# Secret Sharing for NP

Ilan Komargodski Moni Naor Eylon Yogev

Weizmann Institute of Science

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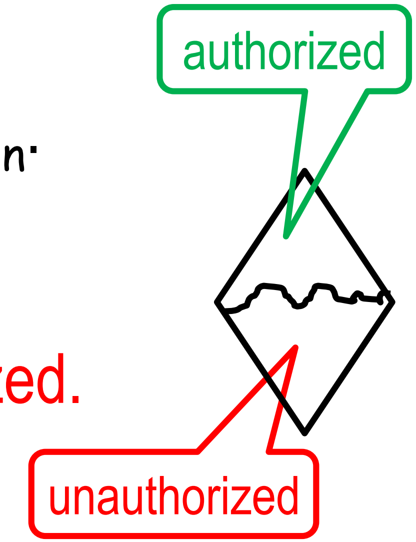
# Secret Sharing

- **Dealer** has secret $S$.
- Gives to users $P_1, P_2, \ldots, P_n$ **shares** $\Pi_1, \Pi_2, \ldots, \Pi_n$.
  - The shares are a **probabilistic function of** $S$.

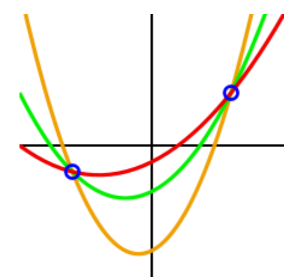



- A subset of users $X$ is either authorized or unauthorized.

$\Pi(X,S)$

**Goal**:
- An authorized $X$ can reconstruct $S$ based on their shares.
- An unauthorized $X$ **cannot** gain *any* knowledge about $S$.

- Introduced by Blakley and Shamir in the late 1970s.
  - Threshold secret sharing

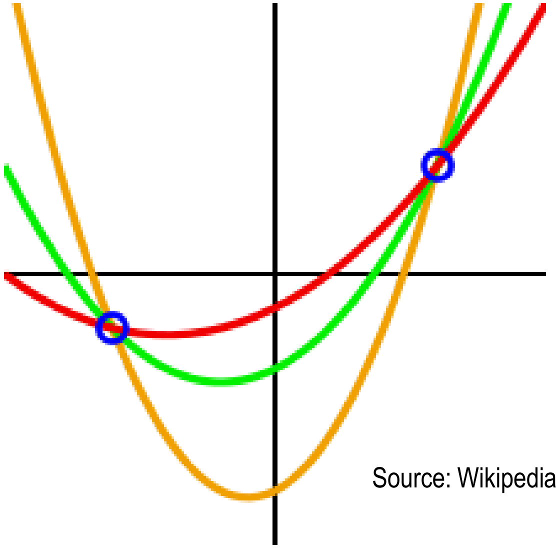


Source: Wikipedia

# Example - Threshold

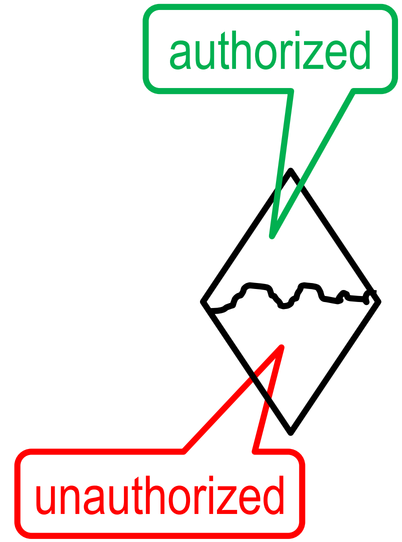
- Shamir's famous example - Threshold Secret Sharing
  - Authorized: any $k$ out of the $n$ parties.
  - Unauthorized: any set of less than $k$ parties.
- Solution: based on a random degree $k$-1 polynomial $q$, s.t.:
  - $q(0) = S$.
  - $\Pi_i = q(i)$.

Example $k=3$:



Source: Wikipedia

# Access Structures




Access Structure $M$:

- An indicator function of the authorized subsets.

- To make sense: $M$ should be monotone:
  if $X' \subset X$ and $M(X')=1$ then $M(X)=1$

**Perfect secret sharing scheme**:

- For any two secrets $S_0$, $S_1$, subset $X$ s.t. $M(X)=0$:

$$\mathrm{Dist}(\Pi(X,S_0)) = \mathrm{Dist}(\Pi(X,S_1)).$$

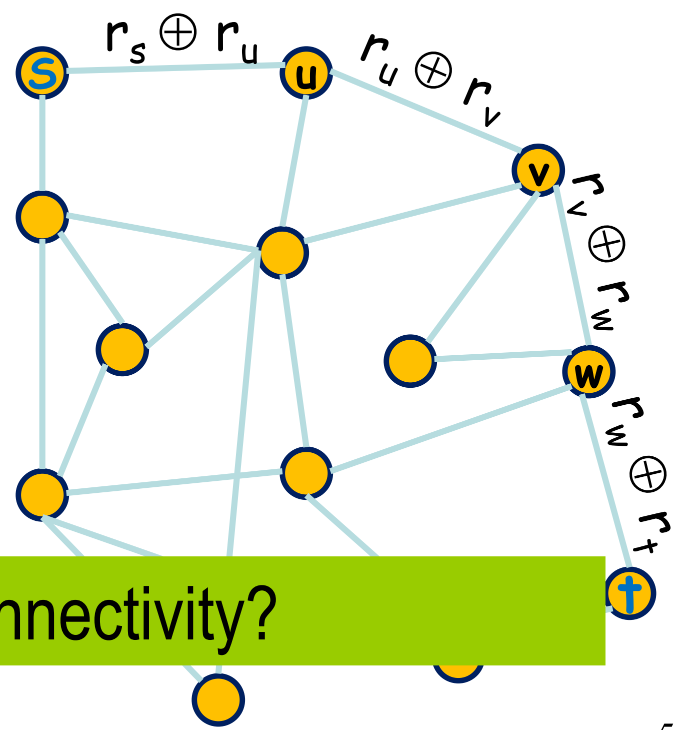
Or equivalently: for any distinguisher $A$:

$$|\Pr[A(\Pi(X,S_0)) = 1] - \Pr[A(\Pi(X,S_1)) = 1]| = 0$$

The **complexity** of the scheme: the **size** of the largest share.

# Example – undirected connectivity

- Parties correspond to edges in a graph G.
- Two special nodes: $s$,$t$.
- Authorized sets: those graphs containing a **path from s to t.**
- Solution:
  - Give vertices random values $r_1,\dots,r_n$.
  - Set $r_t = S \oplus r_s$.
  - For edge $\Pi_{u,v} = r_u \oplus r_v$.
- Reconstruction:
  - XOR all shares.




What about directed connectivity?

# Known Results

**Theorem** [Ito, Saito and Nishizeki 1987] :

For every $\mathcal{M}$ there exists a perfect secret sharing scheme

- might have exponential size shares in the number of parties.

**Theorem** [Benaloh-Leichter 1988] :

If $\mathcal{M}$ is a **monotone formula** $\Phi$: there is a perfect secret sharing scheme where the size of a share is proportional to $|\Phi|$.

Karchmer-Wigderson generalized this results to **monotone span programs** [1993]

**Major question**: can we prove **a lower bound on the size** of the shares for ***some*** access structure?

– Even a non constructive result is interesting

# Computational Secret Sharing

- **Perfect** secret sharing scheme:

  Any unauthorized subset $X$ gains absolutely ***no*** information:

  – For any $A$, secrets $S_0$, $S_1$, subset $X$ s.t. $M(X)=0$:

  $$|\Pr[A(\Pi(X,S_0)) = 1]-\Pr[A(\Pi(X,S_1)) = 1]|=0.$$

- **Computational** secret sharing scheme:

  Any unauthorized subset $X$ gains no ***useful*** information:

  $$\Pi(X,S_0) \approx \Pi(X,S_1)$$

  In the **indistinguishability** of encryption style:

  For any PPT $A$, two secrets $S_0$, $S_1$, subset $X$ s.t. $M(X)=0$:

  $$|\Pr[A(\Pi(X,S_0)) = 1] - \Pr[A(\Pi(X,S_1)) = 1]| < neg$$

This is a non-uniform definition

# Computational Secret Sharing

**Theorem [Yao~89]**:
If $M$ can be computed by a **monotone** poly-size circuit $C$ then:

There is a **computational** secret sharing scheme for $M$.

- Size of a share is proportional to $|C|$.
- Assuming one-way functions.

Construction similar to Yao's **garbled circuit**

- What about monotone access structure that have small **non**-monotone circuits?
  - Matching:
    - Parties correspond to edges in the complete graph.
    - Authorized sets: the subgraphs containing a **perfect matching.**

Open problem: do all monotone functions in $P$ have computational secret sharing schemes?

# Secret Sharing for NP

Rudich circa 1990

What about going beyond *P*?

- Efficient **verification** when the authorized set proves that it is authorized
  - Provide a witness

Example:

- Parties correspond to edges in the **complete graph**.
- Authorized sets: subgraphs containing a **Hamiltonian Cycle.**
- The **reconstruction** algorithm should be provided with the **witness**: a cycle.

# Secret Sharing and Oblivious Transfer

**Theorem**:

If one-way functions exist and a computationally secret sharing scheme for the Hamiltonian problem exists then:

**Oblivious Transfer** Protocols exist.

- In particular Minicrypt = Cryptomania
- Construction is non-blackbox

- No hope ***under standard assumptions*** for perfect or statistical scheme for Hamiltonicity

# Witness Encryption
# [Garg, Gentry, Sahai, Waters 2013]

Includes y

- A witness encryption ($\mathrm{Enc}_L$, $\mathrm{Dec}_L$) for a language $L \in NP$:
  - Encrypt **message** m relative to **string** y: $ct = \mathrm{Enc}_L(x,m)$
  - For any $y \in L$: let $ct = \mathrm{Enc}_L(y,m)$ and let w be **any** witness for x. Then $\mathrm{Dec}_L(ct,w) = m$.
  - For any $y \notin L$: $ct = \mathrm{Enc}_L(y,m)$ computationally hides the message m.

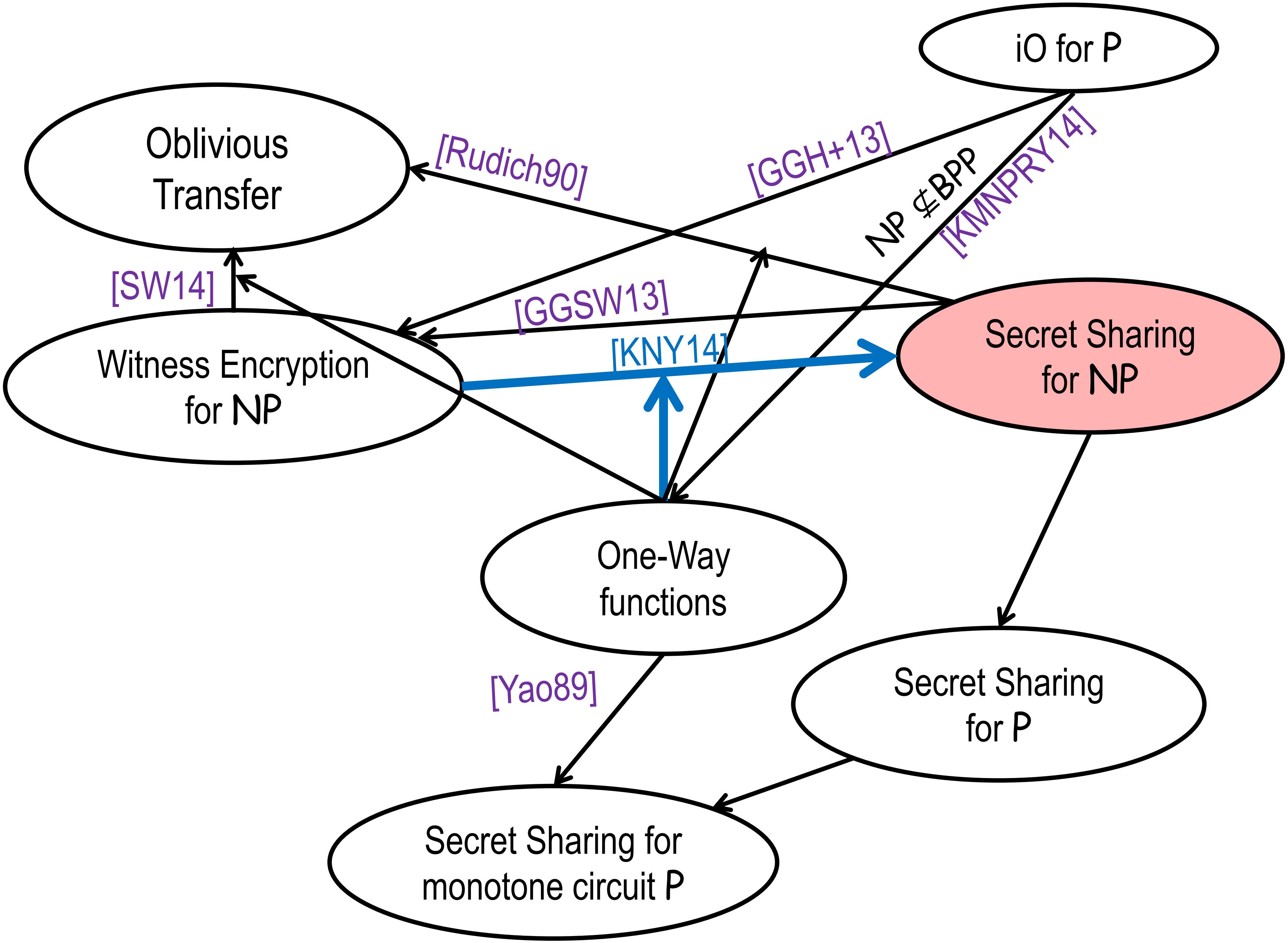
- Gave a candidate construction for witness encryption.
- Byproduct: a candidate construction for secret sharing for a specific language in NP (Exact Cover).

Multilinear Maps, Indistinguishability Obfuscation (iO)…

# Our Results

If one-way functions exist then:

- Secret Sharing for $\mathcal{NP}$ and **Witness Encryption** for $\mathcal{NP}$ are (existentially) **equivalent**.
- If there is a **secret sharing** scheme for **one** $\mathcal{NP}$-complete language, then there is one for **all** languages in $\mathcal{NP}$.



iO for P
Oblivious Transfer
[Rudich90]
[GGH+13]
NP ⊄ BPP
[KMNPRY14]
[SW14]
[GGSW13]
Witness Encryption for NP
[KNY14]
Secret Sharing for NP
One-Way functions
[Yao89]
Secret Sharing for P
Secret Sharing for monotone circuit P

# Definition of secret sharing for $\mathcal{NP}$

Let $\mathcal{M}$ be a monotone access structure in $\mathcal{NP}$.

- **Completeness**:
  For any $X$ s.t. $M(X)=1$, any witness $w$ (for $X$), and any secret $S$:

  $$\mathrm{recon}(\Pi(X,S),w) = S.$$

  - All operations polytime

# Definition of secret sharing for NP: Security

- Let $M$ be a monotone access structure in NP.

**Security**:
For any adversary $A=(A_{samp},A_{dist})$ such that $A_{samp}$ chooses two secrets $S_0,S_1$ and a subset $X$ it holds that:

$$|\Pr[M(X)=0 \wedge A_{dist}(\Pi(S_0,X)) = 1] - \Pr[M(X)=0 \wedge A_{dist}(\Pi(S_1,X)) = 1]| < neg.$$

This is a static and uniform definition

- A weaker possible definition is to require that $X$ is **always** unauthorized.

# The Construction

For access structure $M \in NP$.

- Define a new language $M' \in NP$:
  - Let $c_1, \ldots, c_n$ be $n$ strings.
  - Then $M'(c_1,\ldots,c_n) = 1$ iff $M(X) = 1$ where:

$$X_i = \begin{cases} 1 \text{ if exist } r_i \text{ s.t. } c_i = com(i, r_i) \\ 0 \text{ otherwise} \end{cases}$$

Computationally hiding: $com(x_1) \approx com(x_2)$
Perfect Binding: $com(x_1)$ and $com(x_2)$ have disjoint support.

Can be constructed from one-way functions in the CRS model with high probability.

# The Construction...

String $y$

Message $m$

**Dealer($S$):**

- Choose $r_1, \ldots, r_n$ uniformly at random.
- For $i \in [n]$, compute $c_i = com(i, r_i)$.
- Compute $ct$ = $WE.Enc_{M'}((c_1, \ldots, c_n), S)$.
- Set $\Pi_i = (r_i, ct)$.

Shared by all

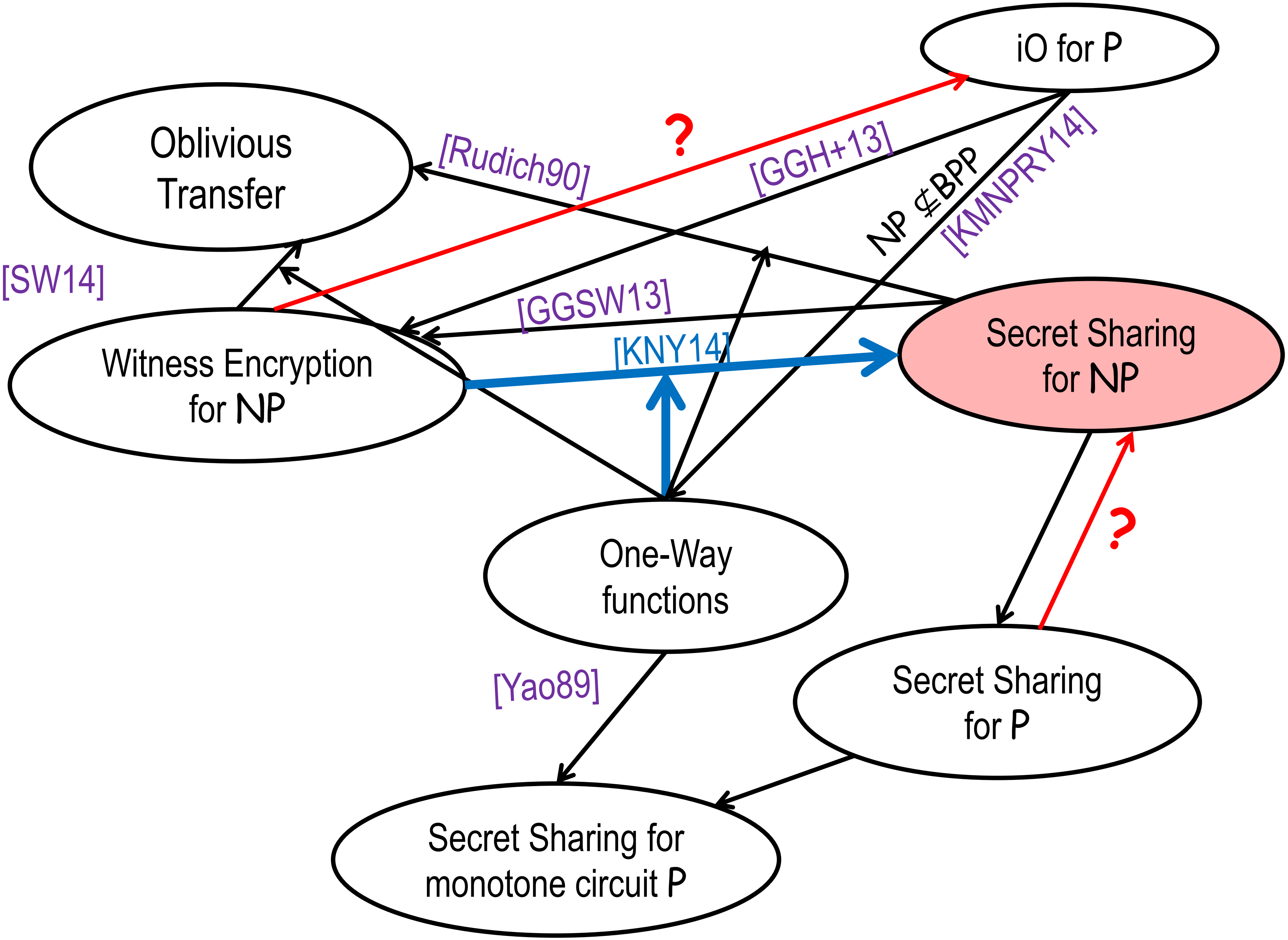
**Reconstruction**: authorized subset $X$ of parties: $M(X)=1$ and witness $w$ witness for $X$.

- Witness for $M'$ consists of openings $r_i$ such that $X_i=1$.
- Set $w' = (r'_1, \ldots, r'_n, w)$.
- Compute $S$ = $WE.Dec_{M'}(ct, w')$.

# Security

Suppose an adversary $A=(A_{samp}, A_{dist})$ breaks the system.

- Construct an algorithm $D$ that breaks the commitment scheme:
  - For a list of commitments $c_1, \ldots, c_n$ distinguish between two cases:
    - They are commitments of $1, \ldots, n$.
    - They are commitments of $n+1, \ldots, 2n$.



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# Open Problems

Brakerski: diO

- **Adaptive** choice of the set X.
- Perfect Secret-Sharing Scheme for directed connectivity.
  - How to cope with the fan-out
- Computational Secret Sharing Scheme for Matching.
  - How to cope with negation?
- A secret sharing scheme for *P* based on less heavy cryptographic machinery.