point. Unfortunately, for this purpose, Lemma 1 does not provide a usable expression for $c_i'(t)$. Nor can we apply the simpler Theorem 2, which is valid only for a single value $t = t_0$.

An interesting exercise is to completely characterize the $n = 3$ case, where

$$y = p_t(x) = A \prod_{i=1}^{3} (x - r_i(t)).$$

It is not hard to explain what happens when roots collide, or to show that a critical point can change directions at most once. When two roots collide, Theorem 2 implies that the critical point between the two roots will move away from the collision in the direction of the fastest moving root. We can describe the triple root collision qualitatively, despite the fact that Theorem 2 does not apply in this case. Indeed, r_1, r_2, r_3, and c are all odd functions of t. The fact that a critical point changes direction at most once, which follows as $\frac{dc}{dt}$ is monotonic when $n = 3$, was a complete surprise to us: we thought that it would be possible to find velocities and initial positions of the roots that would send fast-moving roots shooting past the critical point at different times, from opposite directions, producing at least two changes in direction. What can one say in the degree-n case?

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From Enmity to Amity

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Abstract. Sloane's influential On-Line Encyclopedia of Integer Sequences is an indispensable research tool in the service of the mathematical community. The sequence A001611 listing the "Fibonacci numbers + 1" contains a very large number of references and links. The sequence A000071 for the "Fibonacci numbers −1" contains an even larger number. Strangely, resentment seems to prevail between the two sequences; they do not acknowledge each other's existence, though both stem from the Fibonacci numbers. Using an elegant result of Kimberling, we prove a theorem that links the two sequences amicably. We relate the theorem to a result about iterations of the floor function, which introduces a new game.

doi:10.4169/000298910X496787

1. INTRODUCTION Sloane's On-Line Encyclopedia of Integer Sequences [**5**] is well known. It is of major assistance to numerous mathematicians and fuses together diverse lines of mathematical research. For example, searching for 2, 3, 4, 6, 9, 14, 22, 35, 56, 90, 145, ... leads to sequence A001611, the "Fibonacci numbers + 1," listing about $\aleph_0$ comments, references, links, formulas, Maple and Mathematica programs, and cross-references to other sequences. Everybody can see that Sequence A000071, which lists the "Fibonacci numbers − 1," has even more material, so it must contain at least $\aleph_1$ comments, references, links, formulas, Maple and Mathematica programs, cross-references to other sequences, and extensions.

Though there are two (unpublished) links common to the two sequences, the respective lists of references of the two sequences have an empty intersection; even in the "adjacent sequences," the sequences do not acknowledge each other. Moreover, there is no cross-reference from one sequence to the other. This is astonishing, bordering on the offensive, since both sequences stem from the same source, the Fibonacci numbers. Are they antagonistic to each other? Our purpose is to show that there should be no animosity between the two sequences; both coexist peacefully in some applications.

2. KIMBERLING'S THEOREM. Let $F_{-2} = 0$, $F_{-1} = 1$, $F_n = F_{n-1} + F_{n-2}$ ($n \geq 0$) be the Fibonacci sequence. (For technical reasons we use an indexing that differs from the usual.) Let $a(n) = \lfloor n\tau \rfloor$ and $b(n) = \lfloor n\tau^2 \rfloor$, where $\tau = (1 + \sqrt{5})/2$ denotes the golden section. We consider iterations of these sequences. An example of an iterated identity is $a(b(n)) = a(n) + b(n)$. It can be abbreviated as $ab = a + b$, where the suppressed variable n is assumed to range over all positive integers, unless otherwise specified. Consider the word $w = \ell_1\ell_2 \cdots \ell_k$ of length k over the binary alphabet $\{a, b\}$, where the product means iteration (composition). The number m of occurrences of the letter b is the *weight* of w. Recently, Clark Kimberling [**4**] proved the following nice and elegant result:

Theorem I. *For $k \geq 2$, let $w = \ell_1\ell_2 \cdots \ell_k$ be any word over $\{a, b\}$ of length k and weight m. Then $w = F_{k+m-4}a + F_{k+m-3}b - c$, where $c = F_{k+m-1} - w(1) \geq 0$ is independent of n.*

Notice that in the theorem—where $w(1)$ is w evaluated at $n = 1$—only the weight m appears, not the locations within w where the bs appear. Their locations, however, obviously influence the behavior of w. This influence is hidden in the "constant" $c = c_{k,m,w(1)}$.

Examples.

(i) Consider the case $m = 0$. Theorem I gives directly $a^k = F_{k-4}a + F_{k-3}b - F_{k-1} + 1$, since $\lfloor\tau\rfloor = 1$, so $w(1) = \lfloor\tau \ldots \lfloor\tau\lfloor\tau\rfloor\rfloor \ldots\rfloor = 1$.

(ii) $m = 1$, $w = ba^{k-1}$. Then $w(1) = \lfloor\tau^2\lfloor\tau \ldots \lfloor\tau\lfloor\tau\rfloor\rfloor \ldots\rfloor = 2$, since $\lfloor\tau^2\rfloor = 2$. Hence $ba^{k-1} = F_{k-3}a + F_{k-2}b - F_k + 2$.

(iii) $m = 1$, $w = a^{k-1}b$. Then $w(1) = a^{k-1}b(1) = \lfloor\tau \ldots \lfloor\tau\lfloor\tau^2\rfloor\rfloor \ldots\rfloor$. What's the value of of this expression? The answer is given in the next section.

3. AN APPLICATION.

Theorem 1. *Suppose that $k \geq 1$, and let $w = a^{k-1}b$. Then $w(1) = a^{k-1}b(1) = F_{k-1} + 1$; thus $c_{k,m,w(1)} = F_{k-2} - 1$, and $w = a^{k-1}b = F_{k-3}a + F_{k-2}b - (F_{k-2} - 1)$.*

We see, in particular, that in a single theorem we have both "Fibonacci numbers + 1" (for $w(1)$) and "Fibonacci numbers − 1" (for $w = w(n)$), coexisting amicably.

Proof. The ratios F_k/F_{k-1} are the convergents of the simple continued fraction expansion of $\tau = [1, 1, 1, \dots]$. Therefore $0 < \tau F_{2k+1} - F_{2k+2} < F_{2k+1}^{-1}$ and $-F_{2k}^{-1} < \tau F_{2k} - F_{2k+1} < 0$ (see, e.g., [**3**, Ch. 10]). We may thus write

$$\tau F_{2k+1} - F_{2k+2} = \delta_1(k), \text{ where } 0 < \delta_1(k) < F_{2k+1}^{-1},$$

and

$$\tau F_{2k} - F_{2k+1} = \delta_2(k), \text{ where } -F_{2k}^{-1} < \delta_2(k) < 0.$$

We note that $b(1) = \lfloor \tau^2 \rfloor = 2 = F_0 + 1$, $ab(1) = \lfloor \tau \lfloor \tau^2 \rfloor \rfloor = \lfloor 2\tau \rfloor = 3 = F_1 + 1$, and $a^2 b(1) = \lfloor 3\tau \rfloor = 4 = F_2 + 1$. To complete the proof, we proceed by induction. Suppose that $a^j b(1) = F_j + 1$ for some $j \geq 2$.

We consider two cases. If j is even, then $j = 2k$ for some $k \geq 1$. Using the induction hypothesis, we get

$$a^{j+1}b(1) = a(a^{2k}b(1)) = \lfloor \tau(F_{2k} + 1) \rfloor = F_{2k+1} + 1 + \lfloor \tau^{-1} + \delta_2(k) \rfloor = F_{2k+1} + 1,$$

since for $k \geq 1$, $F_{2k} \geq F_2 = 3$ so $-1/3 < \delta_2(k) < 0$, and $0.6 < \tau - 1 = \tau^{-1} < 0.62$.

Similarly, if j is odd then $j = 2k + 1$ for some $k \geq 1$, and we get

$$\begin{aligned} a^{j+1}b(1) &= a(a^{2k+1}b(1)) = \lfloor \tau(F_{2k+1} + 1) \rfloor \\ &= F_{2k+2} + 1 + \lfloor \tau^{-1} + \delta_1(k) \rfloor = F_{2k+2} + 1, \end{aligned}$$

since for $k \geq 1$, $F_{2k+1} \geq 5$, so $0 < \delta_1(k) < 1/5$. ∎

The word $a^{k-1}b$ features in many identities proved in [**2**]. In particular, b, ab, a^2b—as well as a^3—play a prominent role in the *Flora* game defined and analyzed there.

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