Binary Searching a Tree

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MIT, CSAIL

Joint work with
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Krzysztof Onak (MIT)

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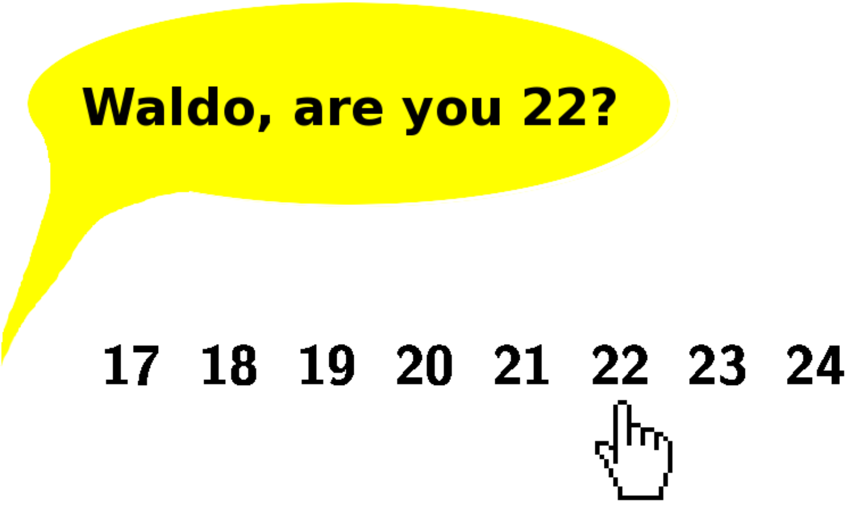

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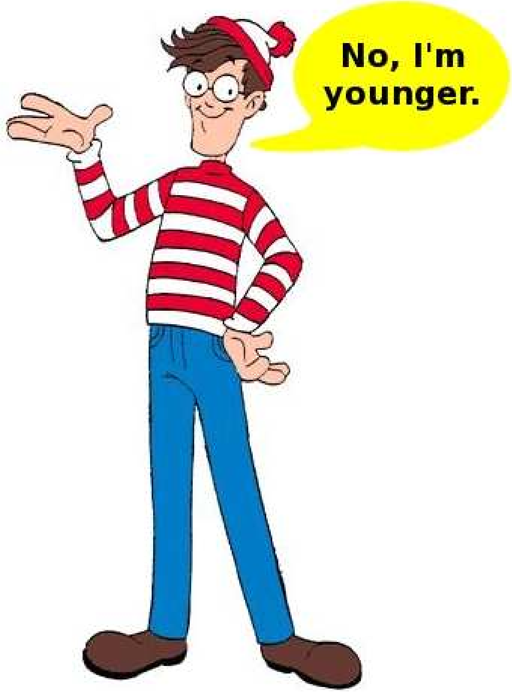

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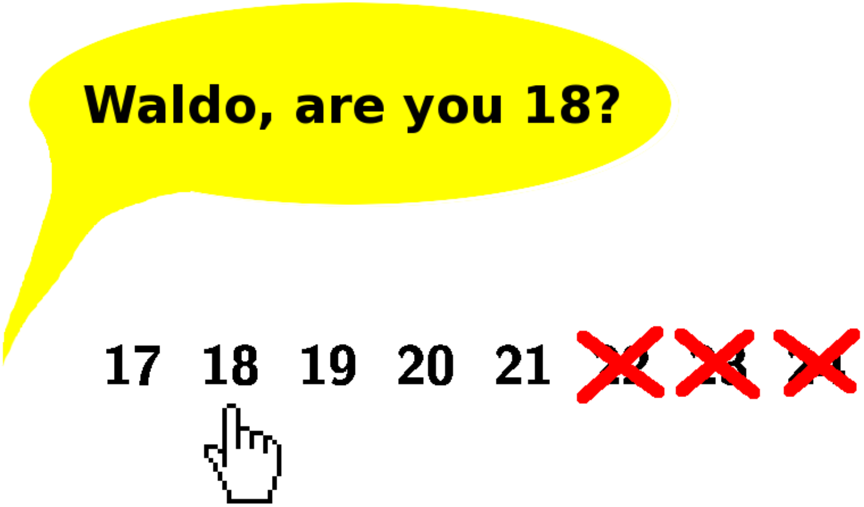

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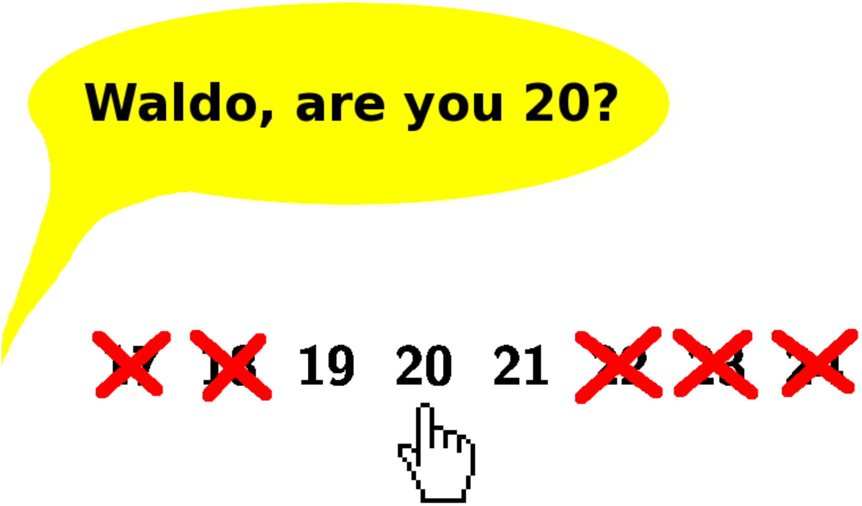

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You must be 19!

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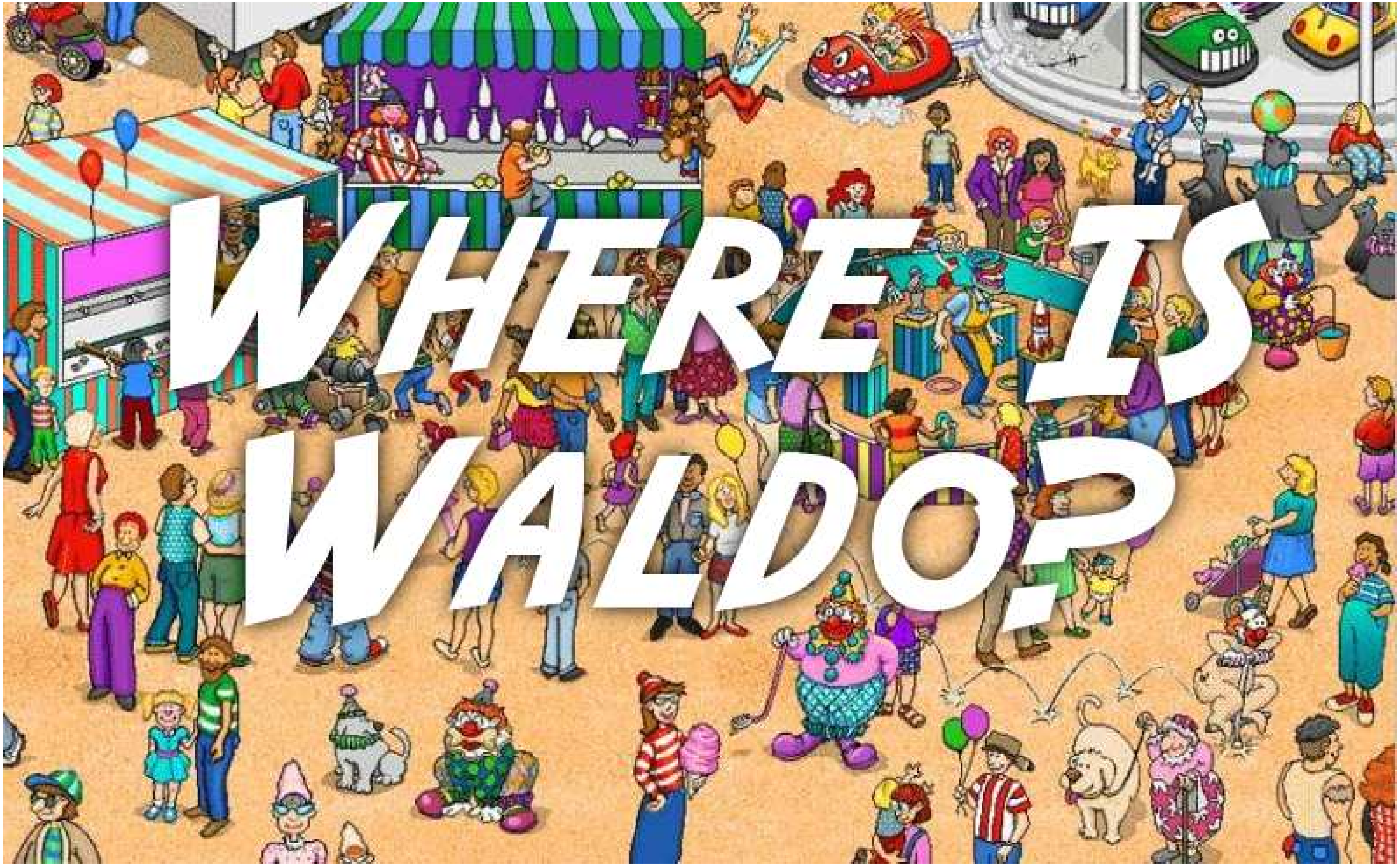
- But there is a greater challenge to face!



WHERE IS WALDO?

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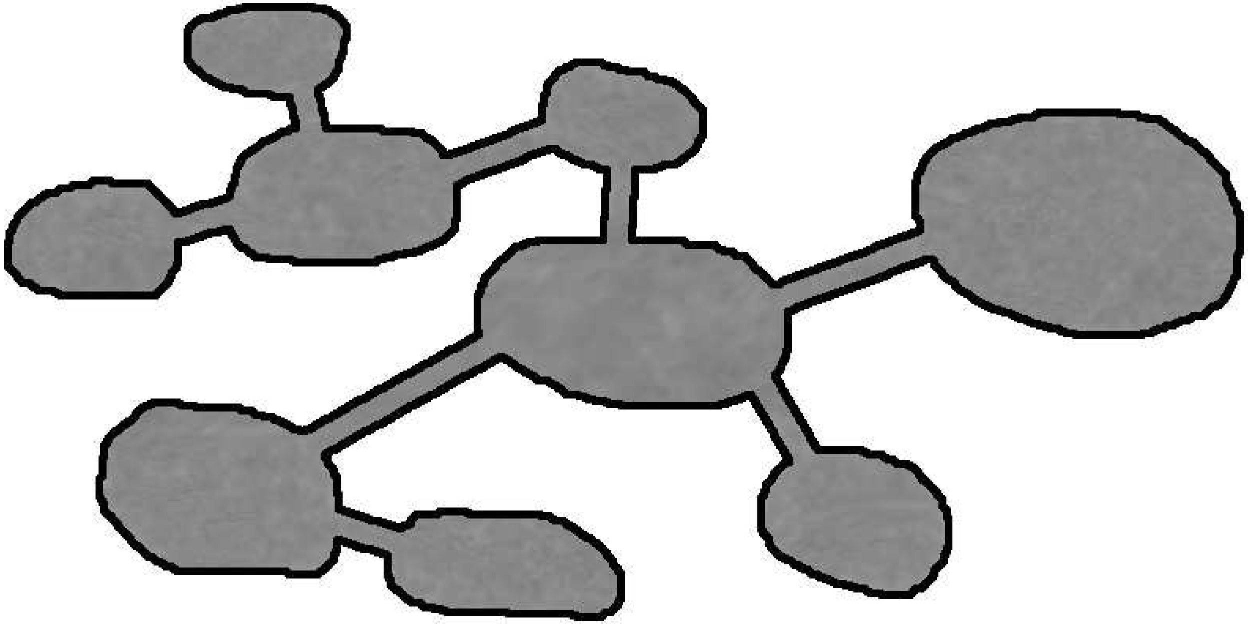

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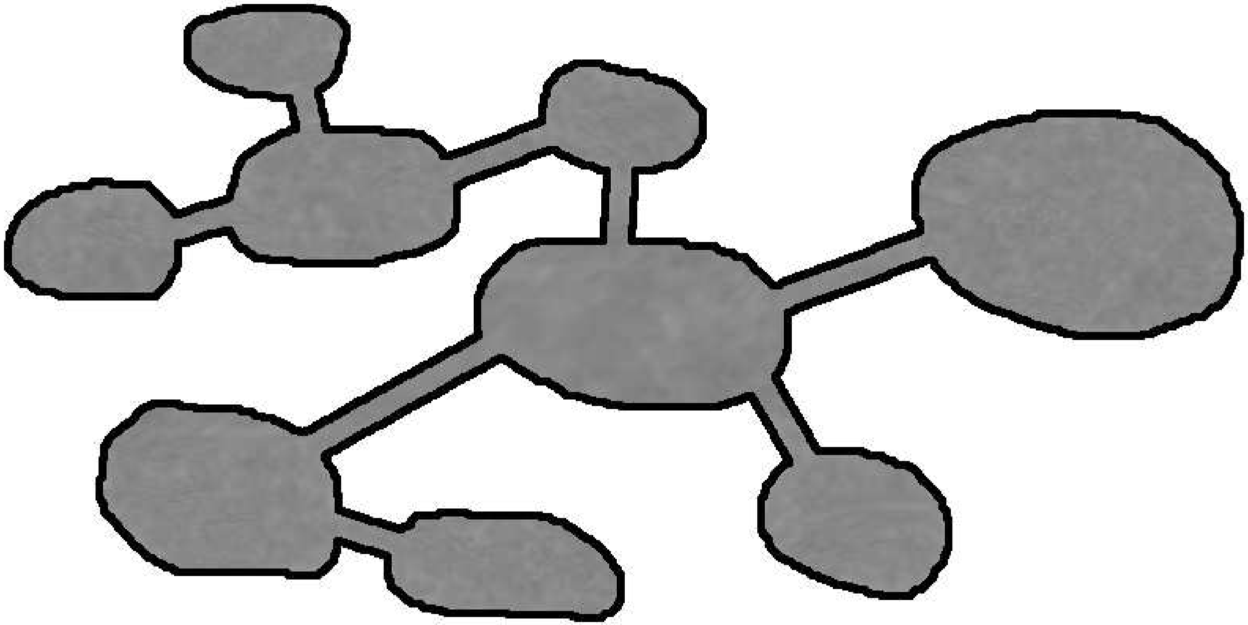
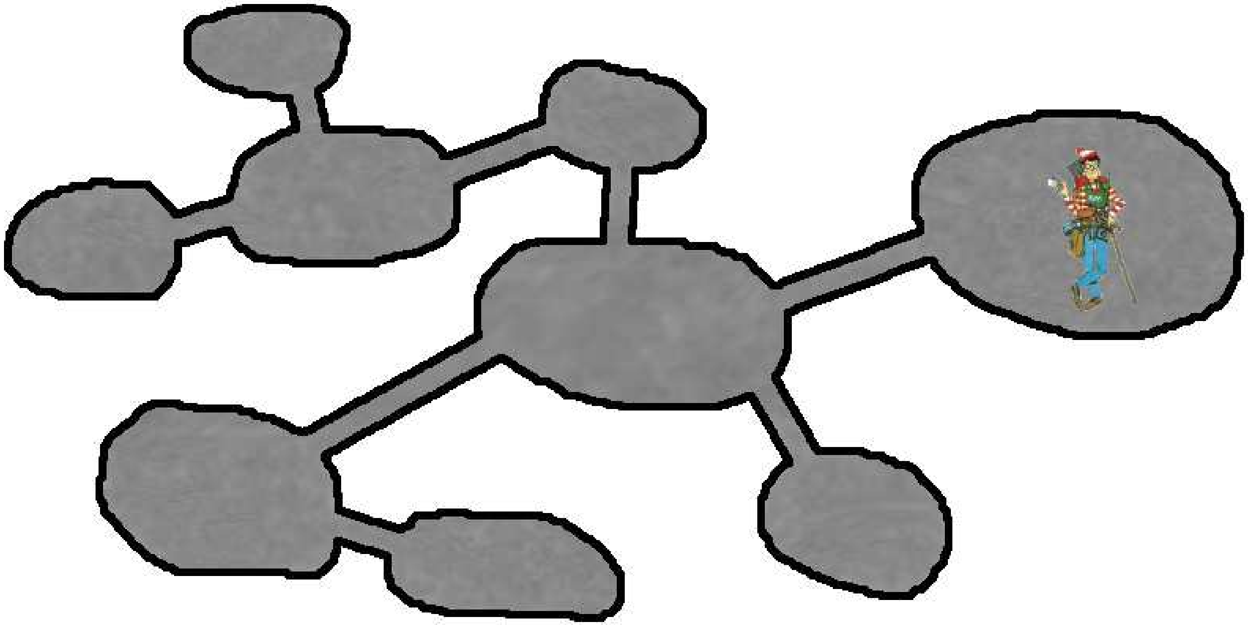


Searching in ~~caves~~ trees

- Waldo hides in a cave.
- The cave consists of chambers and corridors.
- The graph of the cave is a tree.
- Goal: Figure out which chamber Waldo is in.



Two query models

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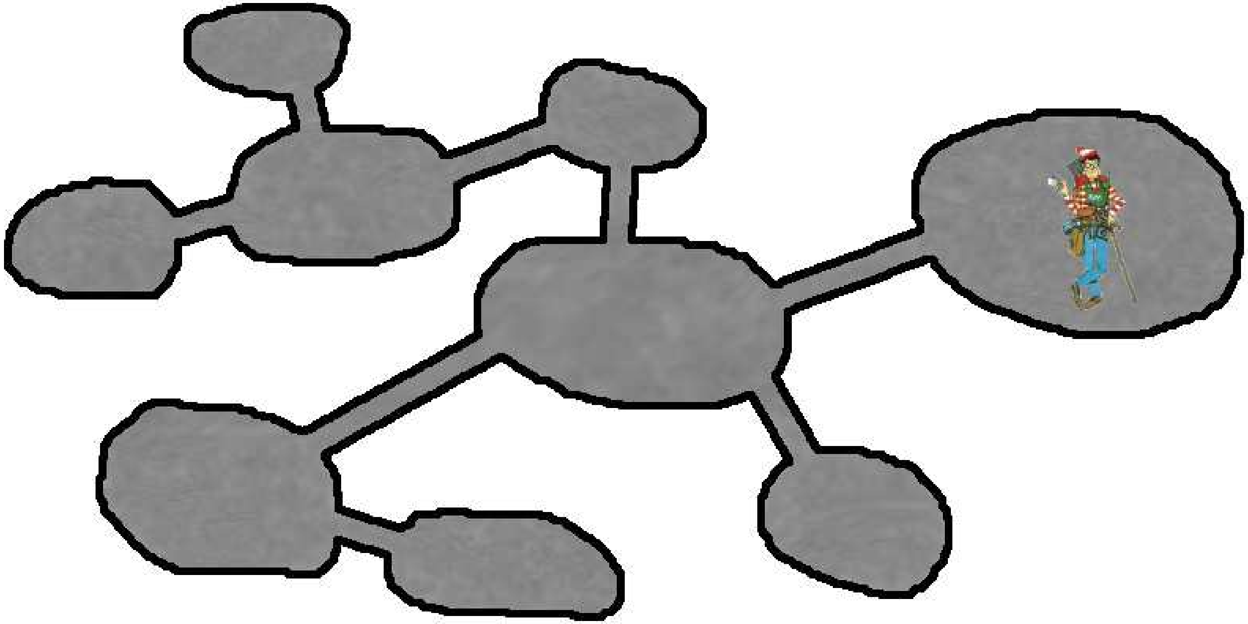

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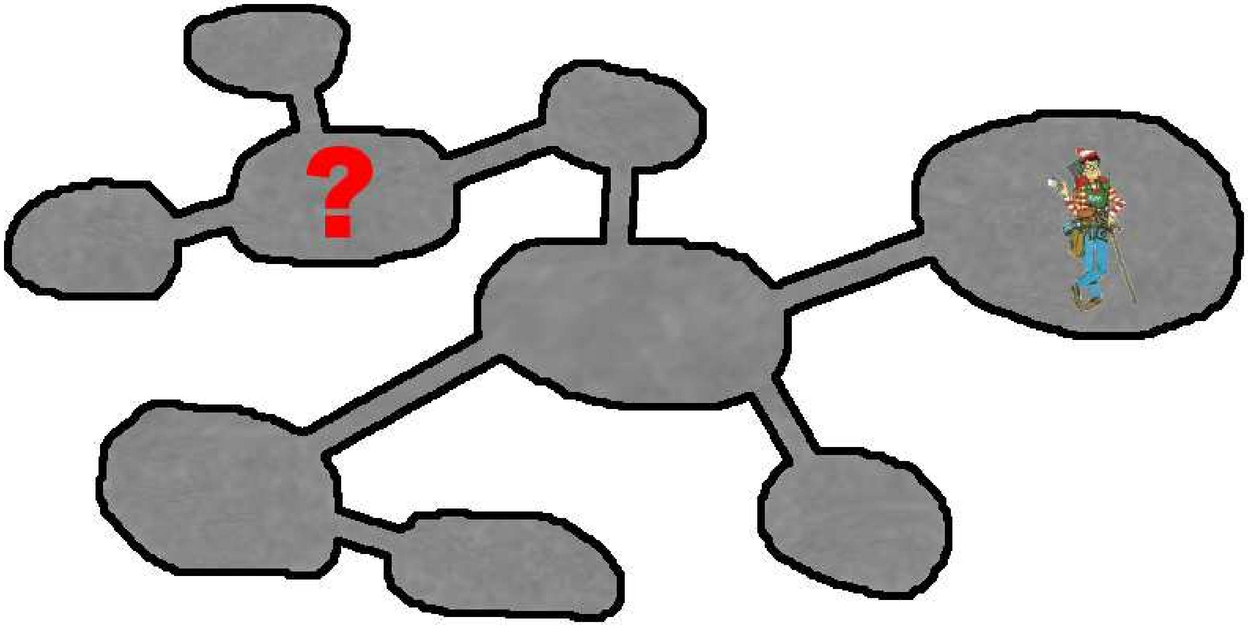

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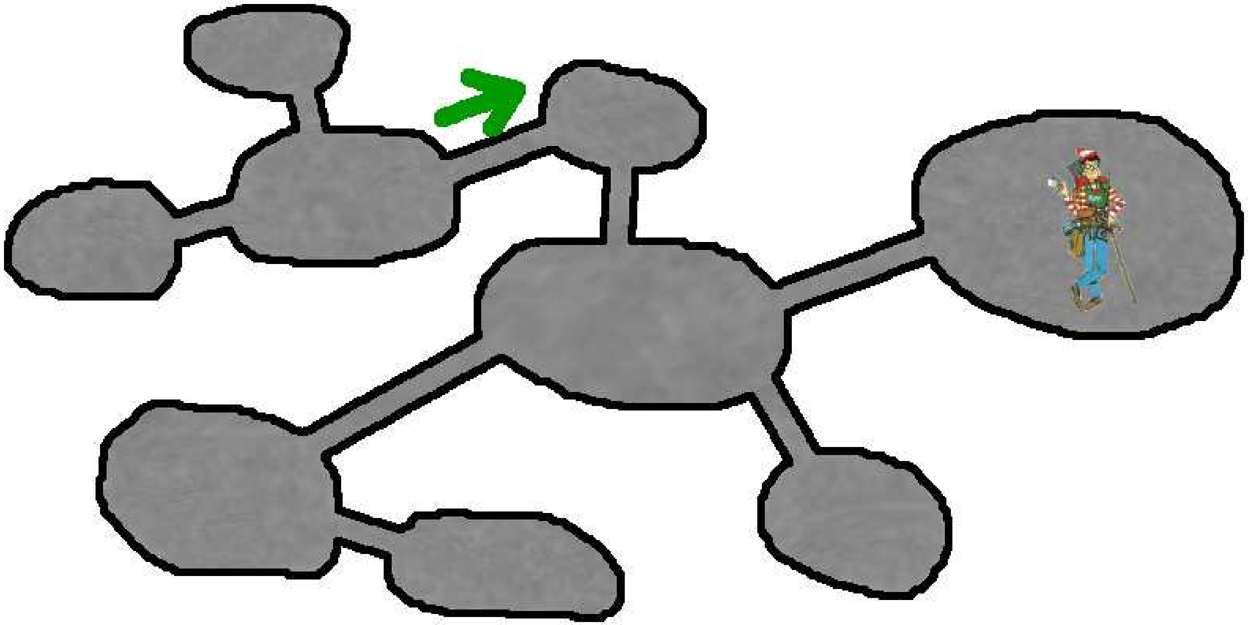

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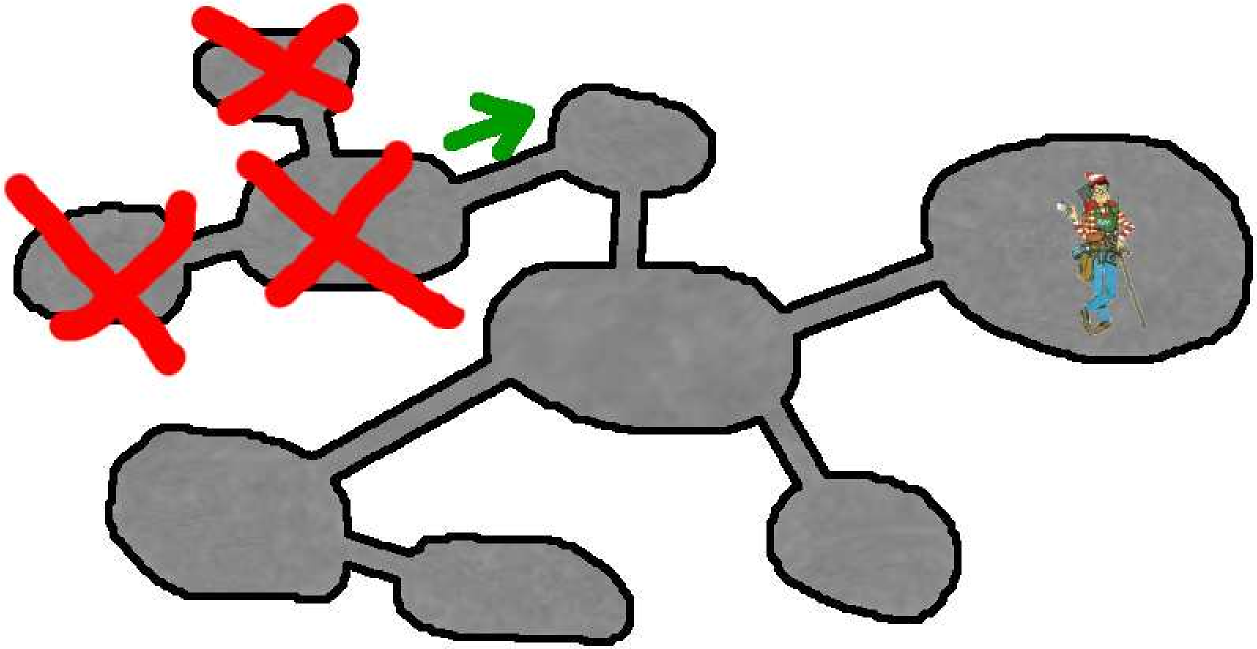

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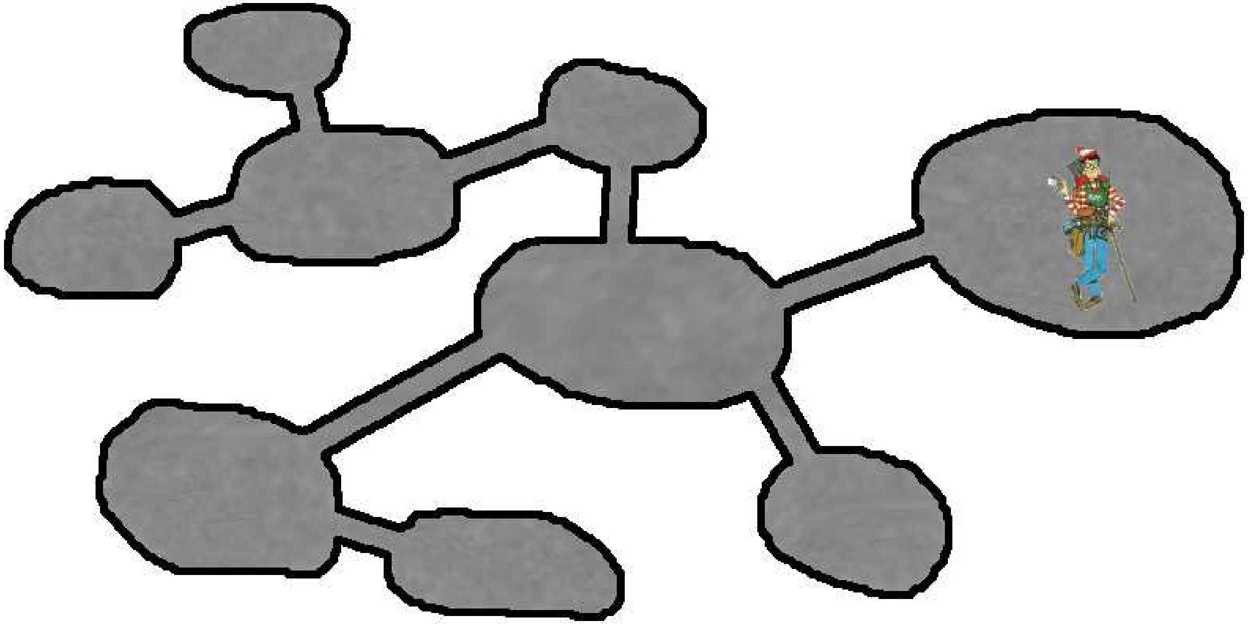

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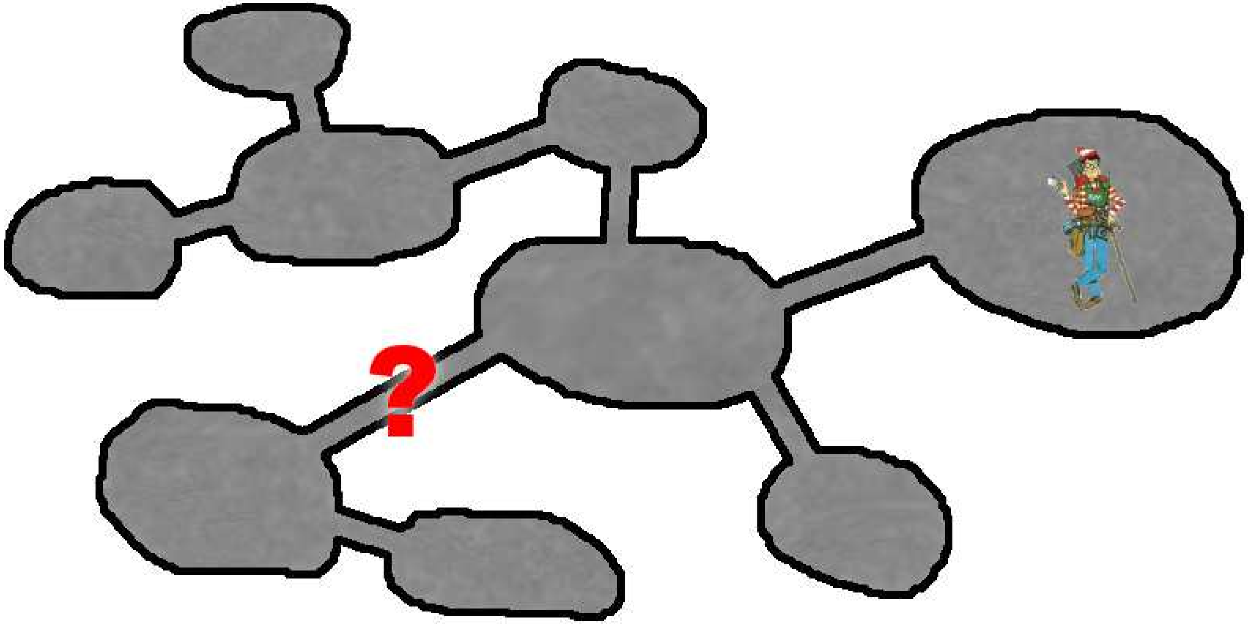

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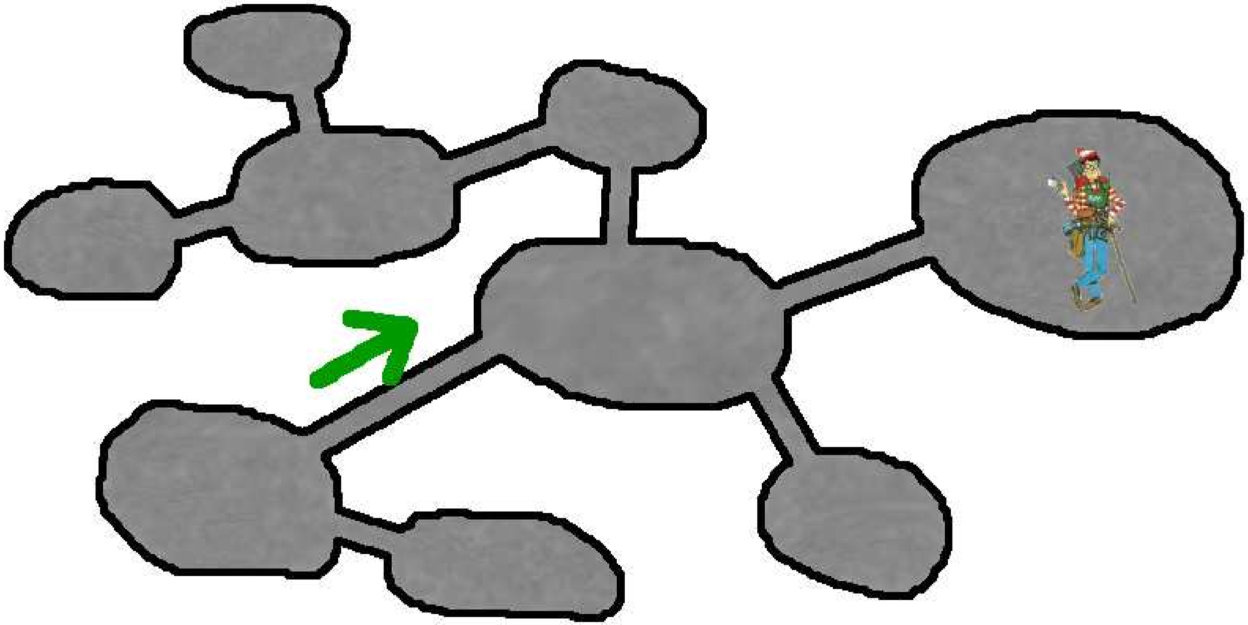

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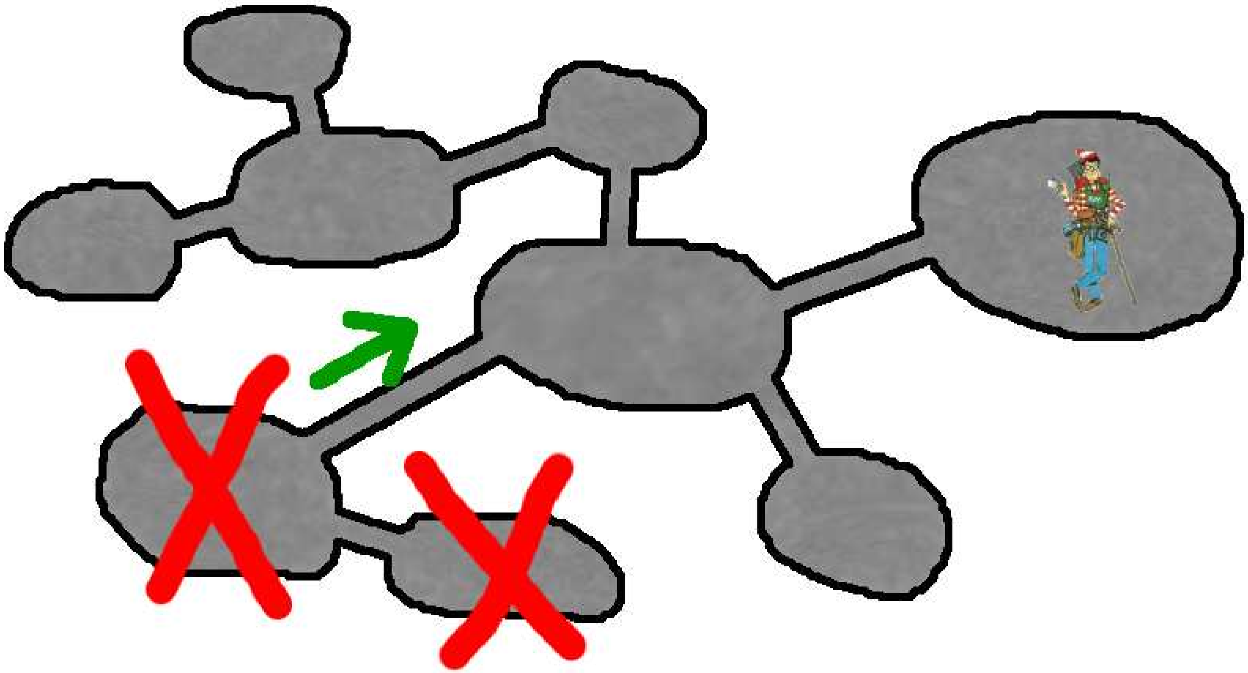

Searching in partial orders

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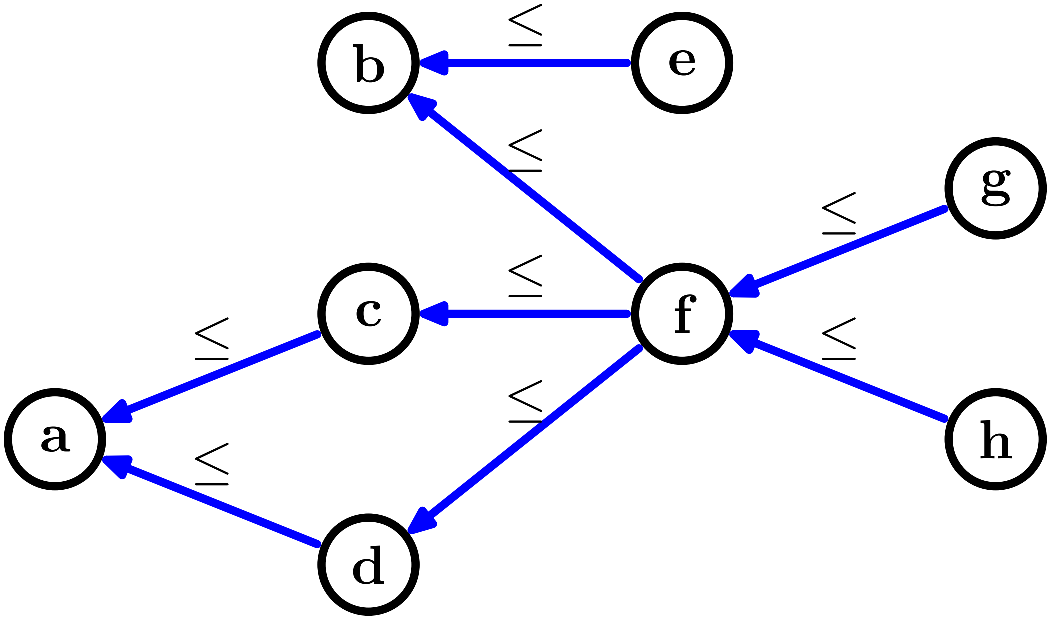

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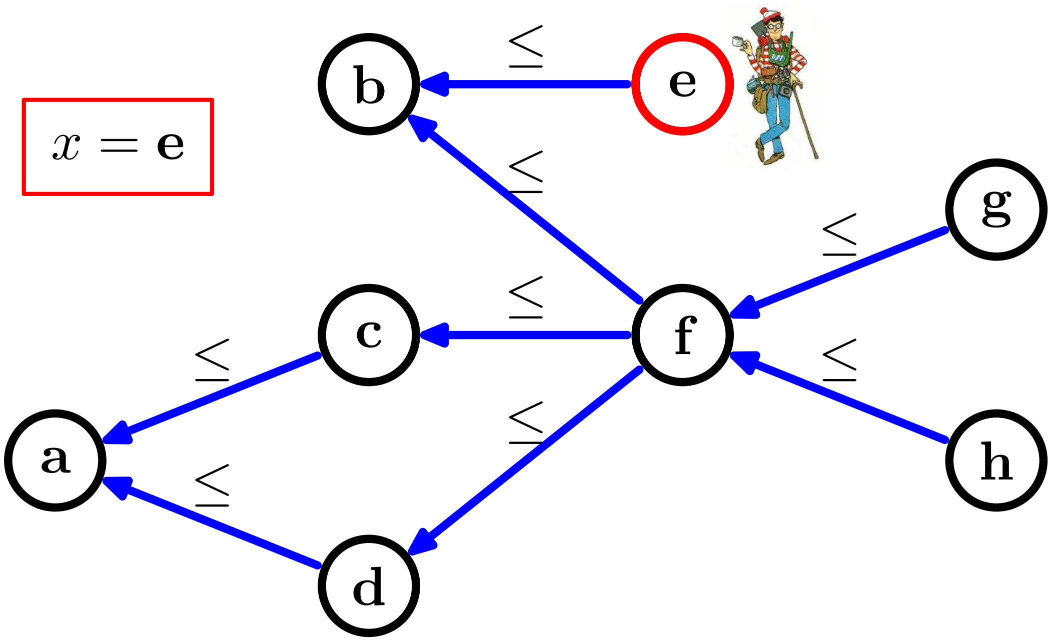

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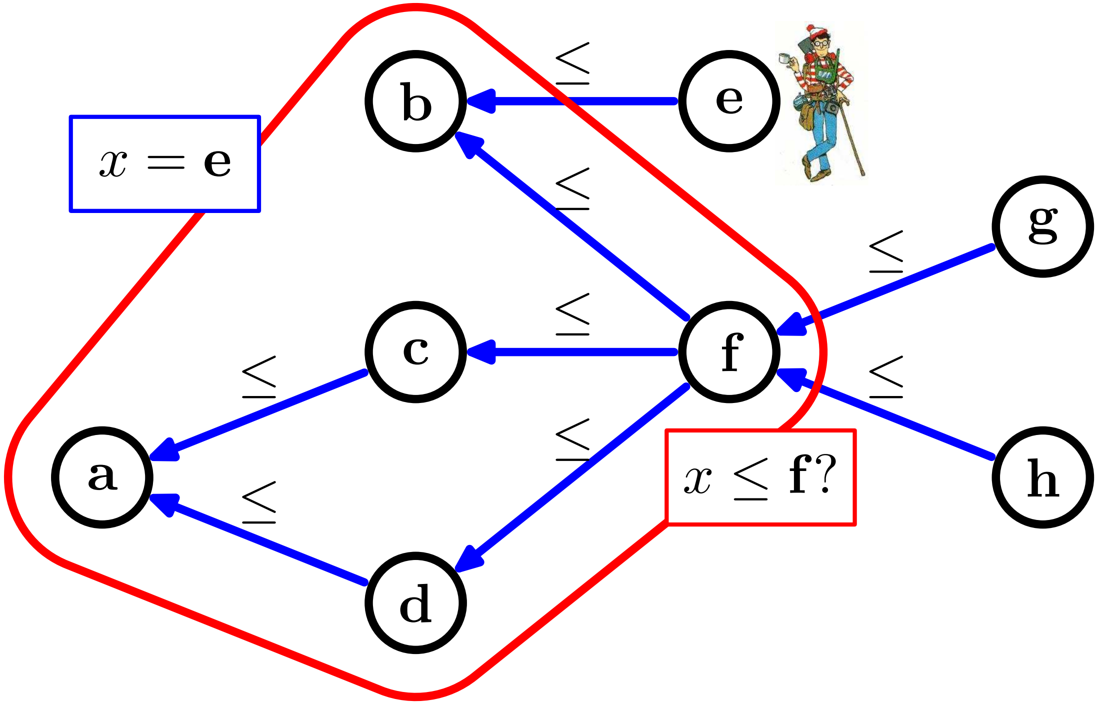

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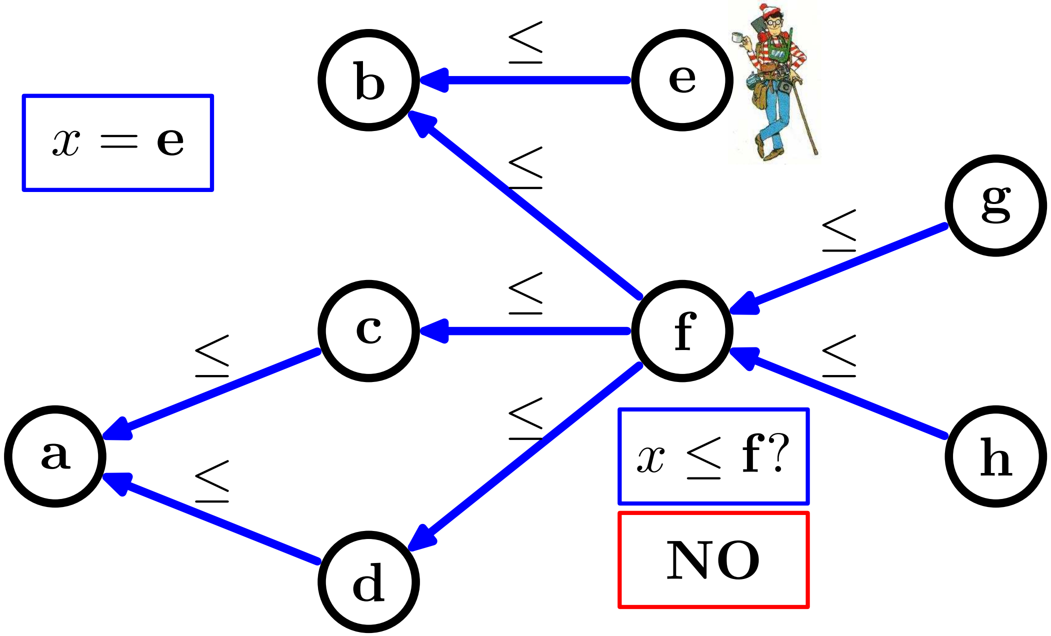

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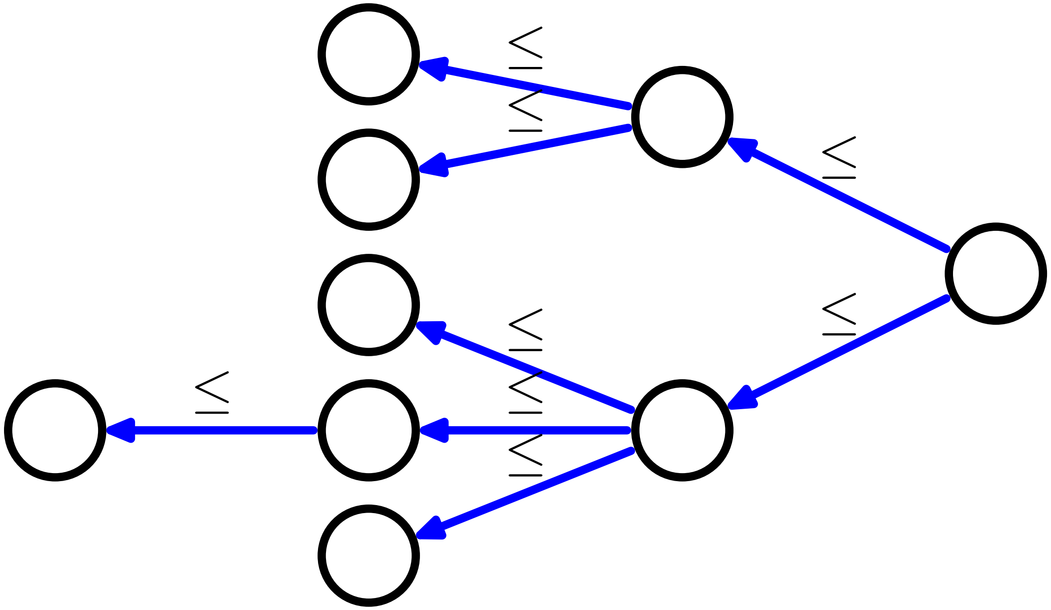
- For some posets the problem is identical to searching in trees in the edge-query model.



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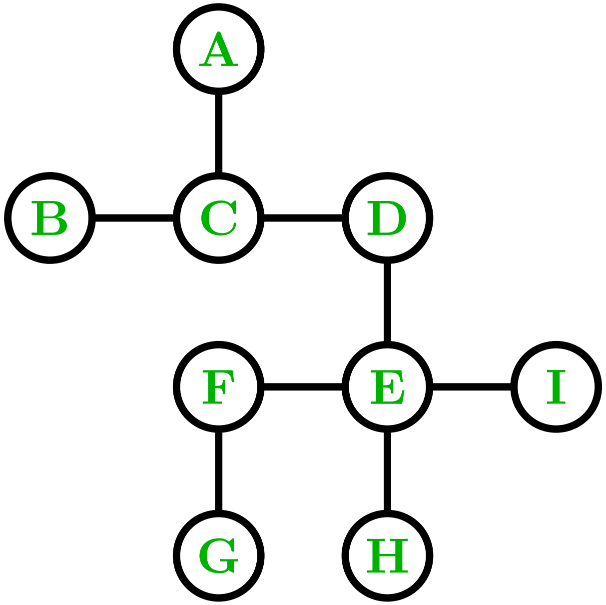
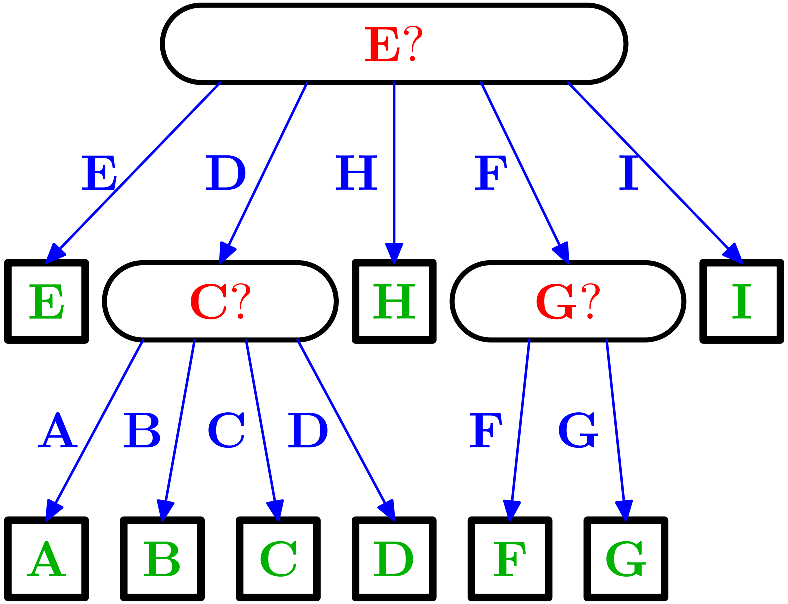
- A sample optimal strategy in the vertex-query model:





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- Onak, Parys [FOCS 2006]:
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  - vertex-query model: optimal strategy in $O(n)$

Our Results

- $O(n)$ in the edge-query model [SODA 2008]
  - novel bottom-up construction algorithm
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- Applications
  - file system synchronization
  - bug detection

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We start with the vertex-query model.

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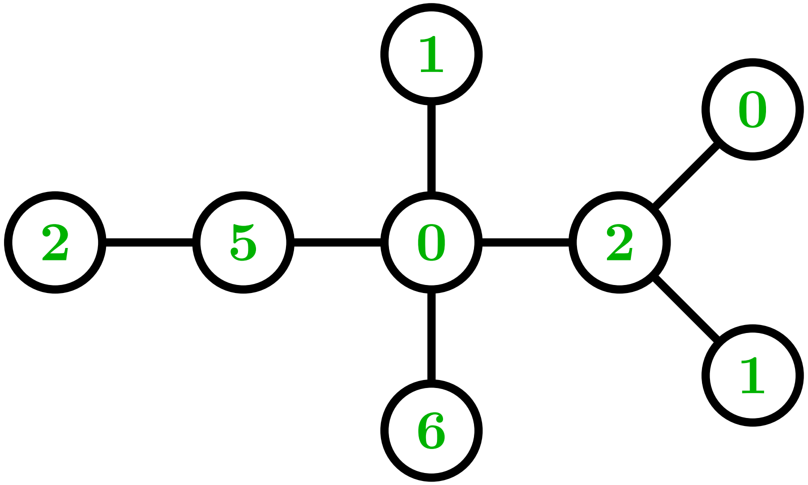

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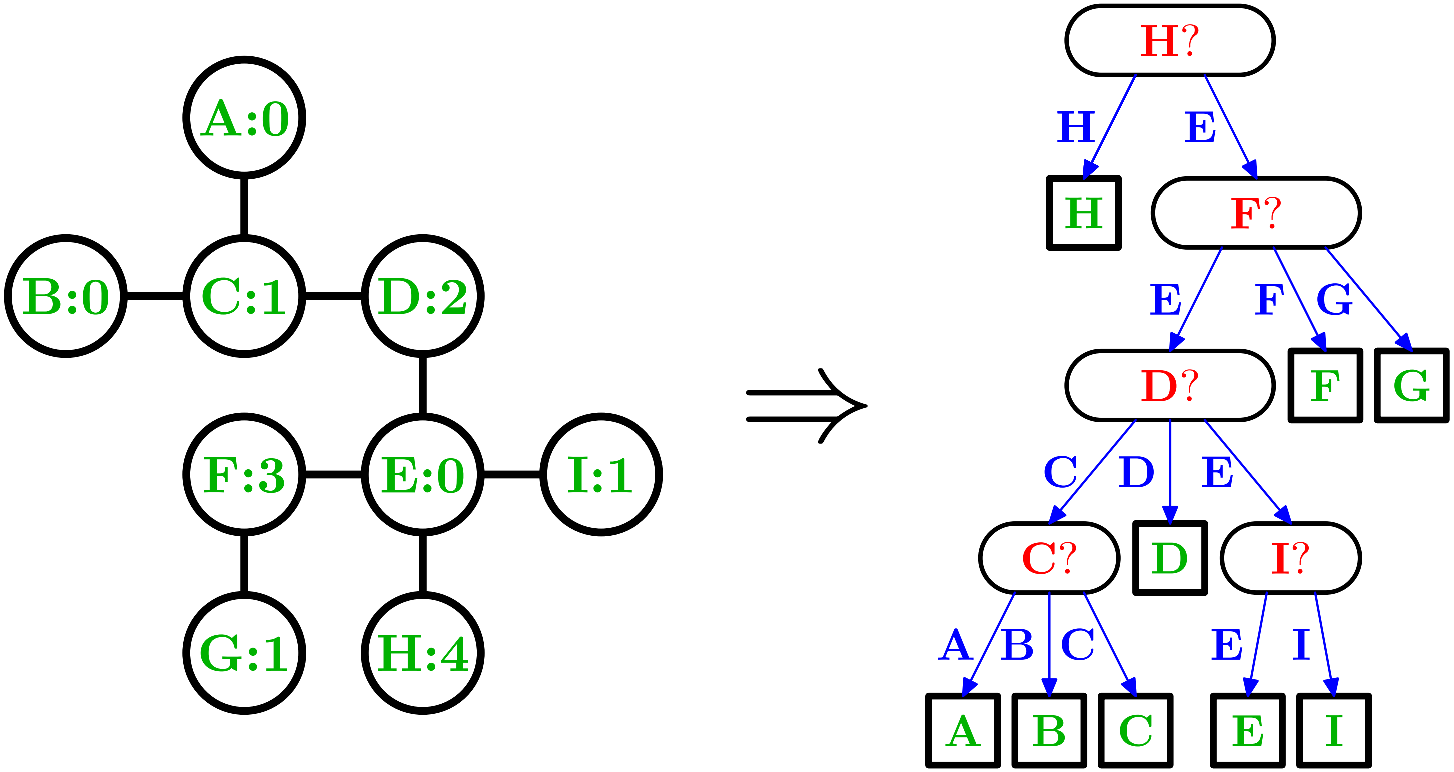

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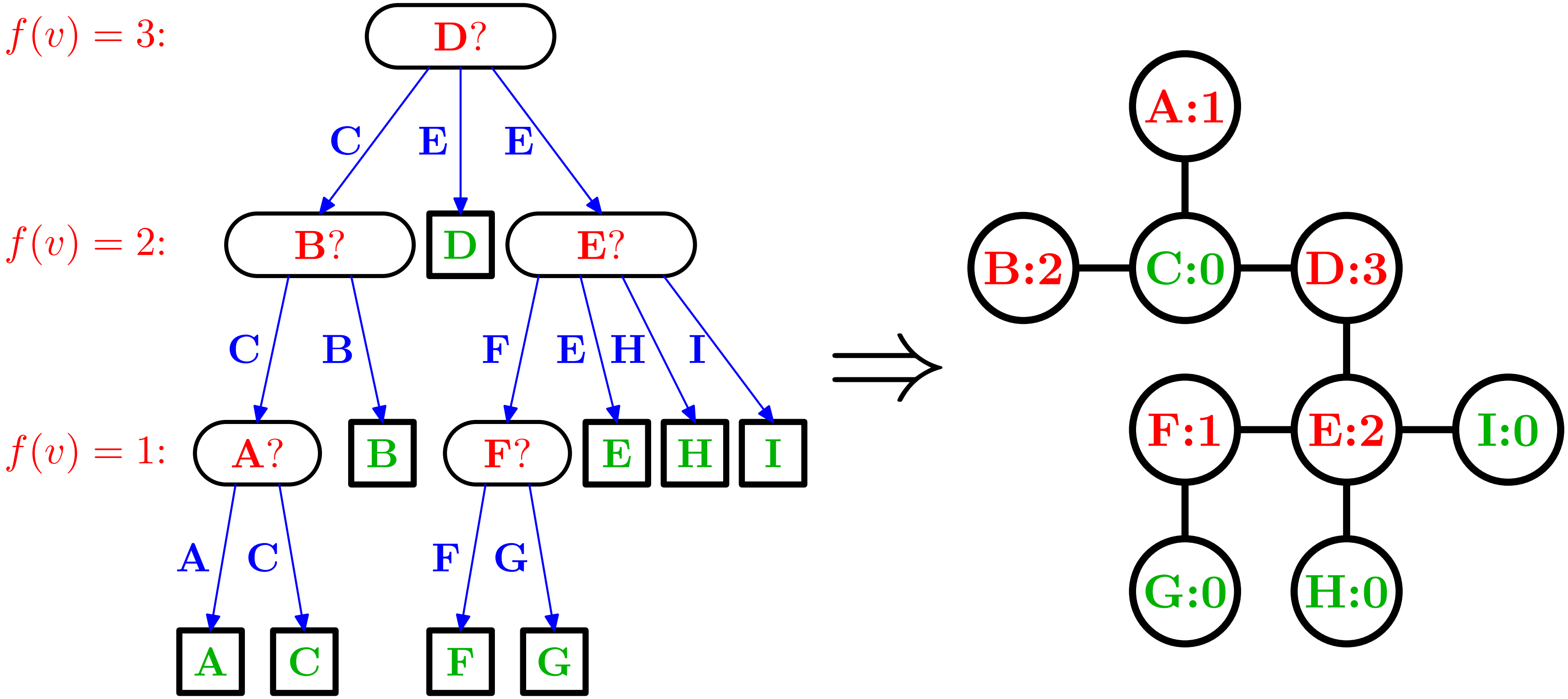

Conclusion

It suffices to construct a strategy function of the least maximum!

Visibility

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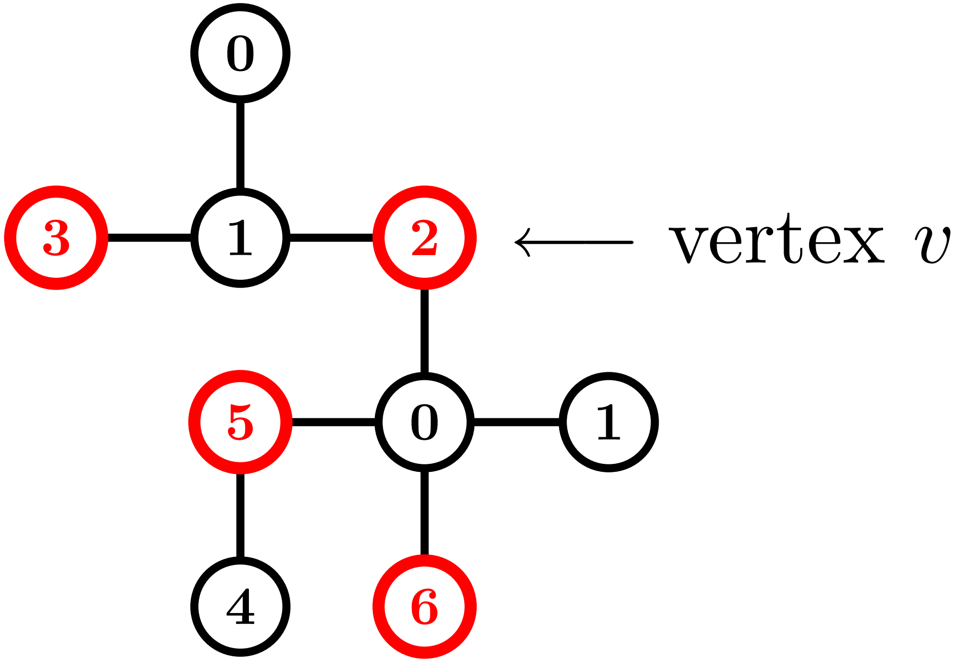

Values visible from v: 3, 2, 5, 6

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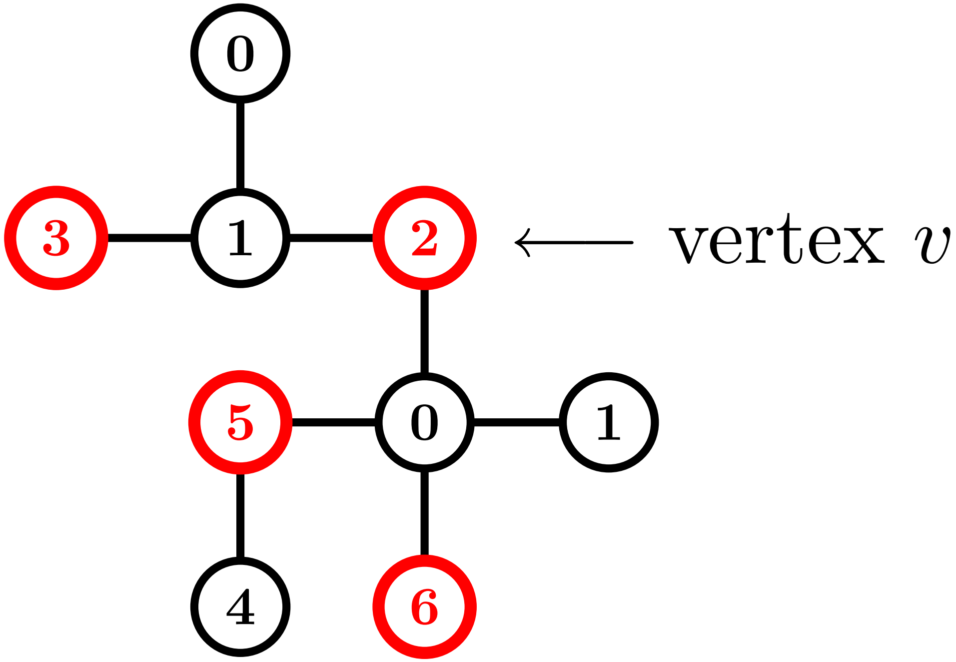

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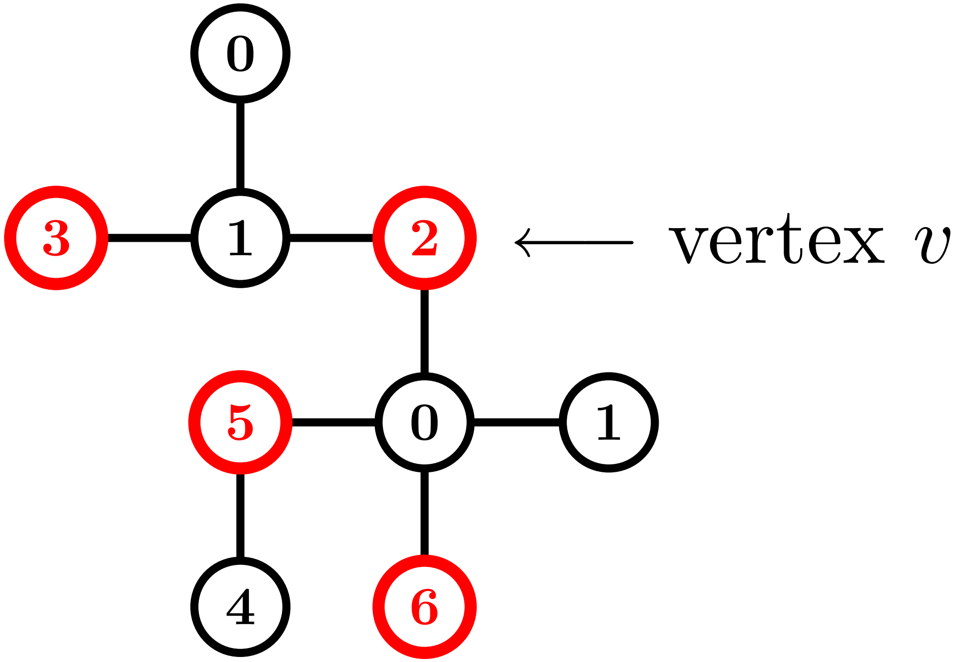

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The visibility sequences are ordered lexicographically. For instance, $(8, 4, 3, 2) > (7, 6, 4, 2, 1)$.

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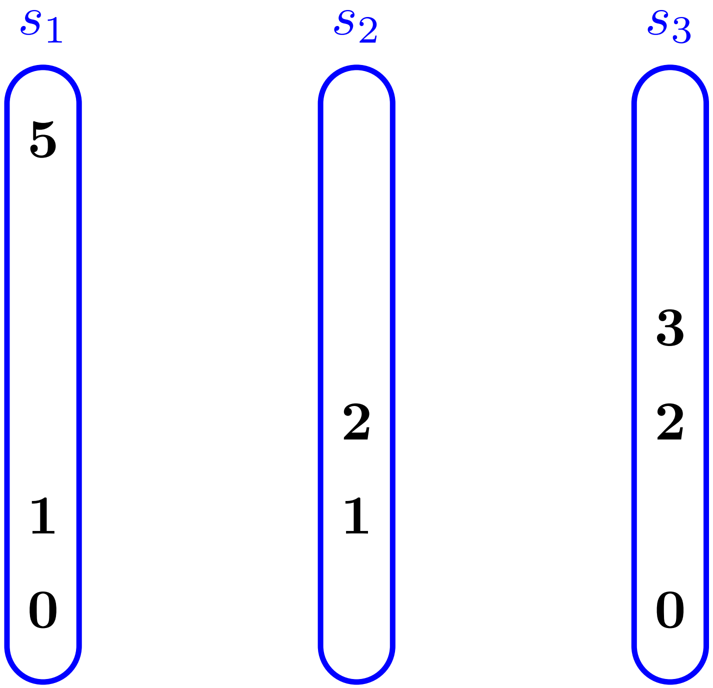
- An extension operator is a procedure that takes those visibility sequences, extends the function, and returns the visibility sequence from v in the subtree rooted at v.

An Optimal Extension

- A *minimizing* extension is one that gives the lexicographically smallest visibility sequence at v.
  - *minimizing extensions* accumulate to an optimal solution [OP 2006].

Vertex-query model

- An extension operator $\mathbb{V}$ for a vertex v:



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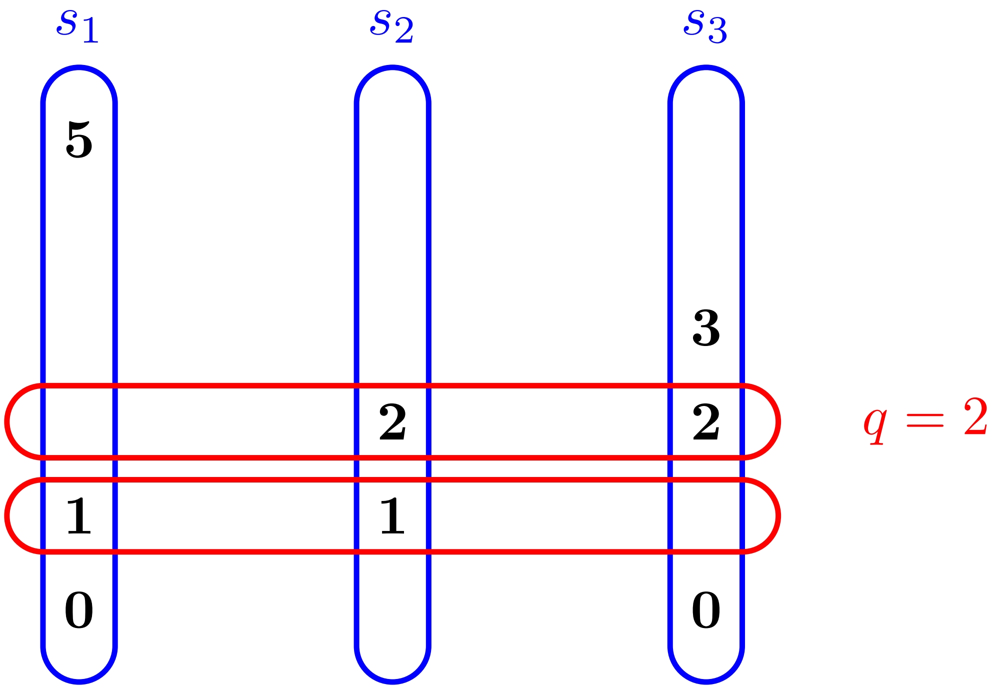

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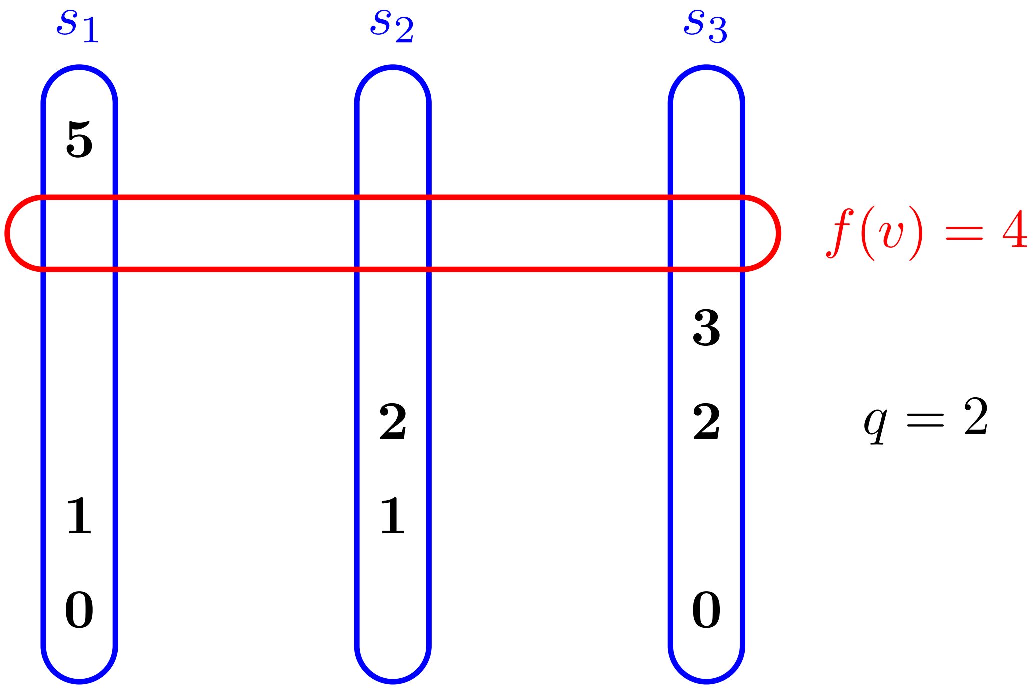

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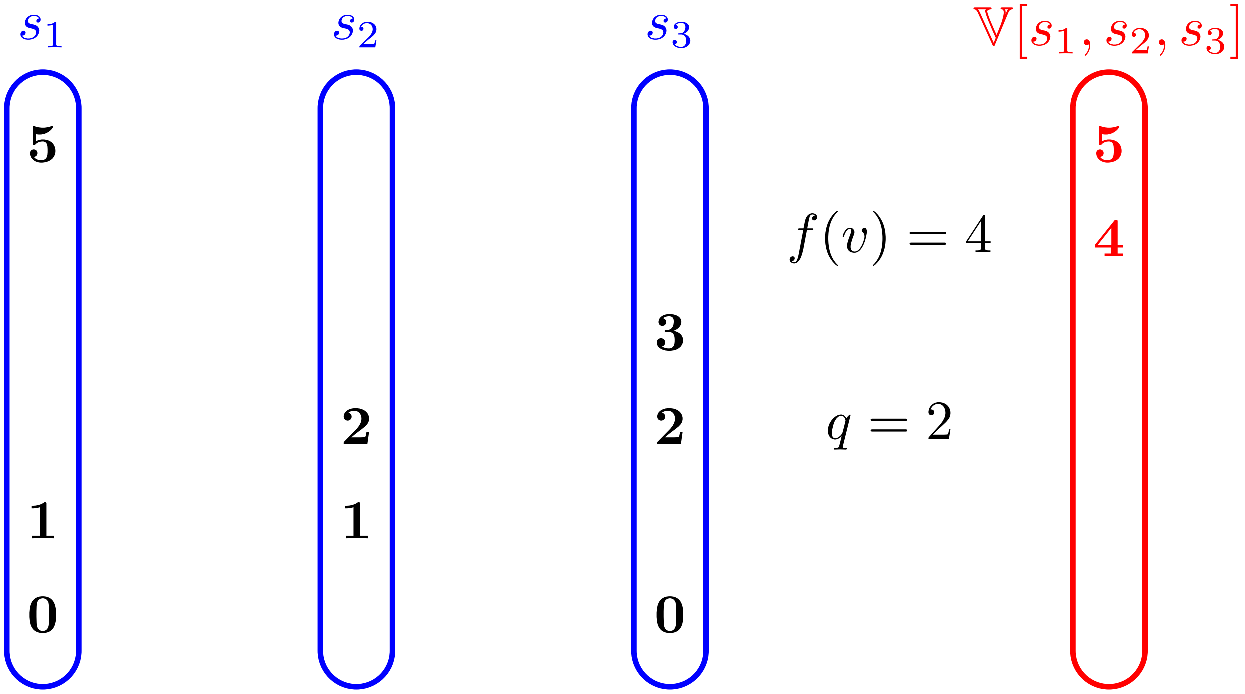

s_1	s_2	s_3		$\mathbb{V}[s_1, s_2, s_3]$
5				**5**
			$f(v) = 4$	**4**
		3		
	2	**2**	$q = 2$	
1	**1**			
0		**0**		

Vertex-query model

- One can show that $\mathbb{V}$ is minimizing.

Vertex-query model

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- The whole computation takes $O(n \log n)$ time, as in the vertex-query model the required vertex can always be located in at most $\lfloor \log_2 n \rfloor$ queries.

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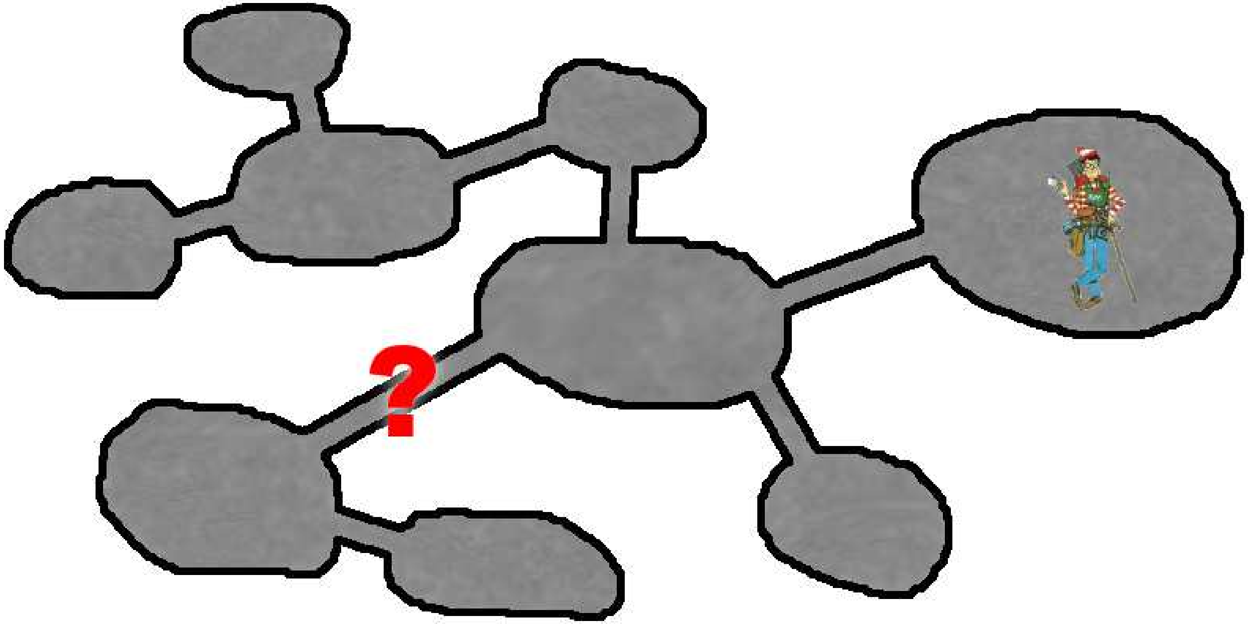
- The running time can be improved to $O(n)$ fairly simple.

Edge-query model

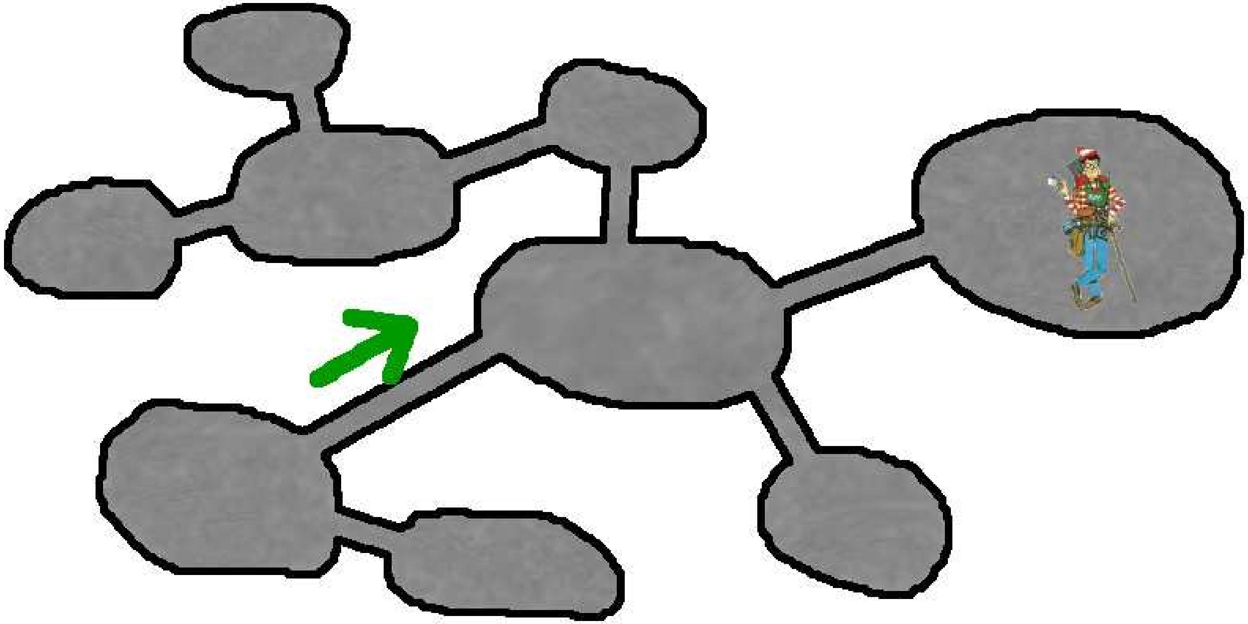
Edge-query model

- Questions about edges.



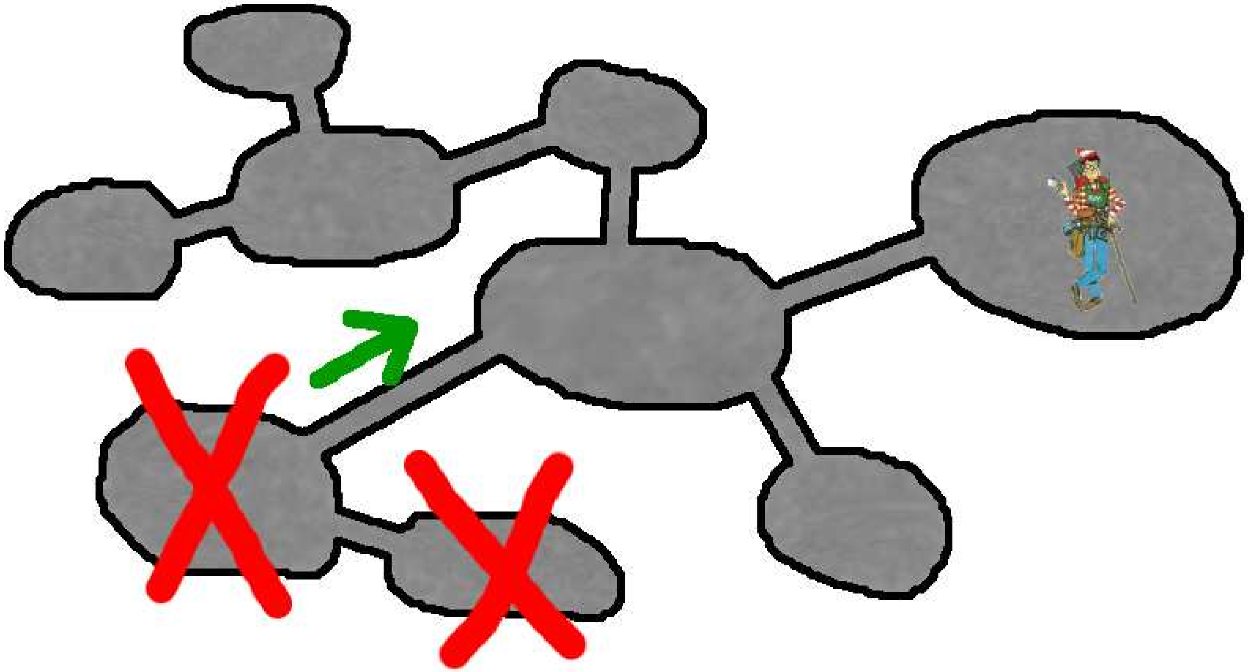
Edge-query model

- Questions about edges.
    - Ask about an edge e.



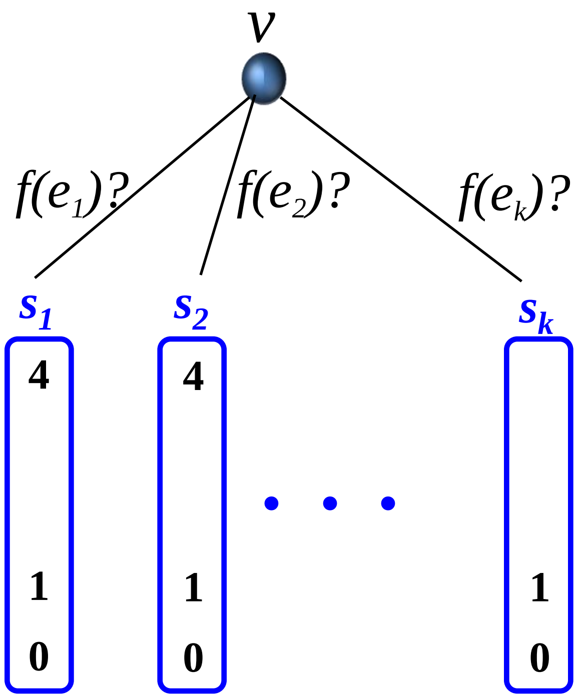
Edge-query model

- Questions about edges.
  - Ask about an edge e.
  - Learn which endpoint of e is closer to Waldo.



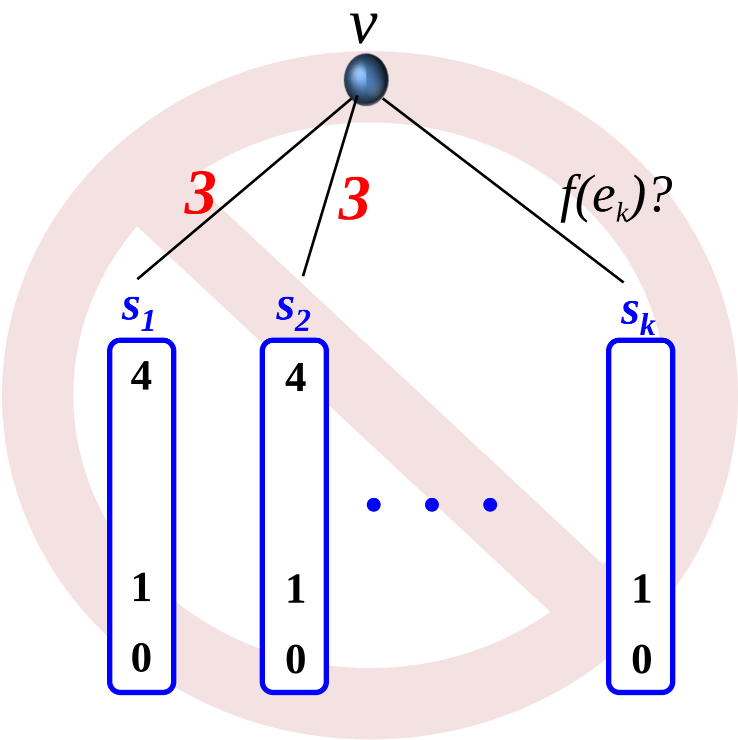
Edge-query model

- An extension assigns all $f(e_i)$'s



Edge-query model

- An extension assigns all $f(e_i)$'s
  - $f(e_i) \neq f(e_j)$



Edge-query model

- An extension assigns all $f(e_i)$'s
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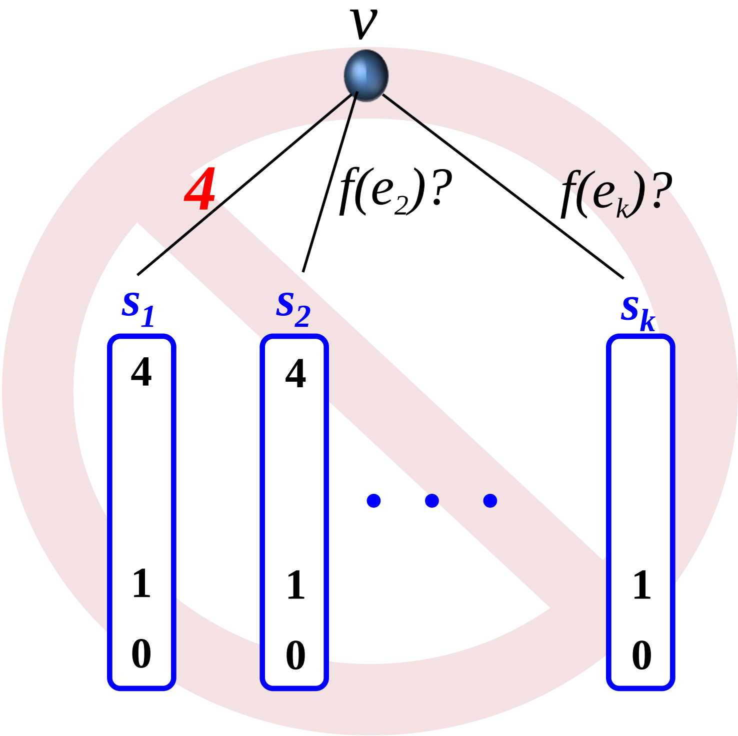

Edge-query model

- An extension assigns all $f(e_i)$'s
  - $f(e_i) \neq f(e_j)$
  - $f(e_i)$ is not in s_i
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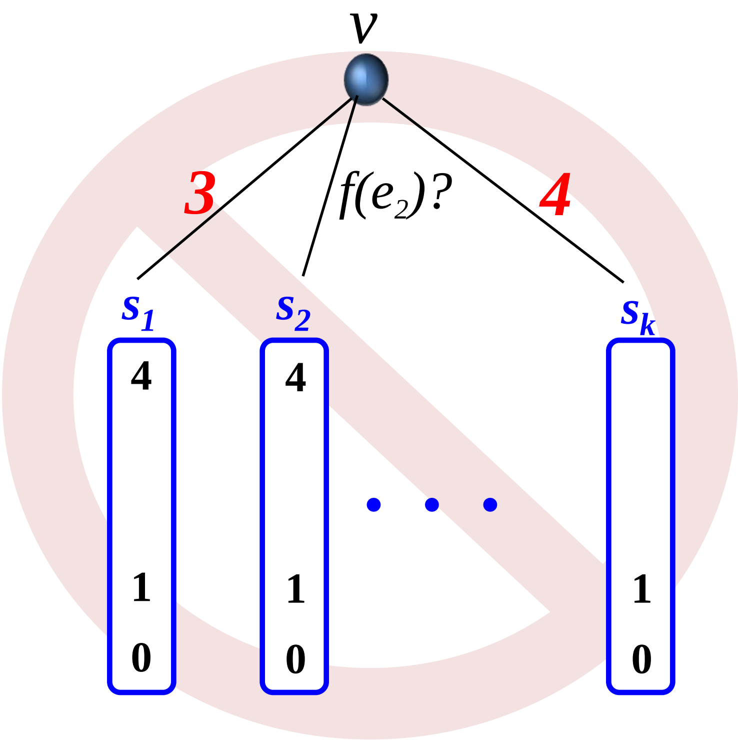

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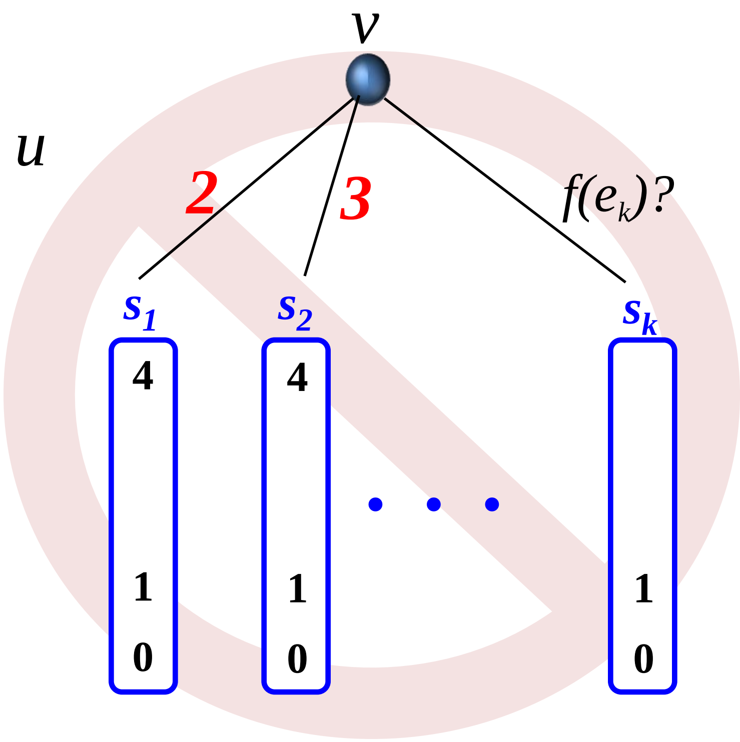
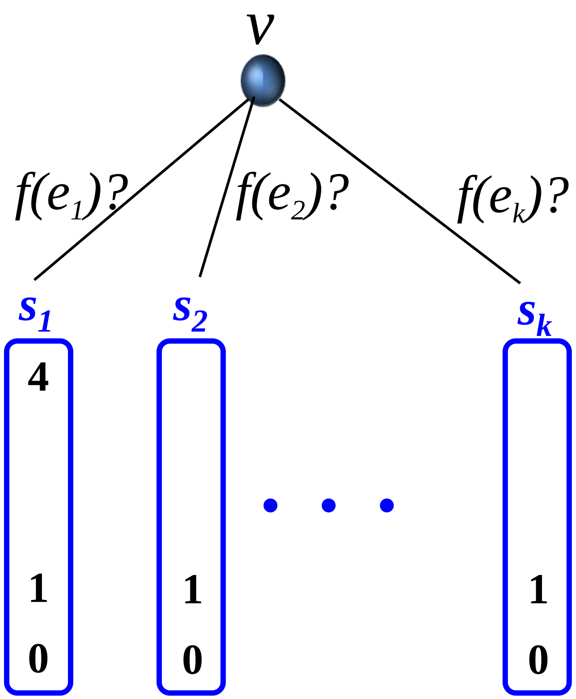
  - $f(e_i)$ is in s_j ⇨ $f(e_j) > f(e_i)$
  - u is in s_i and s_j ⇨ $max\{f(e_i), f(e_j)\} > u$



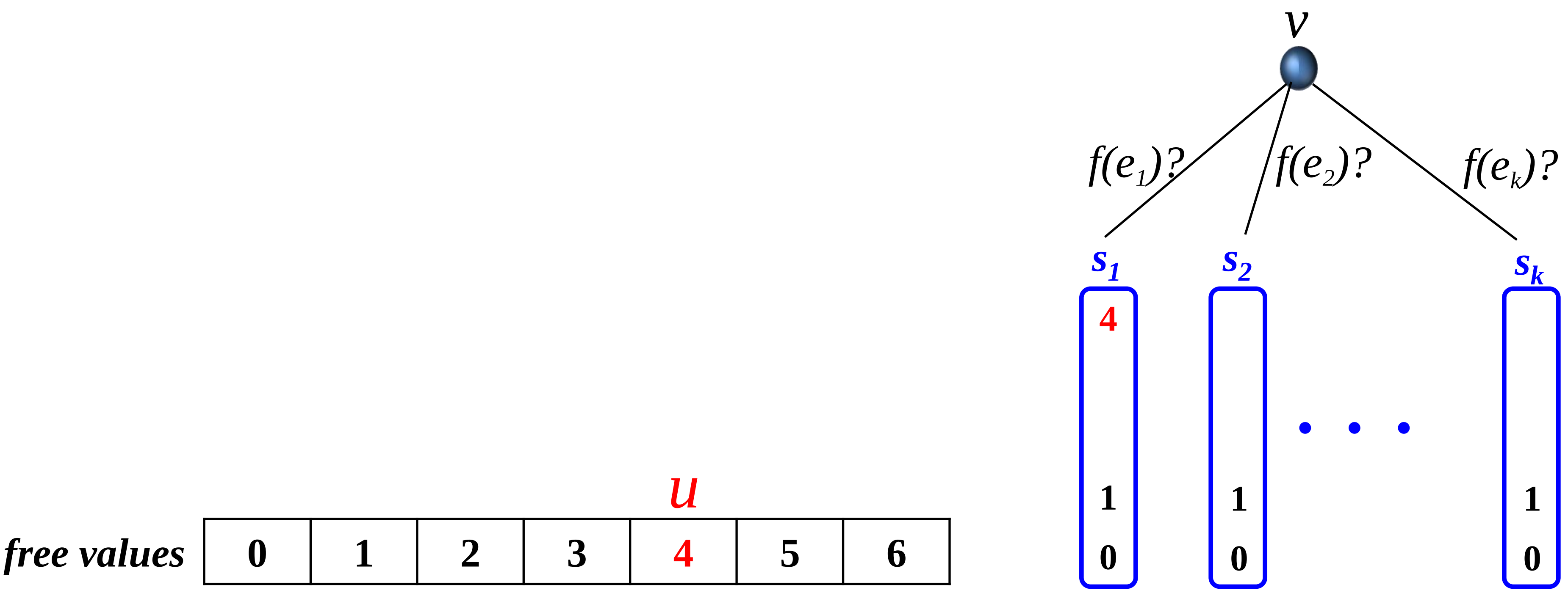
Algorithm Outline



free values	0	1	2	3	4	5	6

Algorithm Outline

- set $u = max\{s_i\}$



Algorithm Outline

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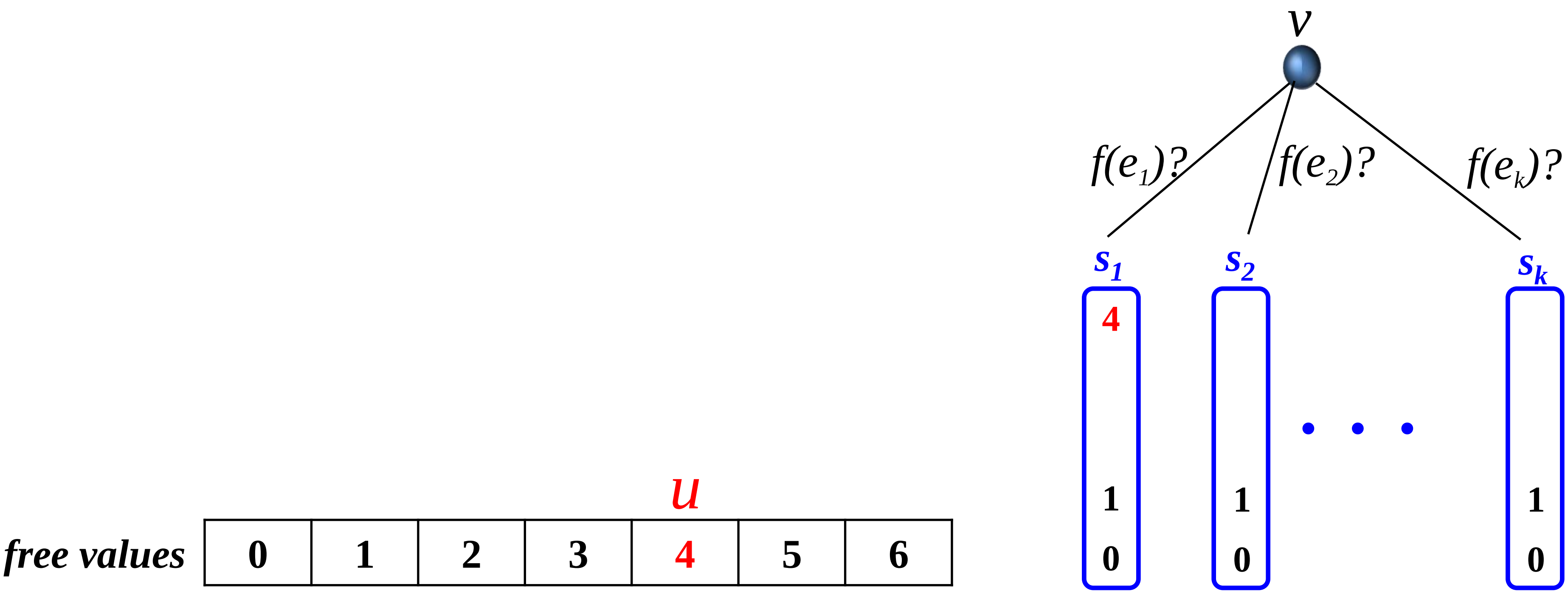

Algorithm Outline

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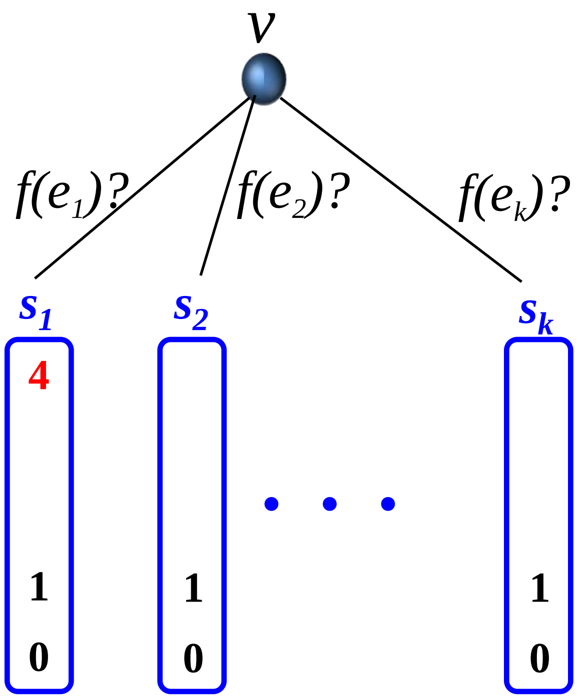

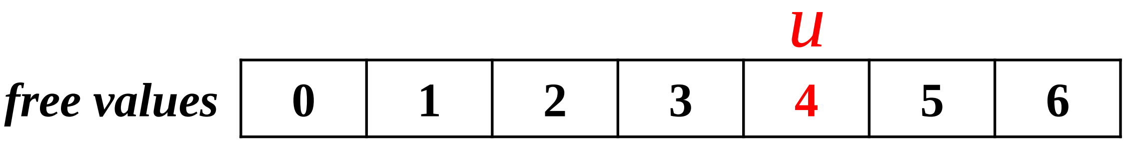

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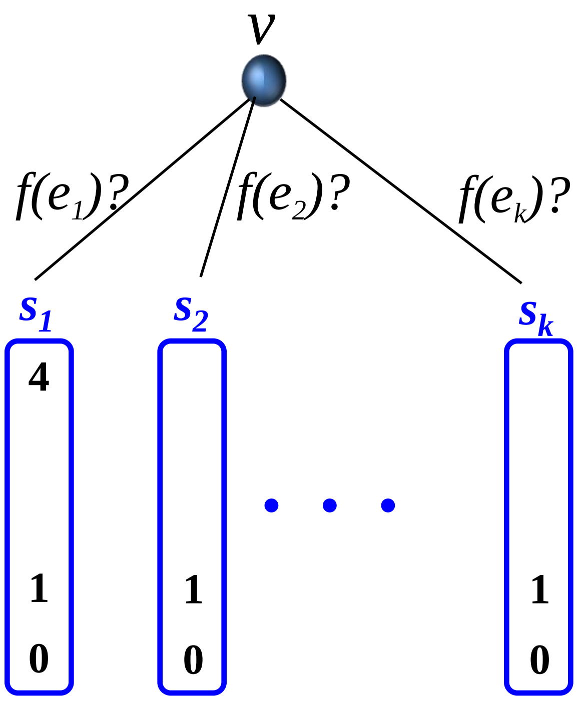

free values	**0**	**1**	**2**	**3**		**5**	**6**

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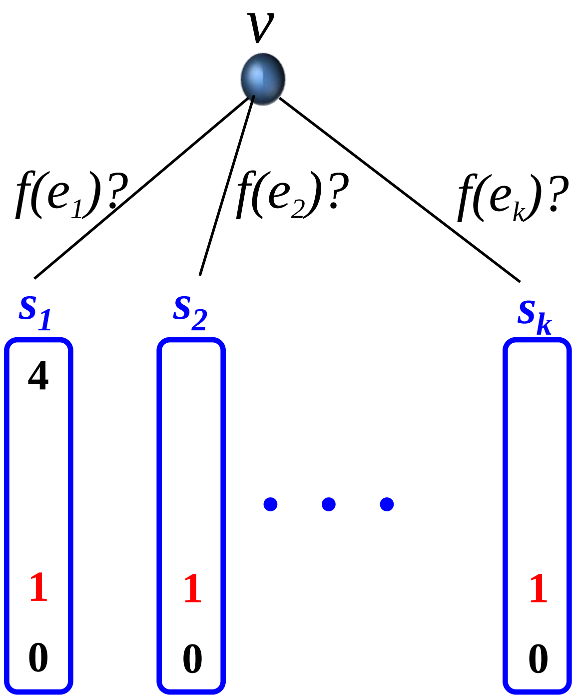

u

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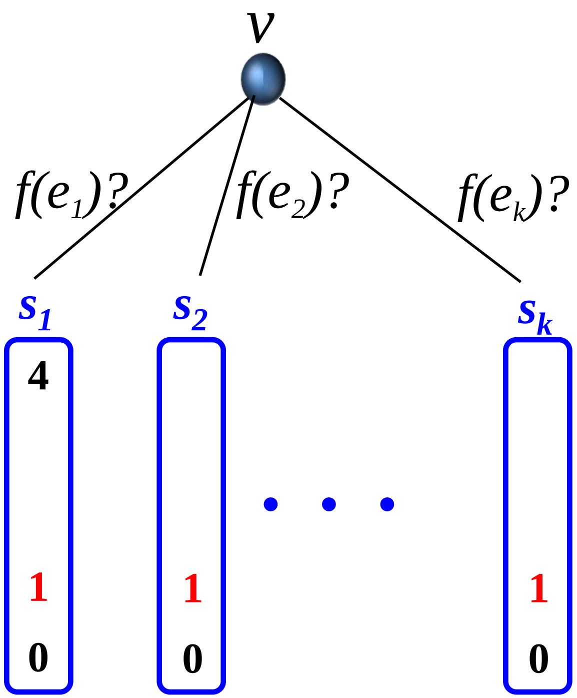

		u	w				
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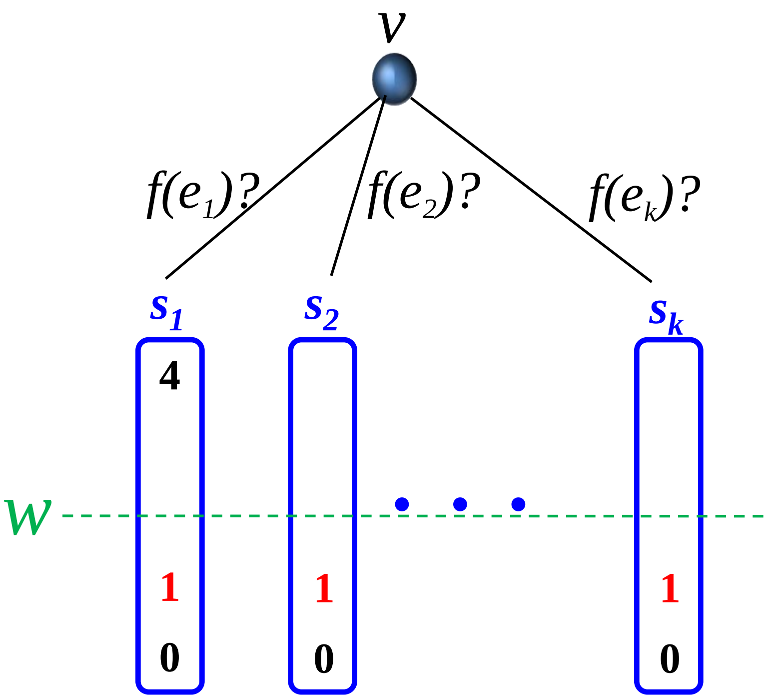

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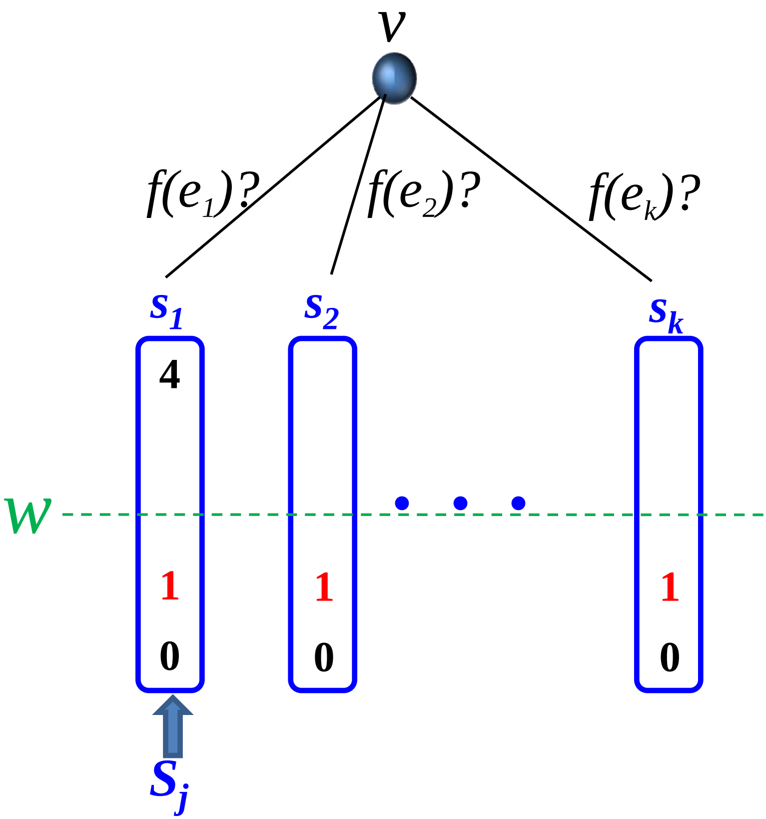

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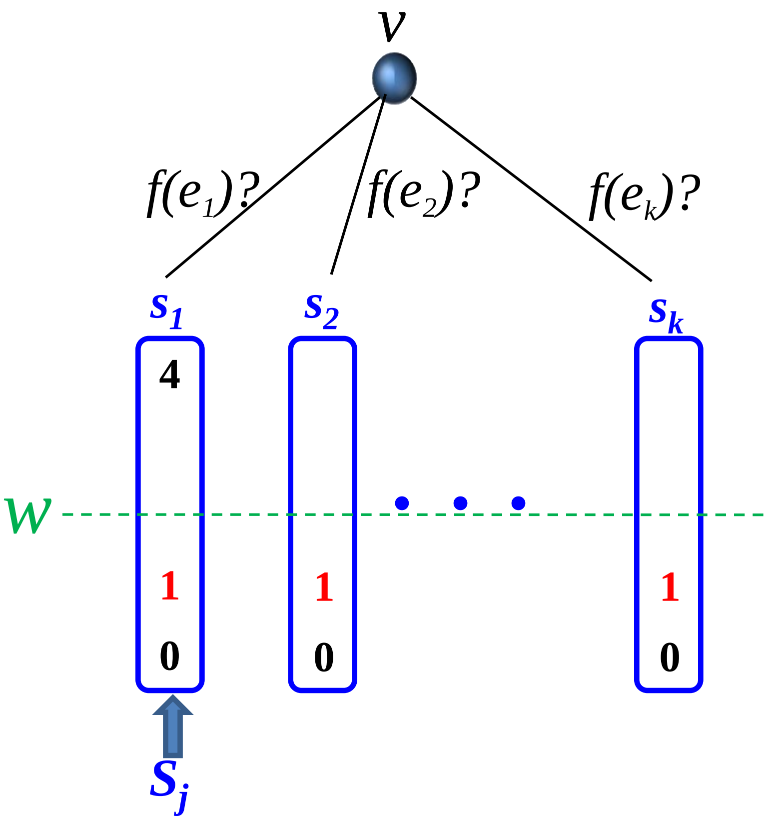

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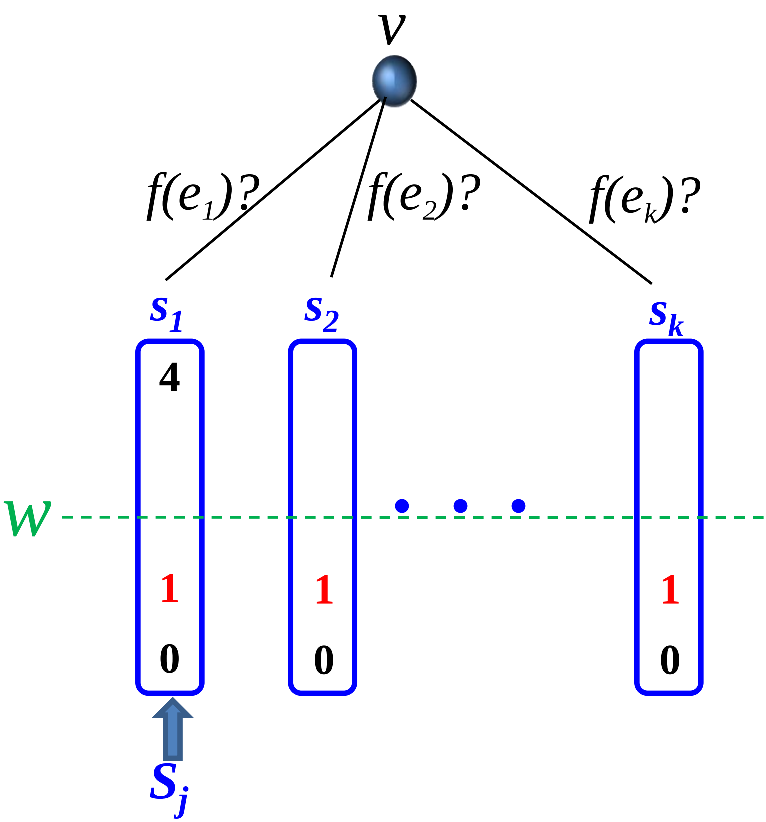

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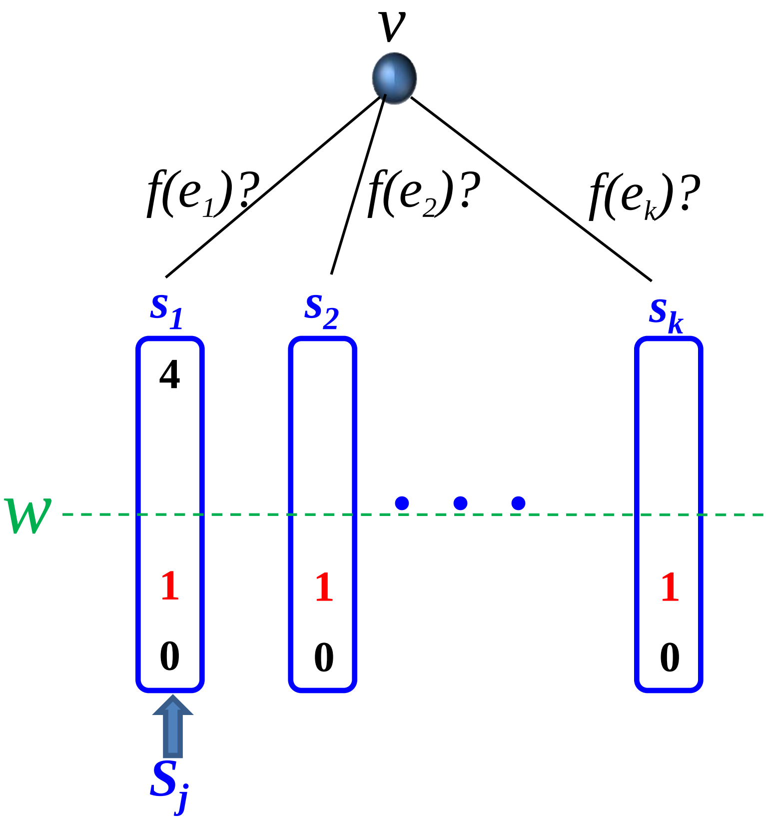

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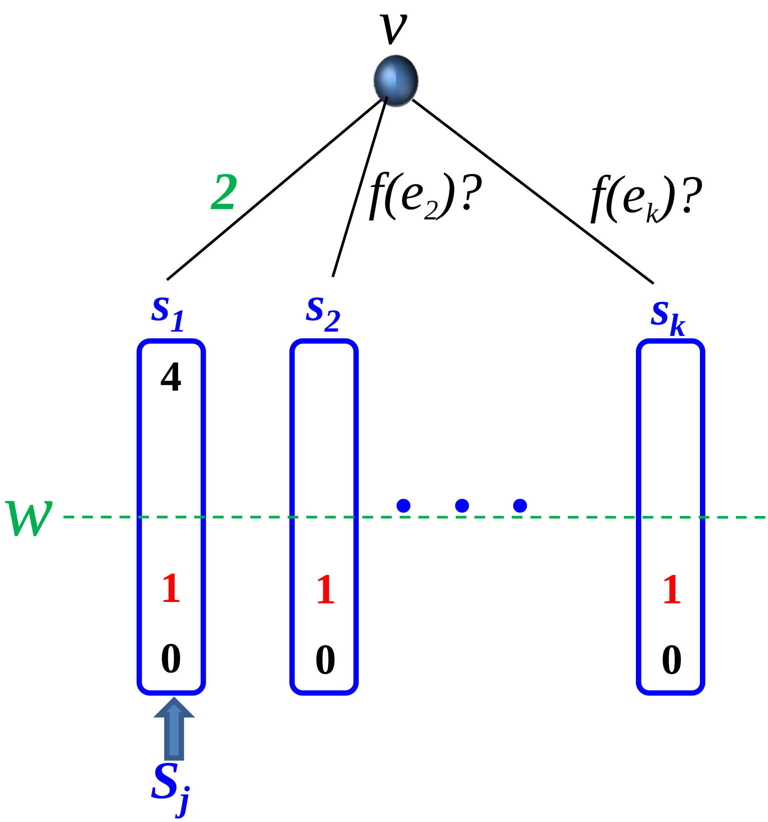

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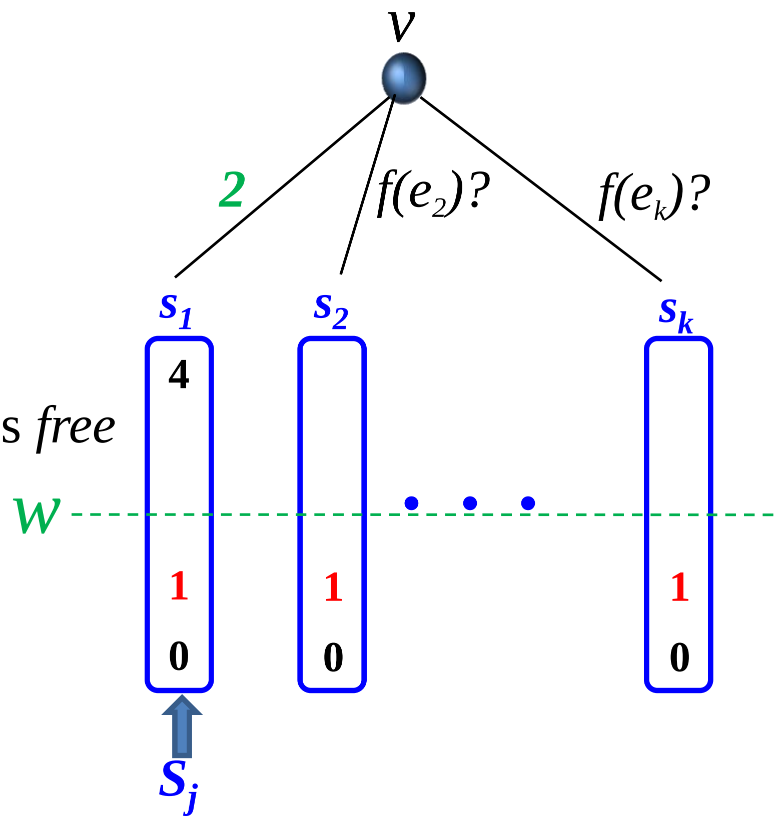

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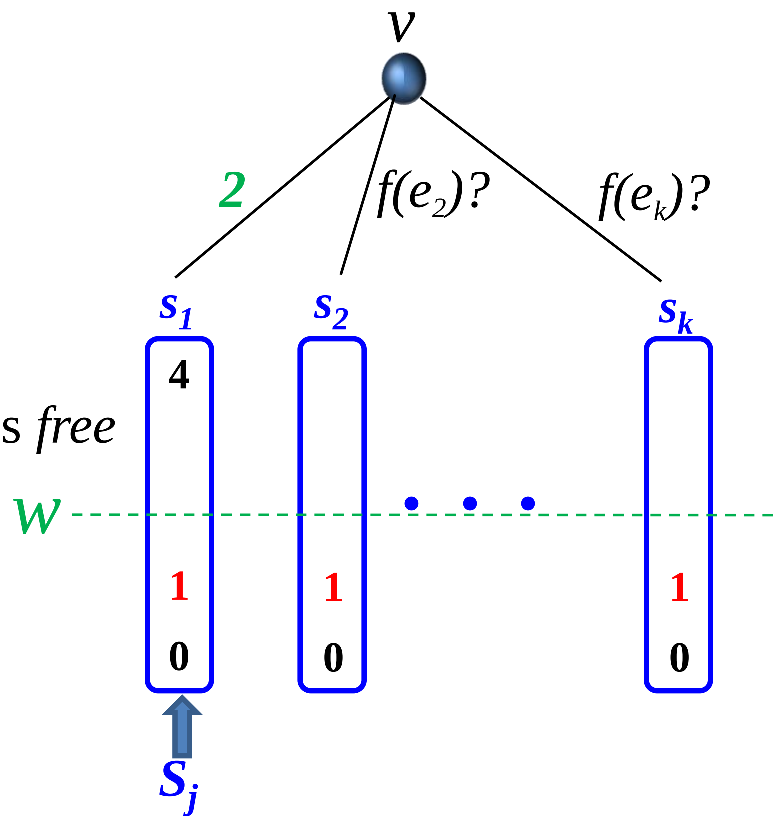
free values	0	1		3		5	6



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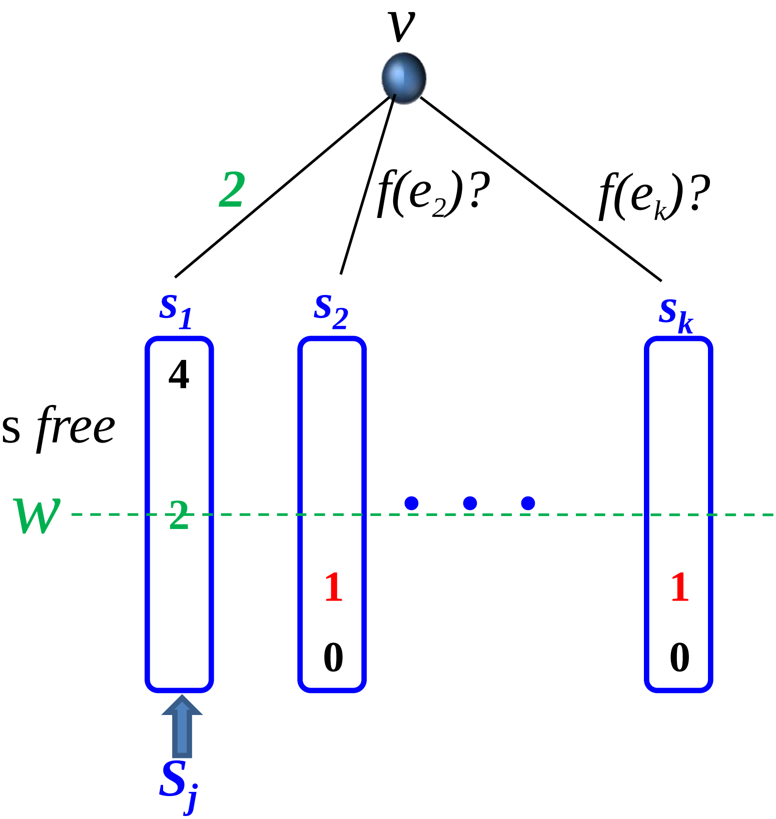

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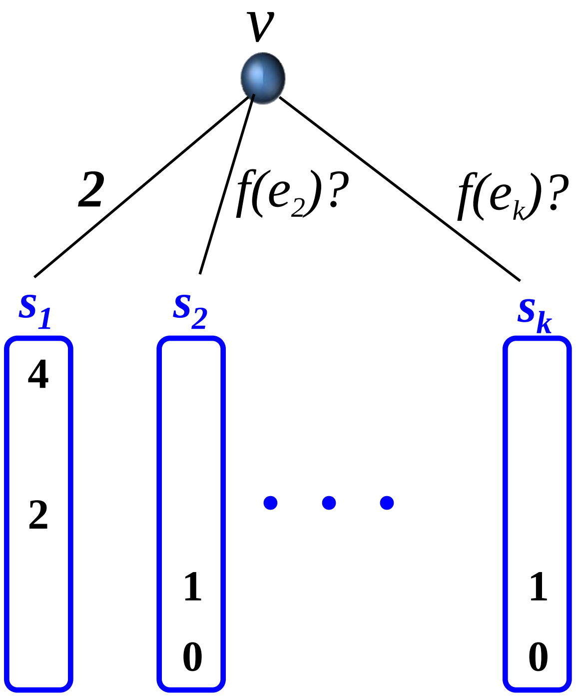

free values

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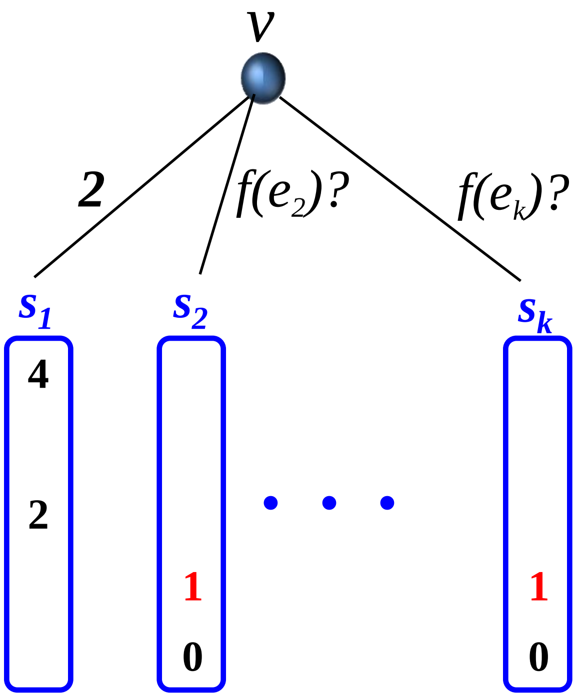

u

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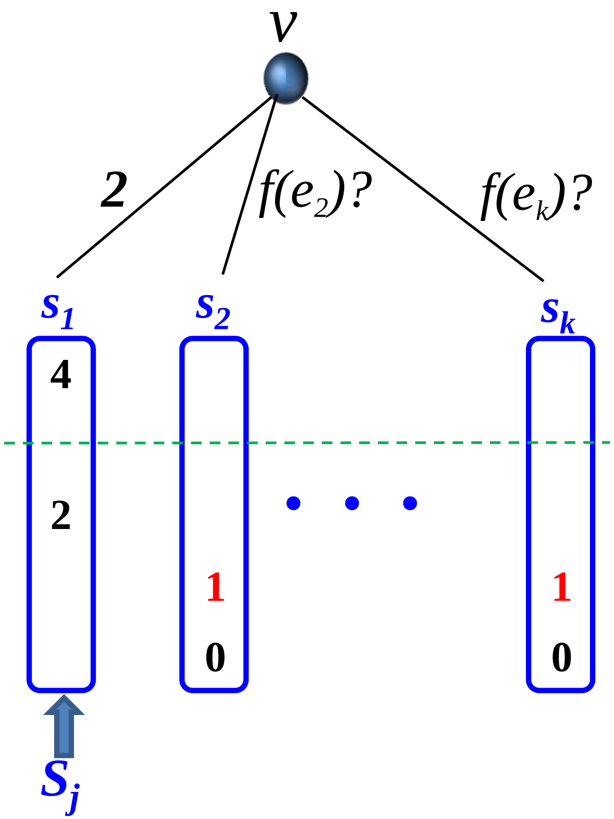

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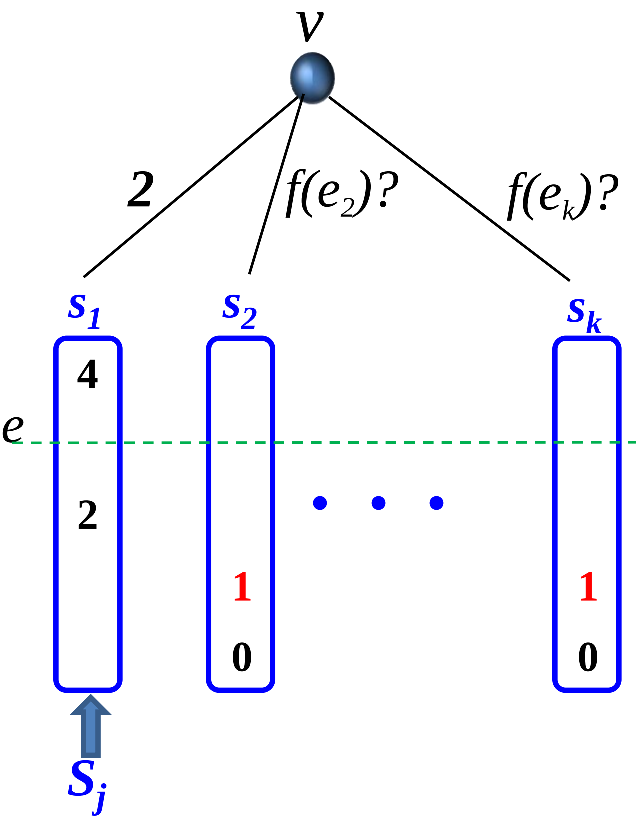
		u		w			
free values	**0**	**1**				**5**	**6**



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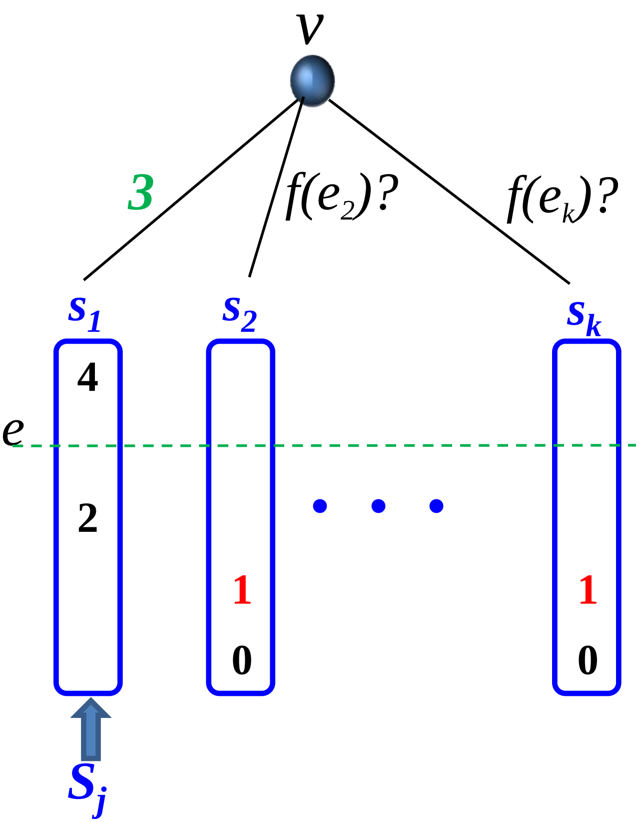

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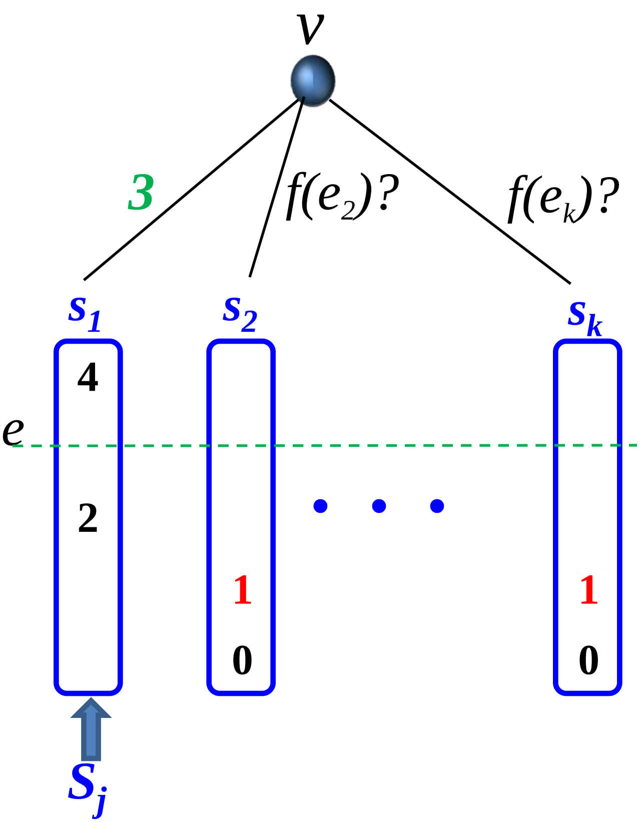
		u		w			
free values	0	1	2			5	6



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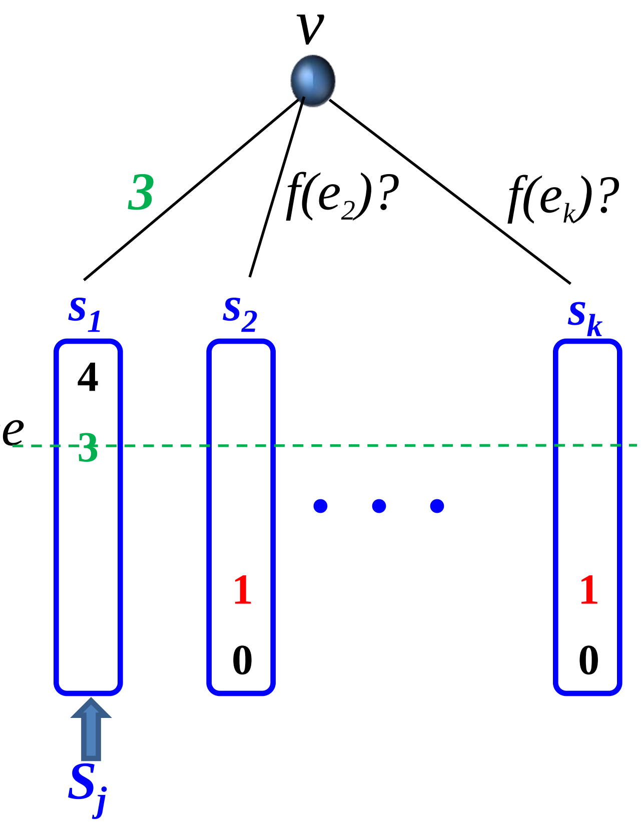

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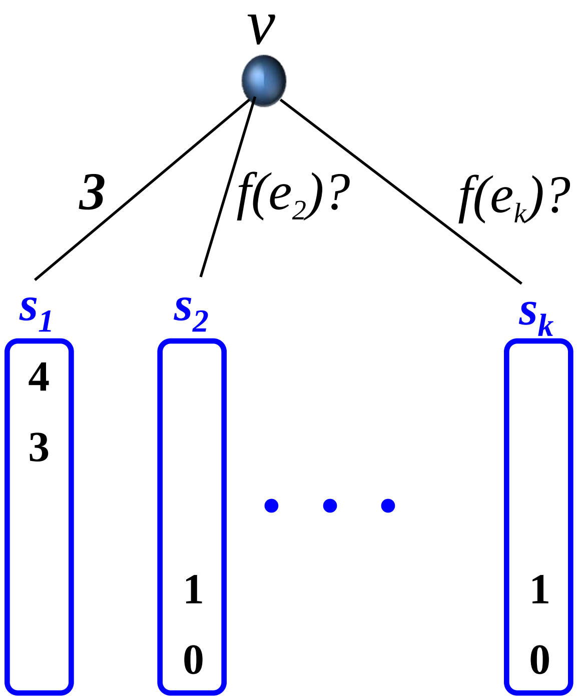

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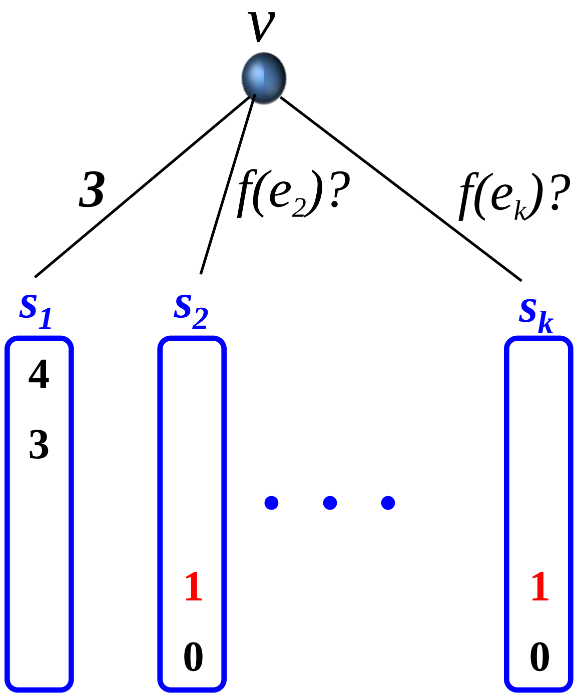

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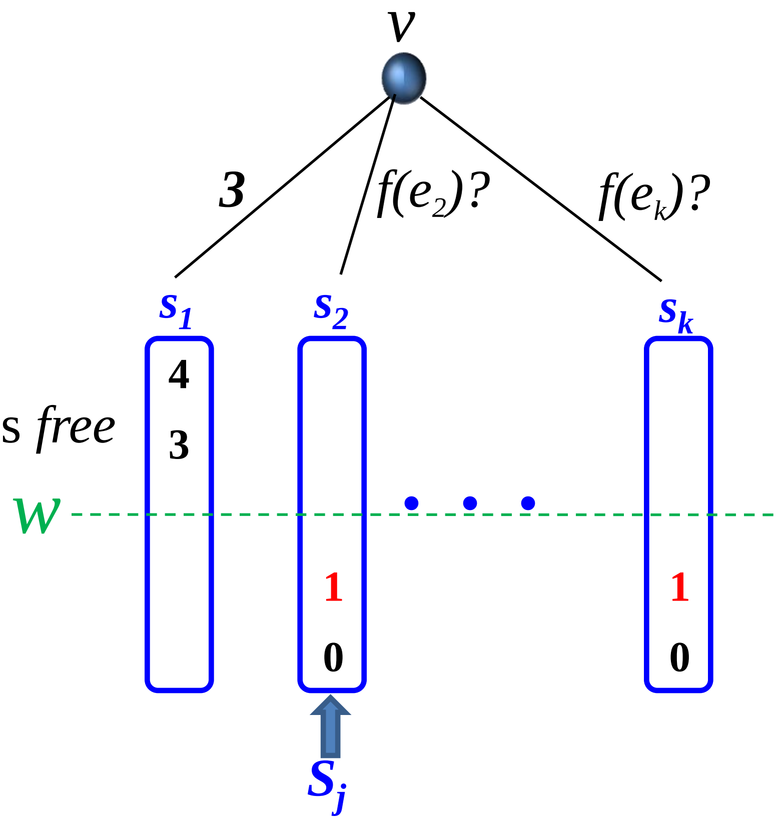

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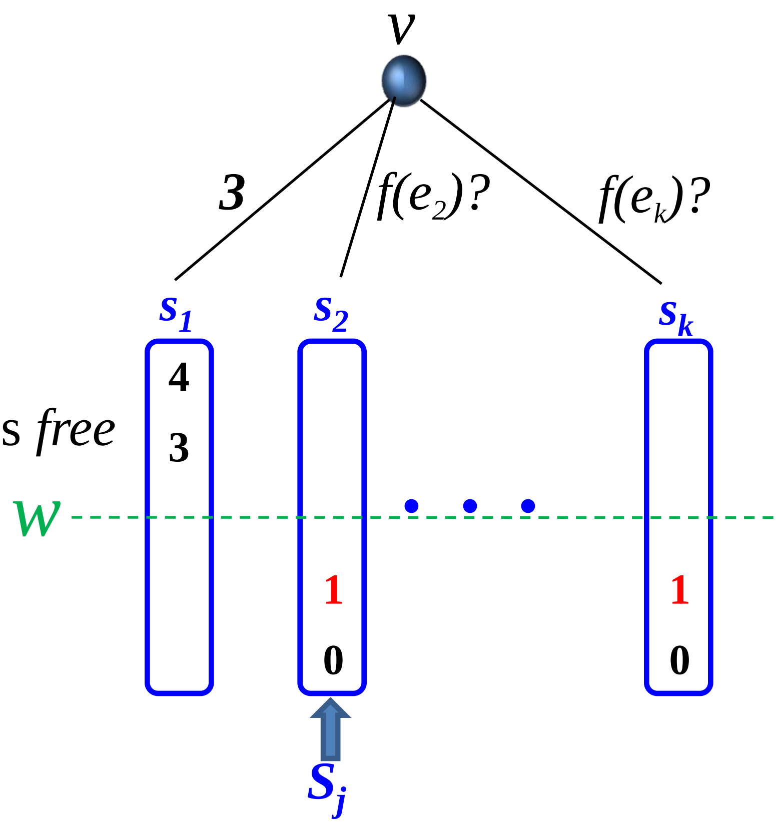

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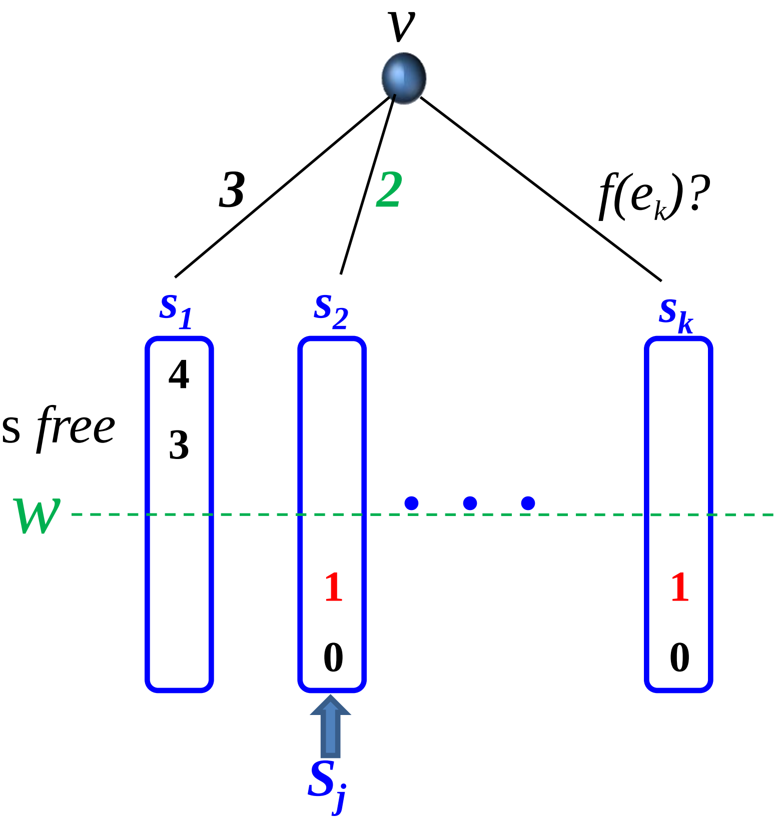

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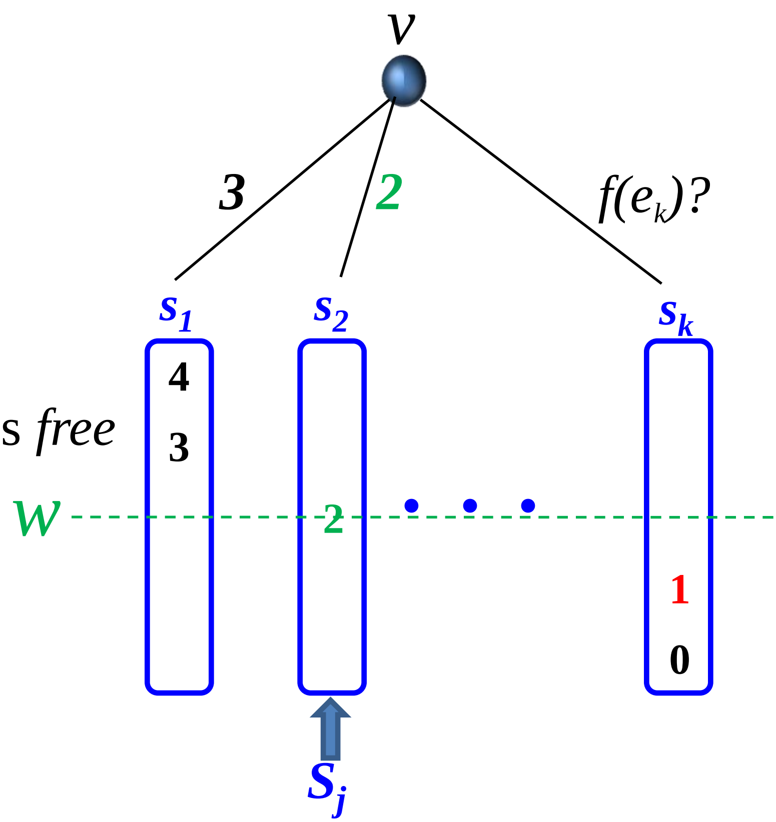
    - remove all values < w from S_j

		u	w				
free values	**0**	**1**				**5**	**6**



Algorithm Outline

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  - if u appears once mark u as *not free*, move to next largest u
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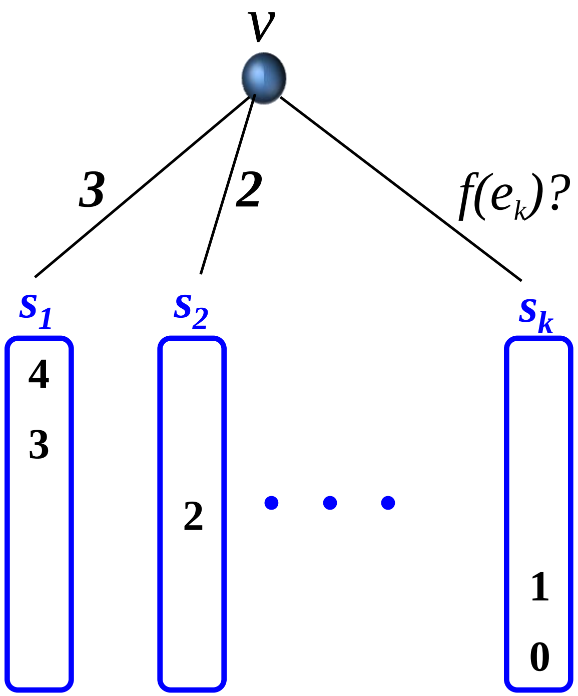

free values	**0**	**1**				**5**	**6**

Algorithm Outline

- set $u = max\{s_i\}$
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  - if u appears once mark u as *not free*, move to next largest u
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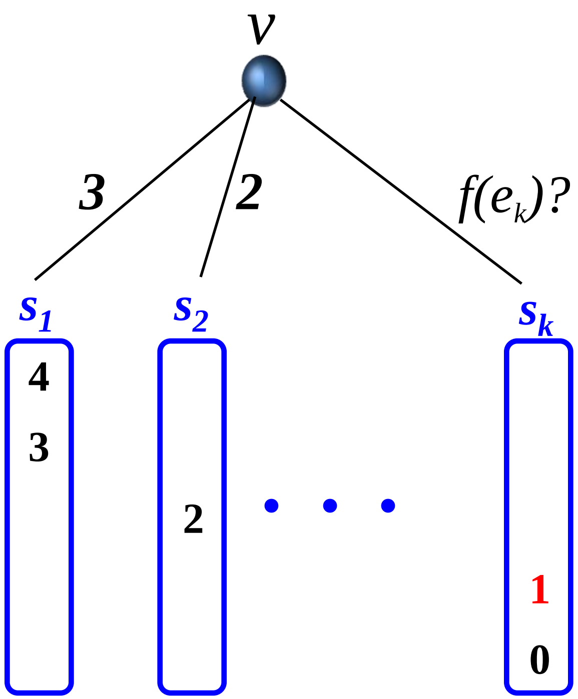

u

free values	**0**	**1**				**5**	**6**

Algorithm Outline

- set $u = max\{s_i\}$
- while not all edges assigned
  - if u appears once mark u as *not free*, move to next largest u
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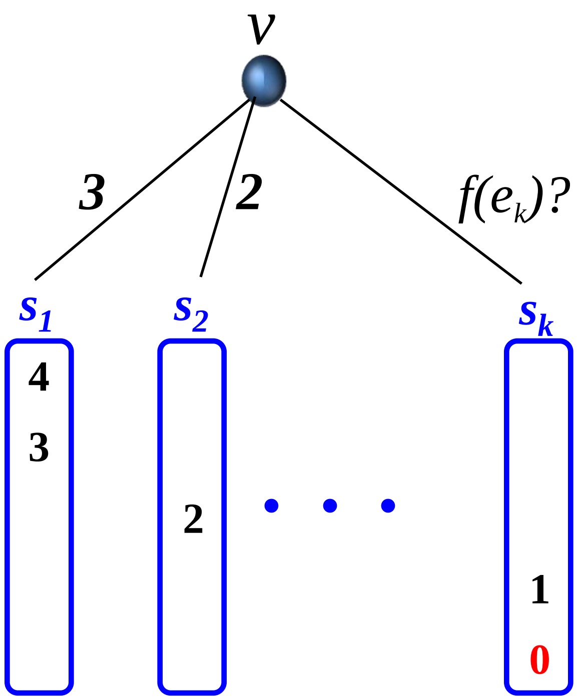
    - remove all values < w from S_j

u

free values	**0**					**5**	**6**



Algorithm Outline

- set $u = max\{s_i\}$
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  - if u appears $\underbrace{\text{once mark}}_{\text{and } u \neq 0}$ u as *not free*, move to next largest u
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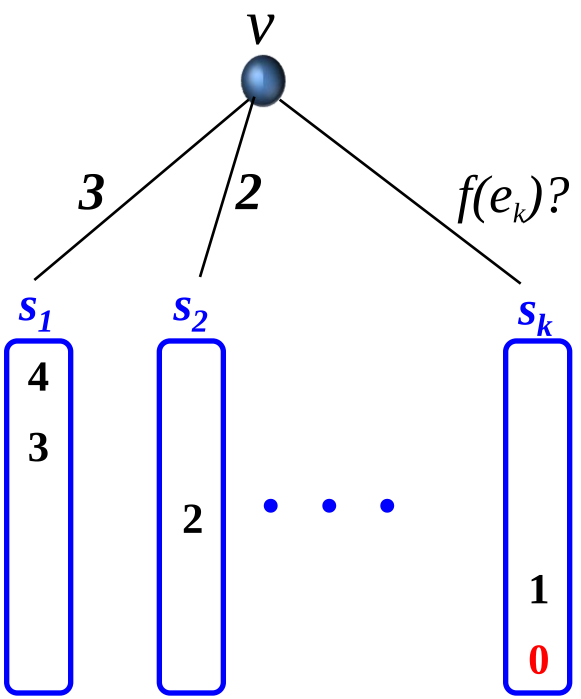

u

free values	**0**					**5**	**6**

Algorithm Outline

- set $u = max\{s_i\}$
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  - if u appears once $\underbrace{\ }_{\text{and } u \neq 0}$ mark u as *not free*, move to next largest u
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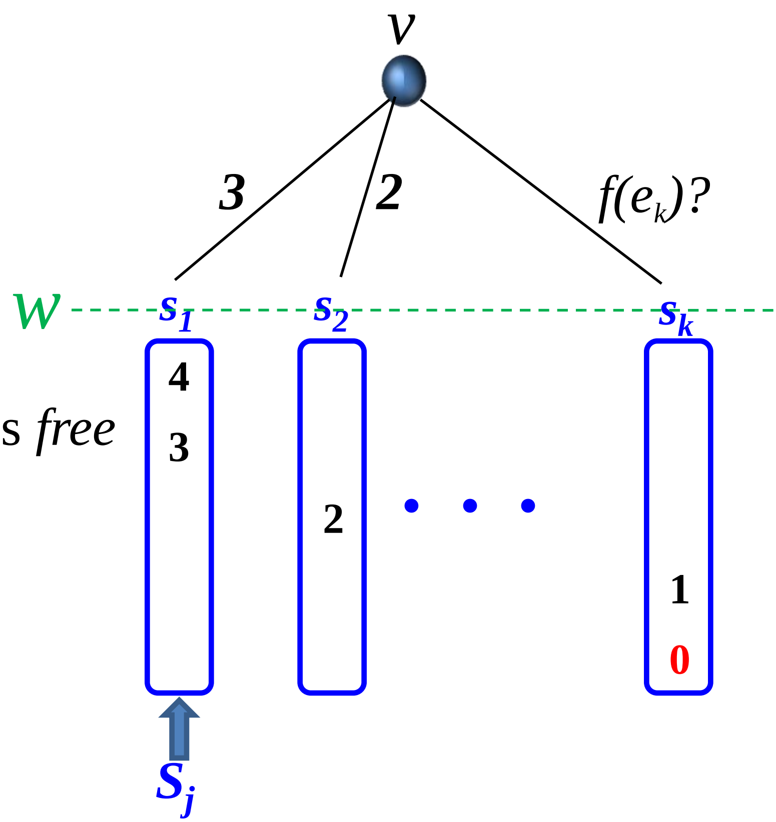
	u					w	
free values	**0**					**5**	**6**



Algorithm Outline

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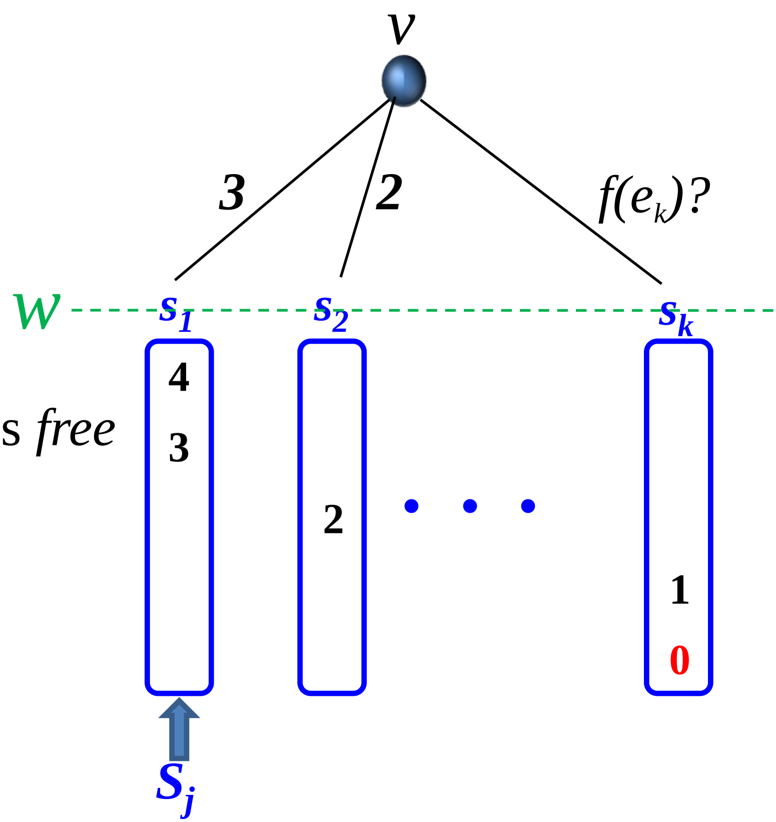
	u					w	
free values	**0**						**6**



Algorithm Outline

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  - if u appears once $\underbrace{\ }_{\text{and } u \neq 0}$ mark u as *not free*, move to next largest u
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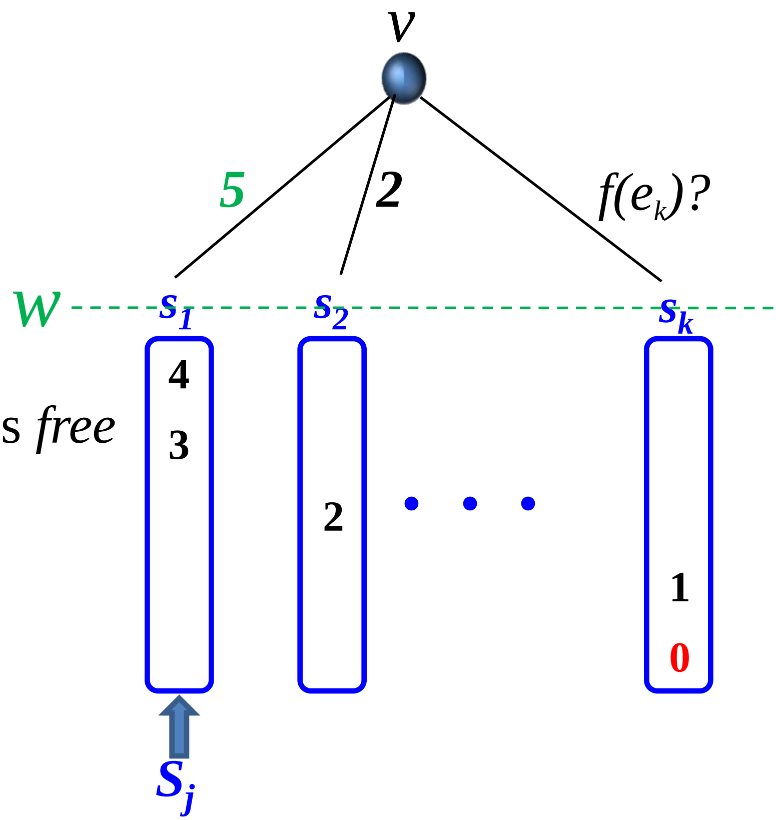
    - remove all values < w from S_j

	u					w	
free values	**0**						**6**



Algorithm Outline

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- while not all edges assigned
  - if u appears once $\underbrace{\ }_{\text{and } u \neq 0}$ mark u as *not free*, move to next largest u
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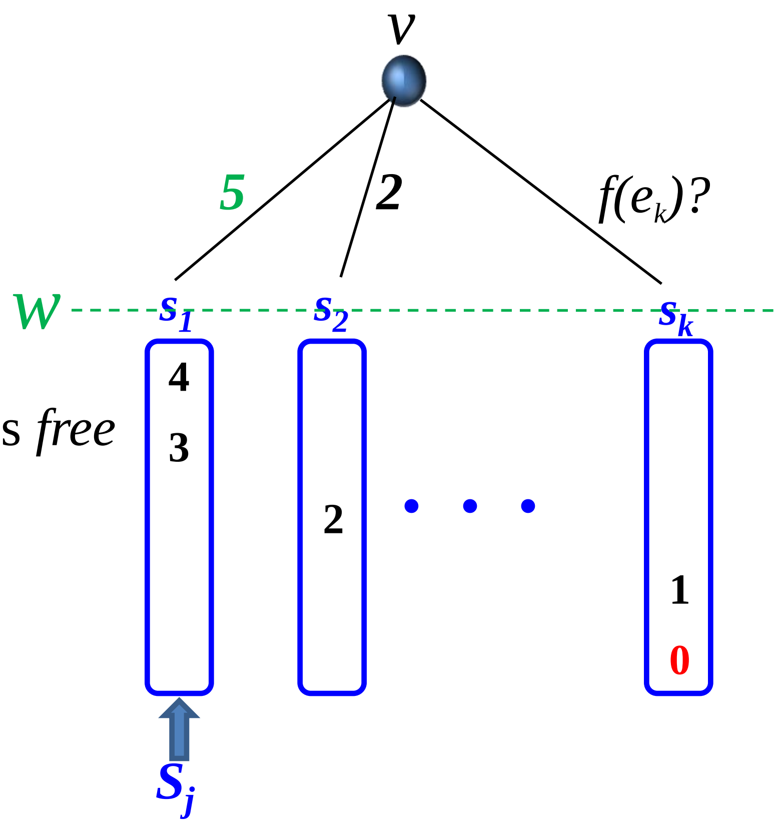
    - remove all values < w from S_j

	u					w	
free values	**0**			**3**	**4**		**6**



Algorithm Outline

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  - if u appears $\underbrace{\text{once}}_{\text{and } u \neq 0}$ mark u as *not free*, move to next largest u
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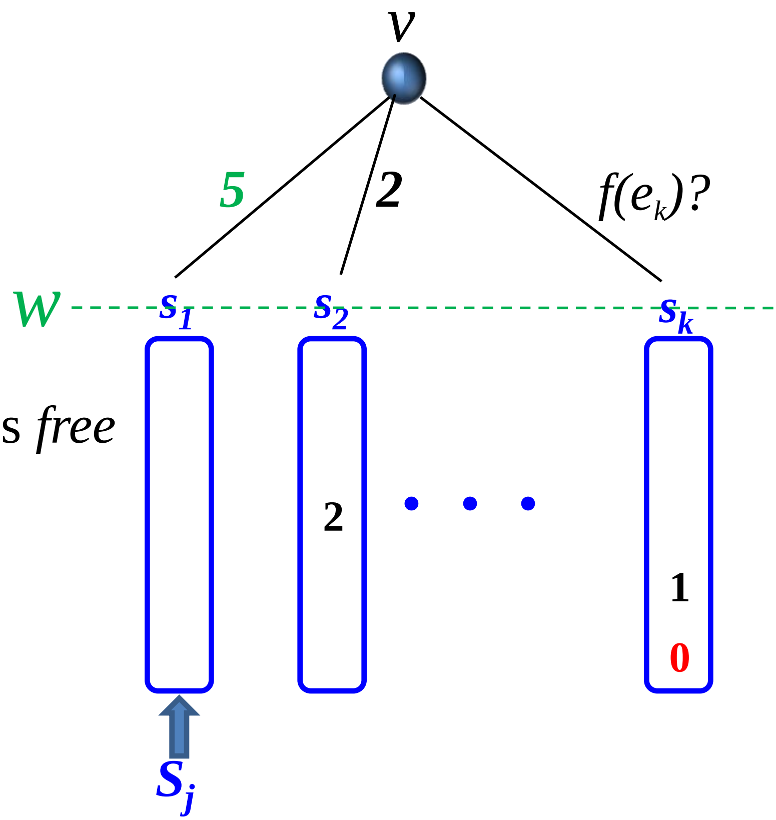
	u					w	
free values	**0**			**3**	**4**		**6**



Algorithm Outline

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  - if u appears once $\underbrace{}_{\text{and } u \neq 0}$ mark u as *not free*, move to next largest u
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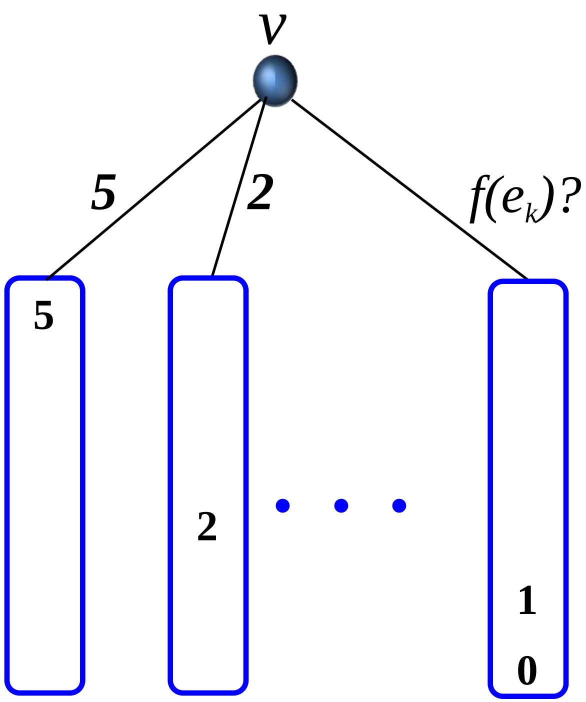

free values	**0**			**3**	**4**		**6**

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- while not all edges assigned
  - if u appears $\underbrace{\text{once}}_{\text{and } u \neq 0}$ mark u as *not free*, move to next largest u
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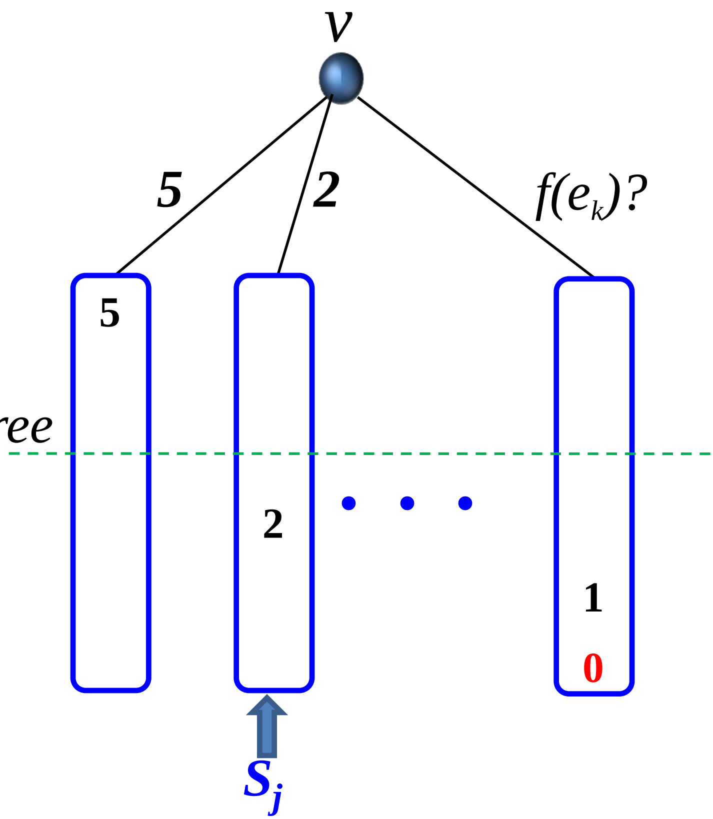
    - remove all values < w from S_j

	u			w			
free values	**0**			**3**	**4**		**6**



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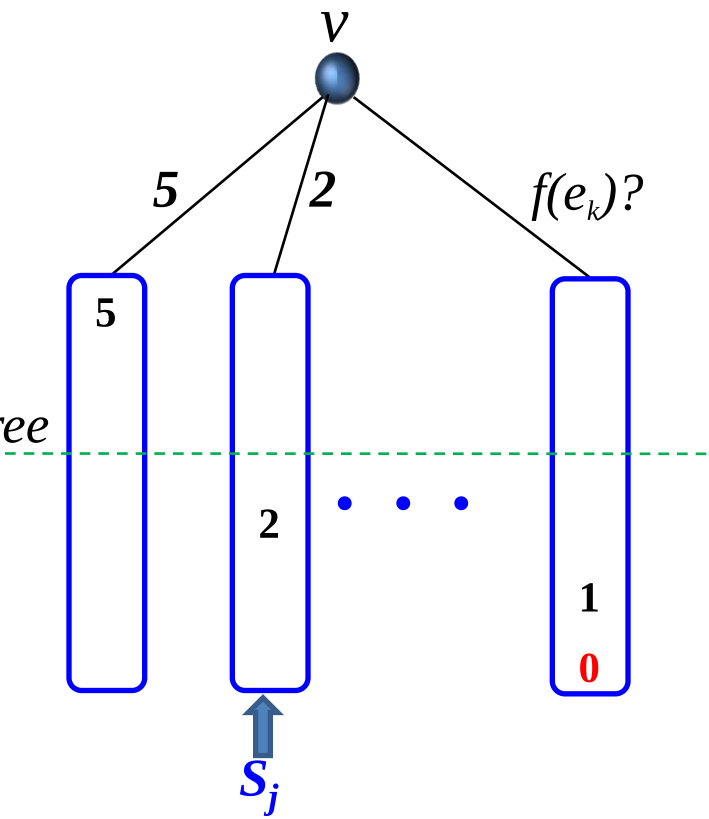
	u			w			
free values	**0**				**4**		**6**



Algorithm Outline

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  - if u appears once mark u as *not free*, move to next largest u

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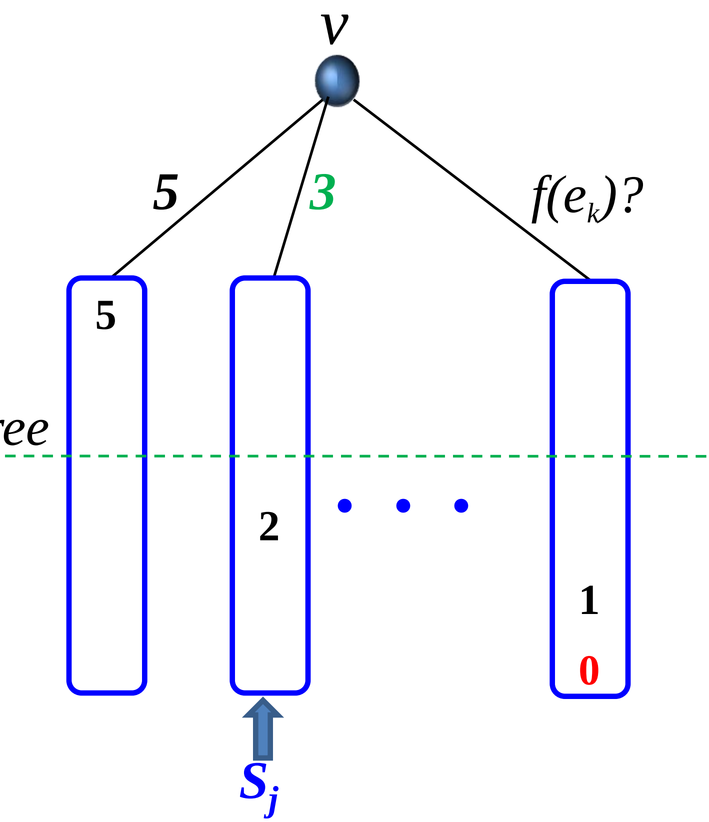
    - remove all values < w from S_j

	u			w			
free values	**0**				**4**		**6**



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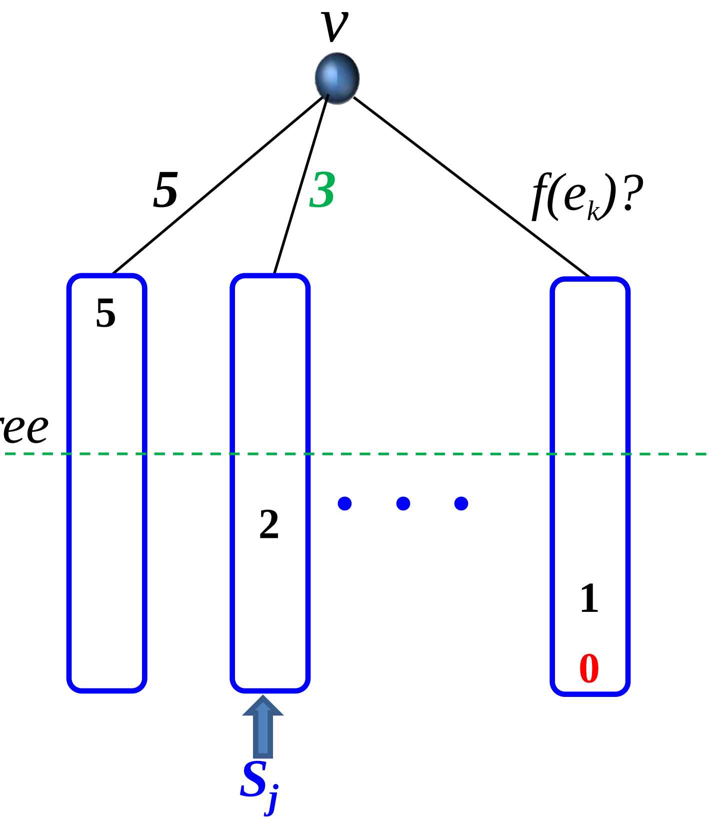
    - mark w as *not free*
    - set current $f(e_j) = w$
    - mark all S_j values between u and w as *free*
    - remove all values < w from S_j

	u			w			
free values	**0**		**2**		**4**		**6**



Algorithm Outline

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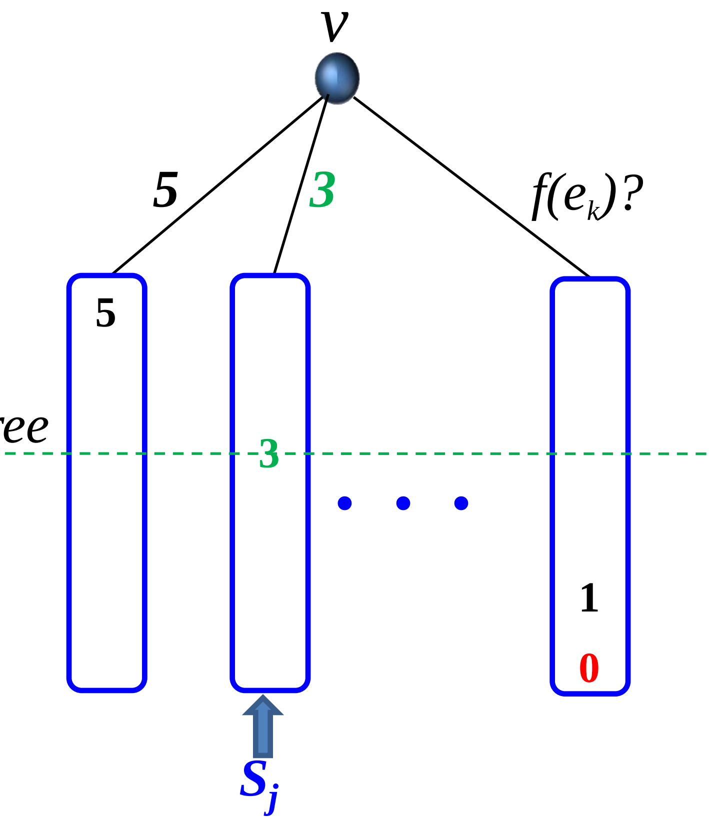
    - remove all values < w from S_j

	u			w			
free values	**0**		**2**		**4**		**6**



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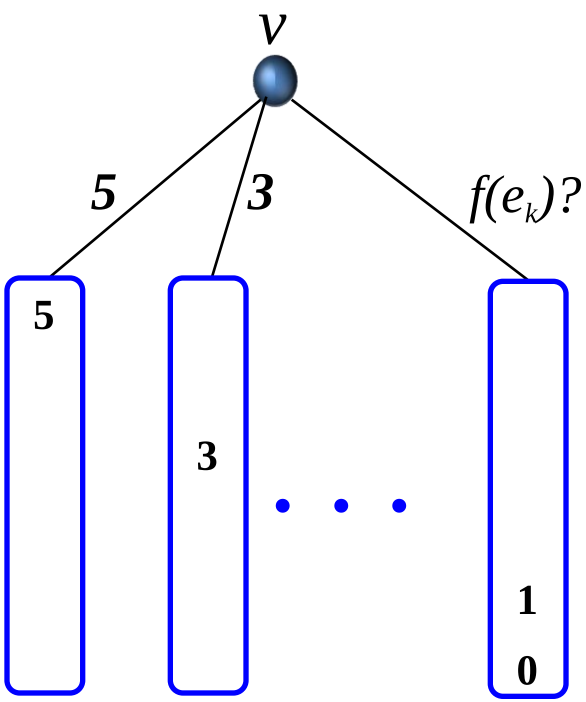

free values	**0**		**2**		**4**		**6**

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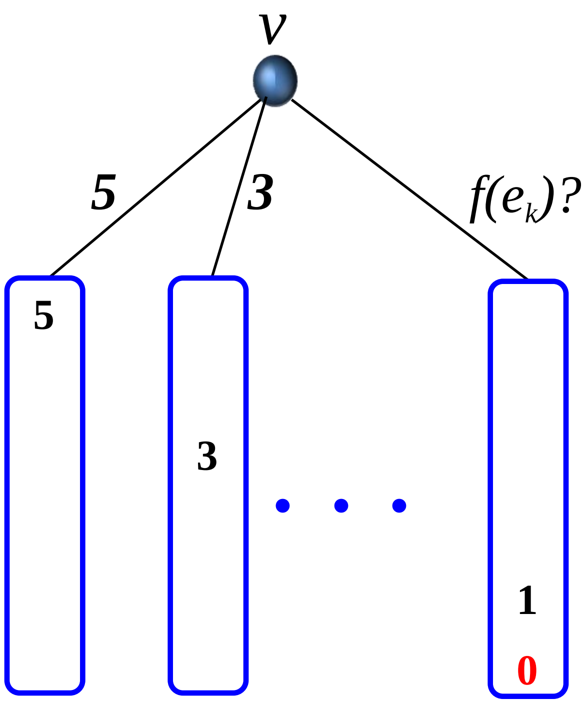
u

free values	**0**		**2**		**4**		**6**



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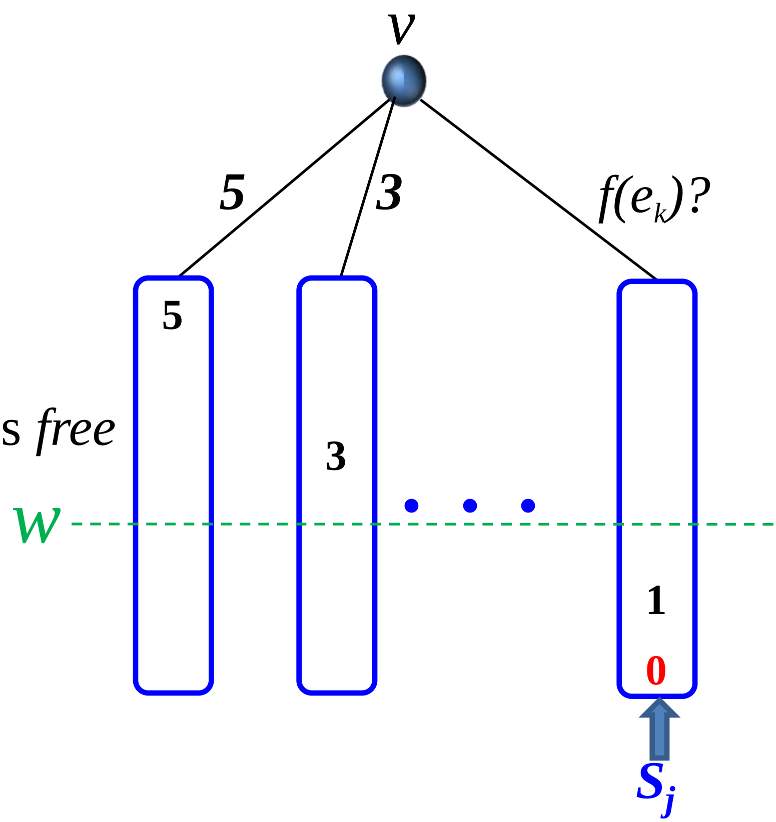
	u		w				
free values	**0**		**2**		**4**		**6**



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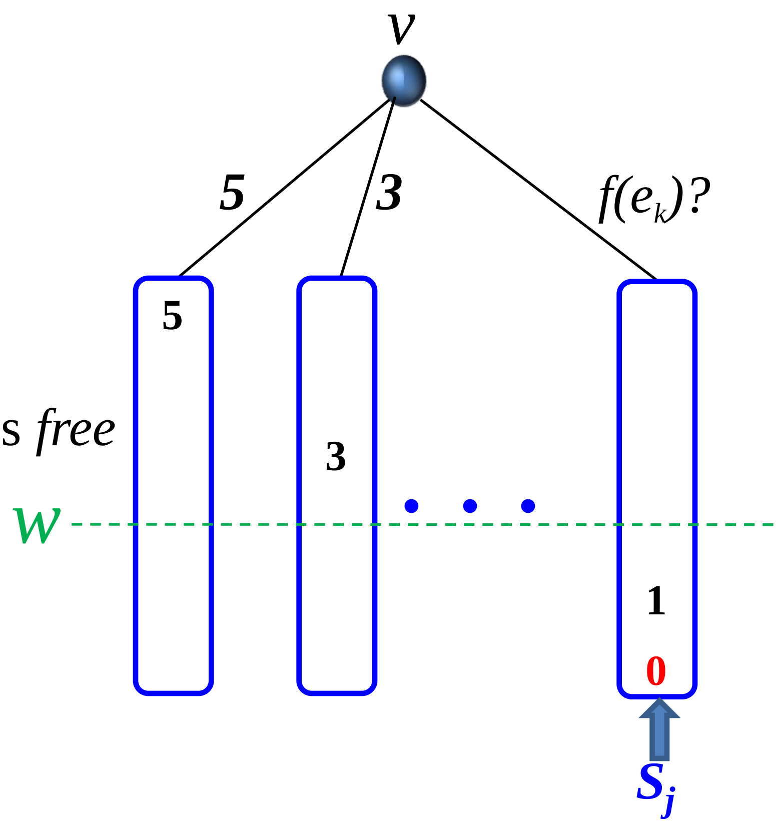
	u		w				
free values	**0**				**4**		**6**



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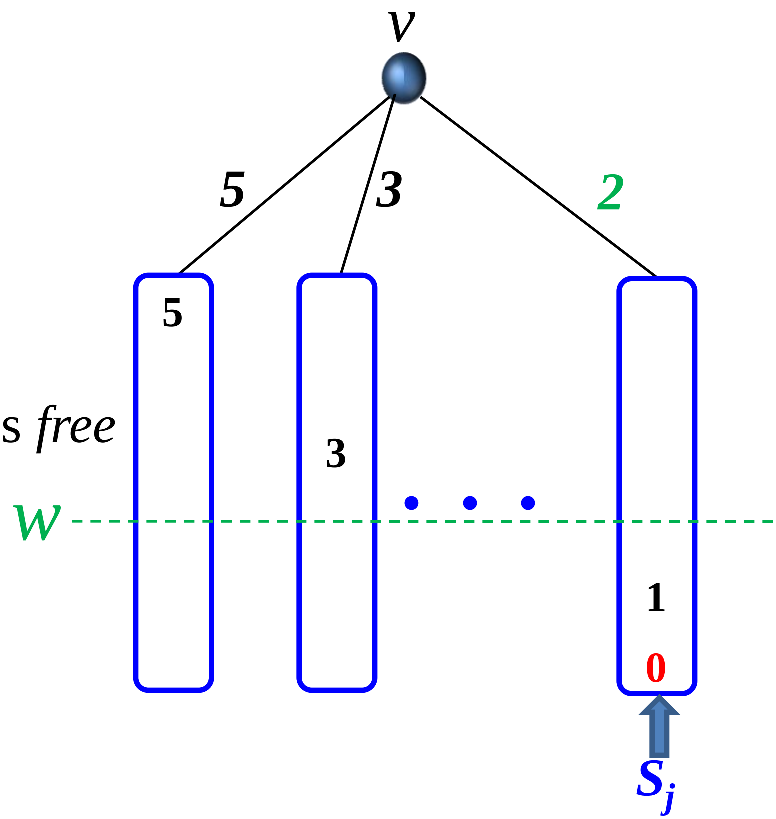
	u		w				
free values	**0**				**4**		**6**



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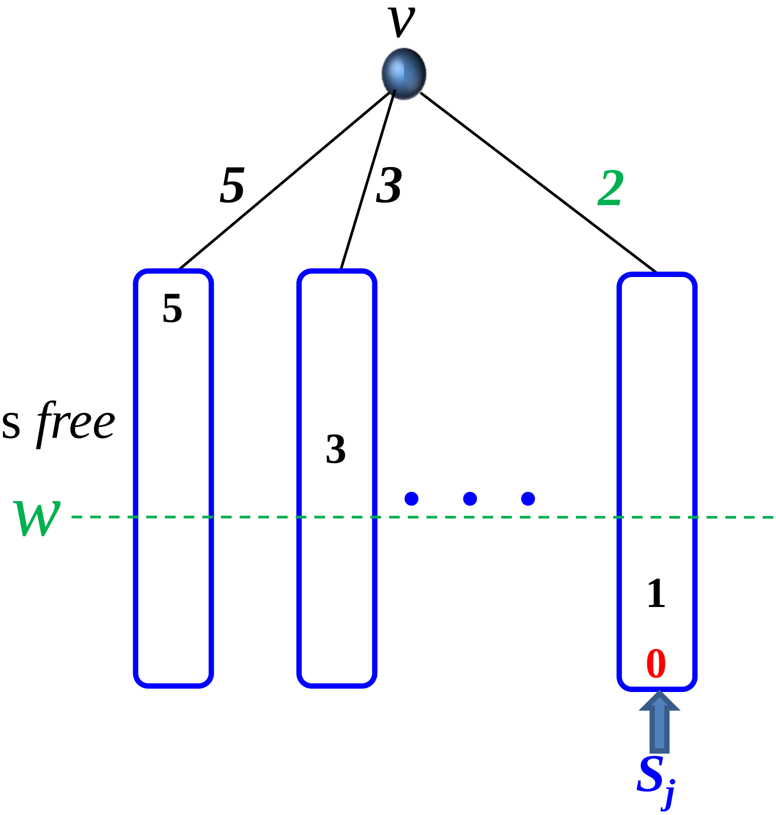
    - remove all values < w from S_j

	u		w				
free values	**0**	**1**			**4**		**6**



Algorithm Outline

- set $u = max\{s_i\}$
- while not all edges assigned
  - if u appears once mark u as *not free*, move to next largest u
    and $u \neq 0$
  - otherwise:
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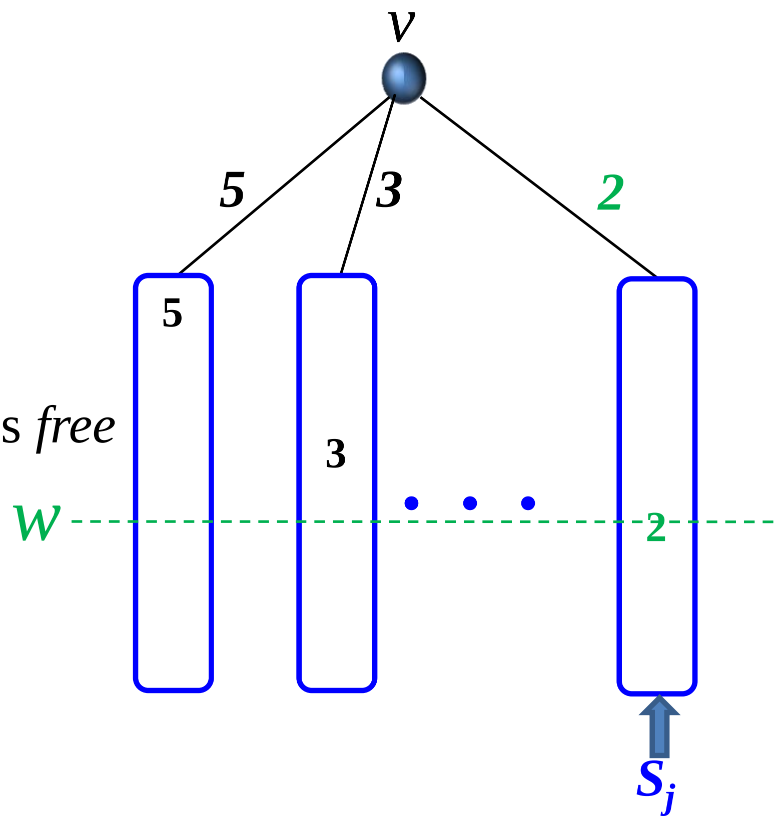
	u		w				
free values	**0**	**1**			**4**		**6**



Algorithm Outline

That's it!

- set $u = max\{s_i\}$
- while not all edges assigned
  - if u appears once mark u as *not free*, move to next largest u

    and $u \neq 0$
  - otherwise:
    - w = smallest free value > u
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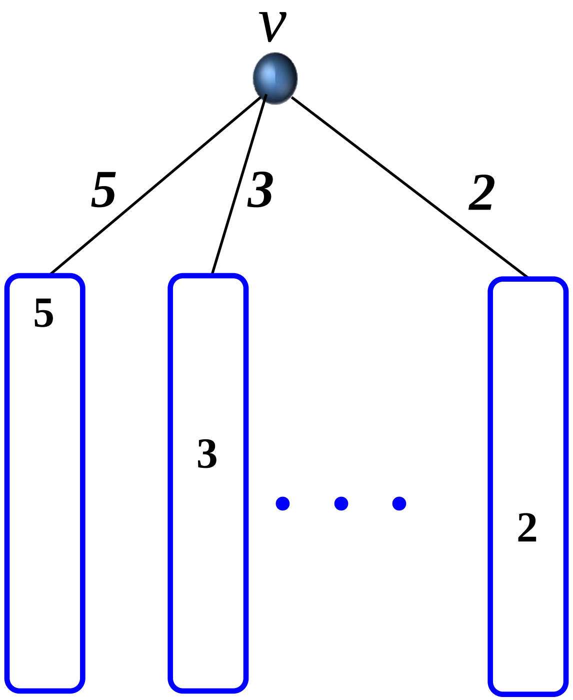

free values	**0**	**1**			**4**		**6**

Algorithm Outline

That's it!

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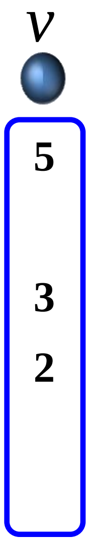
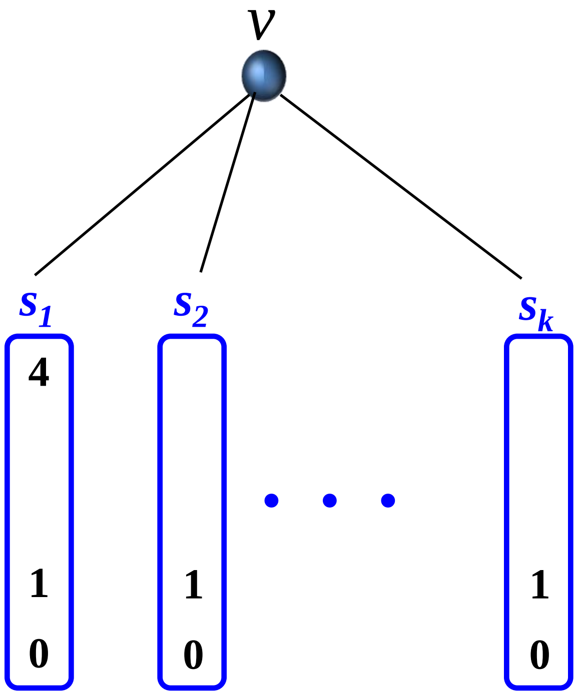


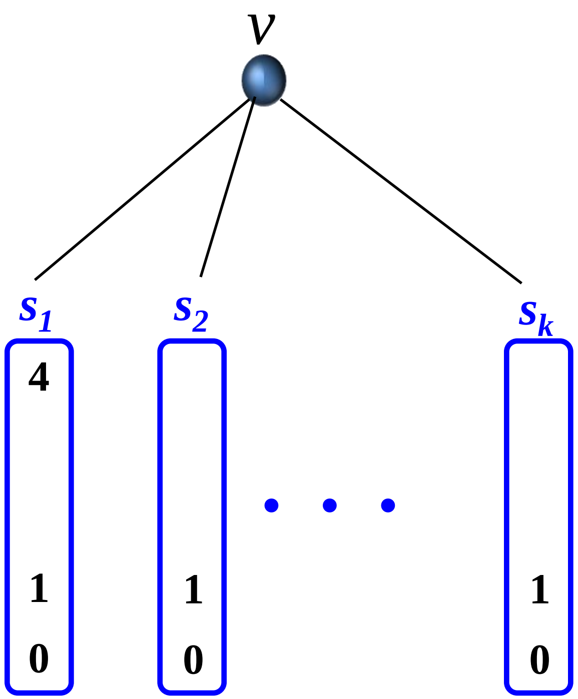
free values	**0**	**1**			**4**		**6**

Running Time



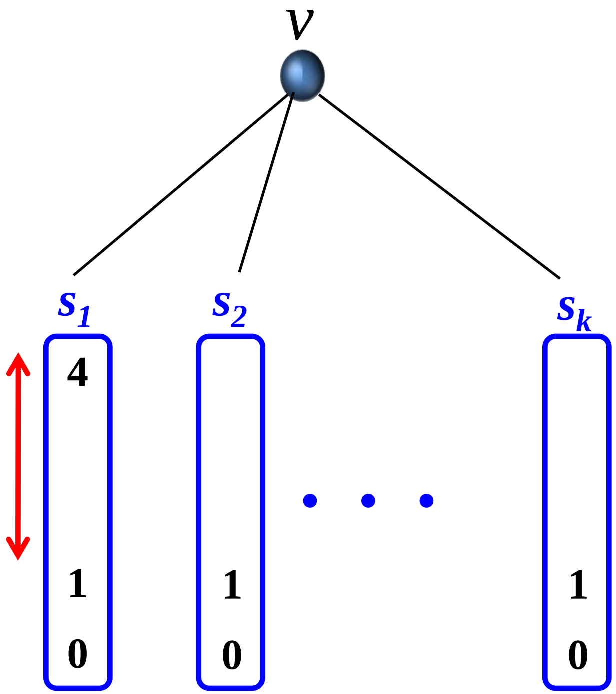
Running Time

- $|S_1| + |S_2| + \ldots + |S_k|$ is not a lower bound !



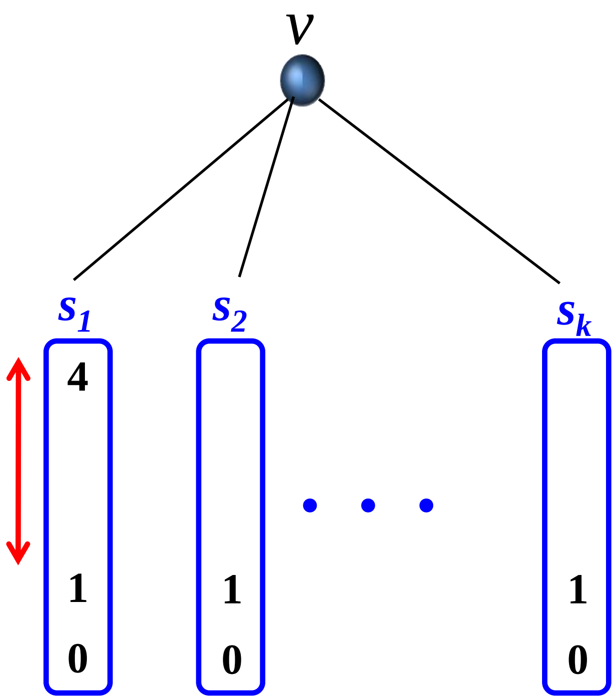
Running Time

- $|S_1| + |S_2| + \ldots + |S_k|$ is not a lower bound !
- in many cases, the largest values of the largest visibility sequence are unchanged at *v* itself



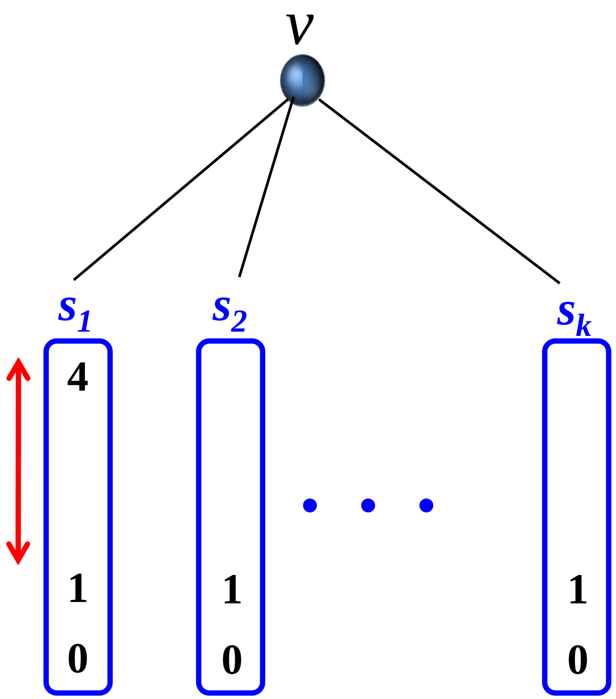
Running Time

- $|S_1| + |S_2| + \ldots + |S_k|$ is not a lower bound !
- in many cases, the largest values of the largest visibility sequence are unchanged at *v* itself
  - $k(v)$ = #*v*'s children



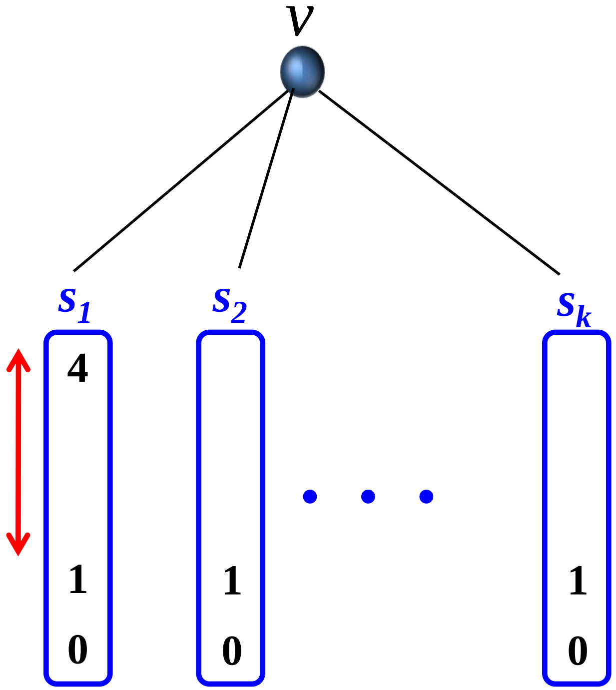
Running Time

- $|S_1| + |S_2| + \ldots + |S_k|$ is not a lower bound !
- in many cases, the largest values of the largest visibility sequence are unchanged at *v* itself
  - $k(v)$ = #*v*'s children
  - $q(v) = |S_2| + \ldots + |S_k|$



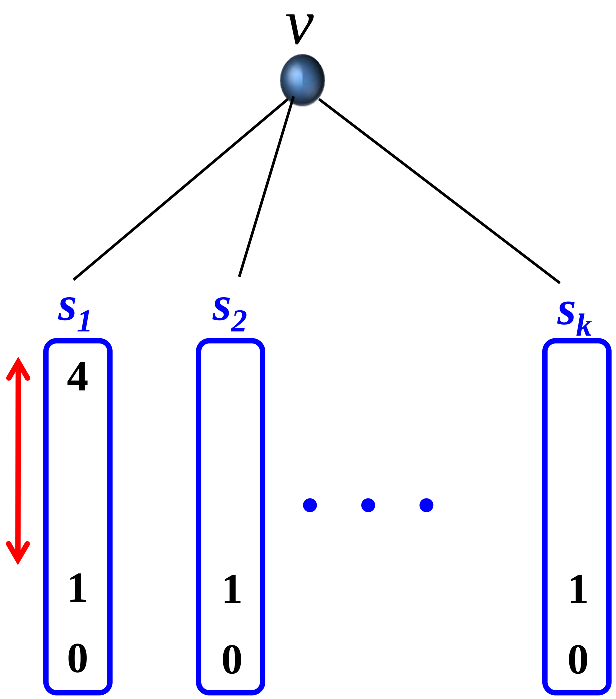
Running Time

- $|S_1| + |S_2| + \ldots + |S_k|$ is not a lower bound !
- in many cases, the largest values of the largest visibility sequence are unchanged at *v* itself
  - $k(v)$ = #*v*'s children
  - $q(v) = |S_2| + \ldots + |S_k|$
  - $t(v)$ = largest value that appears in S_v but not in S_1



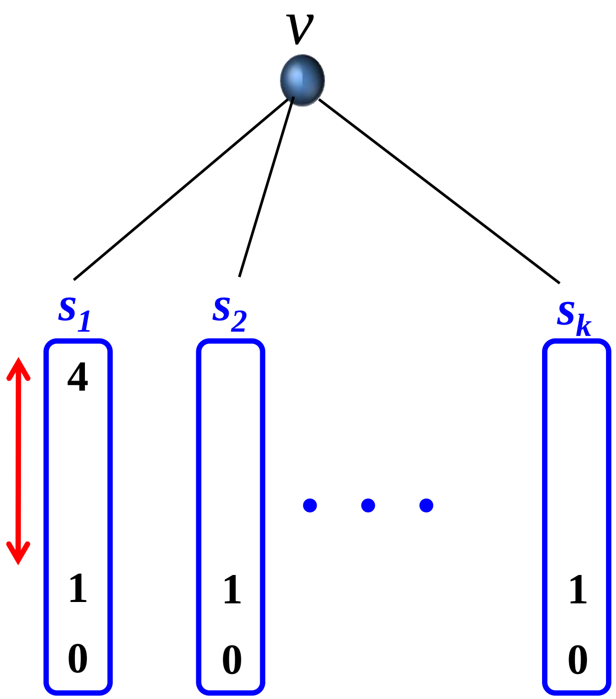
Running Time

- $|S_1| + |S_2| + \ldots + |S_k|$ is not a lower bound !
- in many cases, the largest values of the largest visibility sequence are unchanged at *v* itself
  - $k(v)$ = #*v*'s children
  - $q(v) = |S_2| + \ldots + |S_k|$
  - $t(v)$ = largest value that appears in S_v but not in S_1
- an extension can be computed in $O(\, k(v)+q(v)+t(v)\,)$



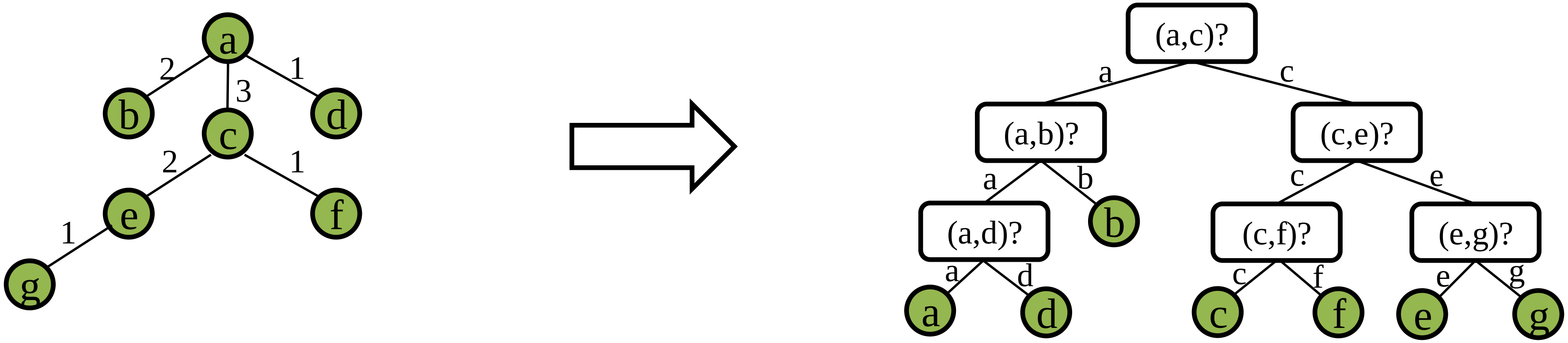
Running Time

- $|S_1| + |S_2| + \ldots + |S_k|$ is not a lower bound !
- in many cases, the largest values of the largest visibility sequence are unchanged at *v* itself
  - $k(v)$ = #*v*'s children
  - $q(v) = |S_2| + \ldots + |S_k|$
  - $t(v)$ = largest value that appears in S_v but not in S_1
- an extension can be computed in $O(\ k(v)+q(v)+t(v)\ )$
- $\sum_v k(v)+q(v)+t(v) = O(n)$

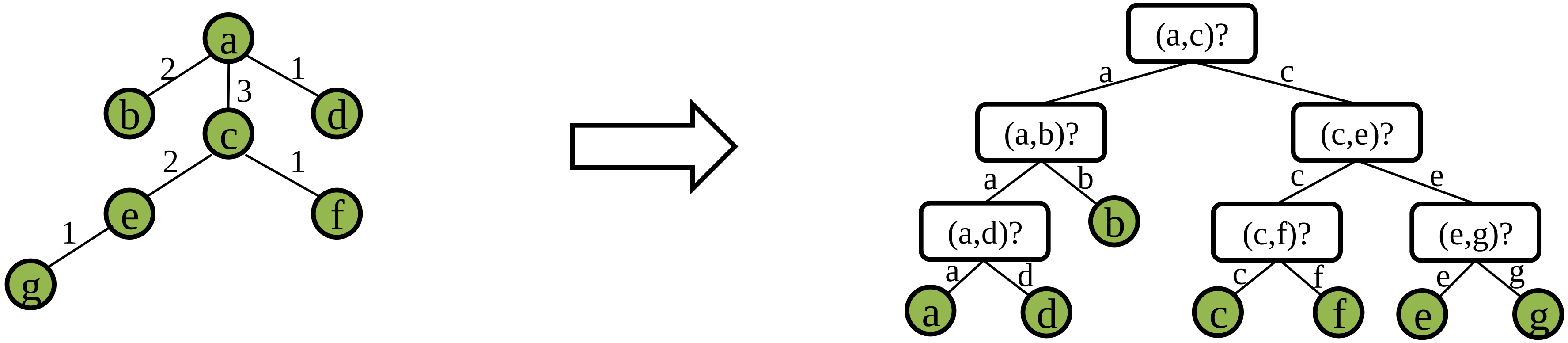


From Strategy Function to Decision Tree in $O(n)$ Time

From Strategy Function to Decision Tree in *O*(*n*) Time



From Strategy Function to Decision Tree in *O(n)* Time



- For all edges *e*
  - let *s* = visibility sequence at *bottom(e)*
  - if *s* contains no values smaller than *f(e)*
    - set *bottom(e)* as the solution when the query on *e* returns *bottom(e)*
  - else, let $v_1 < \ldots < v_k < f(e)$ in *s*, let e_i be the edge v_i is assigned to
    - set e_k as the solution when the query on *e* returns *bottom(e)*
    - for every $1 \le i < k$ set e_i as the solution when the query on e_{i+1} returns $top(e_{i+1})$
    - set $top(e_1)$ as the solution when the query on e_1 returns $top(e_1)$

Thank you !!