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Finding the Shortest Move-Sequence in the Graph-Generalized 15-Puzzle is NP-Hard

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Abstract

Following Wilson, Johnson, and Kornhauser, Miller and Spirakis, we consider the following one-party game. The game consists of moving distinct pebbles along the edges of an undirected graph. At most one pebble may be present in each vertex at all time. It is only allowed to move one pebble at a time (clearly it must be moved to a previously empty vertex).

We show that the problem of finding the shortest sequence of moves from one "graph configuration" to another is NP-Hard.

Keywords: NP-Completeness, Games' Complexity, Computational Group Theory.

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1. Introduction

The following generalization of the "15-Puzzle" appeared in [W], [Jo] and [KGS]:

Board: A finite, undirected, simple graph. The graph will be denoted by $G(V, E)$.

Pebbles: There exist $|V| - 1$ distinct pebbles. Let us mark the pebbles by $1, 2, ..., |V| - 1$.

Legal Board Configuration: Every vertex contains at most one pebble. $BC: V \mapsto \{0, 1, 2, .., |V| - 1\}$ is a *legal board configuration* if it is one-to-one and onto. $BC(\cdot)$ can be interpreted as follows: if $BC(v) \neq 0$ then vertex v contains pebble $BC(v)$ if $BC(v) = 0$ then vertex v is empty.

A Legal Move consists of moving a pebble, along one of the edges of the graph to an empty vertex. Clearly, a legal move is a transformation on the set of legal configurations. Let $BC(\cdot)$ be a legal configuration and $BC'(\cdot)$ be the configuration which results from $BC(\cdot)$ after a legal move. Then there exist two vertices, $u, v \in V$, such that: $BC(v) = BC'(u)$, $BC(u) = BC'(v) = 0$, $(u, v) \in E$ and $BC(w) = BC'(w)$ for all $w \in V - \{u, v\}$. In this move $BC(v)$ is moved from vertex v to vertex u.

A Sequence of (t) Moves is a sequence, $BC_0(\cdot), BC_1(\cdot), BC_2(\cdot), ..., BC_t(\cdot)$, of legal board configurations such that $BC_i(\cdot)$ is the result of applying a legal move to $BC_{i-1}(\cdot)$, $1 \leq i \leq t$. The configuration $BC_0(\cdot)$ is called the *beginning configuration* of the above sequence. The configuration $BC_t(\cdot)$ is called the *finishing configuration* of the above sequence.

A Solution: A pair of legal board configurations is said to have a solution if there exists a sequences of moves beginning at one and finishing at the other.

Kornhauser, Miller and Spirakis [KMS] showed that, for any nonseparatable graph $G(V, E)$, *if* a pair of legal board configurations has a solution *then* it has a solution by $O(|V|^3)$ moves. Furthermore, they showed that such a solution (by $O(|V|^3)$ moves) can be found in $O(|V|^3)$ time.

A natural algorithmic question arises:

Is it feasible to find the shortest solution, for a pair of legal board configurations which has a solution ?

We answer this question negatively, proving that finding such a solution is NP-Hard.

Remark: It should be noted that (P1) *the problem of finding the shortest solution to a given pair of board configurations* is a special case of (P2) *the problem of finding the shortest generator sequence realyzing a given permutation.* The latter problem (P2) has been formulated and shown to be NP-Hard in [EG]. Lately, (P2) has been shown to be PSpace-Complete ([Je]).

2. The NP-CompletenessResult

In order to discuss the problem of finding the shortest solution to a solvable pair of legal board configurations, we introduce the following decision problem (herafter referred to as *Shortest Move Sequence (SMS)* Problem):

Input: A nonseparatable, simple, undirected graph $G(V,E)$; a pair, $B(\cdot)$ and $F(\cdot)$, of legal board configuration; and an integer K.

Question: Is there a sequence of K (or less) legal moves beginning at $B(\cdot)$ and finishing at $F(\cdot)$?

Theorem: The Shortest Move Sequence (SMS) problem is NP-Complete.

proof: First note that SMS$\in NP$ (w.l.o.g $K = O(|V|^3)$). We prove that SMS is complete by reducing the 3-Exact-Cover (3XC) problem to it. The *3XC problem* is defined as follows:

Input: A set $U = \{e_i\}_{i=1}^{3n}$ and a collection $S = \{s_j\}_{j=1}^{m}$ of 3-element subsets (3-subsets) of U.

Question: Is there a subcollection, $S' \subseteq S$, such that every element is U occurs in exactly one member of S' ?

If existing, such a collection, S', is called an *exact cover.*

Karp has proved that 3XC$\in NPC$ (see [GJ]).

Given an instance, $(U = \{e_i\}_{i=1}^{3n}, S = \{s_j\}_{j=1}^{m})$, of the 3XC, we construct the following SMS instance:

Set $V = U^0 \cup U^1 \cup S \cup \{T\}$, where $U^\sigma = \{e^\sigma : e \in U\}$ for $\sigma \in \{0,1\}$. The vertices e^0 and e^1 will be associated with the element $e \in U$. The vertex s will be associated with the 3-subset $s \in S$. The vertex T will be called the *temporary vertex.*

Set $E = E_{3XC} \cup \{(T,s) : s \in S\} \cup \{(e^0,e^1) : e \in U\}$, where $E_{3XC} = \{(e^\sigma, s) : e \in U \wedge s \in S \wedge \sigma \in \{0,1\} \wedge e \in s\}$. The edges in E_{3XC} consists of the description of the 3XC instance. Note that $(e^\sigma, s) \in E_{3XC}$ iff the element $e \in U$ appears in the 3-subset $s \in S$.

Set $B(e_i^\sigma) = 2i - 1 + \sigma$, for $1 \le i \le 3n$ and $\sigma \in \{0,1\}$. Set $B(s_j) = 6n + j$, for $1 \le j \le m$. Set $B(T) = 0$. In the begin configurations T is empty while the pebbles are placed in a "canonical" order. The pebbles $2i-1$ and $2i$ will be associated with the element e_i, $1 \le i \le 3n$. The pebble $6n+j$ will be associated with the 3-subset s_j, $1 \le j \le m$.

Set $F(e_i^\sigma) = 2i - \sigma$, for $1 \le i \le 3n$ and $\sigma \in \{0,1\}$. Set $F(s_j) = 6n + j$, for $1 \le j \le m$. Set $F(T) = 0$. In the finish configurations T is still empty and the pebbles in the vertices which are associated with the 3-subsets remain invariant w.r.t the begin configuration. The pebbles associated with an element are switched w.r.t. the begin configuration.

Set $K = 2 \cdot n + 3 \cdot 3n = 11n$.

Having presented our reduction it remains to show that it is indeed valid.

Assume that the 3XC instance has an exact cover $S' = \{s_{i_j}\}_{j=1}^{n}$. Let f: $\{1, 2, ..., n\} \times \{1, 2, 3\} \mapsto \{1, 2, ..., 3n\}$, such that $e_{f(j,k)}$ is the k-th element in the 3-subset s_{i_j} (order on the elements in each 3-subset can be induced by an ordering of U). Note that $s_{i_j} = \{e_{f(j,1)}, e_{f(j,2)}, e_{f(j,3)}\}$ and $U = \{e_{f(j,k)}: 1 \leq j \leq n \wedge 1 \leq k \leq 3\}$. Then the following is a solution to the corresponding SMS instance:

```
for j = 1 to n do begin;
  move 6n + i_j from s_{i_j} to T;
  for k = 1 to 3 do begin;
    move 2f(j,k) - 1 from e^0_{f(j,k)} to s_{i_j};
    move 2f(j,k) from e^1_{f(j,k)} to e^0_{f(j,k)};
    move 2f(j,k) - 1 from s_{i_j} to e^1_{f(j,k)};
    [ comment: At this stage 2f(j,k) - 1 and 2f(j,k) are switched.]
  end
  move 6n + i_j from T to s_{i_j};
  [ comment: At this stage all pebbles associated to elements
    in s_{i_j} are switched and all the pebbles associated with
    3-subsets are back in place.]
end
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One can easily verify that the following procedure transforms in $11n = K$ moves the begin configuration into the finish configuration.

Assume, on the other hand that the SMS instance has a solution in no more than $K = 11n$ moves. Let us denote this solution (sequence of moves) by Q. The following facts concerning Q can be easily verified:

Fact 1: Switching pebble $2i - 1$ with pebble $2i$ $(1 \leq i \leq 3n)$, requires at least 2 moves of one of these pebbles and 1 move of the other. Furthermore, this switching requires that at least one of these pebbles passes through a vertex associated with a 3-subset which contain the element e_i.

Fact 2: If some pebble passes through a 3-subset vertex, s_j $(1 \leq j \leq m)$, during Q then the pebble $6n + j$ must have been moved during Q.

Let M denote the set of pebbles, associated with 3-subsets, which moved during Q.

Fact 3: $3 \cdot 3n + 2 \cdot |M|$ is a lower bound on the number of moves in Q.

Recall that $K = 11n$ is an upper bound on the number of moves (in Q). Thus,

Fact 4: $|M| \leq n$.

Fact 5: The collection $C = \{s_j: 6n + j \in M\}$ constitutes a cover of the set U. I.e. for every element $e \in U$, there exists a 3-subset $s \in C$ such that $e \in s$. (Note that $2i - 1$ has been switched with $2i$, for each $1 \leq i \leq 3n$.)

Combining Facts 4 and 5, we conclude that C is an exact cover of the 3XC instance.

QED

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4. References

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