VI)(c)(2)], make the move $((x,y)\to(I_p(w)-1, I_p(w)+ad-1)\in P)$ $[(x,y)\to(I_p(w)+1, I_p(w)+ad+2)\in P]$. **End.**

Verification. We shall verify the algorithm for misère play only, since the argument for normal play is very similar.

In Step 2 we have $x=H_n$ for some $n\geqslant 0$ by Property 1. The node w constructed in Step 2 is $R_p(E_n)$ by the inverse of Property 2. In Step 3 we have $x=E_n$ by Property 1. The inverse of Property 3 implies that $I_q(v)=n$; and $H_n=x+an+1$ by Theorem 2.1. The correctness of Step 4 is obvious. The correctness of Step 5 follows from Property 3 and Case 2 of part (ii) of the proof of Theorem 2.1. It only remains to show that there is a node w in $C^0_{m,1}$ satisfying $w=R_q(d)$. This follows from

$$d=\left\lfloor\frac{y-x-1}{a}\right\rfloor\leqslant\frac{y-x-1}{a}<\frac{H_n-E_n-1}{a}\qquad\{\text{since } E_n=x<y<H_n\}$$

$$=n\leqslant E_n\qquad\{\text{since } \alpha>1\Rightarrow E_n=\lfloor n\alpha\rfloor\geqslant n\}$$

$$=x<p_m=q_m+q_{m-1}\qquad\{\text{by the last part of Theorem 6.1(iv)}\}$$

$$\leqslant aq_m+q_{m-1}=q_{m+1}.$$

Hence $d\leqslant q_{m+1}-1$, so d appears in $\tau^0_{m,1}$ by Theorem 6.1(iii), hence $R_q(d)$ appears in $C^0_{m,1}$.

The complexity of the algorithm is the same as that of the algorithm based on the arithmetic characterization, and the space requirement is also limited to $O(\log x)$.

Note. An earlier paper of the present author [3] originally contained the analysis of normal Wythoff games and the basic theory of cedar trees and their use for consolidating normal play. Since the two referees of [3] recommended to publish the cedar tree part separately, the present paper resulted, to which we have added, however, the misère analysis, which, as shown above, is also consolidated by means of cedar trees.

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