Euclid and Wythoff games

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Abstract

Two characterizations of the Sprague-Grundy function values of Euclid's game, in terms of the winning strategy of the generalized Wythoff game, are given.

Keywords: Euclid's game, Wythoff's game, Sprague-Grundy function

Both Euclid and the generalized Wythoff game with parameter $n \in \mathbb{Z}_{>0}$ (abbreviated as GW_n) are two-person games played on two integers (x, y). Their moves, however, are quite different. In Euclid, the integers remain positive throughout; a move consists of decreasing the larger number by any positive multiple of the smaller, as long as the result remains positive. The player first unable to move loses [4]. A polynomial-time algorithm for computing the Sprague-Grundy function (g-function) for Euclid is given in [5].

The game GW_n is played over $\mathbb{Z}_{\geq 0}$. The moves are of two types: removal of any positive integer from *one* of the numbers; or removal of $k > 0$ from one and $\ell > 0$ from the other, subject to $|k - \ell| < n$ for fixed $n \in \mathbb{Z}_{>0}$. The player unable to move because the position is $(0, 0)$ loses. See [3] for three algorithms for the 0s of the g-function, two of which are polynomial in the logarithmic size of the two piles. The case $n = 1$ is the classical Wythoff game, see e.g., [7], [2], [8].

For both games we assume throughout, without loss of generality, that $x \leq y$, except when stated otherwise.

For $n \in \mathbb{Z}_{\geq 0}$, let ϕ_n be the larger of the two roots of $z - z^{-1} = n$. Then $\phi_n = (n + \sqrt{n^2 + 4})/2$, $\phi_n^{-1} = (-n + \sqrt{n^2 + 4})/2$. It follows from [5] that for Euclid, $g(x, y) = n$ precisely for all y in the closed integer interval $[\lceil \phi_n x \rceil, \lfloor \phi_{n+1} x \rfloor]$. For computing the g-function it thus suffices to compute the boundary points $\lfloor \phi_n x \rfloor$. (For $n \in \mathbb{Z}_{>0}$, $\lceil \phi_n x \rceil = 1 + \lfloor \phi_n x \rfloor$; $\phi_0 = 1$, ϕ_1 the golden section.)

In this note we present two characterizations of the extremal points $\lfloor \phi_n x \rfloor$, which also reveal, incidentally, a curious connection between the strategies of GW_n and Euclid.

From the identity $(1+z)^{-1}+(1+z^{-1})^{-1}=1$, which holds for all reals $z\notin\{0,-1\}$, it follows that the two sequences $\cup_{k=1}^{\infty}\lfloor k(1+z)\rfloor$ and $\cup_{k=1}^{\infty}\lfloor k(1+z^{-1})\rfloor$ are complementary with respect to $\mathbb{Z}_{>0}$ if z is a positive irrational number. For $n\in\mathbb{Z}_{>0}$, let $\alpha_n=1+\phi_n^{-1}$, $\beta_n=1+\phi_n$. Then $\alpha_n^{-1}+\beta_n^{-1}=1$, $\beta_n-\alpha_n=n$, so $\{(A_x,B_x): x\in\mathbb{Z}_{\geq 0}\}$ is the set of all 0-values of g for GW_n, where $A_x=\lfloor\alpha_n x\rfloor$, $B_x=\lfloor\beta_n x\rfloor$ ([3], §3). They are the second-player win positions of GW_n, also called P-positions. Further,

$$A_x=\mathrm{mex}\{A_i,B_i: 0\leq i<x\},\ B_x=A_x+nx\quad\forall x\in\mathbb{Z}_{\geq 0},\tag{1}$$

where $\mathrm{mex}(S)$ for any set $S\subset\mathbb{Z}_{\geq 0}$, $S\neq\mathbb{Z}_{\geq 0}$, is the smallest nonnegative integer not in S. We have proved,

Theorem 1 *Let $n,x\in\mathbb{Z}_{>0}$. Then $\lfloor\phi_n x\rfloor=B_x-x$, where B_x is given by* (1).

Thus the g-value $n-1$ of Euclid, $n\in\mathbb{Z}_{>0}$, is given in terms of the P-positions (1) of the generalized Wythoff game with parameter n. (Using (1) we can also compute $B_{x+1},B_{x+2},\ldots$ via $A_{x+1},A_{x+2},\ldots$.)

The second characterization of $\lfloor\phi_n x\rfloor$ is obtained by means of an exotic numeration system. It could be done in terms of the numeration system induced by α_n, as in [3]. But the result is nicer using a numeration system based on ϕ_n. For stating it, we recall a few basic facts from the theory of continued fractions [6]. Let α be an irrational number with simple continued fraction expansion $\alpha=[a_0,a_1,a_2,\ldots]$ where $a_0\in\mathbb{Z}$, $a_i\in\mathbb{Z}_{>0}$ for $i>0$. Its *convergents* $[a_0,a_1,\ldots,a_k]$ satisfy

$$p_{-1}=1,\ p_0=a_0,\ p_k=a_kp_{k-1}+p_{k-2}\quad(k\geq 1),$$

$$q_{-1}=0,\ q_0=1,\ q_k=a_kq_{k-1}+q_{k-2}\quad(k\geq 1).$$

The *q-numeration system* is the numeration system based on the denominators q_i of the convergents of α. For any $N\in\mathbb{Z}_{>0}$, N can be written uniquely in the q-system in the form:

$$N=\sum_{i=0}^{m}d_iq_i,\ 0\leq d_0<a_1,\ 0\leq d_i\leq a_{i+1},\ d_i=a_{i+1}\Longrightarrow d_{i-1}=0\ (i\geq 1).\tag{2}$$

For the present case we introduce also an *alternate* q-numeration system, denoted by q'-system, which is identical to the q-system, except that $d_0<a_1$ is replaced by $d_0\leq a_1$. Then some integers have two representations: one with $d_0<a_1$, and the other with $d_0=a_1$. In these cases we always select the latter. Then any $N\in\mathbb{Z}_{>0}$ can be written uniquely in the form (2) with $d_0<a_1$ replaced by $d_0\leq a_1$.

For $i\geq -1$, let $D_i=\alpha q_i-p_i$. From the theory of continued fractions it is known that

$$-1=D_{-1}<D_1<D_3<\ldots<0<\ldots<D_4<D_2<D_0=\alpha-a_0.$$

Lemma 1 *For* $j \geq -1$, $m \geq 1$, $D_j + \sum_{i=1}^{m} a_{j+2i} D_{j+2i-1} = D_{j+2m}$.

This follows from the fact that $D_j + a_{j+2}D_{j+1} = D_{j+2}$, and D_{j+2} can be added to the next term $a_{j+4}D_{j+3}$ to give D_{j+4}, etc. A formal proof is given in [3], §5.

Note that for $n \in \mathbb{Z}_{>0}$, $\phi_n = [n, n, n, \ldots]$ is the simple continued fraction expansion of the irrational number ϕ_n.

Extending a notation of [1], a number in the q- or q'-system of ϕ_n $(n > 0)$ is called *evil* if it ends in an even number of 0s. Otherwise it is *odious*. Given a number $N = \sum d_i q_i$ in the q- or q'-system, the number $\mathcal{L}(N) = \sum d_i q_{i+1}$ is the *left shift* of N. By induction we see that $q_m = p_{m-1}$ for $m \geq 1$.

Lemma 2 *Let* $n \in \mathbb{Z}_{>0}$. *Every positive integer is evil in the* q'*-system of* ϕ_n.

Proof. It is readily seen that for every $m \geq 0$, $n \sum_{i=0}^{m} q_{2i} = q_{2m+1}$. Hence any odious representation ending in $d_{2m+1}, 0, \ldots, 0$ $(2m+1$ trailing 0s), $d_{2m+1} \neq 0$, can also be represented in the evil form ending in $(d_{2m+1} - 1), n, 0, n, 0, \ldots, 0, n$, and only the latter is legitimate in the q'-system. ■

Theorem 2 *Let* $n, x \in \mathbb{Z}_{>0}$. *If* $x = \sum_{i=0}^{m} d_i q_i$ *in the* q'*-system for suitable* $m \geq 0$, *then* $\lfloor \phi_n x \rfloor = \sum_{i=0}^{m} d_i p_i = \sum_{i=0}^{m} d_i q_{i+1}$. *In particular,* $\lfloor \phi_n x \rfloor$ *is odious.*

Proof. By Lemma 2, every $x \in \mathbb{Z}_{>0}$ has an evil representation of the form $x = \sum_{i=0}^{m} d_i q_i$, $0 \leq d_i \leq n$ in the q'-system. It suffices to show that $0 < x\phi_n - \sum_{i=0}^{m} d_i p_i < 1$, since then $\lfloor \phi_n x \rfloor = \sum_{i=0}^{m} d_i p_i$. By Lemma 1, for $t = \lceil m/2 \rceil$ we have, $x\phi_n - \sum_{i=0}^{m} d_i p_i = \sum_{i=0}^{m} d_i D_i \leq n \sum_{i=0}^{t} D_{2i} = D_{2t+1} - D_{-1} < -D_{-1} = 1$. Further, $\sum_{i=0}^{m} d_i D_i \geq n \sum_{i=0}^{t} D_{2i+1} - D_1 + D_0 = D_{2t+2} - D_1 > 0$. ■

Consequences. (i) For $n, x \in \mathbb{Z}_{>0}$, represent x in the q'-system of ϕ_n. Then $y = \mathcal{L}(x)$ is the largest lattice point satisfying $g(x, y) = n - 1$, and $y_1 = 1 + \mathcal{L}(x)$ is the smallest lattice point such that $g(x, y_1) = n$. The set $\mathcal{L}(x)$, $x \geq 1$, is the set of all odious numbers in the q-system of ϕ_n.

(ii) For computing $y = \lfloor \phi_n x \rfloor$ it suffices to express x in the q'-system of ϕ_n. Its left shift is y (which can also be expressed in terms of the x-th P-position of GW_n). This computation is clearly polynomial in the succinct input size $O(\log x)$. The points $\mathcal{L}(x)$, $x \geq 1$, and $1 + \mathcal{L}(x)$ straddle the ray $y = \phi_n x$ from below and above respectively with all the lattice points closest to that ray.

(iii) To win in a sum of games from a position (x, y), $x \leq y$, the winner has to move, at each stage, to (x, y') (possibly $y' < x$), satisfying $g(x, y') = n$ for specified n (0 for a single game). Since x, n are given, the above method yields immediately the value $y' = y - kx$, where k is the smallest positive integer such that $y - kx \leq \mathcal{L}(x)$ and x is represented in the q'-system of ϕ_{n+1}.

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