# Longest common extensions in trees

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Preprocess a given string $s[1..n]$ for computing the longest common prefix of $s[i..n]$ and $s[j..n]$.

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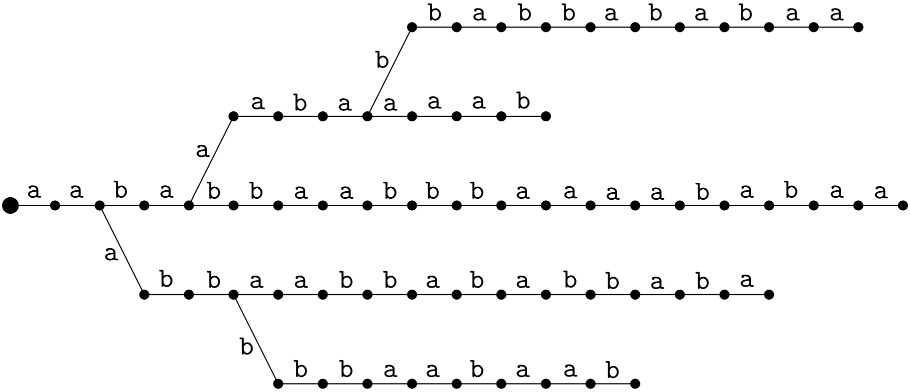


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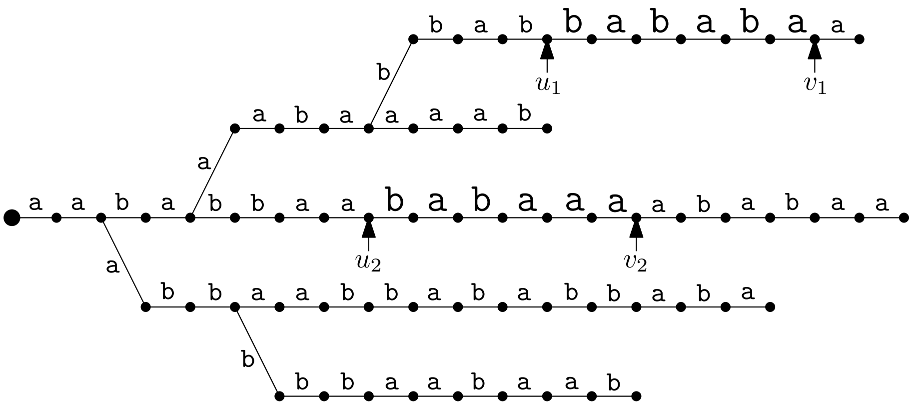


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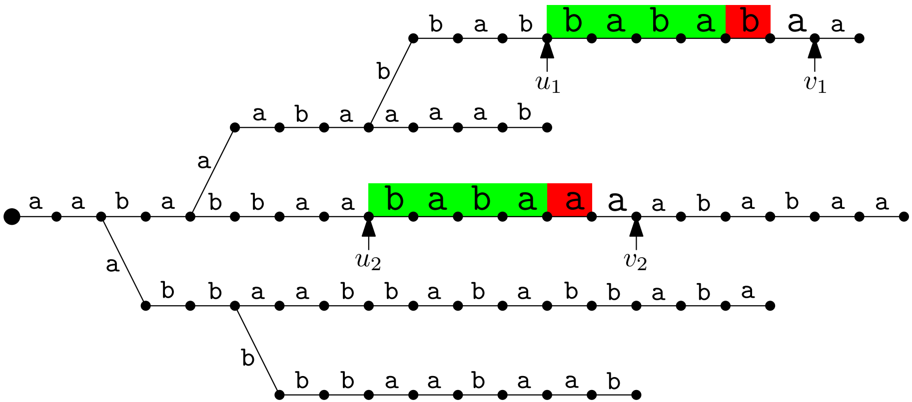


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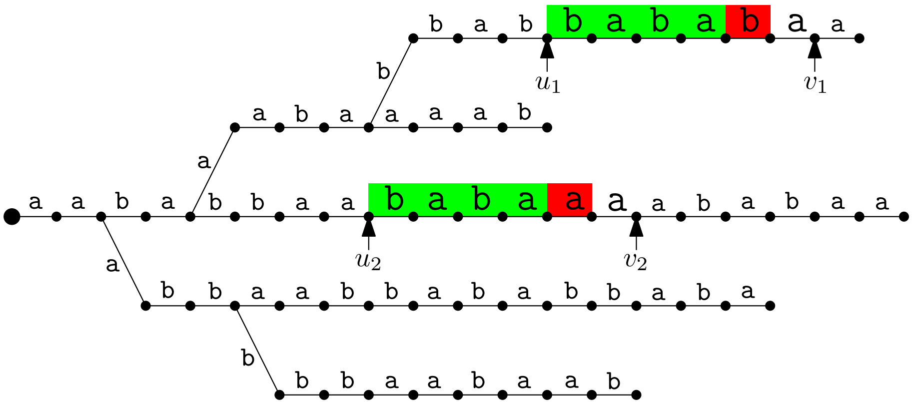

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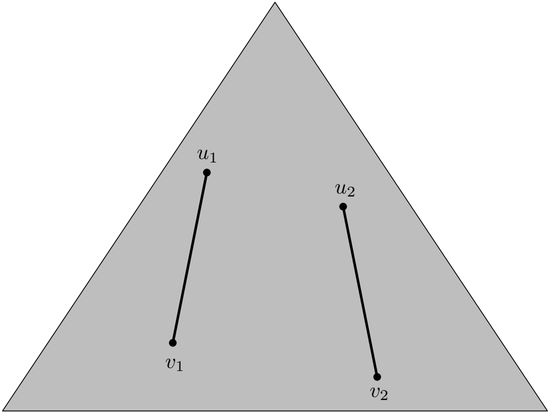
# Path-path queries

Given nodes $u_1, v_1, u_2, v_2$ such that $u_1$ is an ancestor of $v_1$ and $u_2$ is an ancestor of $v_2$, report the longest matching prefix of paths $u_1 \rightsquigarrow v_1$ and $u_2 \rightsquigarrow v_2$.




Without losing the generality, the paths are of the same length.

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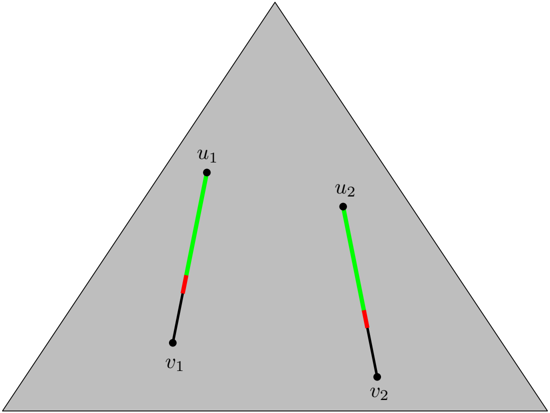


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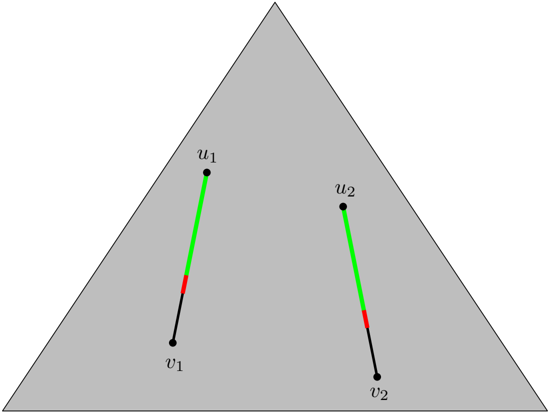


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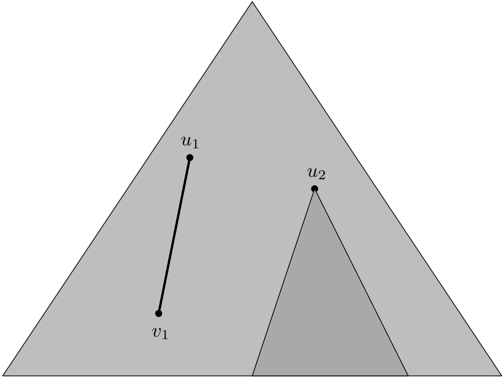


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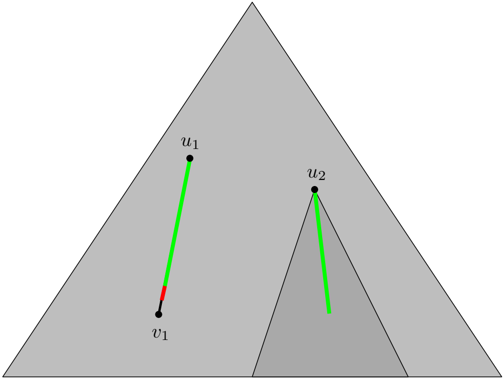


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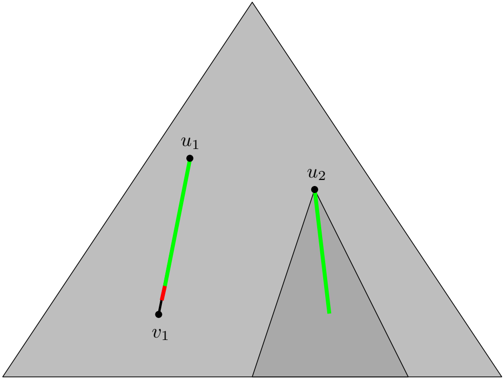


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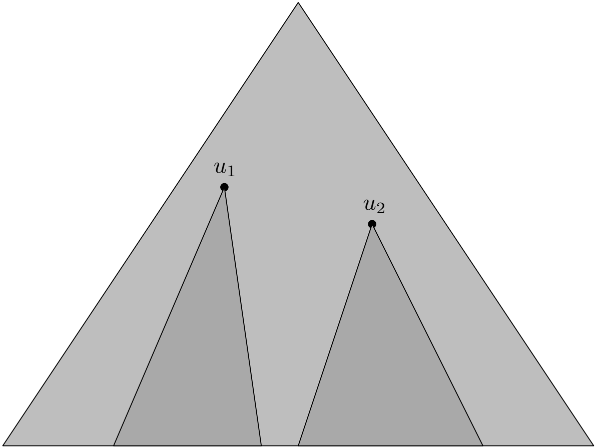
# Tree-tree queries

Given nodes $u_1, u_2$, report the longest matching prefix of paths $u_1 \leadsto v_1$ and $u_2 \leadsto v_2$, where $v_1$ is a descendant of $u_1$ and $v_2$ is a descendant of $u_2$.



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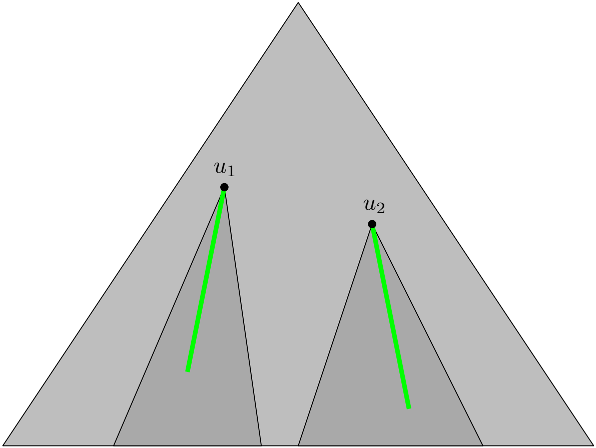

# Results

| problem | query | space | lowerbound |
|---|---|---|---|
| path-path | $\mathcal{O}(\log^* n)$ | $\mathcal{O}(n)$ | |
| path-tree | $\mathcal{O}((\log \log n)^2)$ | $\mathcal{O}(n)$ | predecessor hard |
| tree-tree | $\mathcal{O}(n/\tau)$ | $\mathcal{O}(n \cdot \tau)$ | set-intersection hard |

# Simple solution for path-path queries

We start with a simple $\mathcal{O}(1)$ query $\mathcal{O}(n \log n)$ space solution. For every $k = 0, 1, \ldots, \log n$, we build a separate structure of size $\mathcal{O}(n)$ allowing us to answer queries for paths of length $2^k$.

## Structure for paths of length $2^k$

There are only $n$ such paths. We sort them (lexicographically) and store the longest common prefix between any two paths adjacent in the sorted order. Longest common prefix of any two paths can be computing with a range minimum query on the stored numbers.

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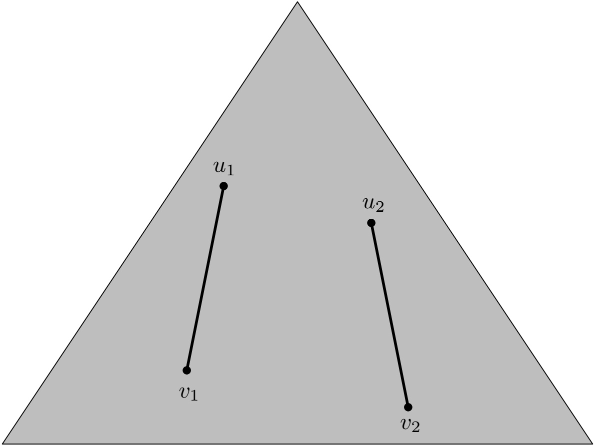
Now, given an arbitrary query where the paths are of (the same) length $\ell$, we can reduce it to two queries where the paths are of length $2^k$, where $k = \lfloor \log \ell \rfloor$.



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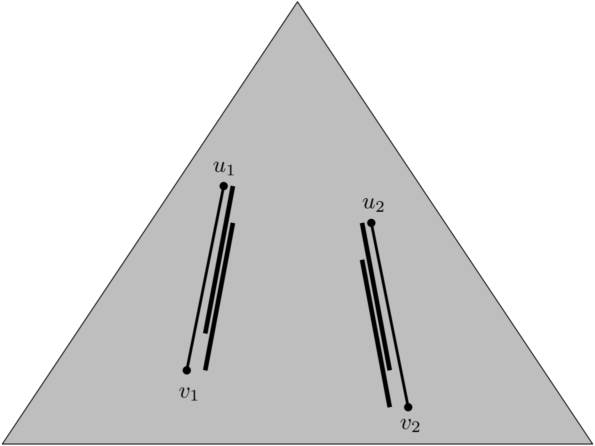

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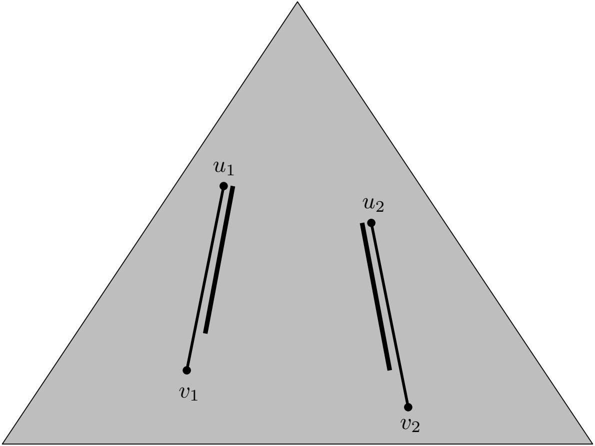

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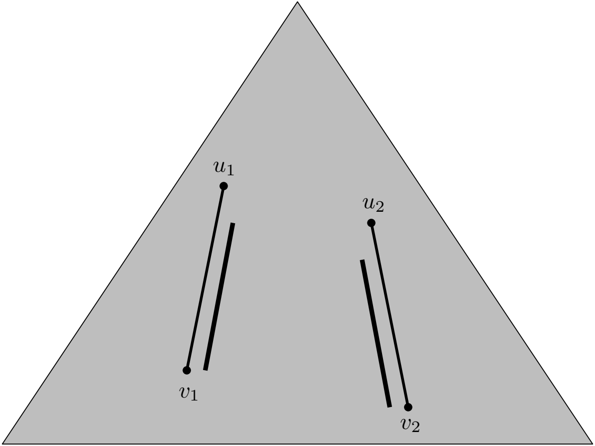

# Reducing the space

The natural approach is to store just some of the paths of length $2^k$, for every $k$. To choose which paths to store, we introduce the notion of **difference covers for trees**.

## Difference covers for trees

For any tree on $n$ nodes and a parameter $x$, it is possible to mark $\frac{2n}{x}$ nodes, so that for any $u, v$ at depths $\geq x^2$, there exists $\Delta \leq x^2$ such that the $\Delta$-th ancestors of both $u$ and $v$ are marked.

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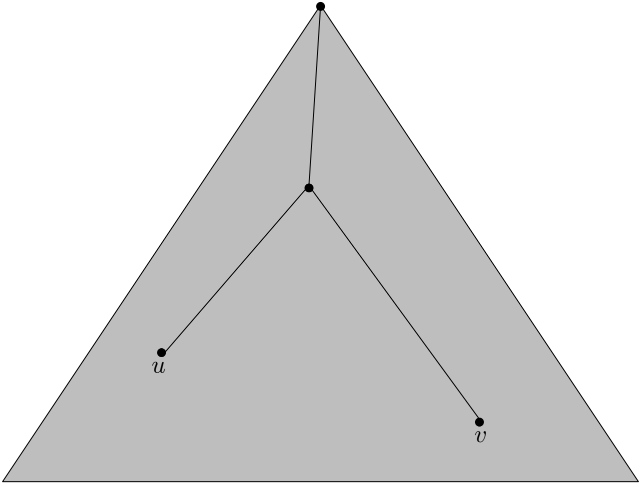
$\frac{2n}{x}$ marked nodes, $\Delta \leq x^2$



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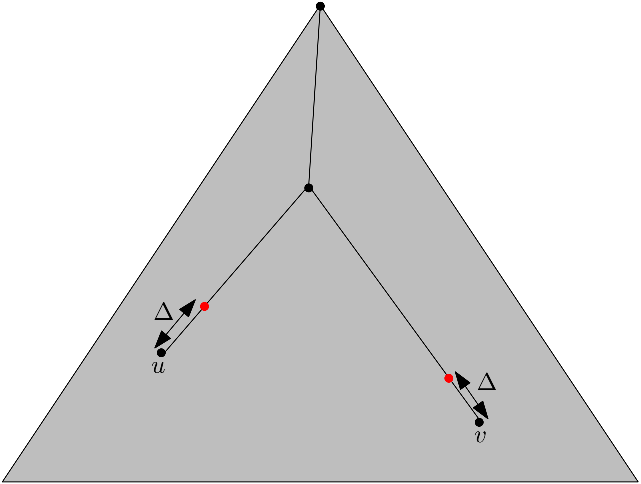

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Find a difference cover with $x = \log n$. Then, for every $k = 0, 1, \ldots, \log n$ preprocess all paths of length $2^k$ ending at marked nodes. Additionally, preprocess all paths of length $\log^2 n$.

Any query can be reduced in $\mathcal{O}(1)$ time to computing the longest common prefix of two paths of length $\leq \log^2 n$.

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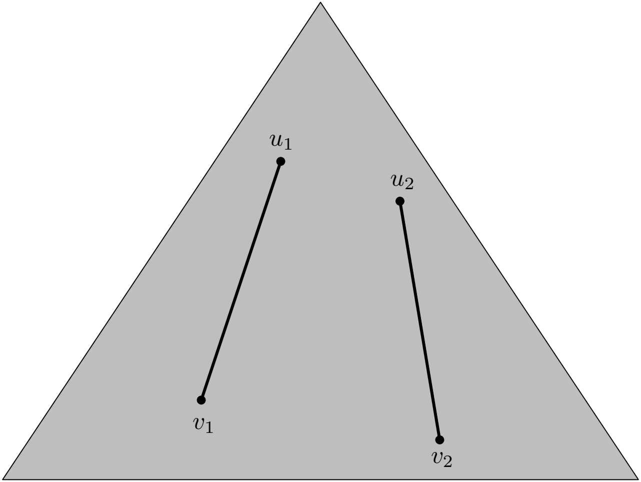

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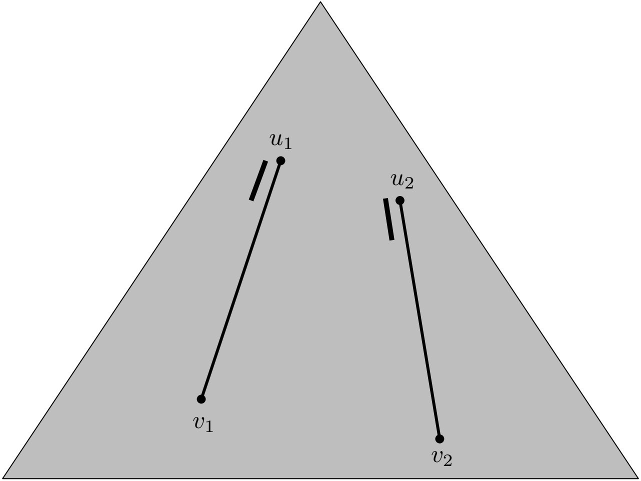

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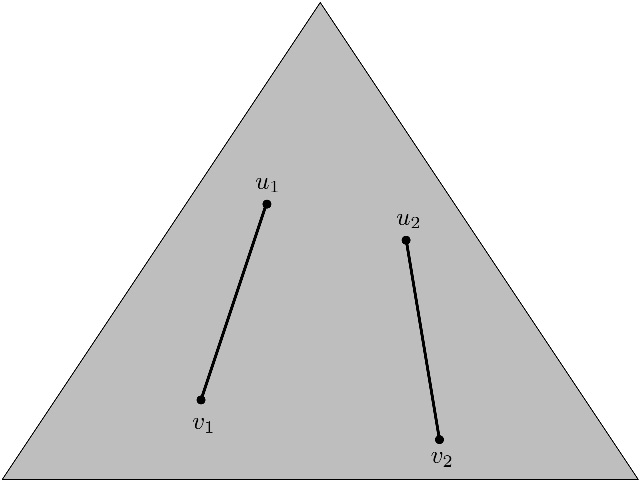
Cover the remaining part with paths of length $2^k$.



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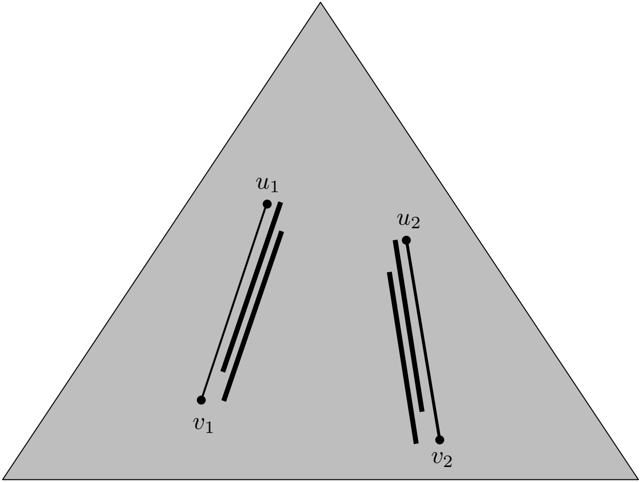

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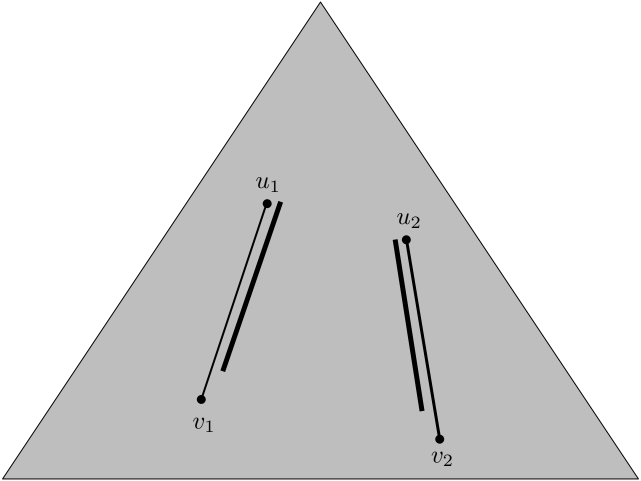
Slide the first pair of paths up so that they end at marked nodes.



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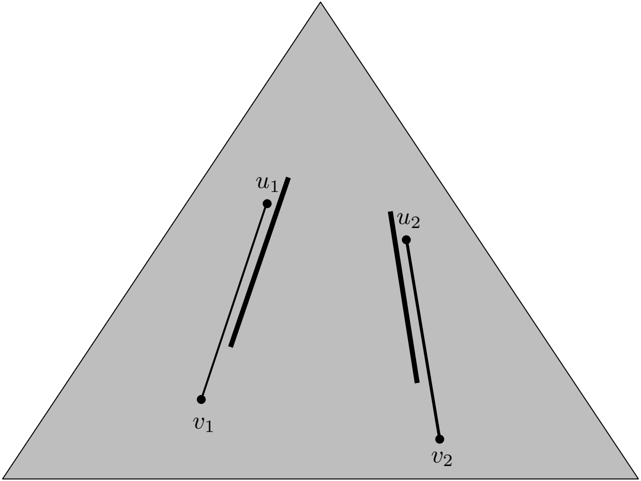

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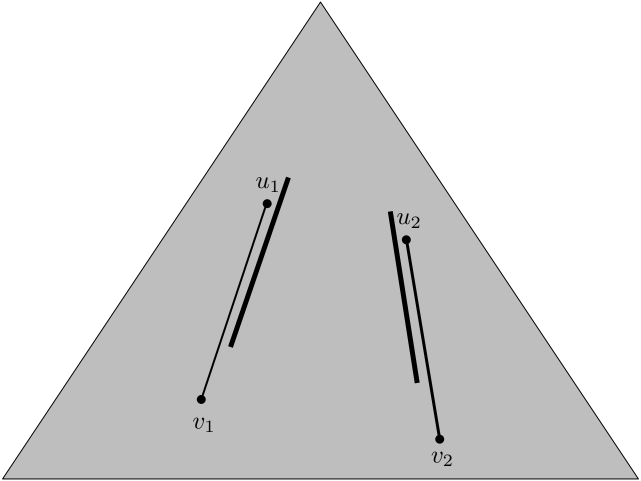
Process the second pair of paths similarly.



# Reducing the space

After $\mathcal{O}(n)$ preprocessing we can reduce any query to computing the longest common prefix of two paths of length $\leq \log^2 n$. Now the reasoning can be iterated, so in the next step we get two paths of length $\leq \log^2(\log^2 n)$, and so on.

The number of iterations is at most $\mathcal{O}(\log^* n)$, so after $\mathcal{O}(n \log^* n)$ preprocessing we can answer any query in $\mathcal{O}(\log^* n)$ time.

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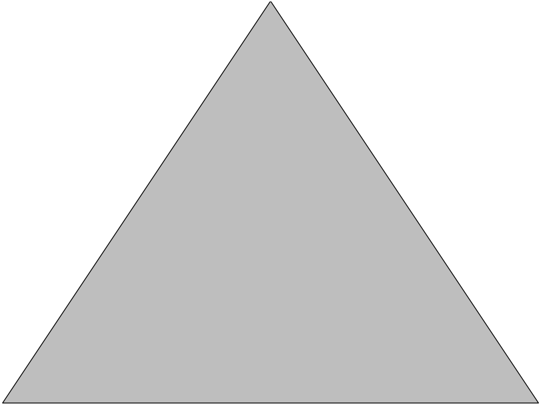
Generalizing the construction used for strings, the first step is to mark every node at depth $0, x, 2x, 3x, \ldots$.



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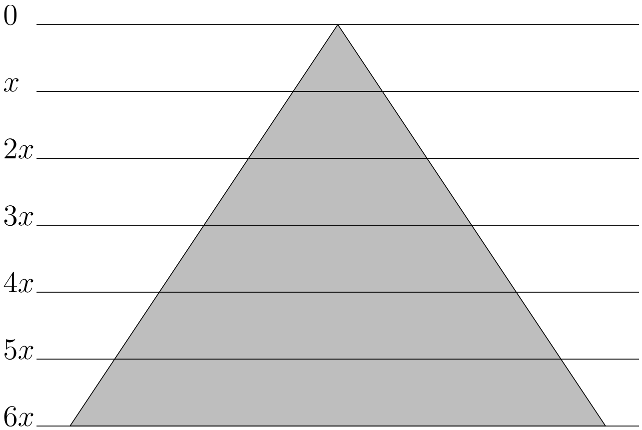

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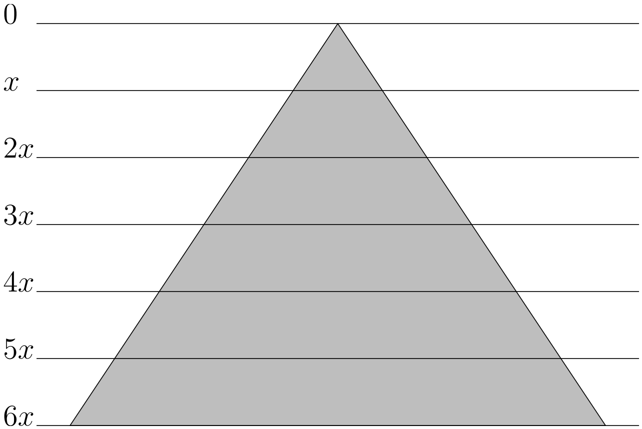
Doesn't quite work, because we cannot guarantee that there are $\frac{n}{x}$ such nodes.



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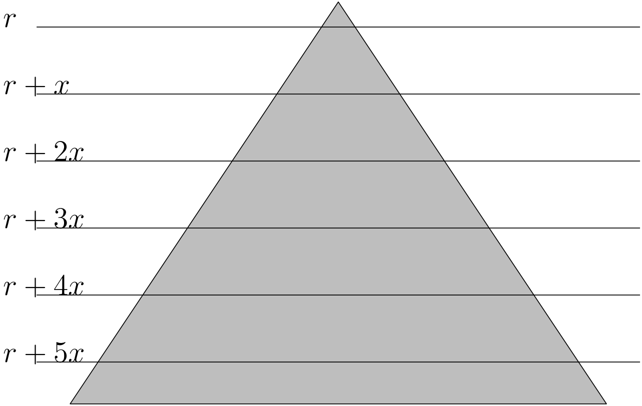
But marking every node at depth $r, r+x, r+2x, r+3x, \ldots$ is also enough for our purposes, where $r \in \{0, 1, \ldots, x-1\}$.



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For at least one $r \in \{0, 1, \ldots, x-1\}$ the number of marked nodes is at most $\frac{n}{x}$.

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Also uses difference covers for trees, but now there are $\log\log n$ iterations, and every of them needs predecessor search, hence the query time is $\mathcal{O}((\log\log n)^2)$.

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By reduction from the predecessor problem, we show that any structure of size $\tilde{\mathcal{O}}(n)$ needs $\Omega(\log\log n)$ time to answer a query.

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# Open problems

1. Remove $\mathcal{O}(\log^* n)$ from the path-path complexity.
2. Decrease $\mathcal{O}((\log\log n)^2)$ to $\mathcal{O}(\log\log n)$ for path-tree queries.
3. Reduce tree-tree queries to set-intersection queries.
4. Other applications of difference covers for trees?

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