The Stackelberg Minimum Spanning Tree Game

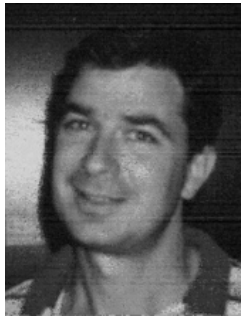
Jean Cardinal

Erik Demaine

Samuel Fiorini

Gwenaël Joret

Stefan Langerman



Ilan Newman

Oren Weimann

G3t R1CH Qu1CK with FTL N3tw0rk1ng!!!

Donald Duck

Uncle Scrooge

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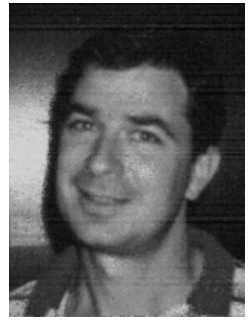
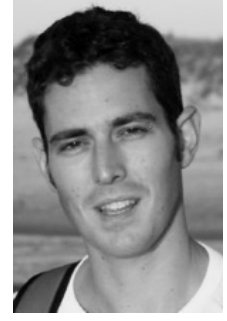


Ilan Newman



Oren Weimann

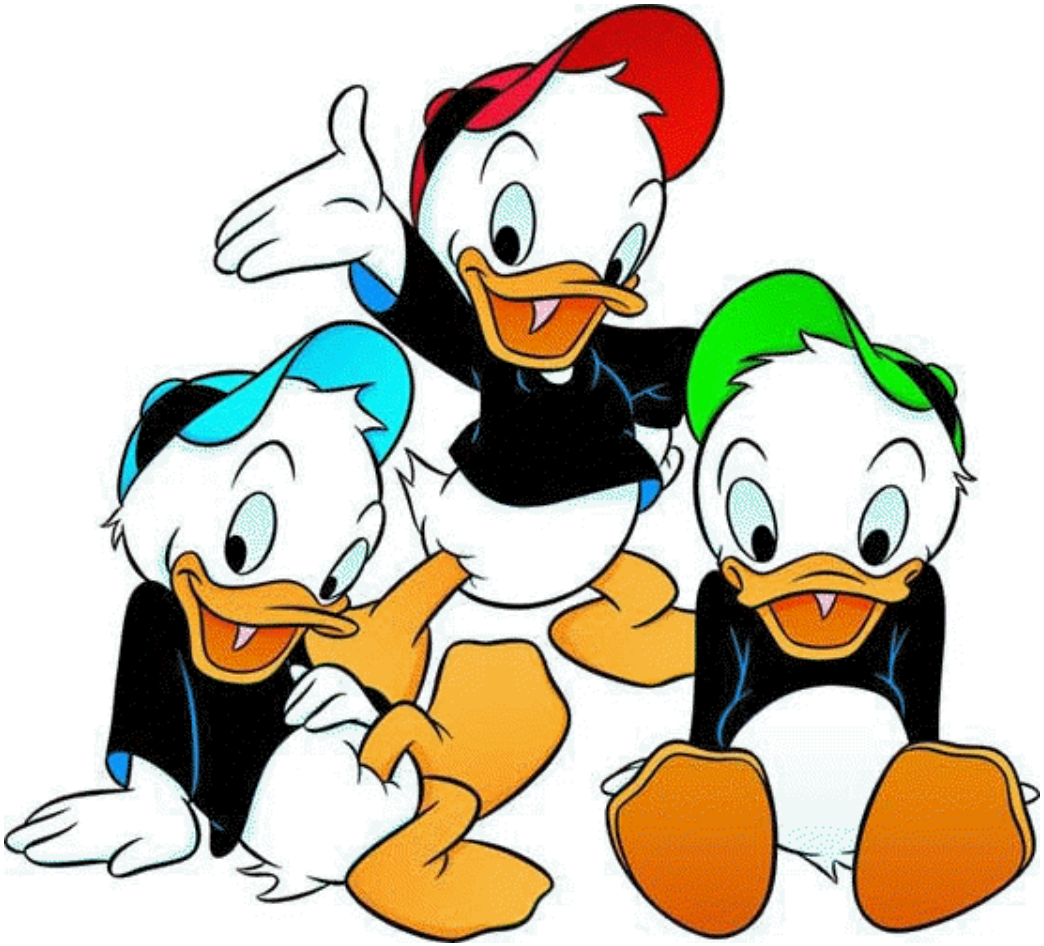
Once upon a time . . .

MIT scientists have invented a revolutionary device to communicate faster than the speed of light.

Gyro Gearloose with Randall L. Stephenson

Scrooge's Goal: Infinite profit!

Donald's Goal: Build a network for his nephews



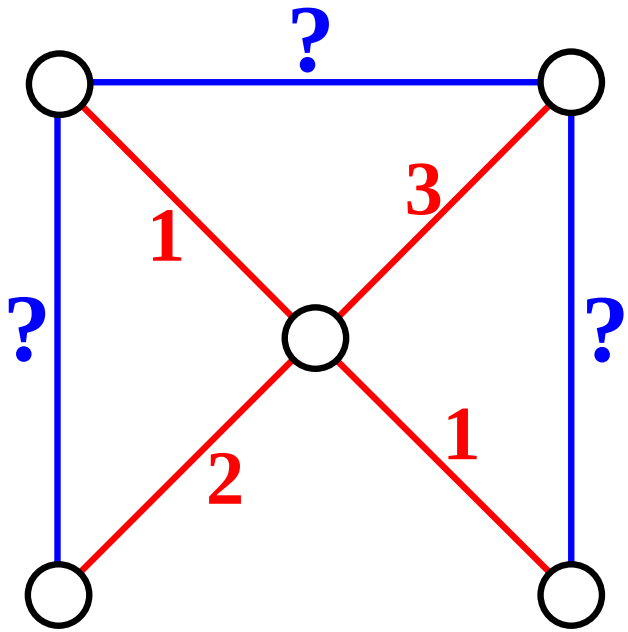
Scrooge's Goal: Infinite profit!

But suddenly. . . a competitor appears!

Gladstone Gander
(to follow John E. Pepper, Jr. as chairman of Verizon)

Example

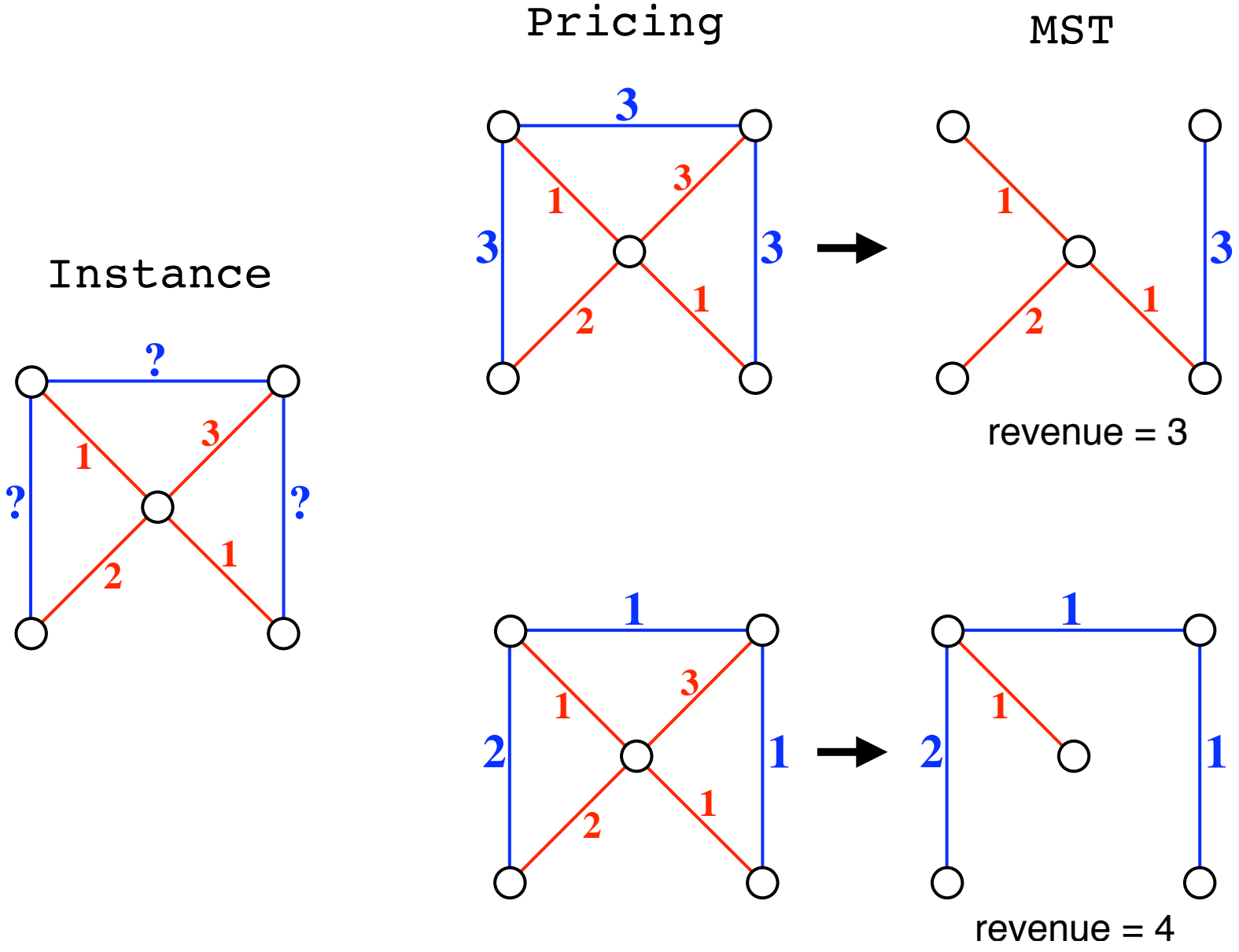
Gladstone Gander



Uncle Scrooge

Donald Duck buys MST

Example



Assumption: blues have priority over reds of same weight

Problem Statement

Given a graph G with red and blue edges

- Each red edge has a fixed cost $c(e)$ [Gladstone Gander]
- The leader [Uncle Scrooge] has to set a price $p(e)$ for each blue edge
- The follower [Donald Duck] then computes a MST with the resulting weights

Goal: maximize total weight of the blue edges in a MST

Context for Stackelberg MST (StackMST)

Stackelberg game:

- Leader makes one move
- Follower makes one move dependent on leader's move

Related work:

- "Highway" problems – pricing shortest paths: Labbé, Marcotte, and Savard (Management Science, 1998), Grigoriev, van Hoesel, Kraaij, Uetz, and Bouhtou (WAOA 05), survey by Stan van Hoesel
- Bundle pricing: Guruswami, Hartline, Karlin, Kempe, Kenyon, and McSherry (SODA 05), Hartline and Koltun (WADS 05), Grigoriev, van Loon, Sitters, and Uetz (WG 06)
- Many other games and mechanisms on spanning trees

Hardness I

Proposition

StackMST is NP-hard, even if $c(e) \in \{1, 2\}\ \forall e$

Proof idea:

Reduction from SetCover

$\longrightarrow$ sets $S_1, S_2, \ldots, S_m$
$\longrightarrow$ elements $u_1, u_2, \ldots, u_n$

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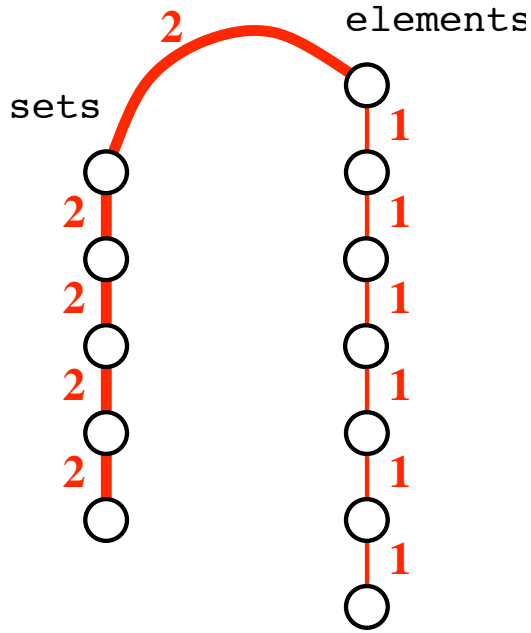

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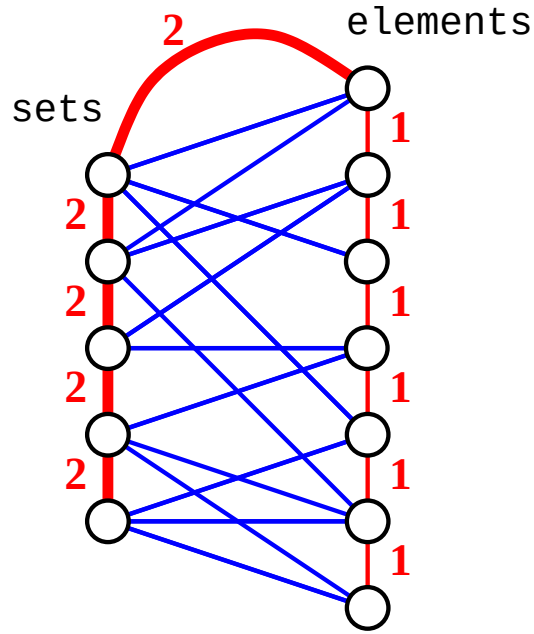

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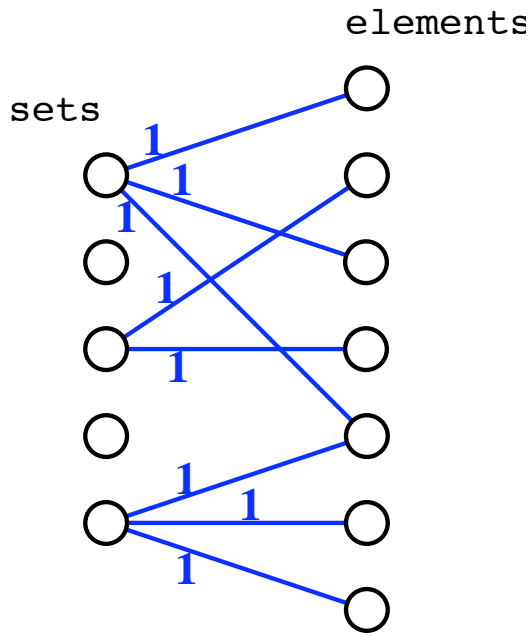

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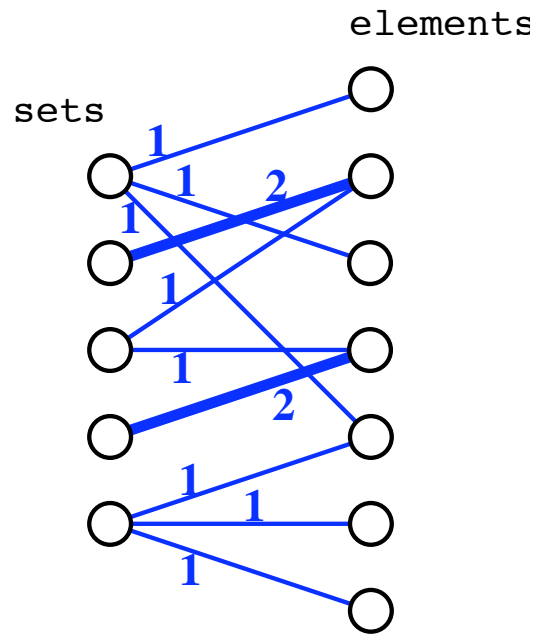

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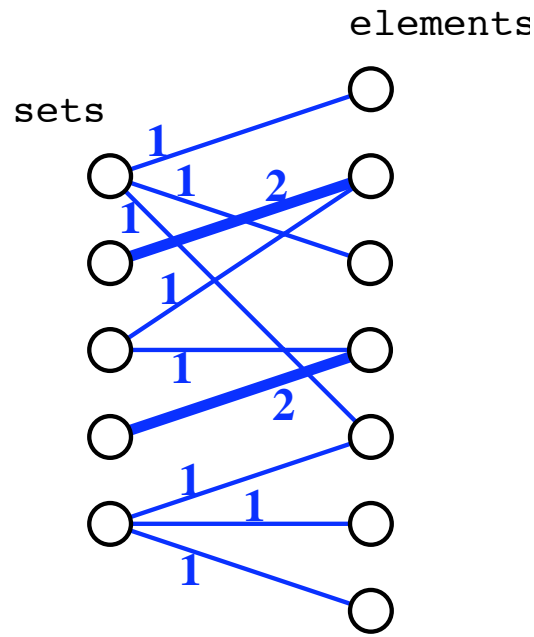


$\exists$ cover of size $\leq t \iff \exists$ pricing giving $\geq (n + t - 1) + 2(m - t)$

Hardness II
Inapproximability result

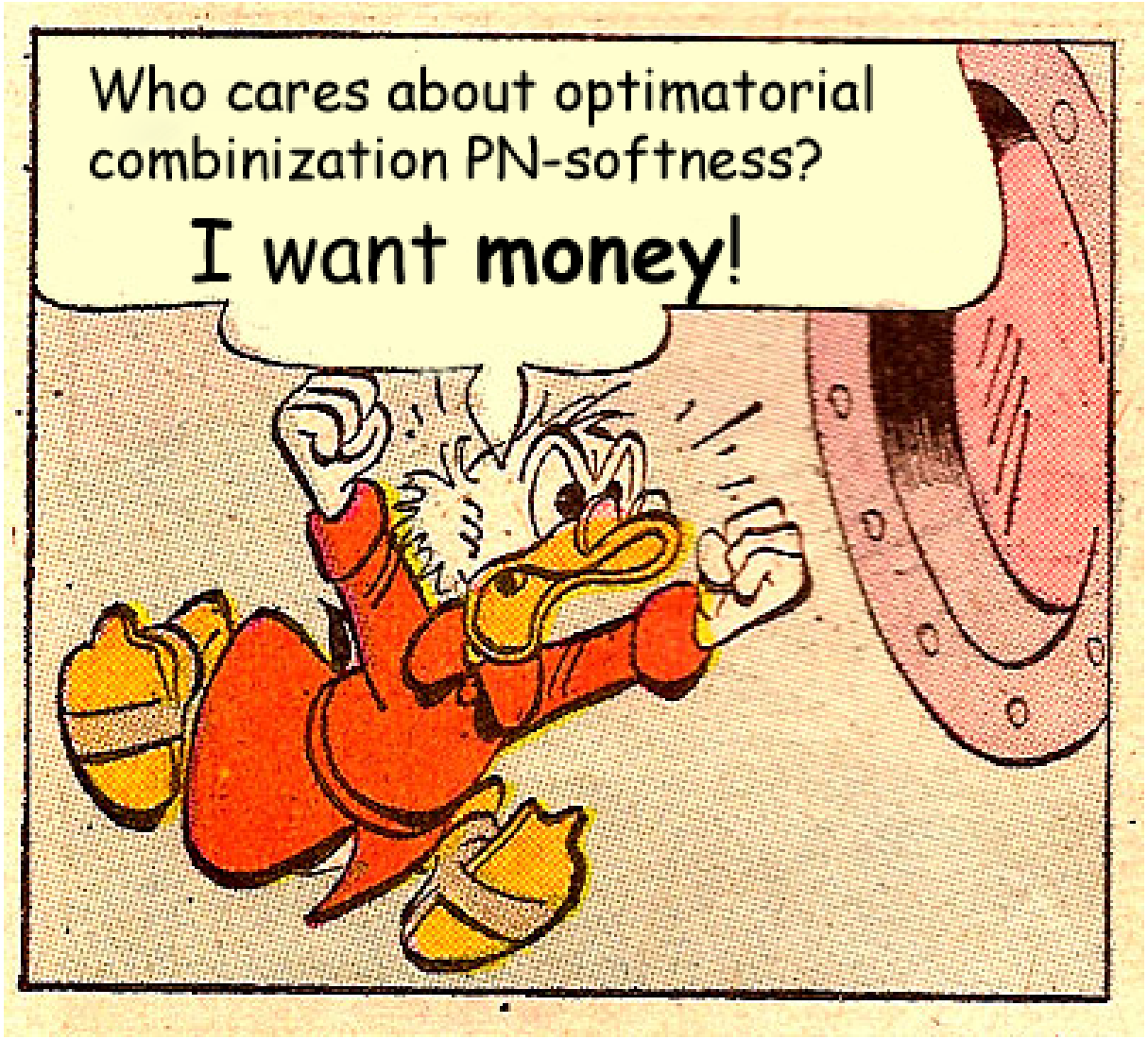
Proposition

StackMST is APX-hard, even if $c(e) \in \{1, 2\}\ \forall e$

Proof idea:

- reduce from VertexCover in graphs with $\Delta \leq 3$
- then everything is linear in n
- $\exists\ (1 - \epsilon)$-approx for StackMST $\Rightarrow$
  $\exists\ (1 + 8\epsilon)$-approx for VertexCover in graphs with $\Delta \leq 3$

Bad News for Uncle Scrooge

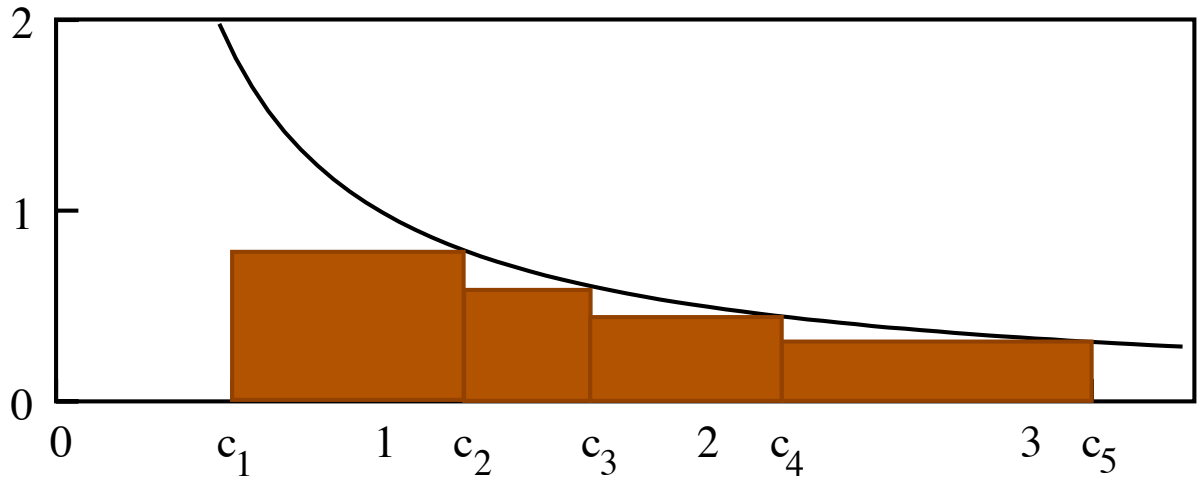


Approximation I

Assume that the red costs are $c_1 < c_2 < \cdots < c_k$

Let $\rho := 1 + \frac{c_2 - c_1}{c_2} + \cdots + \frac{c_k - c_{k-1}}{c_k}$

We have: $\rho \leq k$ and $\rho \leq 1 + \int_{c_1}^{c_k} \frac{1}{t}\,\mathrm{d}t = 1 + \ln \frac{c_k}{c_1}$



Theorem

StackMST is ρ-approximable $\Rightarrow \min\{k, 1 + \ln \frac{c_k}{c_1}\}$*-approximable*

Approximation II

The Best-out-of-k algorithm

Algorithm Best-out-of-k (uniform pricing):

- For $i = 1, \ldots, k$: assign price c_i to all blue edges
- Pick i maximizing revenue of the corresponding MST

Approximation II
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$$revenue = MST - MST \cap R \leq MST^{\infty} - MST^{0}$$

$$\Rightarrow OPT \leq MST^{\infty} - MST^{0} = \sum_{i=1}^{k} c_i m_i^{\infty} - \sum_{i=1}^{k} c_i m_i^{0}$$

Where

- m_i^{∞} := #red edges of cost c_i in MST^{∞}
- m_i^{0} := #red edges of cost c_i in MST^{0}

Approximation II

Analysis: wavefront argument

b_i := #blue edges (of cost c_i) in i-th tree

Claim

$m_i^\infty - m_i^0 = b_i - b_{i+1}$ *for all* i

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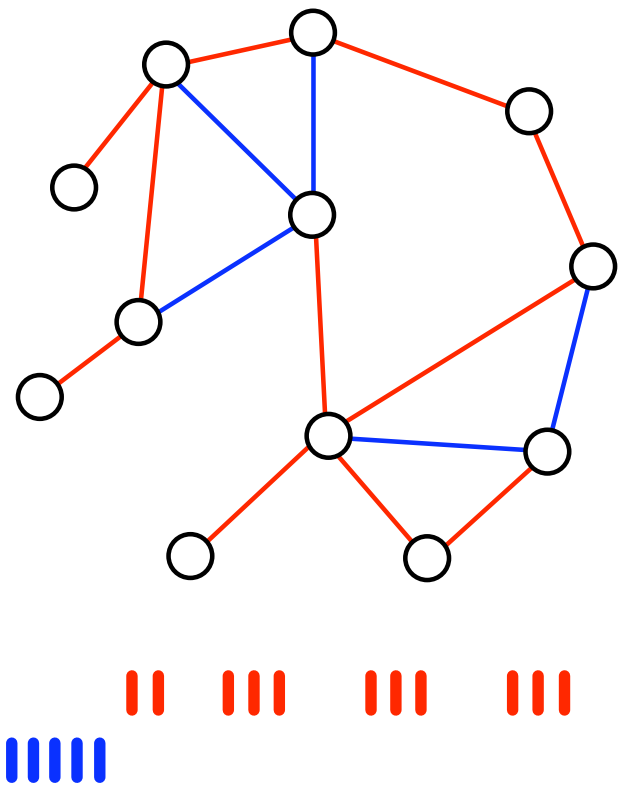

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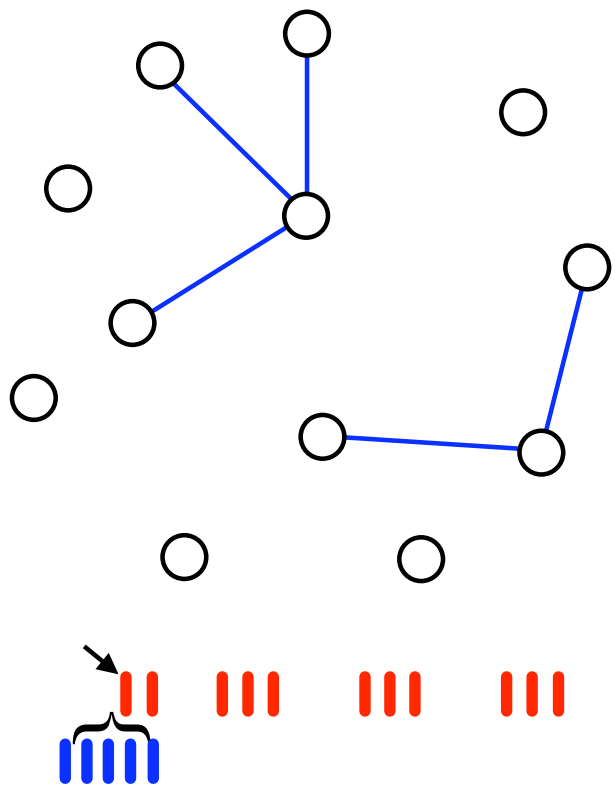

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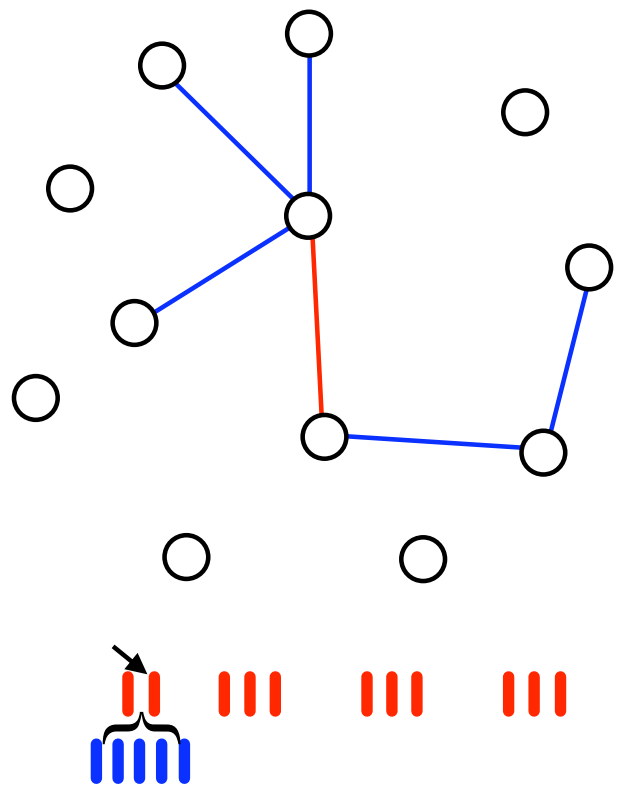

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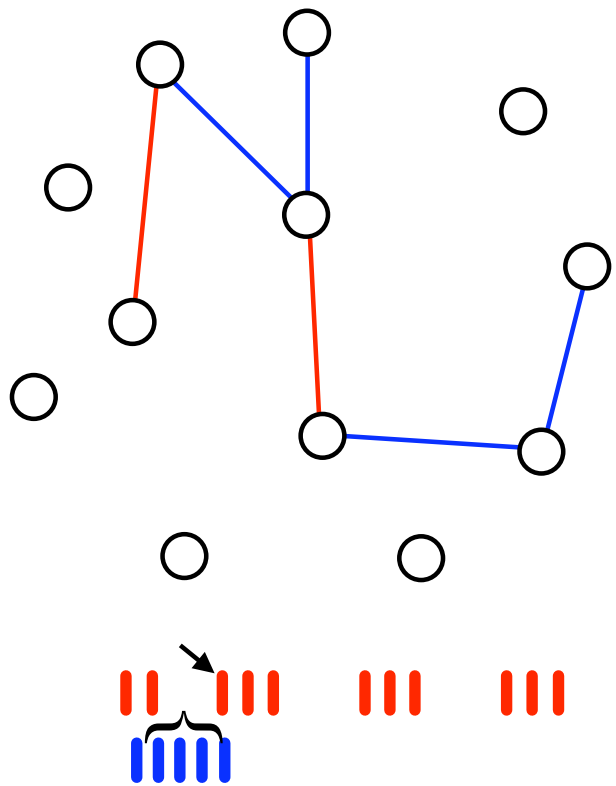

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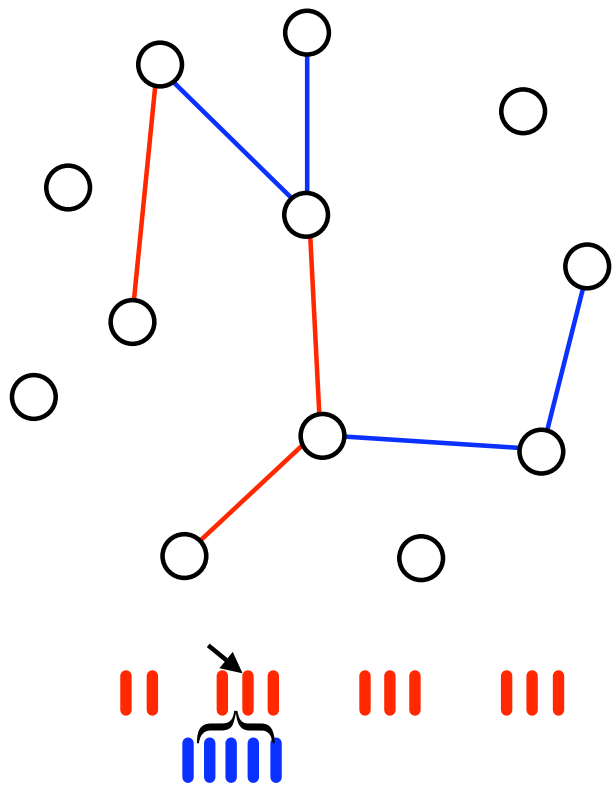

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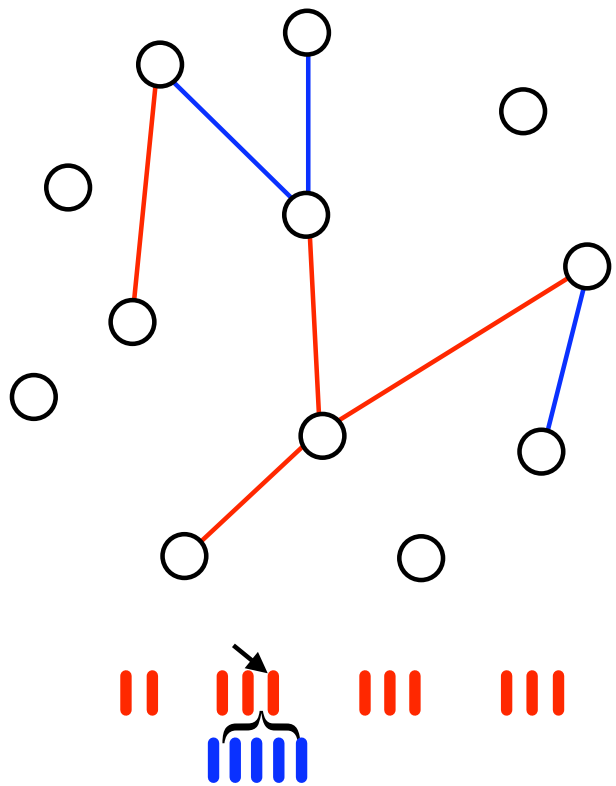

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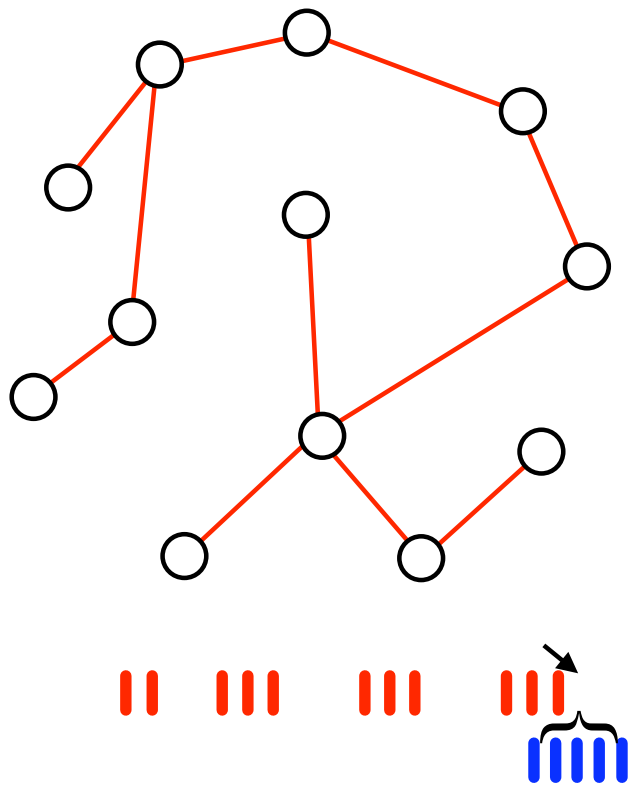

Approximation III

Analysis: rewriting OPT

Denote the revenue of the i-th tree by $q_i := c_i b_i$

From claim, we get

$$m_i^\infty - m_i^0 = \frac{q_i}{c_i} - \frac{q_{i+1}}{c_{i+1}} \text{ for } 1 \leq i \leq k-1 \text{ and } m_k^\infty - m_k^0 = \frac{q_k}{c_k}$$

$\Rightarrow$ Can rewrite upper bound on OPT:

$$\begin{aligned}\sum_{i=1}^{k} c_i(m_i^\infty - m_i^0) &= c_1\left(\frac{q_1}{c_1} - \frac{q_2}{c_2}\right) + c_2\left(\frac{q_2}{c_2} - \frac{q_3}{c_3}\right) + \cdots + c_k\frac{q_k}{c_k} \\ &= q_1 + \frac{c_2 - c_1}{c_2}q_2 + \cdots + \frac{c_k - c_{k-1}}{c_k}q_k\end{aligned}$$

Approximation IV

Analysis: final bit

Taking j such that q_j is maximum, we have

$$\begin{aligned}\frac{\mathrm{OPT}}{q_j} &\leq \frac{\sum_{i=1}^{k} c_i(m_i^{\infty} - m_i^{0})}{q_j} \\ &\leq \frac{q_1}{q_j} + \frac{c_2 - c_1}{c_2} \cdot \frac{q_2}{q_j} + \cdots + \frac{c_k - c_{k-1}}{c_k} \cdot \frac{q_k}{q_j} \\ &\leq 1 + \frac{c_2 - c_1}{c_2} + \cdots + \frac{c_k - c_{k-1}}{c_k} = \rho\end{aligned}$$

So Best-out-of-k is a ρ-approximation algorithm

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Can tweak the analysis to get:

Proposition

Best-out-of-k is also a $(3 + 2 \ln b)$*-approximation*

And they lived happily ever after...

IP formulation

$$(\text{IP}) \xrightarrow{\text{relax}} (\text{LP})$$

Proposition (**good news**)

We have $(IP) \leq (LP) \leq MST^{\infty} - MST^{0}$

Proposition (**bad news**)

The integrality gap of (LP) is k *on instances with* k *distinct costs (and with* $\rho \sim k$*)*

Open questions / outlook

Main open question:

- Does StackMST admit a constant-factor approximation?

Generalizations of StackMST:

- Pricing matroids: always ρ-approximable
- Stackelberg Steiner trees: also generalizes highway pricing

The End

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PENNY WISE
DEPTH GAUGE 20 ft
CARL BARKS
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