Acquaintance Time of a Graph

Itai Benjamini* Igor Shinkar* Gilad Tsur*

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Abstract

We define the following parameter of connected graphs. For a given graph $G = (V, E)$ we place one agent in each vertex $v \in V$. Every pair of agents sharing a common edge are declared to be acquainted. In each round we choose some matching of G (not necessarily a maximal matching), and for each edge in the matching the agents on this edge swap places. After the swap, again, every pair of agents sharing a common edge are acquainted, and the process continues. We define the *acquaintance time* of a graph G, denoted by $\mathcal{AC}(G)$, to be the minimal number of rounds required until every two agents are acquainted.

We first study the acquaintance time for some natural families of graphs including the path, expanders, the binary tree, and the complete bipartite graph. We also show that for all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying $1 \le f(n) \le n^{1.5}$ there is a family of graphs $\{G_n = (V_n, E_n)\}_{n \in \mathbb{N}}$ with $|V_n| = n$ such that $\mathcal{AC}(G_n) = \Theta(f(n))$. We also prove that for all n-vertex graphs G we have $\mathcal{AC}(G) = O\left(\frac{n^2}{\log(n)/\log\log(n)}\right)$, thus improving the trivial upper bound of $O(n^2)$ achieved by sequentially letting each agent perform depth-first search along some spanning tree of G.

Studying the computational complexity of this problem, we prove that for any constant $t \ge 1$ the problem of deciding that a given graph G has $\mathcal{AC}(G) \le t$ or $\mathcal{AC}(G) \ge 2t$ is $\mathcal{NP}$-complete. That is, $\mathcal{AC}(G)$ is $\mathcal{NP}$-hard to approximate within multiplicative factor of 2, as well as within any additive constant factor.

On the algorithmic side, we give a deterministic polynomial time algorithm that given an n-vertex graph G distinguishes between the cases $\mathcal{AC}(G) = 1$ and $AC(G) \ge n - O(1)$. We also give a randomized polynomial time algorithm that distinguishes between the cases $\mathcal{AC}(G) = 1$ and $AC(G) = \Omega(\log(n))$ with high probability.

*{itai.benjamini,igor.shinkar,gilad.tsur}@weizmann.ac.il Faculty of Mathematics and Computer Science, Weizmann Institute of Science, Rehovot, Israel.

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1 Introduction

In this work we deal with a problem where agents walk on a graph meeting each other, and our goal is to make every pair of agents meet as fast as possible. Specifically, we introduce the following parameter of connected graphs. For a given graph $G = (V, E)$ we place one agent in each vertex of the graph. Every pair of agents sharing a common edge are declared to be acquainted. In each round we choose some matching of G (not necessarily a maximal matching), and for each edge in the matching the agents on this edge swap places. After the swap, again, every pair of agents sharing a common edge are acquainted, and the process continues. We define the *acquaintance time* of a graph G, denoted by $\mathcal{AC}(G)$, to be the minimal number of rounds required until every two agents are acquainted with each other.

Several problems of similar flavor have been studies in the past. One such problem is the Routing Permutation on Graphs via Matchings studied by Alon, Chung, and Graham in [ACG94], where the input is a graph $G = (V, E)$ and a permutation of the vertices $\sigma : V \to V$, and the goal is to route all agents to their respective destinations according to σ; that is the agent sitting originally in the vertex v should be routed to the vertex $\sigma(v)$ for all $v \in V$. In our setting we encounter a similar routing problem, where we route the agents from some set of vertices $S \subseteq V$ to some $T \subseteq V$ without specifying the target location in T of each of the agents. More related problems include the well studied problems of Gossiping and Broadcasting (see the survey of Hedetniemi, Hedetniemi, and Liestman [HHL88] for details), and the Target Set Selection Problem (see, e.g., [KKT03, Che09, Rei12]).

In order to get some feeling regarding this parameter note that if for a given graph G a list of matchings $(M_1, \ldots, M_t)$ is a witness-strategy for the assertion that $\mathcal{AC}(G) \leq t$, then the inverse list $(M_t, \ldots, M_1)$ is also a witness-strategy for this assertion. We remark that in general a witness-strategy is not commutative in the order of the matchings.[1] For a trivial bound of $\mathcal{AC}(G)$ we have $\mathcal{AC}(G) \geq \mathrm{diam}(G)/2$, where $\mathrm{diam}(G)$ is the maximal distance between two vertices of G. It is also easy to see that for every graph $G = (V, E)$ with n vertices it holds that $\mathcal{AC}(G) \geq \frac{\binom{n}{2}}{|E|} - 1$. Indeed, before the first round exactly $|E|$ pairs of agents are acquainted. Similarly, in each round at most $|E|$ new pairs get acquainted. This implies that $|E| + \mathcal{AC}(G) \cdot |E| \geq \binom{n}{2}$, since in any solution the total number of pairs that met up to time $\mathcal{AC}(G)$ is $\binom{n}{2}$.

[1]For example, let $G = (V = \{1, 2, 3, 4\}, E = \{(1, 2), (2, 3), (3, 4\}$ be the path of length 4. Then, the sequence $(M_1 = \{(1, 2)\}, M_2 = \{(3, 4)\}, M_3 = \{(1, 2)\})$ is a strategy for acquaintance in G, whereas, the sequence $(M_1 = \{(1, 2)\}, M_3 = \{(1, 2)\}, M_2 = \{(3, 4)\})$ is not.

For an upper bound, note that for every graph G with n vertices $\mathcal{AC}(G) \le 2n^2$, as every agent can meet all others by traversing the graph along some spanning tree in at most $2n$ rounds.

We also consider the corresponding computational problem. Obviously, for $t \in \mathbb{N}$ the problem of deciding whether a given graph G has $\mathcal{AC}(G) \le t$ is in $\mathcal{NP}$, and the natural $\mathcal{NP}$-witness is a sequence of t matchings that allows every two agents to get acquainted. We prove that the acquaintance time problem is $\mathcal{NP}$-complete, by showing a reduction from the coloring problem. We show that $\mathcal{AC}(G)$ is $\mathcal{NP}$-hard to approximate within multiplicative factor of 2, as well as any additive constant factor. In fact, it is $\mathcal{NP}$-hard to decide whether $\mathcal{AC}(G) = 1$ for a given input graph G. On a higher level, this problem seems to differ from the classical $\mathcal{NP}$-complete problems, such as graph coloring or vertex cover in the sense that $\mathcal{AC}(G)$ is a "dynamic" parameter that studies some evolution in time.

We also study the algorithmic aspect of the problem when restricted to graphs G with $\mathcal{AC}(G) = 1$. We show that there is a deterministic polynomial time algorithm that given an n-vertex graph G distinguishes between the cases $\mathcal{AC}(G) = 1$ and $AC(G) \ge n - O(1)$. In addition, we give a randomized polynomial time algorithm that distinguishes with high probability between the cases of $\mathcal{AC}(G) = 1$ and $AC(G) = \Omega(\log(n))$.

1.1 Our results

We start this work by providing asymptotic computations of the acquaintance time for some interesting families of graphs. For instance, if P_n is the path of length n, then $\mathcal{AC}(P_n) = O(n)$, which is tight up to a multiplicative constant, since $\mathrm{diam}(P_n) = n$. In particular, this implies that $\mathcal{AC}(H) = O(n)$ for all Hamiltonian graphs H with n vertices. We also prove that for constant degree expanders $G = (V, E)$ on n vertices the acquaintance time is $O(n)$, which is tight, as $|E| = O(n)$ and $\mathcal{AC}(G) = \Omega(\frac{n^2}{|E|})$. More examples include the binary tree, the complete bipartite graph, and the barbell graph.

We then provide examples of graphs with different ranges of the acquaintance time. We show in Theorem 5.1 that for every function $f : \mathbb{N} \to \mathbb{N}$ that satisfies $1 \le f(n) \le n^{1.5}$ there is a family of graphs $\{G_n\}_{n \in \mathbb{N}}$ on n vertices with $\mathcal{AC}(G_n) = \Theta(f(n))$. Another interesting result says that for every graph G with n vertices the acquaintance time is, in fact, asymptotically smaller than the trivial $O(n^2)$ bound. Specifically, we prove in Theorem 5.5 that for every graph G with n vertices $\mathcal{AC}(G) = O\left(\frac{n^2}{\log(n)/\log\log(n)}\right)$. It would be interesting to close this gap, and find the range of $\mathcal{AC}(G)$, when going over all graphs G with n vertices.

We also study the problem of computing/approximating $\mathcal{AC}(G)$ for a given graph G. As explained above, the problem of deciding whether a given graph G has acquaintance time at most t is in $\mathcal{NP}$, and the $\mathcal{NP}$-solution is the sequence of at most t matchings that allow every two agents to meet. Such sequence is called *a strategy for acquaintance in G*. We prove that the acquaintance time problem is $\mathcal{NP}$-complete, by showing a reduction from the coloring problem. Specifically, Theorem 6.1 says that for every $t \geq 1$ it is $\mathcal{NP}$-hard to distinguish whether a given graph G has $\mathcal{AC}(G) \leq t$ or $\mathcal{AC}(G) \geq 2t$. Hence, $\mathcal{AC}(G)$ is $\mathcal{NP}$-hard to approximate within multiplicative constant of 2, as well as any additive constant. In fact, we conjecture that it is $\mathcal{NP}$-hard to approximate $\mathcal{AC}$ within any multiplicative constant.

On the algorithmic side we study graphs whose acquaintance time equals to 1. We show there is a deterministic polynomial time algorithm that when given an n-vertex graph G with $\mathcal{AC}(G) = 1$ finds a strategy for acquaintance that consists of $n - O(1)$ matchings. We also design a randomized polynomial time algorithm that when given an n-vertex graph G with $\mathcal{AC}(G) = 1$ finds with high probability a $O(\log(n))$-rounds strategy for acquaintance.

2 Definitions and Notations

Throughout the paper all graphs are simple and undirected. We use standard notations for the standard parameters of graphs. Given a graph $G = (V, E)$ and two vertices $u, v \in V$ the distance between u and v, denoted by $\mathrm{dist}(u, v)$, is the length of a shortest path from u to v in G. For a vertex v and a set of vertices $U \subseteq V$ the distance of v from U is defined to be $\mathrm{dist}(v, U) = \min_{u \in U} \mathrm{dist}(v, u)$. The diameter of the graph G, denoted by $\mathrm{diam}(G)$, is the maximal distance between two vertices of the graph. For a vertex $u \in V$ the set of neighbors of u is denoted by $N(u) = \{w \in V : (u, w) \in E\}$. Similarly, for a set $U \subseteq V$ the set of neighbors of U is denoted by $N(U) = \{w \in V : \exists u \in U \text{ such that } (u, w) \in E\}$. The *independence number of G*, denoted by $\alpha(G)$, is the cardinality of the largest independent set, that is, a set of vertices in the graph, no two of which are adjacent. The *chromatic number of G*, denoted by $\chi(G)$, is the minimal number $c \in \mathbb{N}$ such that there is a mapping $f : V \to \{1, \ldots, c\}$ of the vertices that satisfies $f(v) \neq f(w)$ for all edges $(v, w) \in E$. The *equi-chromatic number of G*, denoted by $\chi_{\mathrm{eq}}(G)$, is the minimal number $c \in \mathbb{N}$ such that there is a *balanced* mapping $f : V \to \{1, \ldots, c\}$ that satisfies $f(v) \neq f(w)$ for all edges $(v, w) \in E$, where a mapping $f : V \to \{1, \ldots, c\}$ is said to be balanced if $|f^{-1}(i)| = |f^{-1}(j)|$ for all $i, j \in \{1, \ldots, c\}$.

For a given graph $G = (V, E)$ the acquaintance time is defined as follows. We place one agent in each vertex $v \in V$. Every pair of agents sharing a common edge are declared to be acquainted. In each round we choose some matching of G, and for each edge in the matching the agents on this edge swap places. After the swap, again, every pair of agents sharing a common edge are acquainted, and the process continues. A sequence of matchings in the graph is called *a strategy*. A strategy that allows every pair of agents to meet is called *a strategy for acquaintance in* G. The acquaintance time of G, denoted by $\mathcal{AC}(G)$, is the minimal number of rounds required for such strategy.

As hinted in the introduction, this problem is related to certain routing problem similar to the one studied in [ACG94]. Specifically, we are interested in the routing task summarized in the following claim. For a given tree $G = (V, E)$ the claim gives a strategy for fast routing of the agents from some set of vertices $S \subseteq V$ to $T \subseteq V$ without specifying the target location in T of each of the agents.

Claim 2.1 *Let $G = (V, E)$ be a tree. Let $S, T \subseteq V$ be two subsets of the vertices of equal size $k = |S| = |T|$, and let $\ell = \max_{v \in S, u \in T}\{\mathrm{dist}(v, u)\}$ be the maximal distance between a vertex in S and a vertex in T. Then, there is a strategy of $\ell + 2(k-1)$ matchings that routes all agents from S to T.*

Proof Let $G = (V, E)$ be a tree, and let $S, T \subseteq V$ be two subsets of the vertices of G. The proof is by induction on k. For the case of $k = 1$ the statement is trivial, as ℓ rounds are enough to route a single agent.

For the induction step let $k \geq 2$, and assume for simplicity that the only agents in the graph are those sitting in S, and our goal is to route them to T. Let $\mathrm{span}(S)$ be the minimal subtree of G containing all vertices $s \in S$, and define $\mathrm{span}(T)$ analogously. Let us assume without loss of generality that there is some $s^* \in S$ that is not contained in $\mathrm{span}(T)$.[2] Let $t^* \in T$ be a vertex such that $\mathrm{dist}(s^*, t^*) = \mathrm{dist}(s^*, T)$, and let $P = (p_0 = s^*, p_1, \ldots, p_r, p_{r+1} = t^*)$ be the shortest path from s^* to t^* in G (note that $r \leq \ell$ by definition of ℓ). By induction hypothesis, there is a strategy consisting of $\ell + 2(k-2)$ rounds that routes the agents from $S \setminus \{s^*\}$ to $T \setminus \{t^*\}$. In such a strategy after the last step all agents are in $T \setminus \{t^*\}$ and thus the vertices $\{p_1, \ldots, p_r\}$ contain no agents (since $p_i \notin T \setminus \{t^*\}$ for all $i \in [r]$). After round number $(\ell + 2(k-2) - 1)$, i.e., one step before the last, the vertices $\{p_1, \ldots, p_{r-1}\}$ contain no agents, because $\mathrm{dist}(p_i, T) \geq 2$ for all $i \leq r - 1$. Analogously, for all $j \leq r$ the vertices $\{p_1, \ldots, p_{r-j}\}$ contain no agent after round number $(\ell + 2(k-2) - j)$. Therefore, we can augment the strategy by moving the

[2] If this is not the case, then we can consider the problem of routing the agents from T to S, and note that viewing this strategy in the reverse order produces a strategy for routing from S to T.

agent from s^* to t^* along the path P. Specifically, for all $i = 0, \ldots, r$ we move the agent from p_i to p_{i+1} in round $\ell + 2(k-2) - r + i + 2$, which adds two rounds to the strategy. The claim follows. ■

3 Some Concrete Examples

We start with an easy example, showing that for the graph P_n, a path of length n, the acquaintance time is $\Theta(n)$.

Proposition 3.1 ($\mathcal{AC}$ of a path): *Let P_n be a path of length n. Then $\mathcal{AC}(P_n) = \Theta(n)$.*

Proof Clearly $\mathcal{AC}(P_n) \geq \text{diam}(P_n)/2 \geq n/2$. For the upper bound denote the vertices of P_n by $v_1, \ldots, v_n$, where v_i is connected to v_{i+1} for all $i \in \{1, \ldots, n-1\}$, and denote by p_i the agent sitting initially in the vertex v_i. Consider the following strategy that works in $O(n)$ rounds:

1. Apply Claim 2.1 in order to route all agents $p_1, \ldots, p_{\lfloor n/2 \rfloor}$ to the vertices $v_{\lceil n/2 \rceil + 1}, \ldots, v_n$, and route $p_{\lfloor n/2 \rfloor + 1}, \ldots, p_n$ to the vertices $v_1, \ldots, v_{\lceil n/2 \rceil}$. This can be done in $O(n)$ rounds. Note that after this sequence every pair of agents (p_i, p_j) with $1 \leq i \leq \lfloor n/2 \rfloor < j \leq n$ have already met each other.

2. Repeat the above procedure recursively on each of the two halves $(v_1, \ldots, v_{\lceil n/2 \rceil})$ and $(v_{\lceil n/2 \rceil + 1}, \ldots, v_n)$ simultaneously.

To bound the total time $T(n)$ of the procedure, we make $O(n)$ rounds in the first part, and at most $T(\lceil n/2 \rceil)$ in the remaining parts. This gives us a bound of

$$T(n) = O(n) + T(\lceil n/2 \rceil) = O(n),$$

as required. ■

The following corollary is immediate from Proposition 3.1.

Corollary 3.2 *Let G be Hamiltonian graph with n vertices. Then $\mathcal{AC}(G) = O(n)$.*

We next prove that for constant degree expanders the acquaintance time is also linear in the size of the graph. For $\alpha > 0$ a d-regular graph $G = (V, E)$ with n vertices is said to be (n, d, α)-expander if for every subset $S \subseteq V$ of size $|S| \leq |V|/2$ it holds that $|N(S) \setminus S| \geq \alpha \cdot |S|$.

Proposition 3.3 ($\mathcal{AC}$ of expander graphs): *Let $G = (V, E)$ be an (n, d, α)-expander graph for some $\alpha > 0$. Then $\mathcal{AC}(G) = \Theta(n)$, where the multiplicative constant in the $\Theta()$ notation depends only on α and d but not on n.*

Proof Clearly $\mathcal{AC}(G) = \Omega(\frac{n^2}{|E|}) = \Omega(n)$, since the expander is of constant degree. For the upper bound we shall need the following theorem due to Björklund, Husfeldt and Khanna saying that expander graphs contain simple paths of linear length.

Theorem 3.4 ([BHK04, Theorem 4]) *Let G be a (n, d, α)-expander graph. Then, G contains a simple path of length $\Omega(\frac{\alpha}{d} \cdot n)$.*

Let P be a simple path of length ℓ in G, where $\ell = \Omega(n)$ is given in Theorem 3.4. Partition all agents into $c = \lceil 2n/\ell \rceil$ disjoint classes $C_1 \cup \cdots \cup C_c$ each of size at most $\ell/2$. Then, for every pair $i, j \in [c]$ we use the strategy from Claim 2.1 to place the agents from the two classes $C_i \cup C_j$ on P, and then apply the strategy from Proposition 3.1 so that every pair of agents from $C_i \cup C_j$ meet. By repeating this strategy for every $i, j \in [c]$, we make sure that every pair of agents on G meet each other. In order to analyze the total length of the strategy, we note that for a single pair $i, j \in [c]$ the total time is at most $O(n) + O(\ell)$, and hence the total length of the strategy is at most

$$\mathcal{AC}(G) \leq \binom{c}{2} \cdot O(n + \ell),$$

which is linear in n, since $\ell = \Omega(n)$, and $c = O(n/\ell) = O(1)$. ■

Next we upper bound the acquaintance time of the binary tree graph.

Proposition 3.5 ($\mathcal{AC}$ of binary tree): *Let T be the binary tree with n vertices. Then $\mathcal{AC}(T) = O(n \log(n))$.*

Note that $\mathcal{AC}(T) = \Omega(n)$ since the number of edges in T is $n - 1$. It would be interesting to compute the asymptotic behavior of $\mathcal{AC}(T)$.

Proof Denote the vertices of T by $\{v_s : s \in \{0,1\}^{\leq \log(n)}\}$, where $(v_s, v_{s'})$ is an edge in the graph if and only if $|s| = |s'| + 1$ and s is a prefix of s'. That is v_ε is the root, v_0, v_1 are the children of v_ε, and so on. For $s \in \{0,1\}^{\leq \log(n)}$ denote by T_s the subtree rooted at v_s. We also denote for by p_s the agent originally located in v_s, and let P_s be the set of agents who were originally in T_s.

We claim that it is enough to find a strategy of length $O(n \log(n))$ that allows every agent in P_0 meet every agent from P_1. Indeed, suppose we have such strategy. We describe a strategy for acquaintance in T

1. Let the agent p_ε sitting in v_ε meet all other agents by performing a DFS walk on the tree, and return everyone to their original locations by applying the same strategy in the reverse order. This step can be done in $O(n)$ rounds.

2. Apply a strategy of length $O(n\log(n))$ that makes all agents in P_0 to meet all agents in P_1.

3. Return the agents of P_0 to T_0 and return the agents of P_1 to T_1. This can be done in $O(n)$ rounds using Claim 2.1.

4. Apply steps 1-3 recursively on the subtree T_0 and on the subtree T_1 simultaneously. That is, every agent from P_{00} meets every agent from P_{01}, and every agent from P_{10} meets every agent from P_{11}, and so on...

Analyzing the total number of rounds, we have $O(n\log(n))$ rounds in the first 3 steps. Therefore, the total number of rounds is upper bounded by $O(n\log(n))+O(n/2\log(n/2))+O(n/4\log(n/4))\cdots = O(n\log(n))$, as required.

Next, we describe a strategy that makes every agent from T_0 to meet every agent from T_1 in $O(n\log(n))$ rounds.

1. Let the agents p_0 and p_1 meet all other agents, and ignore them from now on. This step can be done in $O(n)$ rounds.

2. Route the agents in P_{00} to the subtree T_{10}, and route the agents in P_{10} to T_{00}. This can be done in $O(n)$ rounds by considering the subtree of T induced by the vertices $T_{00}\cup T_{10}\cup\{v_\varepsilon, v_0, v_1\}$ and applying Claim 2.1.

3. Apply induction on the depth of the tree to make all agents in P_{00} (who are located in T_{10}) to meet all agents in P_{11} (located in T_{11}), and simultaneously make all agents in P_{01} (who are located in T_{00}) to meet all agents in P_{01} (located in T_{01}).

4. Route the agents in P_{01} to the subtree T_{10}, route the agents in P_{11} to T_{11}, route the agents in P_{00} to T_{00}, and route the agents in P_{10} to T_{01}. This can be done in $O(n)$ rounds by applying Claim 2.1 on the appropriate subgraphs.

5. Apply induction on the depth of the tree to make all agents in P_{01} (who are located in T_{10}) to meet all agents in P_{11} (located in T_{11}), and simultaneously make all agents in P_{00} (who are located in T_{00}) to meet all agents in P_{10} (located in T_{01}).

It is clear that in steps 1,3, and 5 all agents from P_0 meet all agents from P_1. For the analysis of the number of rounds let us denote the total number of rounds by $T(n)$. Then, steps 1,2, and 4 contribute $O(n)$ rounds to $T(n)$, and steps 3 and 5 contribute additional $2T(n/2)$ rounds. Therefore, $T(n) = O(n) + 2T(n/2) = O(n\log(n))$. ■

4 Separating $\mathcal{AC}(G)$ From Other Parameters

In this section we provide several results that separate $\mathcal{AC}(G)$ from other parameters of graphs. Our first example shows a graph with low diameter, low clique cover number (that is, $\overline{G}$ has low chromatic number), such that $\mathcal{AC}(G)$ is large.

Proposition 4.1 ($\mathcal{AC}$ of the barbell graph): *Let G be the barbell graph. That is, G consists of two cliques of size n connected by a single edge, called bridge. Then $\mathcal{AC}(G) = \Theta(n)$.*

Proof The upper bound follows from Hamiltonicity of G (see Corollary 3.2). For the lower lower, denote the vertices of the two cliques by A and B, and denote the bridge by (a_0, b_0), where $a_0 \in A$ and $b_0 \in B$. Then, in any strategy for acquaintance either all agents from A visited in a_0, or all agents from B visited in b_0, and the proposition follows. ■

A more interesting example shows existence of a Ramsey graph G with $\mathcal{AC}(G) = 1$. For more details regarding graphs with $\mathcal{AC}(G) = 1$ see Section 7.

Proposition 4.2 (Ramsey graph with $\mathcal{AC}(G) = 1$): *There is a graph G on n vertices that contains neither a cliques nor an independent sets of size $O(\log(n))$ such that $\mathcal{AC}(G) = 1$.*

Proof Let $H = (U = \{u_1, \ldots, u_{n/2}\}, F)$ be a Ramsey graph on $n/2$ vertices that contains neither a cliques nor an independent sets of size $O(\log(n))$. We construct $G = (V, E)$ as follows. The vertices of G are two copies of U, i.e., $V = \{u_1, \ldots, u_{n/2}\} \cup \{u'_1, \ldots, u'_{n/2}\}$. The edges of G are the following.

1. The vertices $\{u_1, \ldots, u_{n/2}\}$ induce a copy of H. That is, $(u_i, u_j) \in E$ if and only if $(u_i, u_j) \in F$.
2. The vertices $\{u'_1, \ldots, u'_{n/2}\}$ induce the complement of H. That is, we set $(u'_i, u'_j) \in E$ if and only if $(u_i, u_j) \notin F$.

3. Add an edge $(u_i, u'_i) \in E$ for all $i \in [n/2]$.

4. For each $i \neq j \in [n/2]$ place one of the edges $(u_i, u'_j), (u_j, u'_i)$ arbitrarily.

By the properties of H it follows that G is also a Ramsey graph. Now, it is straightforward to check that the matching $M = \{(u_i, u'_i) : i \in [n/2]\}$ is a 1-round strategy for acquaintance. ■

The proof of Proposition 3.3 may suggest that small routing number (as defined by Alon et al. [ACG94]) implies fast acquaintance time. The following example shows separation between the two parameters for the complete bipartite graph $K_{n,n}$. It was shown in [ACG94] that for any permutation of the vertices $\sigma : V \to V$ the agents can be routed routed from $v \in V$ to the destination $\sigma(v)$ in 4 rounds. We prove next that $\mathcal{AC}(K_{n,n}) = \Theta(\log(n))$.

Proposition 4.3 ($\mathcal{AC}$ of $K_{n,n}$): *Let $n = 2^r$ for some $r \in \mathbb{N}$. Let $K_{n,n} = (A, B, E)$ be complete bipartite graph with $|A| = |B| = n$ vertices on each side. Then $\mathcal{AC}(K_{n,n}) = \log_2(n)$.*

Proof Assign each agent a string $x = (x_0, x_1, \ldots, x_r) \in \{0,1\}^{r+1}$ such that all agents who started on the same side have the same first bit x_0. We now describe an r-rounds strategy for acquaintance. In the i'th round move all agents with $x_i = 0$ to A and all agents with $x_i = 1$ to B. Now if two agents are assigned strings x and x' such that $x_i \neq x'_i$, then in the i'th round they will be on different sides of the graph, and hence will be acquainted.

We now claim that r rounds are also necessary. Indeed, suppose we have a t-rounds strategy for acquaintance. Assign each agent a string $x = (x_0, x_1, \ldots, x_t) \in \{0,1\}^{t+1}$, where $x_i = 0$ for $i \leq t$ if and only if in the i'th round the agent was in A. Note that two agents met during the t rounds if and only if their strings are different. This implies $2^{t+1} \geq 2n$, and thus $t \geq r$, as required. ■

5 The Range of $\mathcal{AC}(G)$

In this section we provide examples of families of graphs on n vertices whose acquaintance time ranges from constant to $n^{1.5}$.

Theorem 5.1 *For every function $f : \mathbb{N} \to \mathbb{N}$ that satisfies $f(n) \leq n^{1.5}$ there is a family of graphs $\{G_n\}_{n \in \mathbb{N}}$ such that G_n has n vertices and $\mathcal{AC}(G_n) = \Theta(f(n))$.*

The proof of the theorem is divided in two parts. In Proposition 5.2 we take care of $f(n) \leq n$, and Proposition 5.3 takes care of $n \leq f(n) \leq n^{1.5}$.

Proposition 5.2 *For every function $f : \mathbb{N} \to \mathbb{N}$ that satisfies $f(n) \in [1, n]$ there is a family of graphs $\{G_n\}_{n\in\mathbb{N}}$ such that G_n has n vertices and $\mathcal{AC}(G_n) = \Theta(f(n))$.*

Proof Consider the graph $G_{r,\ell} = (V, E)$ with vertices $V = \{v_{i,j} : i \in [r], j \in [\ell]\}$. where there vertices $\{v_{i,j} : i \in [r]\}$ from a clique for all $j \in [\ell]$, and, in addition, for every $i, i' \in [r]$ such that $|i - i'| = 1 \pmod r$ we have $(v_{i,j}, v_{i',j}) \in E$ for all $j \in \ell$. That is, the vertices are divided into r cliques each of size ℓ, and the edges between adjacent cliques on a cycle form a perfect matching.

We claim that $\mathcal{AC}(G_{r,\ell}) = \Theta(r)$. For a lower bound note that $\text{diam}(G_{r,\ell}) = r/2$, and hence $\mathcal{AC}(G_{r,\ell}) = \Omega(r)$. For an upper bound consider first the case of $r = 2$; that is, the graph consisting of two disjoint cliques each of size ℓ, with ℓ edges between them forming a perfect matching. Then, it is easy to see that $\mathcal{AC}(G_{2,\ell}) = O(1)$. This is achieved by swapping $\ell/2$ vertices in one clique with $\ell/2$ vertices in the other clique constant number of times.

The bound $\mathcal{AC}(G_{r,\ell}) = O(r)$ is obtained using the strategy similar to the one for P_r explained in Proposition 3.1, where we consider each clique as a single block, and each swap in P_r corresponds to a swap of the blocks, rather than single vertices. The difference is that even if two blocks of size ℓ are adjacent, it does not imply that all the 2ℓ vertices in the two blocks have met. In order to achieve that we apply the strategy above for the $G_{2,\ell}$ graph. ■

Proposition 5.3 *For every function $f : \mathbb{N} \to \mathbb{N}$ that satisfies $f(n) \in [n, n^{1.5}]$ there is a family of graphs $\{G_n\}_{n\in\mathbb{N}}$ such that G_n has n vertices and $\mathcal{AC}(G_n) = \Theta(f(n))$.*

Proof Consider the graph $O_{r,\ell}$ that consists of r cliques each of size ℓ, and another vertex z called the center. In each each clique there is one vertex connected to the center. We claim $\mathcal{AC}(O_{r,\ell}) = \Theta(\min(n\ell, nr))$. Since the total number of vertices in the graph is $n = r\ell + 1$, by choosing either $r \approx f(n)/n$ and $\ell \approx n^2/f(n)$ we will get that $\mathcal{AC}(O_{r,\ell}) = \Theta(f(n))$, as required.

In order to prove an upper bound of $O(nr)$ note that solving the acquaintance problem on $O_{r,\ell}$ can be reduced to solving $\binom{r}{2}$ problems of Hamiltonian graphs of size $2\ell+1$, where each problem corresponds to a pair of cliques together with the center z. By Hamiltonicity each such problem is solved in $O(\ell)$ rounds.

In order to prove an upper bound of $O(n\ell)$ we can bring every agent to the center, and all other agents will meet him in $O(\ell)$ rounds, using the vertices connected to z.

For the lower bound let us define for every agent p_i and every time $t \in \mathbb{N}$ let $\varphi_t(p_i)$ be the number of agents that p_i has met up to time t. Note that for $h = \mathcal{AC}(O_{r,\ell})$ there is a strategy such that $\sum_{i\in[n]} \varphi_h(p_i) = n \cdot (n-1)$, since every agent met every other agent up to time h. On the other hand, in each time t the sum $\sum_{i\in[n]} \varphi_t(p_i)$ increases by at most $2r + \ell$, as the only agents who could potentially affect the sum are those who moved to the center (contributing at most r to the sum), an agent who moved from the center to one of the cliques (contributing at most ℓ to the sum), and the r neighbors of the center (each contributing at most 1 to the sum). This implies a lower bound of $\mathcal{AC}(O_{r,\ell}) \cdot (2r + \ell) = \Omega(n^2)$, which completes the proof of Proposition 5.3. ■

Building on the lower bound in the proof of Proposition 5.3 we show that bottlenecks in graphs imply high acquaintance time.

Proposition 5.4 *Let $G = (V, E)$ be a graph with n vertices. Suppose there is a subset of vertices $S \subseteq V$ such that when removing S from G each connected component in the remaining graph $G[V \setminus S]$ is of size at most ℓ. Then $\mathcal{AC}(G) = \Omega(\frac{\binom{n}{2}-|E|}{|S|\cdot\ell+\sum_{s\in S}\deg(s)})$.*

Proof Let us define for every agent p_i and every time $t \in \mathbb{N}$ the set $\varphi_t(p_i) \subseteq \{p_j : j \in [n]\}$ to contain all agents that p_i has met up to time t, as well as all agents who shared a connected component in $G[V \setminus S]$ with p_i up to time t. By definition of $\mathcal{AC}$ for $h = \mathcal{AC}(G)$ there is a strategy such that $\sum_{i\in[n]} |\varphi_h(p_i)| = n \cdot (n-1)$, since every agent met every other agent up to time h. Note that on the t'th round the increment to $\varphi_t(p_i)$ compared to $\varphi_{t-1}(p_i)$ is either because p_i entered S and met new agents in S and in its connected components of $G[V \setminus S]$, or because an agent left S and entered one of the connected components. Thus, in each time t the sum $\sum_{i\in[n]} |\varphi_t(p_i)|$ increases by at most $|S| \cdot \ell + \sum_{s\in S} \deg(s)$, where $|S| \cdot \ell$ upper bounds the number of meetings that were added because of agents moving out of S, while the value $\sum_{s\in S} \deg(s)$ bounds the number of meetings that are accounted for by agents that entered S in round t. This implies a lower bound of $\Omega(\frac{\binom{n}{2}-|E|}{|S|\cdot\ell+\sum_{s\in S}\deg(s)})$, which completes the proof of Proposition 5.4. ■

Next, we show that for every graph G with n vertices the acquaintance time is in fact asymptotically smaller than the trivial bound of $2n^2$. Specifically, we prove the following theorem.

Theorem 5.5 *For every graph G with n vertices it holds that $\mathcal{AC}(G) = O\left(\frac{n^2}{\log(n)/\log\log(n)}\right)$.*

The proof of Theorem 5.5 relies on the following two claims.

Claim 5.6 *Let G be a graph with n vertices. If G contains a simple path of length ℓ, then $\mathcal{AC}(G) = O(n^2/\ell)$.*

Claim 5.7 *Let G be a graph with n vertices. If G has a vertex of degree Δ, then $\mathcal{AC}(G) = O(n^2/\Delta)$.*

We postpone the proofs of both claims until later and show how to deduce from them Theorem 5.5.

Proof of Theorem 5.5 Let $k = \Theta\left(\frac{\log(n)}{\log\log(n)}\right)$ be the largest integer such that $k^k \leq n$. For such choice of k the graph G either contains a simple path of length k, or it contains a vertex of degree at least k. In the former case by Claim 5.6 we have $\mathcal{AC}(G) = O(n^2/k)$. In the latter case we use Claim 5.7 to conclude that $\mathcal{AC}(G) = O(n^2/k)$. The theorem follows. ■

We now prove Claims 5.6 and 5.7.

Proof of Claim 5.6: Assume without loss of generality that ℓ is the length of the longest simple path in G. We shall also assume that G is a tree that contains a path of length ℓ (if not, then G has a spanning tree with this property). Then, in particular, we have $\text{dist}(u, v) \leq \ell$ for every two vertices $u, v \in V$.

In order to prove the claim we apply of Claim 2.1 together with Proposition 3.1 similarly to the proof of Proposition 3.3. Divide the agents into $c = \lceil 2n/\ell \rceil$ classes $C_1, \ldots, C_c$ of size $\lfloor \ell/2 \rfloor$ each. For every pair $i, j \in [c]$ we use Claim 2.1 to route the agents from the two classes $C_i \cup C_j$ to a path of length ℓ, and then, apply the strategy from Proposition 3.1 so that every pair of agents from $C_i \cup C_j$ meet. By repeating this strategy for every $i, j \in [c]$, we make sure that every pair of agents on G meet each other.

Since $\text{dist}(u, v) \leq \ell$ for every two vertices $u, v \in V$, by Claim 2.1 the agents $C_i \cup C_j$ can be routed to a path of length ℓ in $O(\ell)$ rounds. Then, using the strategy from Proposition 3.1 every pair of agents from $C_i \cup C_j$ meet in at most $O(\ell)$. Therefore, the acquaintance time of G can be upper bounded by

$$\mathcal{AC}(G) \leq \binom{c}{2} \cdot O(\ell) = O(n^2/\ell),$$

as required. ■

Proof of Claim 5.7: Assume without loss of generality that G is a tree rooted at a vertex r of degree Δ. (This can be done by considering a spanning tree of G) Denote the children of r by $v_1, \ldots, v_\Delta$, and let $p_1, \ldots, p_\Delta$ be the agents originally located at these vertices. We claim that there is a $O(n)$-rounds strategy that allows $p_1, \ldots, p_\Delta$ to meet all agents.

Given such strategy, we apply it on G. We then route the agents $p_1, \ldots, p_\Delta$ to arbitrary Δ leaves of the tree different from $v_1, \ldots, v_\Delta$ (this can be done in $O(n)$ rounds), and ignore them until the end of the process. Now we need to solve the same problem on a tree with $n-\Delta$ vertices. Repeating the process we get that $AG(G) \leq \sum_{i=1}^{n/\Delta} O(n-i\Delta) = O(n^2/\Delta)$, as required.

Next, we describe a $O(n)$-rounds strategy that allows $p_1, \ldots, p_\Delta$ to meet all agents. For any $1 \leq i \leq \Delta$ consider a subtree $T_i = (V_i, E_i)$ of G rooted at v_i. It is enough to show how the agents $p_1, \ldots, p_\Delta$ can meet all agents from T_i in $O(|T_i|)$ rounds. First, let p_i meet all agents in T_i in $O(|T_i|)$ steps. This can be done by running p_i along a DFS of T_i. It is enough now to find a $O(|T_i|)$- rounds strategy that allows all agents of T_i to visit in the root r. This task can be reduced to the routing problem considered in Claim 2.1. Specifically, define a tree on $2|V_i|$ vertices that contains the tree T_i with additional $|V_i|$ vertices each connected only to vertex r. It is easy to see that a strategy that routes all agents from the copy of T_i to the additional vertices can be turned into a strategy that allows all agents of T_i to visit in the root r. By Claim 2.1 such strategy can be implemented using $O(|T_i|)$ rounds. This completes the proof of Claim 5.7. ■

6 $\mathcal{NP}$-Hardness Results

In this section we show that the acquaintance time problem is $\mathcal{NP}$-hard. Specifically, we prove the following theorem.

Theorem 6.1 *For every $t \geq 1$ it is $\mathcal{NP}$-hard to distinguish whether a given graph G has $\mathcal{AC}(G) \leq t$ or $\mathcal{AC}(G) \geq 2t$.*

Before actually proving the theorem, let us first see the proof in the special case of $t = 1$.

Special case of $t = 1$: We start with the following $\mathcal{NP}$-hardness result, saying that for a given graph G it is hard to distinguish between graphs with small chromatic number and graphs with somewhat large independent set. Specifically, Lund and Yanakakis [LY94] prove the following result.

Theorem 6.2 ([LY94, Theorem 2.8]) *For every $K \in \mathbb{N}$ sufficiently large the following gap problem is $\mathcal{NP}$-hard. Given a graph $G = (V, E)$ distinguish between the following two cases:*

- $\chi_{\text{eq}}(G) \leq K$*; i.e., there is a K-coloring of G with color classes of size $\frac{|V|}{K}$ each.*
- $\alpha(G) \leq \frac{1}{2K}$.

We construct a reduction from the problem above to the acquaintance time problem, that given a graph G outputs a graph H so that (1) if $\chi_{\text{eq}}(G) \leq K$, then $\mathcal{AC}(H) = 1$, and (2) if $\alpha(G) \leq \frac{1}{2K}$, then $\mathcal{AC}(H) \geq 2$.

Given a graph $G = (V, E)$ with n vertices $V = \{v_i : i \in [n]\}$, the reduction outputs a graph $H = (V', E')$ as follows. The graph H contains $|V'| = 2n$ vertices, partitioned into two parts $V' = V \cup U$, where $|V| = n$ and $U = U_1 \cup \cdots \cup U_K$ with $|U_j| = n/K$ for all $j \in [K]$. The vertices V induce the complement graph of G. For each $j \in [K]$ the vertices of U_j form an independent set. In addition, we set edges between every pair of vertices $(v, u) \in V \times U$ as well as between every pair of vertices in $(u, u') \in U_j \times U_j'$ for all $j \neq j'$. This completes the description of the reduction.

Completeness: We first prove the completeness part, namely, if $\chi_{\text{eq}}(G) = K$, then $\mathcal{AC}(H) = 1$. Suppose that the color classes of G are $V = V_1 \cup \cdots \cup V_K$ with $|V_j| = n/K$ for all $j \in [K]$. Note that each color-class V_j induces a clique in H. Consider the matching that for each $j \in [K]$ swaps the agents in U_j with the agents in V_j. (This is possible since by the assumption $|V_j| = \frac{n}{K}$, and all vertices of U_j are connected to all vertices of V_j.) In order to verify that such matching allows every pair of agents to meet each other, let us denote by p_v the agent sitting originally in vertex v. Note that before the swap all pairs listed below have already met.

1. For all $j \in [K]$ and every $v, v' \in V_j$ the pair of agents $(p_v, p_{v'})$ have met.
2. For all $j \neq j'$ and for every $u \in U_j$, $u' \in U_{j'}$ the pair of agents $(p_u, p_{u'})$ have met.
3. For all $v \in V$ and $u \in U$ the pair of agents (p_v, p_u) have met.

After the swap the following pairs meet.

1. For all $j \neq j'$ and for every $v \in V_j$, $v' \in V_{j'}$ the agents p_v and $p_{v'}$ meet using an edge between U_j and U_j'.
2. For all $j \in [K]$ and every $u, u' \in V_j$ the agents p_u and $p_{u'}$ meet using an edge in V_j.

This completes the completeness part.

Soundness: For the soundness part assume that $\mathcal{AC}(H) = 1$. We claim that $\alpha(G) > 1/2K$. Note first that if there is a single matching that allows all agents to meet, then for every $j \in [K]$ all but at most K agents from U_j must have moved by the matching to V. (This holds since U does not contain $K+1$ clique.) Moreover, all the agents from U_j who moved to V must have moved to a clique induced by V. This implies that V contains a clique of size at least $n/K - K > n/2K$, which implies that $\alpha(G) > 1/2K$.

This completes the proof of Theorem 6.1 for the special case of $t = 1$. The proof of Theorem 6.1 for general $t \geq 2$ is quite similar, although it requires some additional technical details.

Proof of Theorem 6.1: We start with the following $\mathcal{NP}$-hardness result due to Khot [Kho01], saying that for a given graph G it is hard to distinguish between graphs with small chromatic number and graphs with small independent set. Specifically, Khot proves the following result.

Theorem 6.3 ([Kho01, Theorem 1.6]) *For every $t \in \mathbb{N}$, and for every $K \in \mathbb{N}$ sufficiently large (it is enough to take $K \geq 2^{O(t)}$) the following gap problem is $\mathcal{NP}$-hard. Given a graph $G = (V, E)$ distinguish between the following two cases:*

- $\chi_{\text{eq}}(G) \leq K$.
- $\alpha(G) \leq \frac{1}{4t^{2t+1}K^{2t}}$.

We construct a reduction from the problem above to the acquaintance time problem, that given a graph G outputs a graph H so that (1) if $\chi_{\text{eq}}(G) \leq K$, then $\mathcal{AC}(H) \leq t$, and (2) if $\alpha(G) \leq \frac{1}{4t^{2t+1}K^{2t}}$, then $\mathcal{AC}(H) \geq 2t$.

Given a graph $G = (V, E)$ with n vertices, the reduction $r(G)$ outputs a graph $H = (V', E')$ as follows. The graph H contains $|V'| = (t+1)n$ vertices, partitioned into two parts $V' = V \cup U$, where $|V| = n$ and $U = \cup_{i \in [t], j \in [K]} U_{i,j}$ with $|U_{i,j}| = n/K$ for all $i \in [t], j \in [K]$. The vertices V induce the complement graph of G. For each $j \in [K]$ the vertices of U_j form an independent set. In addition, we set edges between every pair of vertices $(v, u) \in V \times U$ as well as between every pair of vertices in $(u, u') \in U_{i,j} \times U_{i',j'}$ for all $(i, j) \neq (i', j')$. This completes the description of the reduction.

Completeness: We first show that prove the completeness part, namely, if $\chi_{\text{eq}}(G) = K$, then $\mathcal{AC}(H) \leq t$. Suppose that the color classes of G are $V = V_1 \cup \dots \cup V_K$ with $|V_j| = n/K$ for all $j \in [K]$. Note that in each color-class V_k induces a clique in H. We show that $\mathcal{AC}(H) \leq t$, which can be achieved as follows: For all $i \in [t]$ in the i'th round the agents located in vertices $U_{i,j}$ swap places with the agents in V_j for all $j \in [K]$. (This is possible since by the assumption $|U_{i,j}| = |V_j| = \frac{n}{K}$, and all vertices of $U_{i,j}$ are connected to all vertices of V_j.) It is immediate to verify that this strategy allows every pair of agents to meet each other. Indeed, denoting by p_v the agent sitting originally in vertex v the only pairs who did not met each other before the first round are contained in the following two classes:

1. For all $j \neq j' \in [K]$ and for every $v \in V_j$, $v' \in V_{j'}$ the pair of agents $(p_v, p_{v'})$.
2. For each $i \in [t]$ and $j \in [K]$ and every $u, u' \in U_{i,j}$ the pair of agents $(p_u, p_{u'})$.

Then, when the agents move along the described matchings, the pairs from the first class meet after the first round. And for each round $i \in [t]$ the pairs from the second class that correspond to $u, u' \in U_{i,j}$ for some $j \in [K]$ meet after the i'th round. This proves the completeness part of the reduction.

Soundness: For the soundness part assume that $\mathcal{AC}(H) \leq 2t - 1$, and consider the corresponding $(2t-1)$-rounds strategy for acquaintance in G. By a counting argument there are $\frac{n}{2}$ agents who originally were located in U and visited in V at most once. By averaging, there are $\frac{n}{2(2t-1)}$ agents who either never visited in U or visited in V simultaneously, and this was their only visit in V. Let us denote this set of agents by P_0. The following claim completes the proof of the soundness part.

Claim 6.4 *Let P_0 be a set of agents of size $\frac{n}{2(2t-1)}$. Suppose they visited in V at most once simultaneously, and visited in U for at most $2t$ times. If every pair of agents from P_0 met each other during these rounds, then $\alpha(G) \geq \frac{n}{2(2t-1)(tK)^{2t}} > \frac{1}{4t^{2t+1}K^{2t}}$.*

Proof Let us assume for concreteness that P_0 stayed in U until the last round, and then moved to V. Before the first round the set of agents P_0 is naturally partitioned into at most tK clusters, each clusters corresponding to agents in some $U_{i,j}$. That is, the agents meet each other if and only if they are in different clusters. Analogously, after t rounds, there is a natural partition of P_0 into $(tK)^{2t}$, where the agents age in the same cluster if and only if they have not met each other thus far. Thus, at least one of the cluster is of size at least $\frac{|P_0|}{(tK)^{2t}} = \frac{n}{2(2t-1)(tK)^{2t}}$. If we assume that every pair of agents from P_0 met each other eventually, then it must be the case that in the last round each cluster moved to some clique in V, and in particular $\overline{G}$ contains a clique of size $\frac{n}{2(2t-1)(tK)^{2t}}$. The claim follows. ■

We have shown a reduction from the coloring problem to the acquaintance time problem, that given a graph G outputs a graph H so that (1) if $\chi_{\text{eq}}(G) \leq K$, then $\mathcal{AC}(H) \leq t$, and (2) if $\alpha(G) \leq \frac{1}{4t^{2t+1}K^{2t}}$, then $\mathcal{AC}(H) \geq 2t$. This completes the proof of Theorem 6.1. ■

6.1 Towards stronger hardness results

We conjecture that, in fact, a stronger hardness result holds, compared to the one stated in Theorem 6.1.

Conjecture 6.5 *For every constant $t \in \mathbb{N}$ it is $\mathcal{NP}$-hard to decide whether a given graph G has $\mathcal{AC}(G) = 1$ or $\mathcal{AC}(G) \geq t$.*

Below we describe a gap problem similar in spirit to the hardness results of Lund and Yanakakis and that of Khot whose $\mathcal{NP}$-hardness implies Conjecture 6.5. In order to describe the gap problem we need the following definition.

Definition 6.6 *Let $t \in \mathbb{N}$ and $\beta > 0$. A graph $G = (V, E)$ is said to be (β, t)-*intersecting *if for every t subsets of the vertices $S_1, \ldots, S_t \subseteq V$ of size βn and for every t bijections $\pi_i : S_i \to [\beta n]$ there exist $j, k \in [\beta n]$ such that all pre-images of the pair (j, k) are edges in E, i.e., for all $i \in [t]$ it holds that $(\pi_i^{-1}(j), \pi_i^{-1}(k)) \in E$.*

Note that a graph G is $(\beta, 1)$-intersecting if and only if G does not contain an independent set of size βn. In addition, note that if G is (β, t)-intersecting then it is also (β', t')-intersecting for $\beta' \geq \beta$ and $t' \leq t$, and in particular $\alpha(G) < \beta$.

We remark without proof that the problem of deciding whether a given graph G is (β, t)-intersecting is co$\mathcal{NP}$-complete. We make the following conjecture regarding $\mathcal{NP}$-hardness of distinguishing between graphs with small chromatic number and (β, t)-intersecting graphs.

Conjecture 6.7 *For every $t \in \mathbb{N}$ and for all $K \in \mathbb{N}$ sufficiently large it is $\mathcal{NP}$-hard to distinguish between the following two cases for a given graph $G = (V, E)$:*

- $\chi_{\mathrm{eq}}(G) \leq K$.
- *The graph G is $(1/K^t, t)$-intersecting.*

Remark Conjecture 6.7 does not seem to follow immediately from the result of Khot stated in Theorem 6.3. One reason for that is due to the fact that Khot's hard instances for the problem are bounded degree graphs, and we suspect that such graphs cannot be (β, t)-intersecting for arbitrarily small $\beta > 0$ even in the case of $t = 2$.

Theorem 6.8 *Conjecture 6.7 implies Conjecture 6.5.*

We omit the proof of this implication, as it is analogous to the proof of Theorem 6.1. The reduction is exactly the same as described in the proof of Theorem 6.1 for the special case of $t = 1$. The analysis is similar to the proof of Theorem 6.1 for general $t \geq 1$, where instead of using the assumption that $\alpha(G)$ is small we use the stronger assumption in the NO-case of Conjecture 6.7.

7 Graphs with $\mathcal{AC}(G) = 1$

In this section we study the graphs whose acquaintance time equals 1. We start with the following straightforward proposition describing a structure of such graphs.

Proposition 7.1 *A graph $G = (V, E)$ satisfies $\mathcal{AC}(G) = 1$ if and only if there is a partition of the vertices $V = A \cup B \cup C$ with $A = \{a_i\}_{i=1}^k$ and $B = \{b_i\}_{i=1}^k$ for some $k \in \mathbb{N}$, and $C = V \setminus (A \cup B)$ such that the following holds:*

1. *$(a_i, b_i) \in E$ for all $i \in [k]$.*
2. *Either $(a_i, b_j) \in E$ or $(a_j, b_i) \in E$ for all $i \neq j \in [k]$.*
3. *Either $(a_i, a_j) \in E$ or $(b_i, b_j) \in E$ for all $i \neq j \in [k]$.*
4. *The vertices of C induce a clique in G.*
5. *For all $c \in C$ and for all $i \in [k]$ we have either $(c, a_i) \in E$ or $(c, b_i) \in E$.*

Proof Suppose first that $\mathcal{AC}(G) = 1$, and let $M = \{(a_1, b_1), \ldots, (a_k, b_k)\}$ be a matching that witnesses the assertion $\mathcal{AC}(G) = 1$. Denote $A = \{a_i\}_{i=1}^k$, $B = \{b_i\}_{i=1}^k$, and let $C = V \setminus (A \cup B)$. The conclusion follows immediately from the fact that M is a 1-round strategy for acquaintance.

For the other direction, if G satisfies the listed conditions, then the matching $M = \{(a_1, b_1), \ldots, (a_k, b_k)\}$ is 1-round strategy for acquaintance in G. ■

We state several immediate implications from Proposition 7.1.

Corollary 7.2 *Let $G = (V, E)$ be an n-vertex graph that satisfies $\mathcal{AC}(G) = 1$, and let $V = A \cup B \cup C$ be a partition of the vertices with $A = \{a_i\}_{i=1}^k$ and $B = \{b_i\}_{i=1}^k$ for some $k \in \mathbb{N}$, and $C = V \setminus (A \cup B)$ as in Proposition 7.1. Then,*

1. *The partition of the vertices $V = A \cup B \cup C$ is not unique in general. For instance, given such a partition we can define another partition $V = A' \cup B' \cup C$, where $A' = (A \cup \{b_i\}) \setminus \{a_i\}$ and $B' = (B \cup \{a_i\}) \setminus \{b_i\}$ for some $i \in [k]$.*

2. *For all $i \in [k]$ it holds that $\deg(a_i) + \deg(b_i) \geq 2k + |C| = n$. For all $c \in C$ we have $\deg(c) \geq k + |C| - 1 \geq \lfloor n/2 \rfloor$, and $\deg(c) < n/2$ is possible only if $|C| = 1$.*

3. *If $|C| = 1$, then there are at least $\lfloor n/2 \rfloor$ vertices $v \in V$ with $\deg(v) \geq \lceil n/2 \rceil$. Otherwise, there are at least $\lceil n/2 \rceil$ vertices $v \in V$ with $\deg(v) \geq \lceil n/2 \rceil$.*

4. *The number of edges in G is at least $|E| = \frac{1}{2}\sum_{v\in V}\deg(v) \geq \frac{n^2+|C|^2-2|C|}{4} \geq \frac{n^2-1}{4}$. Equality holds only if $|C| = 1$.*

5. *For all $J \subseteq [k]$ and $C' \subseteq C$ the graph induced by the vertices $V' = \{a_i, b_i : i \in J\} \cup C'$ induces a subgraph $G' = G[V']$ with $\mathcal{AC}(G') = 1$.*

6. *For all $J \subseteq [k]$ we have $|\{(a_i, b_j) \in E : i, j \in J\} \geq |J| + \binom{|J|}{2} = \frac{|J|^2+|J|}{2}$. In particular, for every subset $U \subseteq V$ we have $\sum_{u\in U}\deg(u) = \Omega(|U|^2)$, and hence for every $d \in \mathbb{N}$ the number of vertices in G of degree at most d is at most $O(d)$.*

7. *The graph G contains a perfect matching[3], consisting of the edges $\{(a_i, b_i)\}_{i=1}^k$ together with a maximal matching in C.*

Proposition 7.3 *Let $G = (V, E)$ be a graph with n vertices that satisfies $\mathcal{AC}(G) = 1$, and let $U \subseteq V$ be the set of vertices $v \in V$ such that $\deg(v) \geq n/2$. Then, for every $W \subseteq V \setminus U$ there exists a matching of size $|W| - 1$ between U and W.*

Proof Let $V = A \cup B \cup C$ be a partition of the vertices with $A = \{a_i\}_{i=1}^k$ and $B = \{b_i\}_{i=1}^k$ for some $k \in \mathbb{N}$, and $C = V \setminus (A \cup B)$ as in Proposition 7.1. We claim first that $|W \cap (A \cup B)| \geq |W| - 1$. Indeed, it holds trivially if $|C| \leq 1$. Otherwise, by Corollary 7.2 Item 2 all vertices in C have degree at least $k + |C| - 1 \geq n/2$, and thus belong to U.

By Corollary 7.2 Item 2 for every $i \in [k]$ it holds that $\deg(a_i) + \deg(b_i) \geq n$, and thus either a_i or b_i belongs to U. Therefore, the required matching is given by $M = \{(a_i, b_i) : i \in [k]$ such that either $a_i \in W$ or $b_i \in W\}$. ■

The following claim gives additional detail on the structure of graphs with $\mathcal{AC}(G) = 1$. It will be used later for the analysis of a (randomized) approximation algorithm for acquaintance in such graphs (see Theorem 7.7).

[3]A perfect matching in an n-vertex graph is a matching consisting of $\lfloor \frac{n}{2} \rfloor$ edges.

Claim 7.4 *Let $G = (V, E)$ be a graph with n vertices that satisfies $\mathcal{AC}(G) = 1$, and let $u, v \in V$ be two vertices of degree at least $n/2$. Then, either $|N(u) \cap N(v)| = \Omega(n)$ or $|E[N(u), N(v)]| = \Omega(n^2)$, where $E[N(u), N(v)] = \{(a, b) \in E : a \in N(u), b \in N(v)\}$ denotes the set of edges between $N(u)$ and $N(v)$.*

Proof If $|N(u) \cap N(v)| \geq 0.1n$, then we are done. Assume now that $|N(u) \cap N(v)| < 0.1n$. Therefore $|N(u) \cup N(v))| > 0.9n$, as $|N(u)| + |N(v)| \geq n$. Define two disjoint sets $N'(u) = N(u) \setminus N(v)$ and $N'(v) = N(v) \setminus N(u)$, and note that by disjointness we have $|N'(u)| \geq 0.4n$ and $|N'(v)| \geq 0.4n$. It suffices to prove that $|E[N'(u), N'(v)]| = \Omega(n^2)$.

Let $V = A \cup B \cup C$ be a partition of the vertices of G with $A = \{a_i\}_{i=1}^k$ and $B = \{b_i\}_{i=1}^k$ for some $k \in \mathbb{N}$, and $C = V \setminus (A \cup B)$, as in Proposition 7.1. Consider the indices $I = \{i \in [k] : a_i, b_i \in N'(u) \cup N'(v)\}$, and define a partition $I = I_u \cup I_v \cup I_{u,v}$, where $I_u = \{i \in [k] : a_i, b_i \in N'(u)\}$, $I_v = \{i \in [k] : a_i, b_i \in N'(v)\}$, and $I_{u,v} = I \setminus (I_u \cup I_v)$. Also, define $C_u = C \cap N'(u)$, and $C_v = C \cap N'(v)$. Note that $|N'(u)| = |C_u| + 2|I_u| + |I_{u,v}|$, and analogously $|N'(v)| = |C_v| + 2|I_v| + |I_{u,v}|$. Using this partition we have

$$|E[N'(u), N'(v)]| \geq |C_u| \cdot |C_v| + \frac{|I_{u,v}|^2}{2} + |I_u| \cdot |I_v| + |I_u| \cdot |C_v| + |I_v| \cdot |C_u|,$$

where the first term follows from the fact that C induces a clique (Proposition 7.1 Item 4), the second and third terms follow from Item 2 of Proposition 7.1, and the last two terms follow from Item 5 of Proposition 7.1.

Now, if $|I_{u,v}| > 0.2n$, then $|E[N'(u), N'(v)]| \geq \frac{|I_{u,v}|^2}{2} \geq 0.02n^2$, as required. Otherwise, we have $|C_u| + |I_u| \geq 0.1n$ and $|C_v| + |I_v| \geq 0.1n$, and therefore $|E[N'(u), N'(v)]| \geq (|C_u| + |I_u|) \cdot (|C_v| + |I_v|) \geq 0.01n^2$. The claim follows. ■

7.1 Algorithmic results

Recall that (unless $\mathcal{P} = \mathcal{NP}$) there is no polynomial time algorithm that, when given a graph G with $\mathcal{AC}(G) = 1$, finds a single round strategy for acquaintance of G. In this section we provide two approximation algorithms regarding graphs whose acquaintance time equals 1. In Theorem 7.5 we give a deterministic algorithm that finds an $(n-1)$-round strategy for acquaintance in such graphs. In Theorem 7.7 we give a randomized algorithm that finds an $O(\log(n))$-round strategy for acquaintance in such graphs.

We start with the simple deterministic algorithm.

Theorem 7.5 *There is a deterministic polynomial time algorithm that when given as input an n-vertex graph $G = (V, E)$ such that $\mathcal{AC}(G) = 1$ outputs an $(n-1)$-round strategy for acquaintance in G.*

Proof The algorithms works by taking one agent at a time and finding a 1-round strategy that allows this agent to meet all others. Specifically, for a given agent the algorithm works as follows. Suppose that the agent p is located in vertex $v \in V$. We consider two cases:

1. There is a 1-round strategy for acquaintance that leaves p in v.

2. There is a 1-round strategy for acquaintance that moves p to some $u \in V$.

Note that by the hypothesis that $\mathcal{AC}(G) = 1$ one of the two cases must hold. In the former case consider the bipartite graph $H = (A \cup B, F)$, where $A = N(v)$ is the neighborhood of v, and $B = V \setminus (N(v) \cup \{v\})$ is the set of non-neighbors of v. We add an edge $(a, b) \in A \times B$ to F if and only if it is contained in E.

Note that if a matching $M \subseteq E$ is a 1-round strategy for acquaintance in G, then M restricted to the edges of H induces a matching of size $|B|$. It is also easy to see that any matching of size $|B|$ in H is a 1-round strategy that allows the agent p to meet all other agents. Such matching can be found in polynomial time (e.g., using an algorithm for maximum flow), which completes the first case.

The second case is handled similarly. Suppose there is a 1-round strategy for acquaintance $M \subseteq E$ that moves the agent p from v to u. We define a bipartite graph $H = (A \cup B, F)$, where $A = N(v) \cap N(u)$ is the set of common neighbors of v and u, and $B = V \setminus (N(v) \cup N(u))$ is the set of common non-neighbors of v and u. Then, similarly to the first case, in order to find a 1-round strategy that allows this p to meet all others, it is enough to find a matching in H of size $|B|$. This completes the proof of the theorem. ■

The following corollary follows from by slightly modifying the proof of Theorem 7.5.

Corollary 7.6 *There is an algorithm that when given as input $c \in \mathbb{N}$ and an n-vertex graph $G = (V, E)$ with $\mathcal{AC}(G) = 1$ outputs an $(n - c)$-round strategy for acquaintance in G in time $n^{c+O(1)}$.*

Proof Recall, the algorithm in Theorem 7.5 that works by taking one agent at a time and finding a 1-round strategy that allows this agent to meet all others. Instead, we first apply the above strategy on $n - c - 1$ agents $\{p_1, \ldots, p_{n-c-1}\}$. At this point we have a $(n - c - 1)$-round strategy that allowed all agents in $\{p_1, \ldots, p_{n-c-1}\}$ to meet everyone. Therefore, it is enough to find a single matching that will allow the remaining $c+1$ agents to meet. This can be done by going over all possible matchings that involve the vertices in which the $c+1$ agents are located. There are at most n^{c+1} such matchings, and we can enumerate all of them in time $n^{c+O(1)}$, as required. ■

We now turn to a randomized polynomial time algorithm with the following guarantee.

Theorem 7.7 *There is a randomized polynomial time algorithm such that when given a graph G with $\mathcal{AC}(G) = 1$ finds $O(\log(n))$-rounds strategy for acquaintance in G.*

Proof Let $G = (V, E)$ be an n-vertex graph, and let $U \subseteq V$ be the set of vertices of degree at least $n/2$. By Item 3 of Corollary 7.2 we have $|U| \geq \lfloor n/2 \rfloor$. The following lemma describes a key step in the algorithm.

Lemma 7.8 *Let P_U be the agents originally located in U. Then, there exists a polynomial time randomized algorithm that finds a $O(\log(n))$-rounds strategy that allows every two agents in P_U to meet.*

Now, consider all the agents P in G. For every subset of $P' \subseteq P$ of size $|P'| \leq |U|$ we can use the aforementioned procedure to produce a $O(\log(n))$-rounds strategy that allows all agents in P' to meet with high probability. Let us partition the agents P into at most $\lceil \frac{2|V|}{|U|} \rceil \leq 5$ disjoint subsets $P = P_1 \cup \cdots \cup P_5$ with at most $\lfloor |U|/2 \rfloor$ agents each in each P_i, and apply the procedure to each pair $P_i \cup P_j$ separately. By Proposition 7.3, we can transfer any pair $P_i \cup P_j$ to U in one step (with the exception of a single agent, that can be dealt with separately as done in Theorem 7.5). When all pairs have been dealt with, all agents have already met each other. This gives us an $O(\log(n))$-rounds strategy that can be found in randomized polynomial time for the acquaintance problem in graphs with $AC(G) = 1$. ■

We return to the proof of Lemma 7.8.

Proof of Lemma 7.8 We describe a randomized algorithm that finds a $O(\log(n))$-rounds strategy that allows every two agents in P_U to meet. Consider the following algorithm for constructing a matching M.

1. For each vertex $u \in U$ select a vertex $u' \in N(u) \cup \{u\}$ as follows.
   (a) With probability 0.5 let $u' = u$.
   (b) With probability 0.5 pick $u' \in N(u)$ uniformly at random.
2. Select a random ordering $\sigma : \{1, \ldots, |U|\} \to U$ of U.
3. Start with the empty matching $M = \emptyset$.

4. Start with an empty set of vertices $S = \emptyset$. The set will include the vertices participating in M, as well as some of the vertices that will not move.

5. For each $i = 1, \ldots, |U|$ do

   (a) Set $u_i = \sigma(i)$.

   (b) If $u_i \notin S$ and $u'_i \notin S$, then // (u_i, u'_i) will be used in the current step

      i. $S \leftarrow S \cup \{u_i, u'_i\}$.

      ii. If $u_i \neq u'_i$, then $M \leftarrow M \cup \{(u_i, u'_i)\}$.

6. Output M.

The following claim bounds the probability that a pair of agents in P_U meet after a single step of the algorithm.

Claim 7.9 *For every $u, v \in U$, let p_u and p_v be the agents located in u and v respectively. Then,* $\Pr[$*The agents p_u, p_v meet after one step*$] \geq c$ *for some absolute constant $c > 0$ that does not depend on n or G .*

In order to achieve a $O(\log(n))$-rounds strategy that allows every two agents in P_U to meet apply the matching constructed above, and then return the agents to their original positions (by applying the same matching again). Repeating this random procedure independently $\lceil \frac{3\log(n)}{c} \rceil$ times will allow every pair of agents to meet with probability at least $1/n^3$. Therefore, by union bound all pairs of agents $p_u, p_v \in P_U$ will meet with probability at least $1/n$. This completes the proof of Lemma 7.8. ■

Proof of Claim 7.9 Note first that for every $i \leq |U|$ and for every vertex $w \in U \cup N(U)$ the probability that in step 5 of the algorithm the vertex w has been added to S before the i'th iteration is upper bounded by $3i/n$. Indeed,

$$\begin{aligned}
\Pr[\exists i' < i \text{ such that } w \in \{u_{i'}, u'_{i'}\}] &\leq \Pr[w \in \{u_{i'} : i' < i\}] + \Pr[w \in \{u'_{i'} : i' < i\}] \\
&\leq \frac{i}{n} + \sum_{i'=1}^{i} \Pr[u'_{i'} = w] \\
&\leq \frac{i}{n} + i \cdot 2/n
\end{aligned}$$

where the bound $\Pr[u'_{i'} = w] \leq 2/n$ follows from the assumption that $\deg(u_{i'}) \geq n/2$ for all $u_{i'} \in U$, and hence, the probability of picking $u_{i'}$ to be w is $1/\deg(u_{i'}) \leq 2/n$.

Let $T \in \{2, \ldots, |U|\}$ be a parameter to be chosen later. Now, let $i \leq |U|$ be the (random) index such that $\sigma(i) = u$, and let $j \leq |U|$ be the (random) index such that $\sigma(j) = v$. Then,

$$\Pr[i \leq T \text{ and } j \leq T] = \frac{\binom{T}{2} \cdot (|U|-2)!}{|U|!} = \frac{T(T-1)}{2 \cdot |U| \cdot (|U|-1)} \geq \frac{T^2}{4n^2}.$$

Conditioning on this event, the probability that either u or u' have been added in S before iteration i is upper bounded by $\frac{3i}{n} \leq \frac{3T}{n}$, and similarly the probability that either v or v' have been added in S before iteration j is at most $\frac{3T}{n}$. Therefore, with probability at least $\frac{T^2}{4n^2} \cdot (1 - \frac{6T}{n})$ both (u, u') and (v, v') will be used in the current step. Therefore,

$$\Pr[\text{The agents } p_u, p_v \text{ meet after one step}] \geq \frac{T^2}{4n^2} \cdot (1 - \frac{6T}{n}) \cdot \Pr[(u', v') \in E].$$

In order to lower bound $\Pr[(u', v') \in E]$ we use Claim 7.4, saying that for every two vertices $u, v \in U$ it holds that either $N(u) \cap N(v) \geq \alpha n$ or $|E[N(u), N(v)]| \geq \alpha \cdot n^2$ for some constant $\alpha > 0$ that does not depend on n or G. Therefore, for every $u, v \in U$ it holds that $\Pr[(u', v') \in E] \geq \alpha/4$. Letting $T = \alpha n/12$ we get that $\Pr[\text{The agents } p_u, p_v \text{ meet after one step}] = \Omega(\alpha^3)$, as required. ∎

8 Other Variants and Open Problems

There are several variants of the problem that one may consider.

1. The problem of maximizing the number of pairs that met when some predetermined number $t \in \mathbb{N}$ of matchings is allowed. Clearly, this problem is also $\mathcal{NP}$-complete, even in the case of $t = 1$.

2. The rules for moving agents are the same, but the goal is to make every agent visit every vertex of the graph.

3. Instead of choosing a matching in each round, one may choose a vertex-disjoint collection of cycles, and move agents one step along the cycle.

One may also consider a more game-theoretic variant of the problem: Let $G = (V, E)$ be a fixed graph with one agent sitting in each vertex of G. In each round every agent p_u sitting in a vertex $u \in V$ chooses a neighbor $u' \in N(v)$ according to some strategy. Then, for every edge $(v, w) \in E$ the agents p_v and p_w swap places if the choice of the agent p_v was w and the choice of p_w was v. Suppose that the graph is known, but the agents have no information regarding their location in the graph (e.g., G is an unlabeled vertex transitive graph). Find an optimal strategy for the agents so that everyone will meet everyone else as quickly as possible. The question also makes sense in the case where the graph is not known to the agents.

We conclude with a list of open problems.

Problem 8.1 *Find $\mathcal{AC}$ of the Binary Tree. Recall that $\mathcal{AC}(Binary-tree)$ is between $\Omega(n)$ and $O(n\log(n))$, where the lower bound is trivial from the number of edges, and the upper bound is obtained in Proposition 3.5.*

Problem 8.2 *Find $\mathcal{AC}$ of the Hypercube graph. Recall that $\mathcal{AC}(Hypercube)$ is between $\Omega(n/\log(n))$ and $O(n)$, where the lower bound is trivial from the number of edges, and the upper bound follows from Hamiltonicity of the graph (see Corollary 3.2).*

Problem 8.3 *Let $p \geq \frac{\log(n)}{n}$ so that w.h.p. $G(n,p)$ is connected . Compute the expectation/typical behavior of $\mathcal{AC}(G(n,p))$. Note that w.h.p. $\mathcal{AC}(G(n,p)) \geq \Omega(p^{-1})$ by the trivial bound on the number of edges. We conjecture that w.h.p. $\mathcal{AC}(G(n,p)) = O(p^{-1} \cdot \mathrm{poly}\log(n))$ which can be achieved by a random sequence of matchings coming from a reasonable distribution.*

Problem 8.4 *Is it true that for every graph G with n vertices the acquaintance time is bounded by $\mathcal{AC}(G) = O(n^{1.5})$?*

Problem 8.5 *Prove Conjecture 6.5, namely, that for every constant $t \in \mathbb{N}$ it is $\mathcal{NP}$-hard to decide whether a given graph G has $\mathcal{AC}(G) = 1$ or $\mathcal{AC}(G) \geq t$. Recall that it follows from Conjecture 6.7.*

Problem 8.6 *Prove stronger inapproximability results. Is it true that $\mathcal{AC}$ is hard to approximate within a factor of $\log(n)$? How about $n^{0.01}$? How about $n^{1.499}$?*

Problem 8.7 *Derandomize the algorithm given in Theorem 7.7.*

Problem 8.8 *Give a structural result regarding graphs with small constant values of $\mathcal{AC}(G)$ similar to Proposition 7.1. Also, is there an efficient $O(\log(n))$-approximation algorithm for such graphs?*

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