Euclidean vs Graph Metrics

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1. Several comments regarding graph metric approximations of the Euclidean metric and scaling limits of graphs
We will discuss how well the graph metric on bounded degree graphs can approximate the metric of homogeneous manifolds equipped with some invariant length metric.

Recall that the scaling limit of the $\mathbb{Z}^2$ grid is the $L^1$ metric on the plane.

In the first part we will remark on graph approximations of the Euclidean ($L^2$) metric.

In the second part we will look at approximating invariant metrics on manifolds with a given topology.

The last part is about inducing Euclidean structure on graphs.
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Given a graph $G = (V, E)$, the *graph distance* between any two vertices is the length of the shortest path between them.
Slack-isometry

Definition
Two metric spaces $G$ and $H$ are said to be quasi-isometric if there exists a map $f : G \to H$ and two constants $1 \leq C < \infty$ and $0 \leq c < \infty$, such that

$$C^{-1}d_H(f(x), f(y)) - c \leq d_G(x, y) \leq Cd_H(f(x), f(y)) + c$$

for every $x, y \in G$,

- For every $y \in H$ there is an $x \in G$ so that $d_H(f(x), y) < c$.

Two metric spaces are said to be slack-isometric iff they are quasi-isometric with multiplicative constant equal to 1. That is, if we can take $C = 1$ in the definition.
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Question

Is there a bounded degree graph which is slack-isometric to the Euclidean plane?

The Pinwheel tiling, which is a non-periodic tiling defined by Charles Radin (see Wikipedia), is a graph quasi-isometric to the Euclidean plane where the multiplicative constant goes to 1 uniformly in the distance.

The Poisson Voronoi tessellation will almost surely have an asymptotically Euclidean metric. Note that the Euclidean metric underlies the construction.
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Near critical percolation

The Gromov-Hausdorff distance between two metric spaces is obtained by taking the infimum over all the Hausdorff distances between isometric embeddings of the two spaces in a common metric space.

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Near critical percolation

Consider the natural embedding of the square grid in the plane. Dilute the planar square grid by removing edges independently with probability $q < 1/2$. $1/2$ is the critical percolation probability (Kesten (80)). Thus almost surely there is a unique connected dense infinite subgrid left.
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Condition on the origin to be in the infinite connected component and look at large balls rescaled to have diameter 1.

For any fixed $q$ the subadditive ergodic theorem was used in the context of first passage percolation to show that the rescaled large balls around the origin will a.s. converge in the Gromov-Hausdorff distance to a centrally symmetric convex body in the Euclidean plane.
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Conjecture

As $q \to 1/2$ the limiting shape Gromov-Hausdorff converges to an Euclidean ball.

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Symmetric graphs

Definition
A graph $G$ is vertex transitive if for every two vertices in $G$ there is an isometry of $G$ mapping one to the other.
Symmetric graphs

Is there a sequence of finite vertex transitive graphs which Gromov-Hausdorff converges to the sphere $S^2$? (equipped with some invariant length metric).
Symmetric graphs

Let \((G_n)\) be an unbounded sequence of finite, connected, vertex transitive graphs with bounded degree such that 
\[|G_n| = o(diam(G_n)^d)\] for some \(d > 0\).

Theorem (with Hilary Finucane and Romain Tessera)

Up to taking a subsequence, and after rescaling by the diameter, the sequence \((G_n)\) converges in the Gromov Hausdorff distance to a torus of dimension \(< d\), equipped with some invariant length metric.
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In particular if the sequence admits a doubling property at a small scale then the limit will be a torus equipped with some invariant length metric. Otherwise it will not converge to a finite dimensional manifold.

The proof relies on a recent quantitative version of Gromov’s theorem on groups with polynomial growth obtained by Breuillard, Green and Tao and a scaling limit theorem for nilpotent groups by Pansu.
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Symmetric graphs

Establishing quantitative versions will have applications to random walks and percolation on vertex transitive graphs.

For example:
Let $G$ be a finite, $d$-regular connected vertex transitive graph. View $G$ as an electrical network in which each edge is a one Ohm conductor.

Conjecture (with Gady Kozma)

For any two vertices

$$\text{electric resistance}(v, u) < C_d + \frac{\text{diam}^2(G) \log |G|}{|G|}.$$ 

In addition for a sequence of vertex transitive graphs, if the diameter is $o(|G_n|)$ then the electric resistance between any two vertices is $o(\text{diam}(G_n))$. 
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Since finite vertex transitive graphs, when converge to a manifold, converge to tori, it follows that the infimum over all such, of the Gromov Hausdorff distance to $S^n$, is attained. Which one is the closest?

Is the skeleton of the truncated icosahedron (soccer ball) the closest to $S^2$?

"Proof": Otherwise we will have different design for soccer balls.

See also Géode (géométrie) in French Wikipedia.
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See also Géode (géométrie) in French Wikipedia.
A metric space $X$ is $(C, c)$-roughly transitive if for every pair of points $x, y \in X$ there is a $(C, c)$-quasi-isometry sending $x$ to $y$.

If $G_n$ is only roughly transitive and $|G_n| = o(\text{diam}(G_n)^{1+\delta})$ for $\delta > 0$ sufficiently small, we are able to prove, this time by elementary means, that $(G_n)$ converges to a circle.

**Question**

Is there an infinite $(C, c)$-roughly transitive graph, with $C, c$ finite, which is not quasi-isometric to a homogeneous space?

Where a homogeneous space is a space with a transitive isometry group.
Roughly transitive graphs

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Going in the opposite direction

Question

*Which graphs can be realized as the nerve graph of a sphere packing in Euclidean d-dimensional space?*

Where vertices correspond to spheres with disjoint interiors and edges to tangent spheres.
Going in the opposite direction

The rich two dimensional theory started with Koebe, who proved that all planar graphs admits a circle packing.

In higher dimensions, Thurston observed that packability implies order $|G|^{(d-1)/d}$ upper bound on the size of minimal separators. There is an emerging theory with many still open directions.
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Metrics on graphs from packings

Two results and a question.

Theorem (with Oded Schramm)

The grid $\mathbb{Z}^4$, $T_3 \times \mathbb{Z}$ and lattices in hyperbolic 4-space cannot be sphere packed in Euclidean $\mathbb{R}^3$. 
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**Theorem (with Oded Schramm)**

*The grid $\mathbb{Z}^4$, $T_3 \times \mathbb{Z}$ and lattices in hyperbolic 4-space cannot be sphere packed in Euclidean $\mathbb{R}^3$.***
Let \((G_n)\) be a sequence of finite, \((k > 2)\)-regular graphs with girth growing to infinity,

Theorem (with Nicolas Curien)
For every \(d\) there exists an \(N(d)\) such that \(G_n\) is not regularly sphere packed in Euclidean \(d\)-dimensional space for any \(n > N(d)\).

The proof uses sparse graphs limits. By regularly we mean uniform upper bound on the ratio of the radii of neighboring spheres.
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Question

Show that any packing of $\mathbb{Z}^3$ in $\mathbb{R}^3$ has at most one accumulation point in $\mathbb{R}^3 \cup \{\infty\}$. 
2: *Stochastic Hyperbolic Infinite Quadrangulation.*

With Nicolas Curien
Guth, Parlier and Young (2010) studied pants decomposition of random closed surfaces obtained by randomly gluing $N$ Euclidean triangles (with unit side length) together.

They gave bounds on the size of pants decomposition of random compact surfaces with no genus restriction as a function of $N$. Their work indicates that the injectivity radius around a typical point is growing to infinity.

Gamburd and Makover (2002) showed that as $N$ grows the genus will converge to $N/4$ and by Euler’s characteristic the average degree will grow to infinity.
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A limit of finite graphs $G_n$ is a random rooted infinite graph $(G, \rho)$ with the property that neighborhoods of $G_n$ around a random vertex converge in distribution to neighborhoods of $G$ around $\rho$.

Formally, let $(G, o)$ and $(G_1, o_1), (G_2, o_2), \ldots$ be random connected rooted locally finite graphs. We say that $(G, o)$ is the limit of $(G_j, o_j)$ as $j \to \infty$ if for every $r > 0$ and for every finite rooted graph $(H, o')$, the probability that $(H, o')$ is isomorphic to a ball of radius $r$ in $G_j$ centered at $o_j$ converges to the probability that $(H, o')$ is isomorphic to a ball of radius $r$ in $G$ centered at $o$.

A random rooted finite graph $(G, o)$ is unbiased, if given $G$ the root $o$ is uniformly distributed among the vertices of $G$. In particular given a (possibly random) graph we will consider the distribution on rooted graphs obtained by rooting at a random uniform vertex.

Exercise: What is the limit of $n$-levels full binary trees?
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Take a uniform measure on triangulations with $N$ triangles conditioned on the genus to be $CN$ for some fixed $C < 1/4$ and a uniformly chosen root.

We conjecture that as $N$ grows, this random surface converges to a rooted random triangulation of the hyperbolic plane with average degree $6/(1 - 4C)$.

In particular we believe that the local injectivity radius around typical vertex will go to infinity on such a surface as $N$ grows.
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Philippe Chassaing constructed the UIPQ via Schaeffer’s bijection from a labeled critical Galton-Watson tree conditioned to survive.

We propose the study of an infinite random quadrangulation constructed similarly from a labeled super critical Galton-Watson trees.

We conjecture that such a *stochastic hyperbolic infinite quadrangulation* describes the limit of random finite quadrangulation with genus growing linearly in the number of quadrangulation.
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We know that simple random walk on the Shiq has positive speed a.s.

Unlike the zero genus UIPQ which is recurrent (Gurel-Gurevich and Nachmias) and sub diffusive (with Curien), basic properties of the Shiq are still unknown.