Voronoi Diagrams on Planar Graphs and Computing the Diameter in Deterministic $\tilde{O}(n^{5/3})$ time

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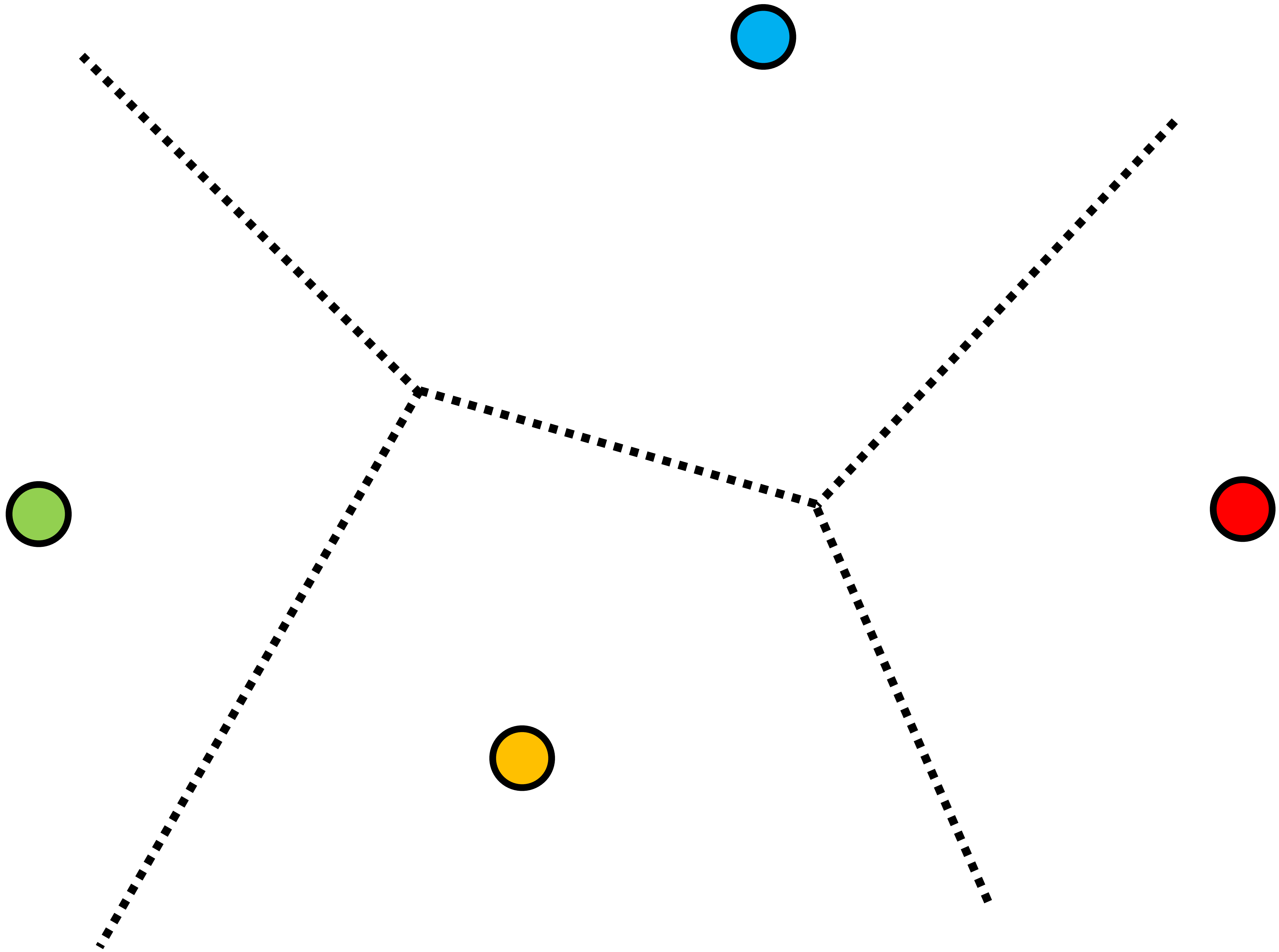
Voronoi diagrams

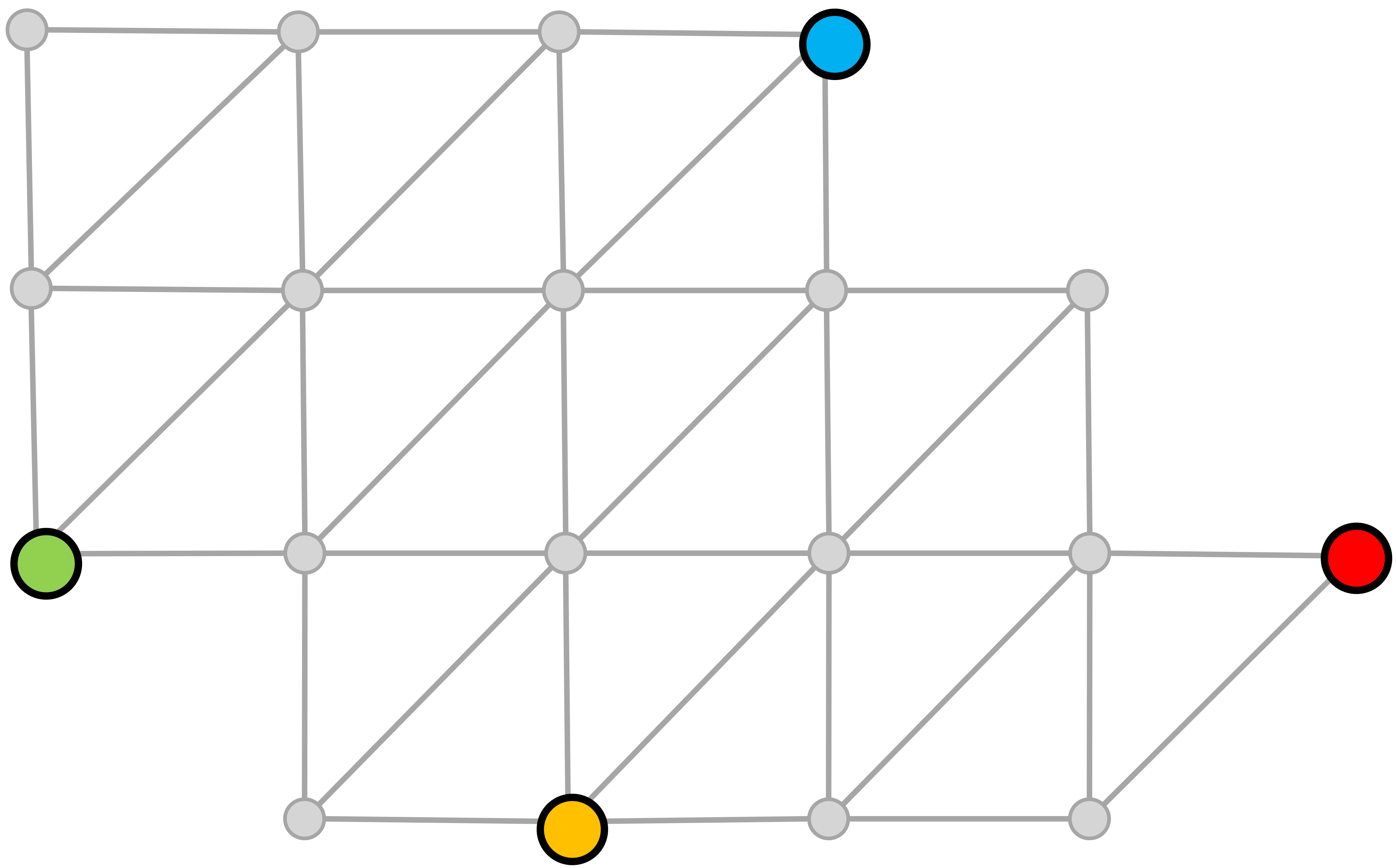


Voronoi
1908

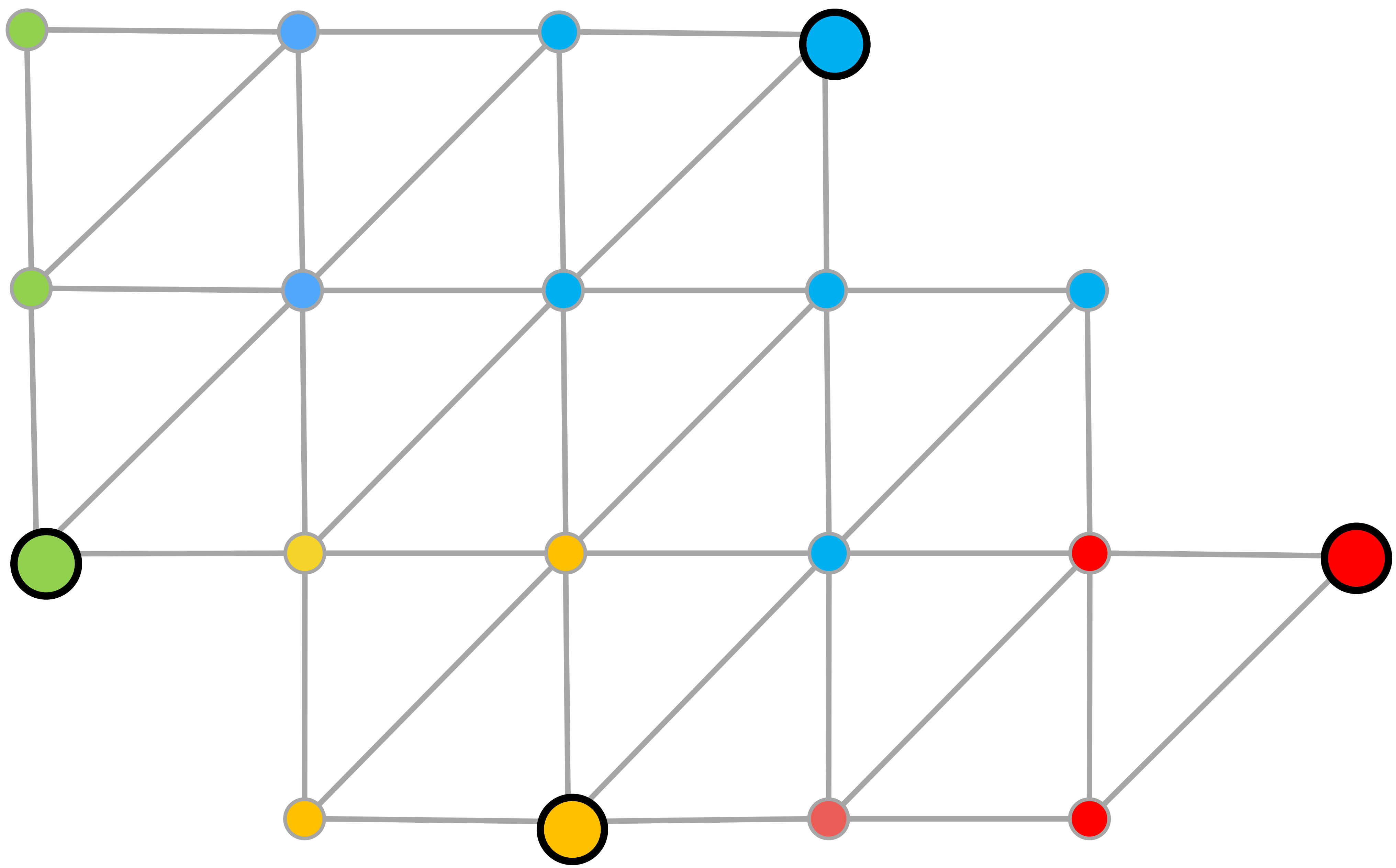
Descartes
1644

Dirichlet
1850

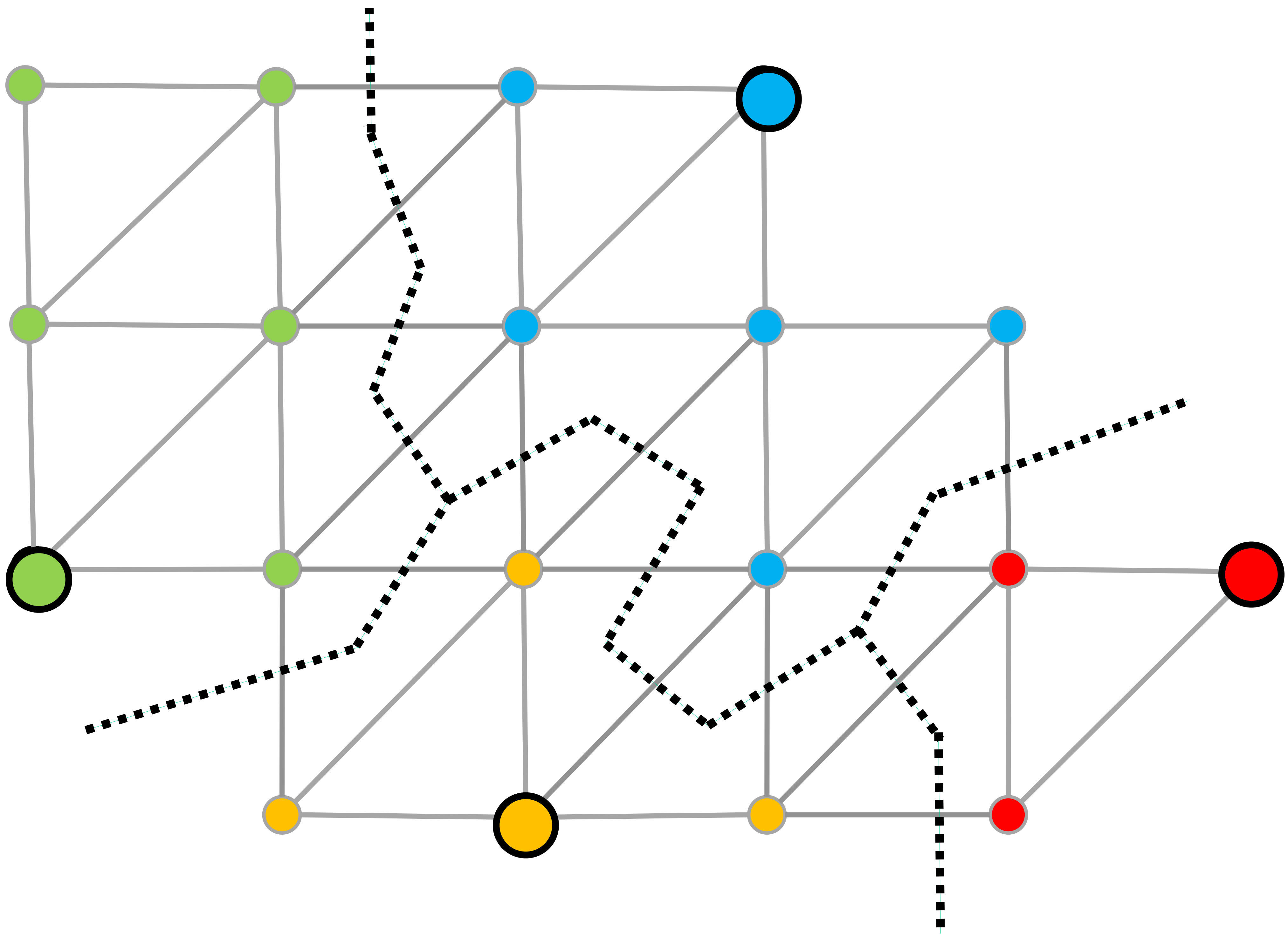
Voronoi diagrams on planar graphs



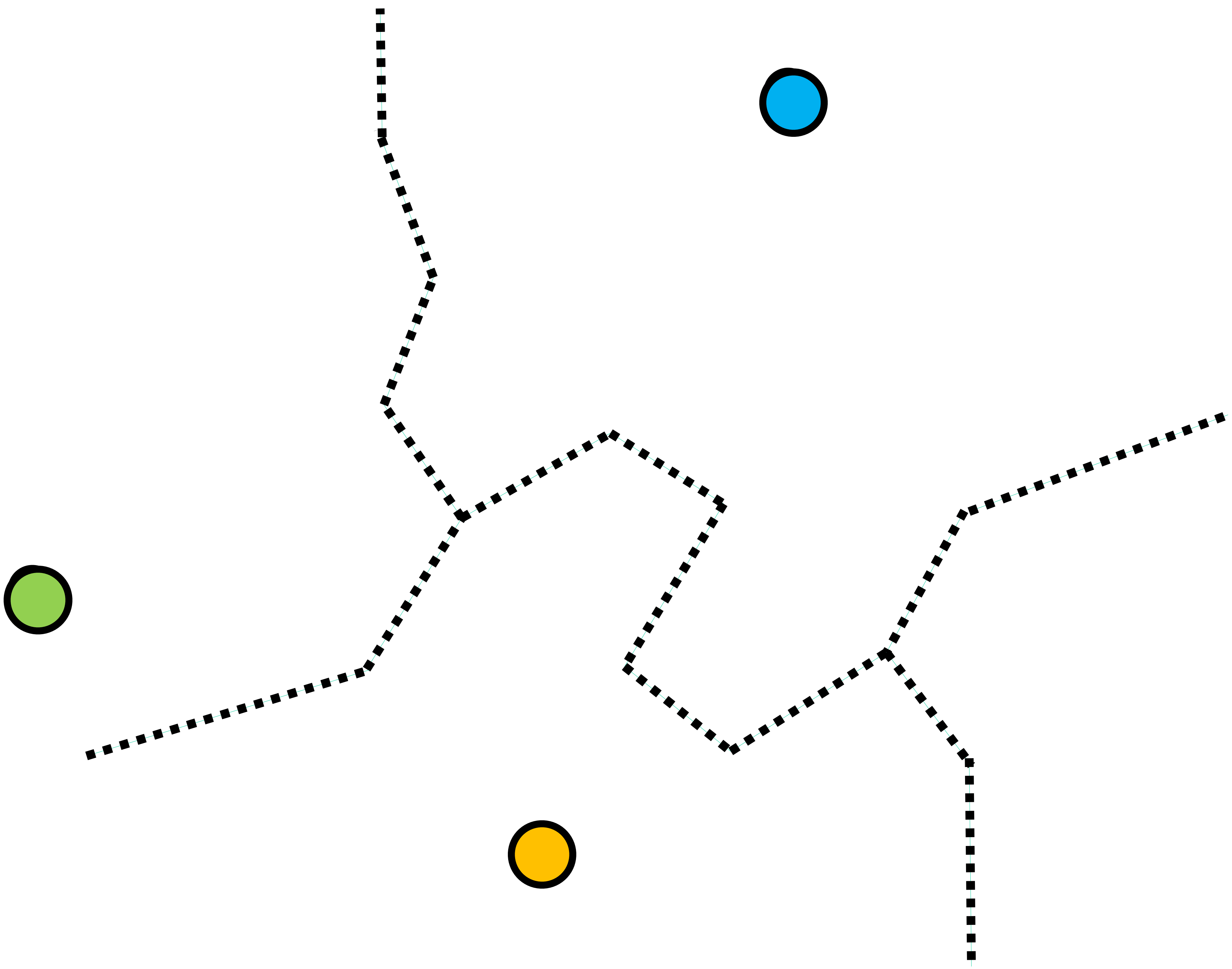
Voronoi diagrams on planar graphs



Voronoi diagrams on planar graphs



Voronoi diagrams on planar graphs



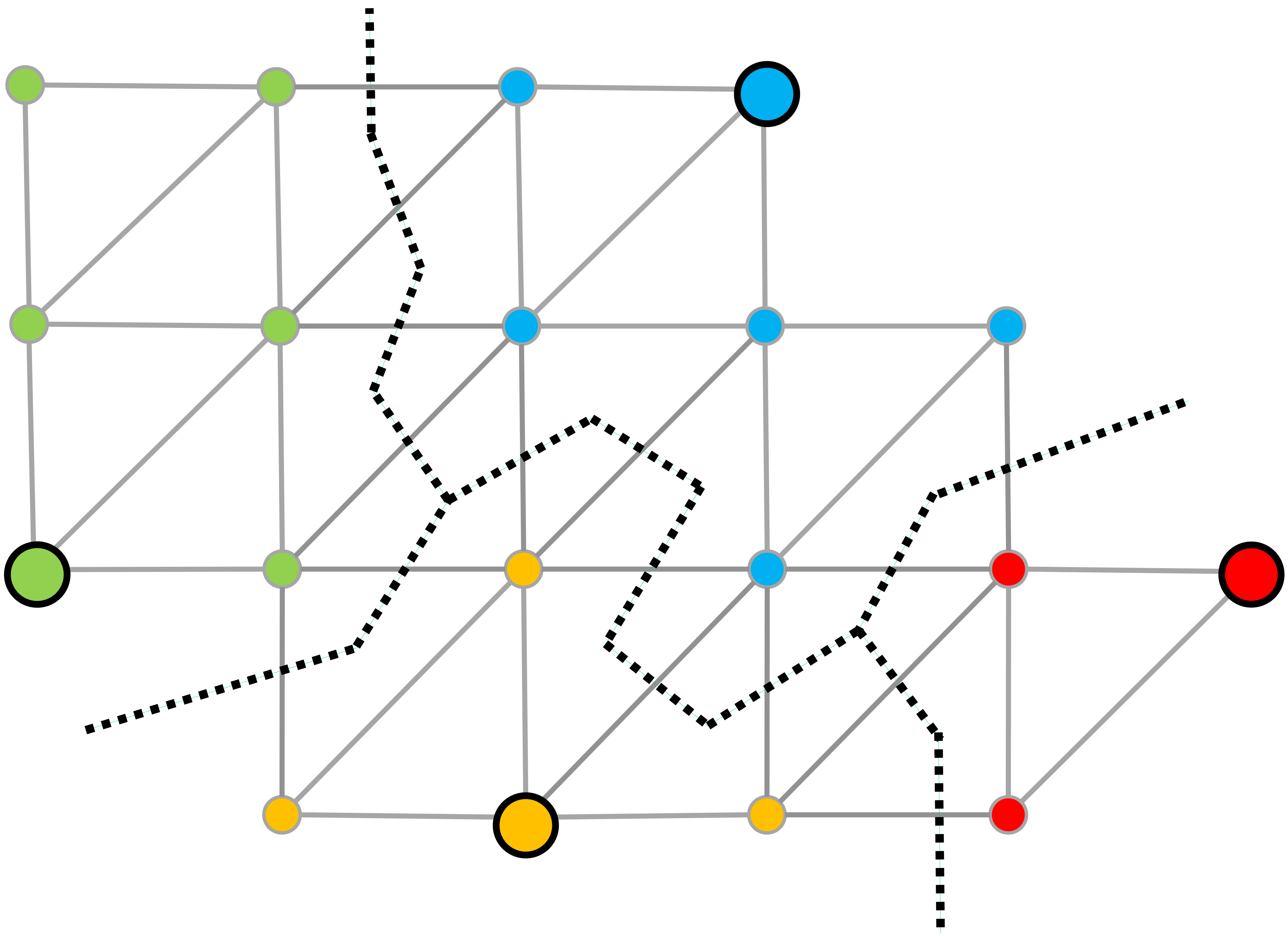
A fresh idea in algorithms for planar graph

- Cabello's breakthrough (best paper in SODA 2017) -
  - can quickly construct Voronoi diagrams on planar graphs
    (using randomized incremental construction of abstract Voronoi diagrams)
  - can use Voronoi diagrams to compute the diameter in sub-quadratic $\tilde{O}(n^{11/6})$ randomized time
- Led to exciting developments in distance oracles for planar graphs [Cohen-Addad et al. FOCS17, next talk]

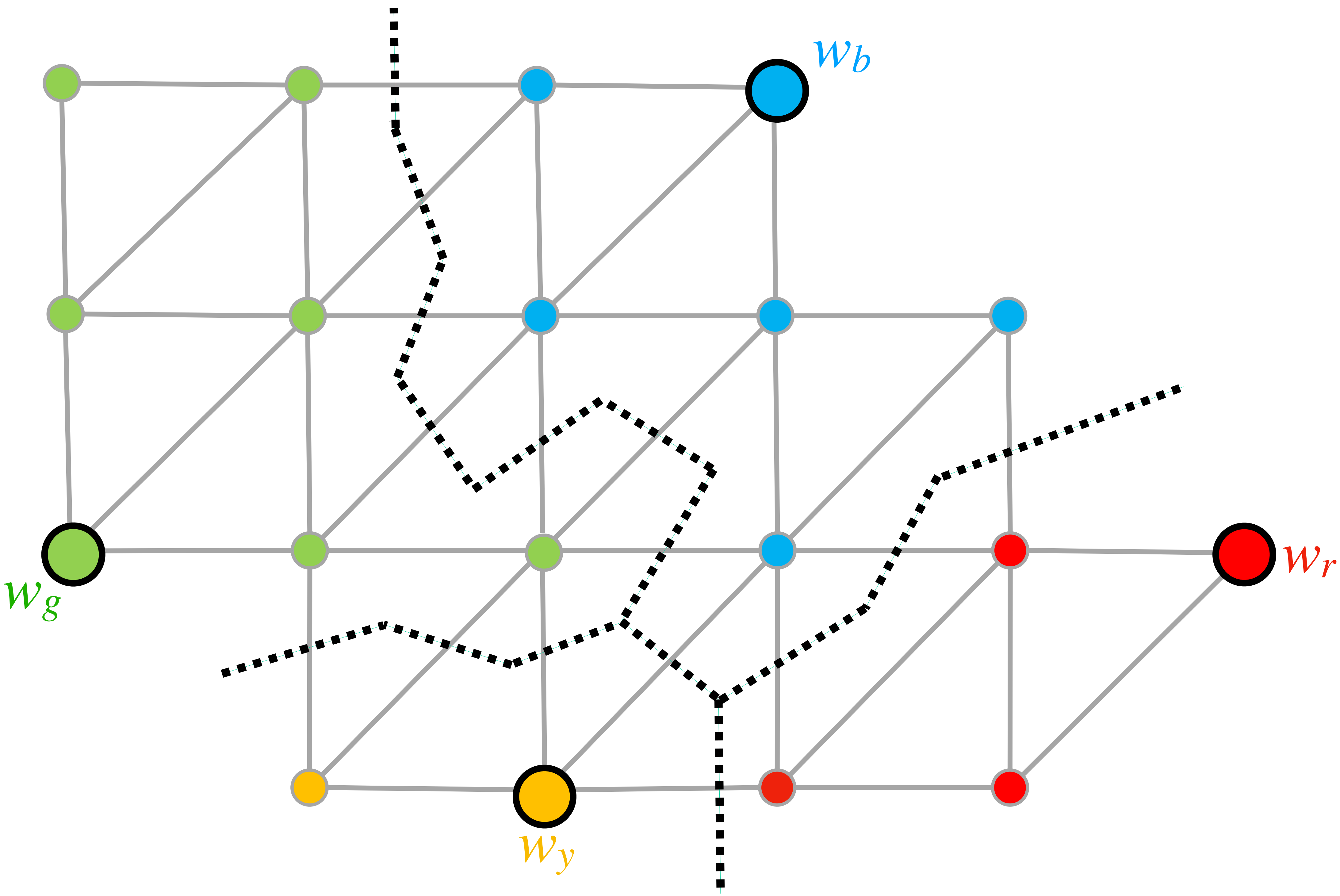
This work

- construction of Voronoi diagrams on planar graphs
  - faster, deterministic, more general
- leads to a faster $O(n^{5/3})$-time algorithm for diameter

Voronoi diagrams on planar graphs

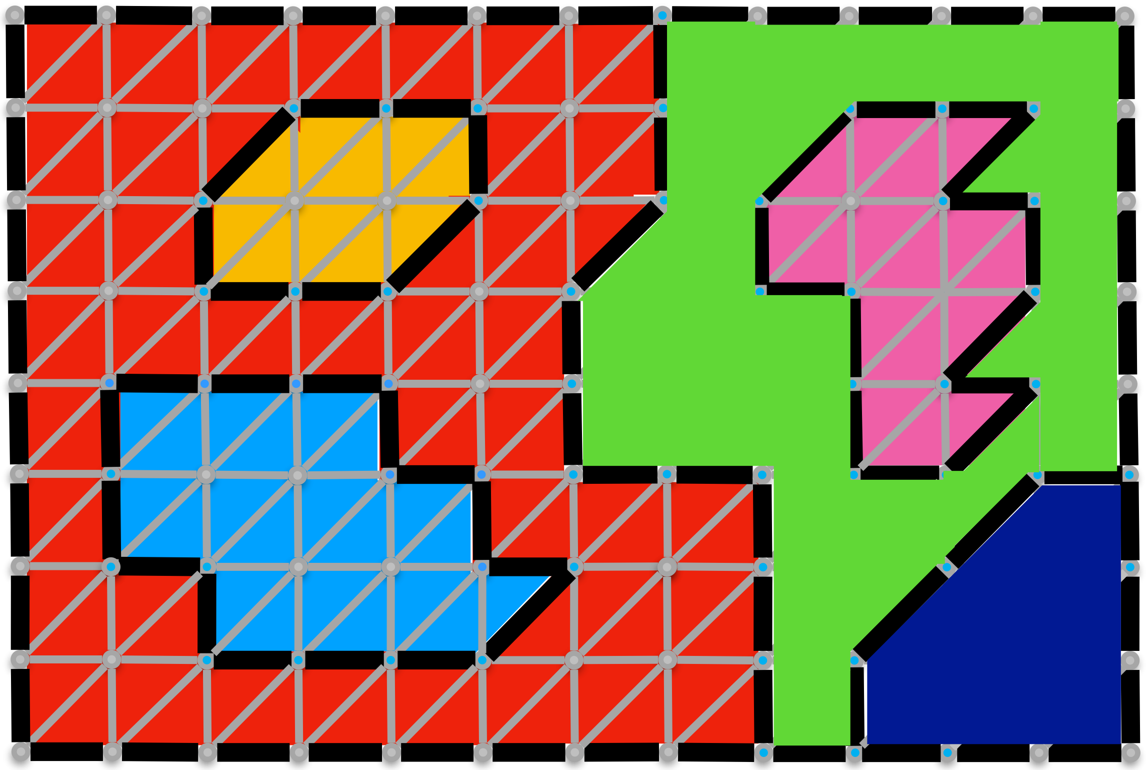


Additively weighted Voronoi diagram



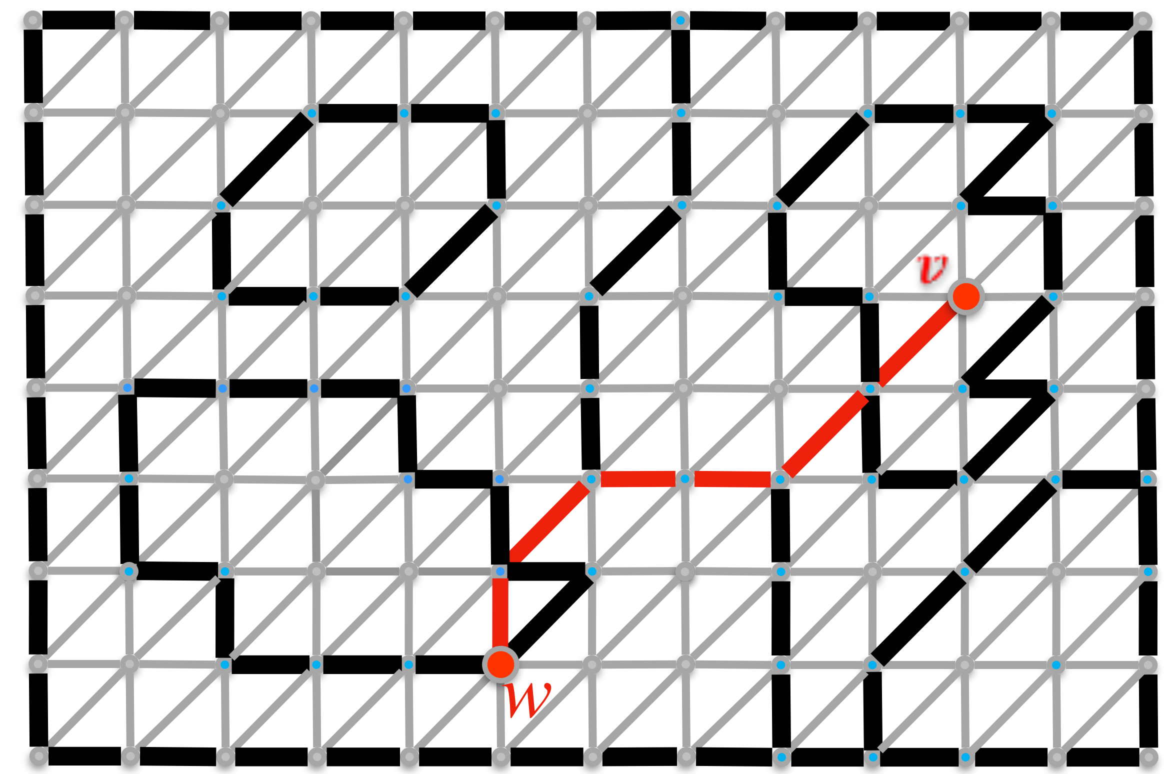
High-level approach for diameter

- compute an r-division: $O(n/r)$ pieces, each with $O(r)$ vertices and $O(r^{1/2})$ boundary vertices



- there are three types of distances:

High-level approach for diameter



- compute an r-division: $O(n/r)$ pieces, each with $O(r)$ vertices and $O(r^{1/2})$ boundary vertices

- there are three types of distances:
  - between a vertex and a boundary vertex

High-level approach for diameter

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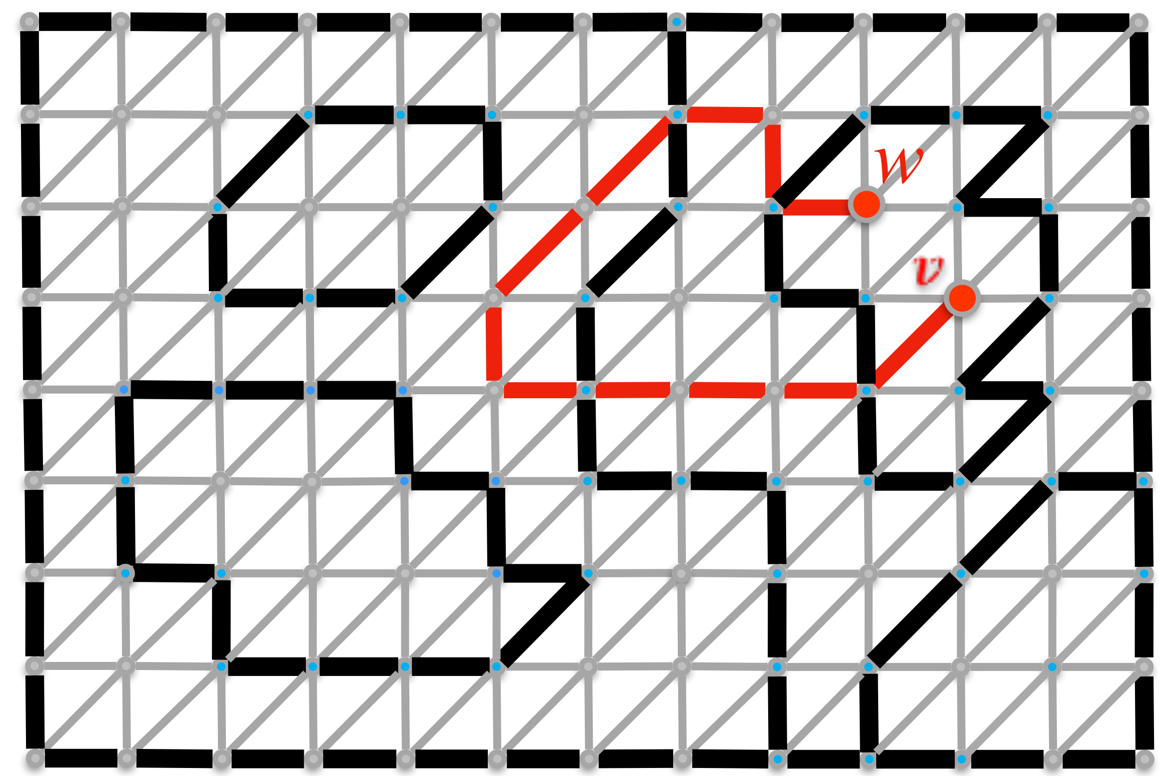

- there are three types of distances:
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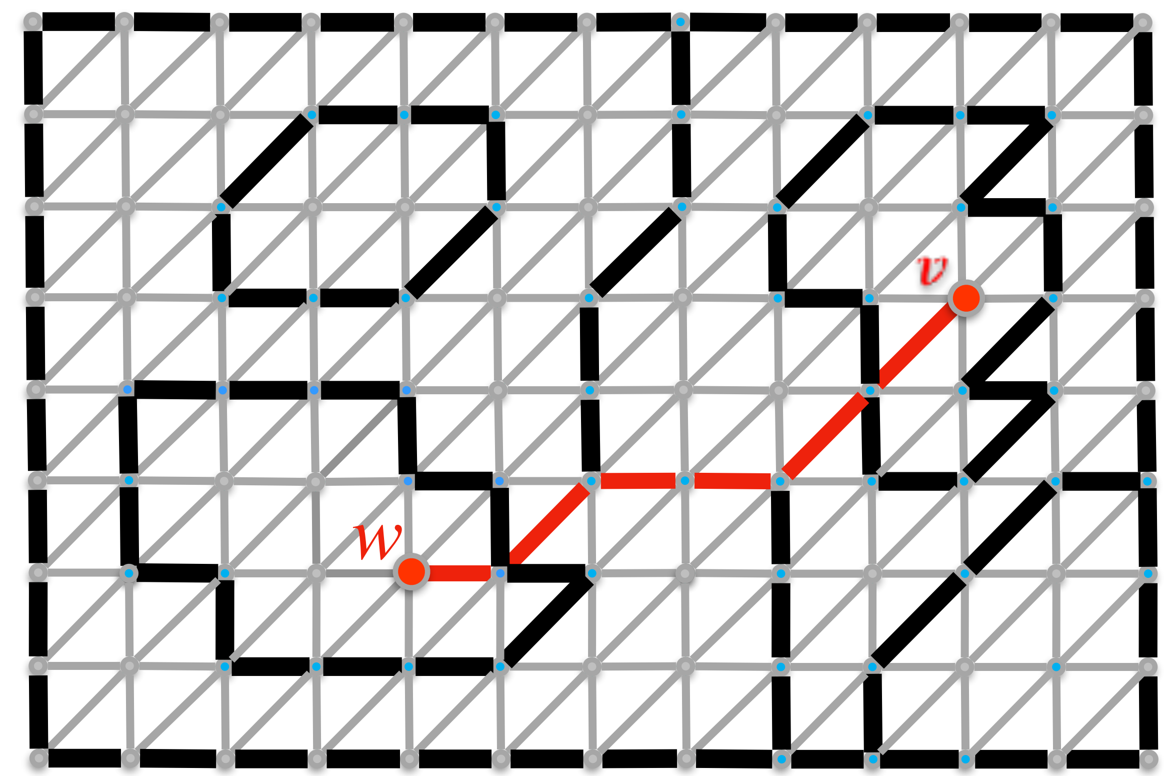
  - between two vertices in the same piece

High-level approach for diameter



- compute an r-division: $O(n/r)$ pieces, each with $O(r)$ vertices and $O(r^{1/2})$ boundary vertices

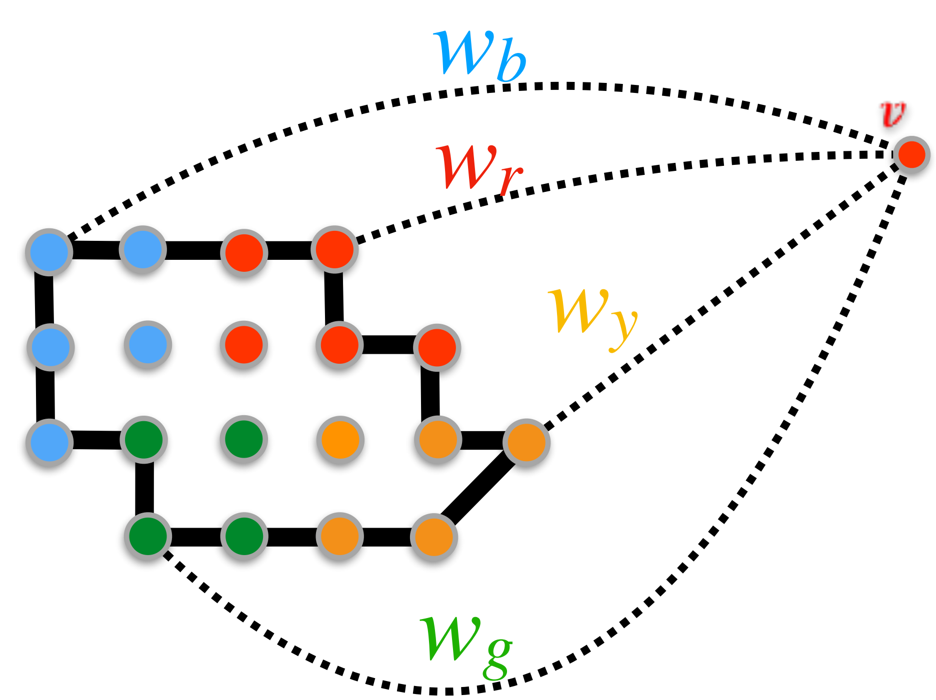
- there are three types of distances:
  - between a vertex and a boundary vertex
  - between two vertices in the same piece
  - between two vertices in different pieces

Dist. between vertices in different pieces

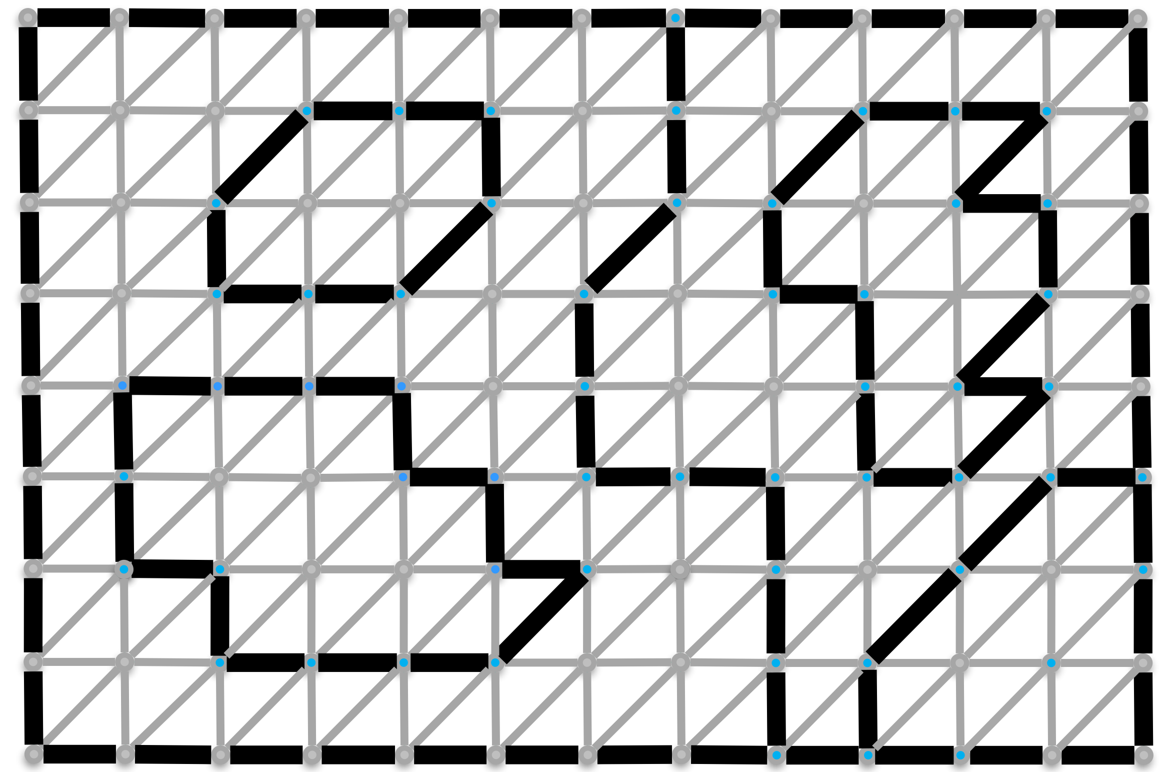
- already computed distances from v to boundary nodes of the other piece
- compute additively weighted Voronoi diagram for the other piece in $\tilde{O}(r^{1/2})$ time
- use Voronoi diagram to return the node furthest from each boundary site in $\tilde{O}(1)$ amortized time per site
- total $\tilde{O}(n \cdot n/r \cdot r^{1/2}) = \tilde{O}(n^2/r^{1/2})$

  # vertices   # pieces

- requires $\tilde{O}(n/r \cdot r^2) = O(nr)$ preprocessing



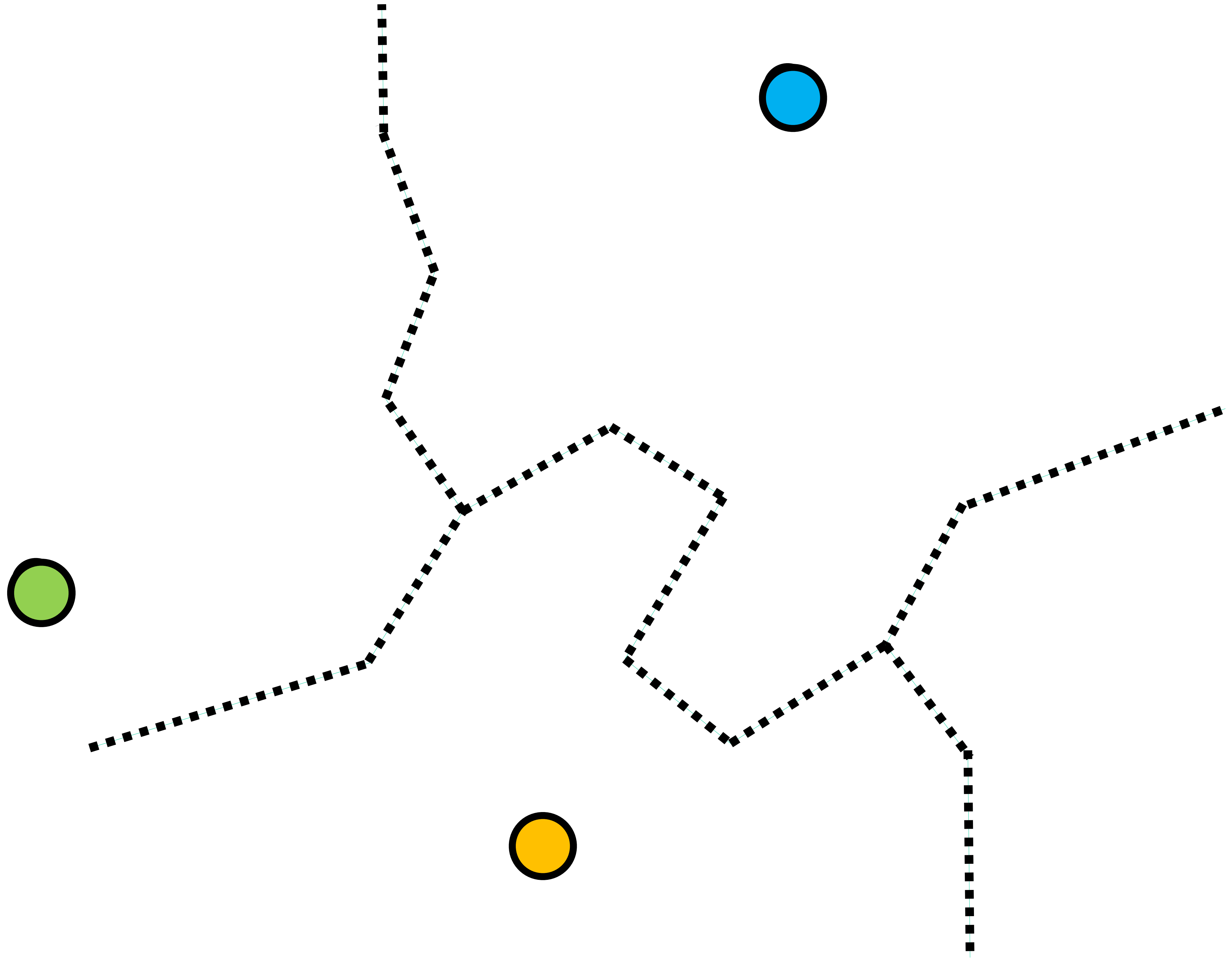
High-level approach for diameter



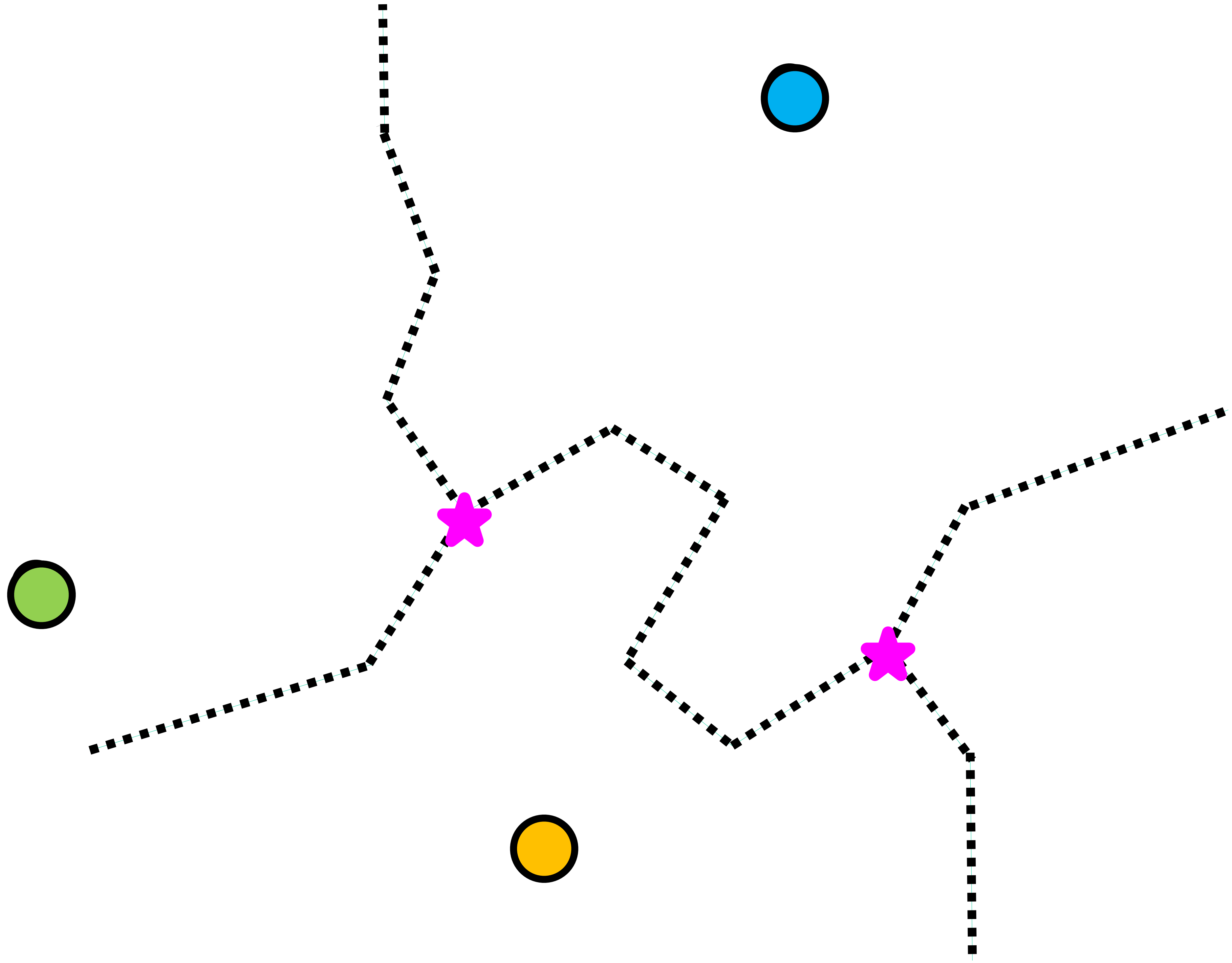
- compute an r-division
- three types of distance:
  - between a vertex and a boundary vertex — $O(n^2/r^{1/2})$ time
  - between two vertices inside the same piece — $O(nr)$ time
  - between two vertices in different pieces — $\tilde{O}(nr + n^2/r^{1/2})$ time

- setting $r = n^{2/3}$ yields total running time of $\tilde{O}(n^{5/3})$

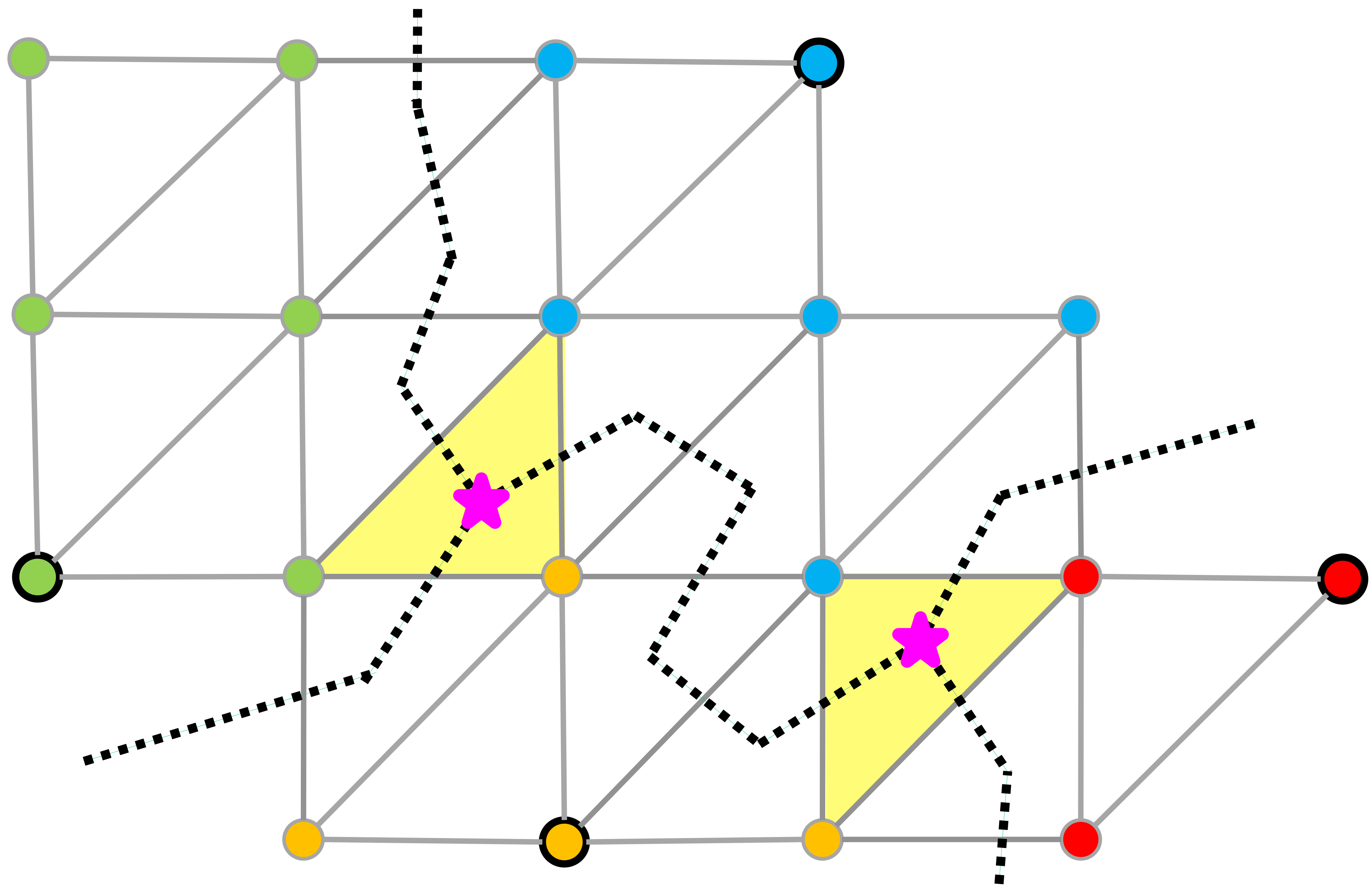
Remainder of the talk: constructing Voronoi diagrams



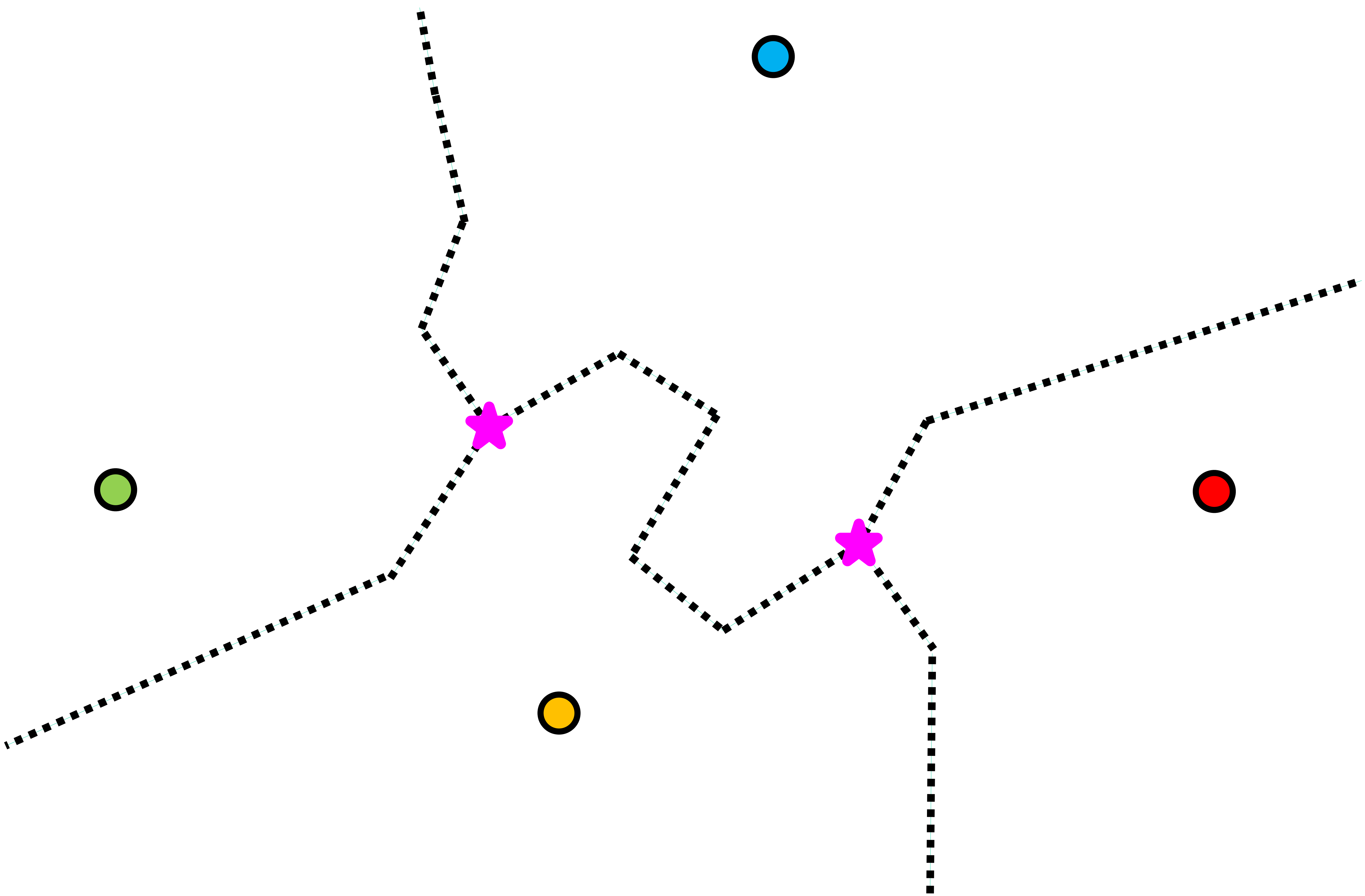
Voronoi vertices - adjacent to three different Voronoi cells



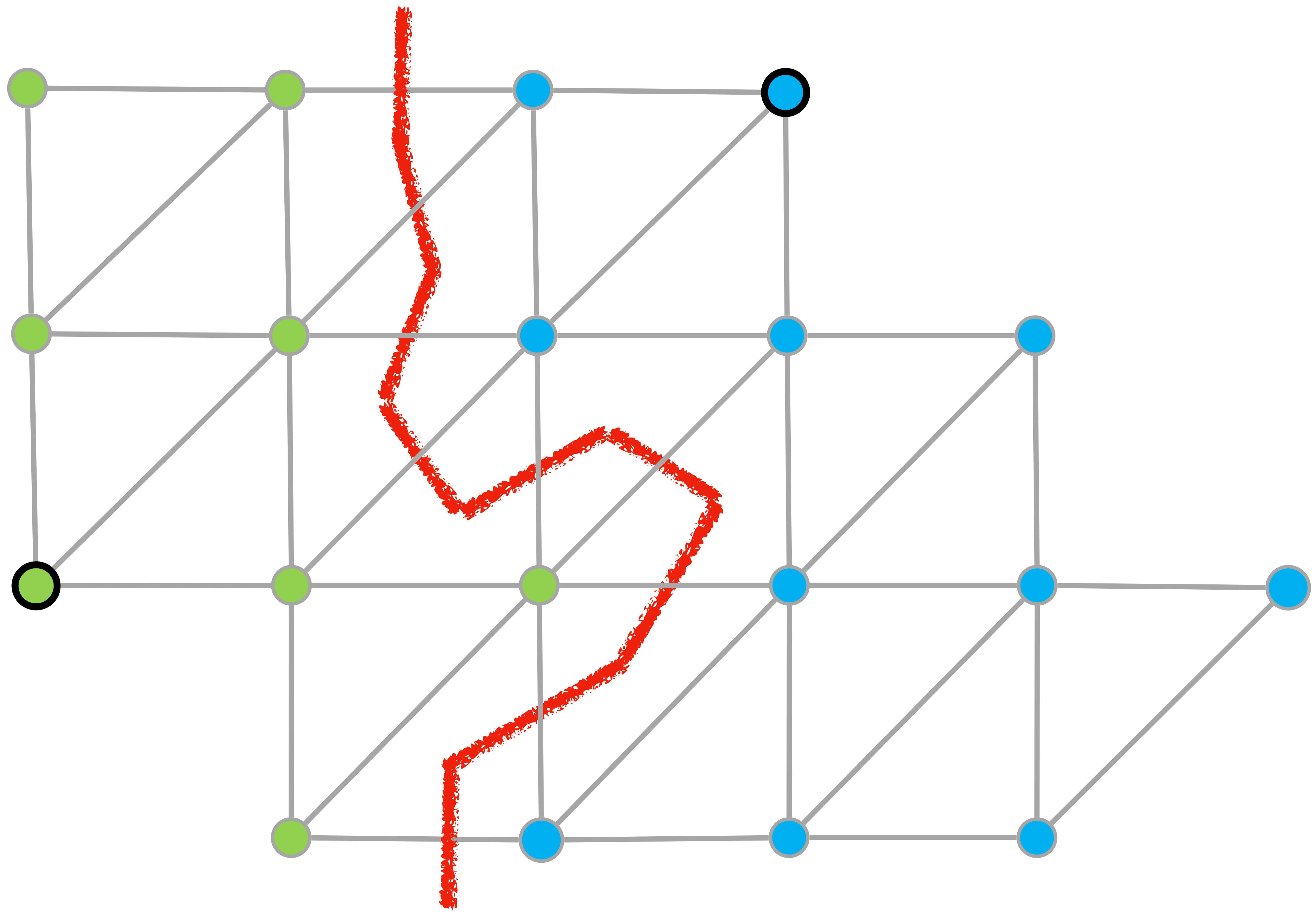
Voronoi vertices - adjacent to three different Voronoi cells (= **trichromatic faces**)



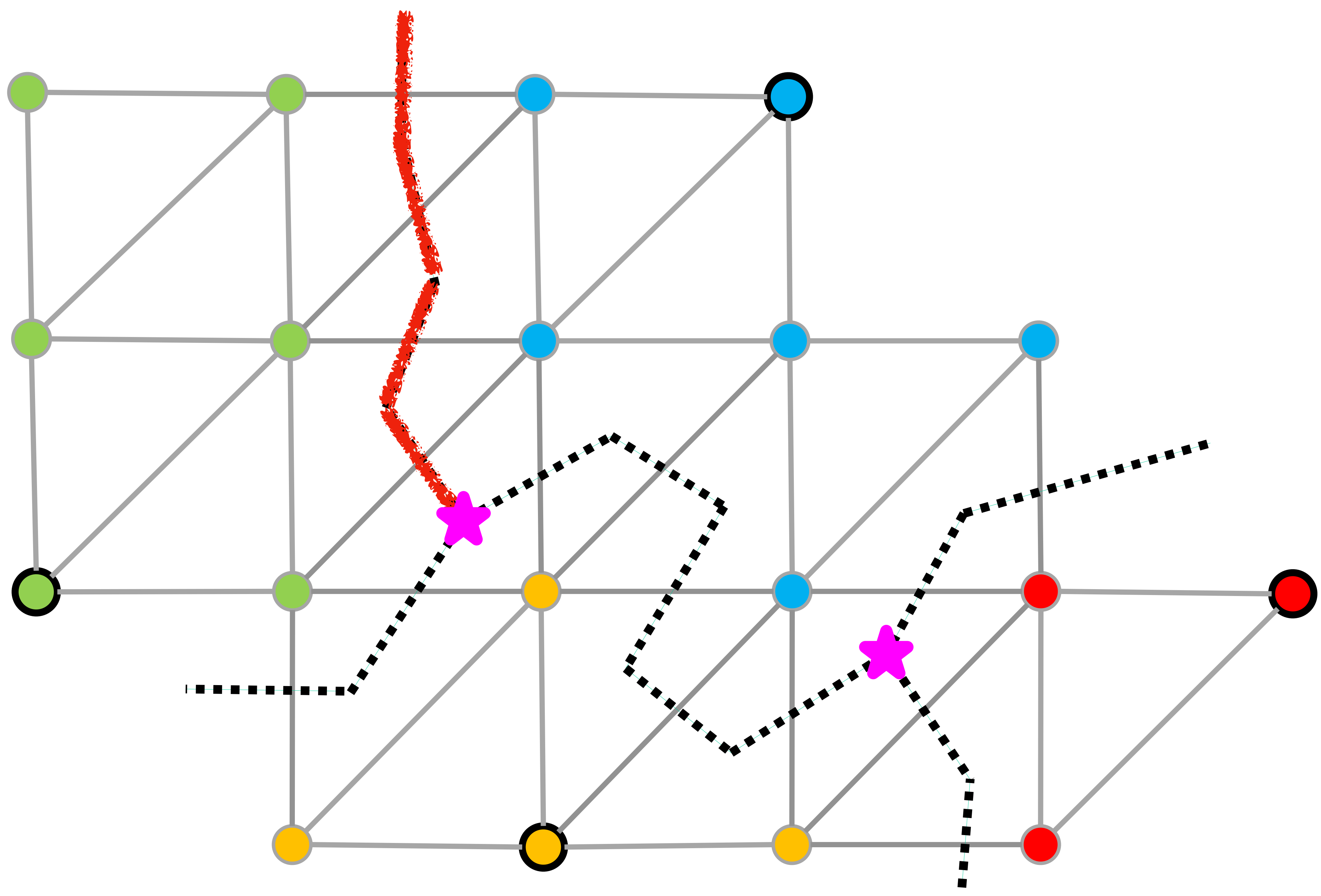
Voronoi diagram with b sites has b cells, $O(b)$ Voronoi vertices, and $O(b)$ Voronoi edges (by Euler's formula).



Bisectors - Voronoi diagram with just two sites

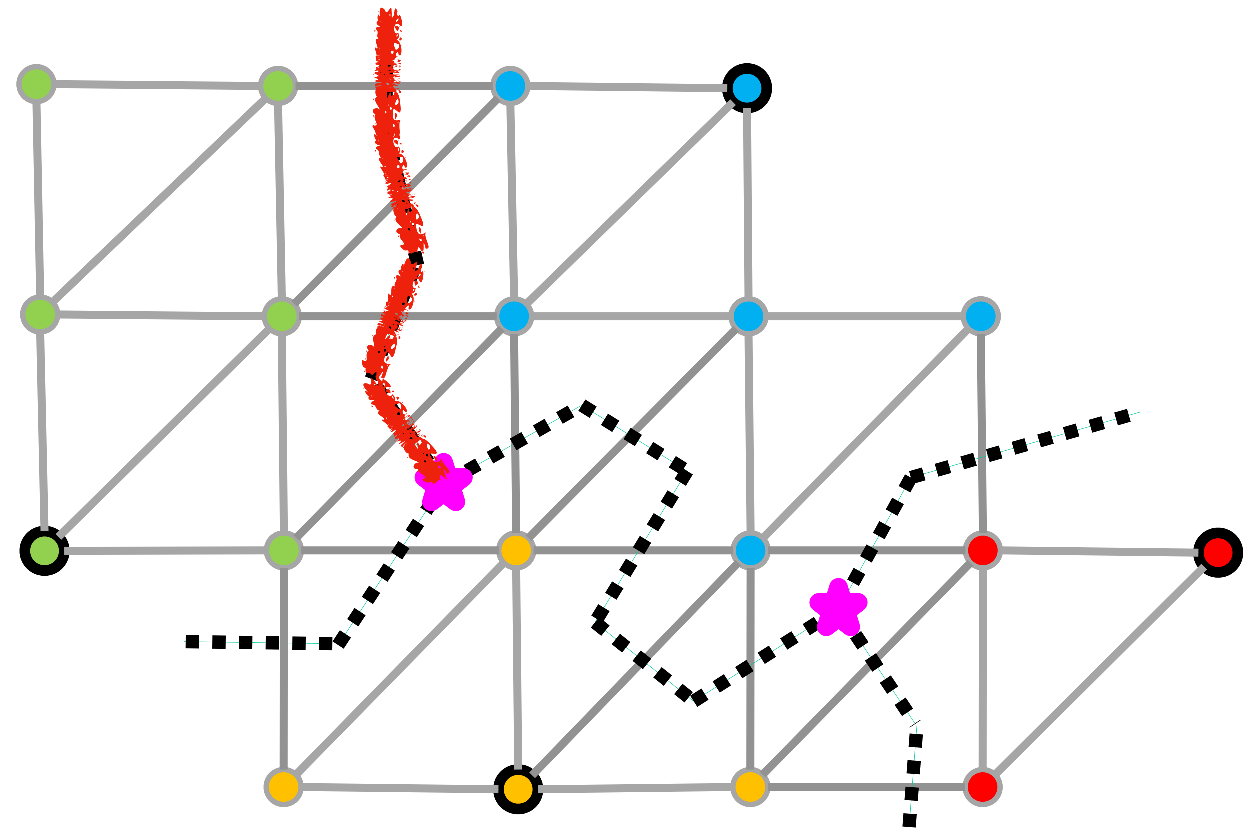


Every Voronoi edge is a subpath of a bisector



Strategy:

- Every Voronoi edge is a subpath of a bisector
- Every Voronoi vertex is the intersection of two bisectors



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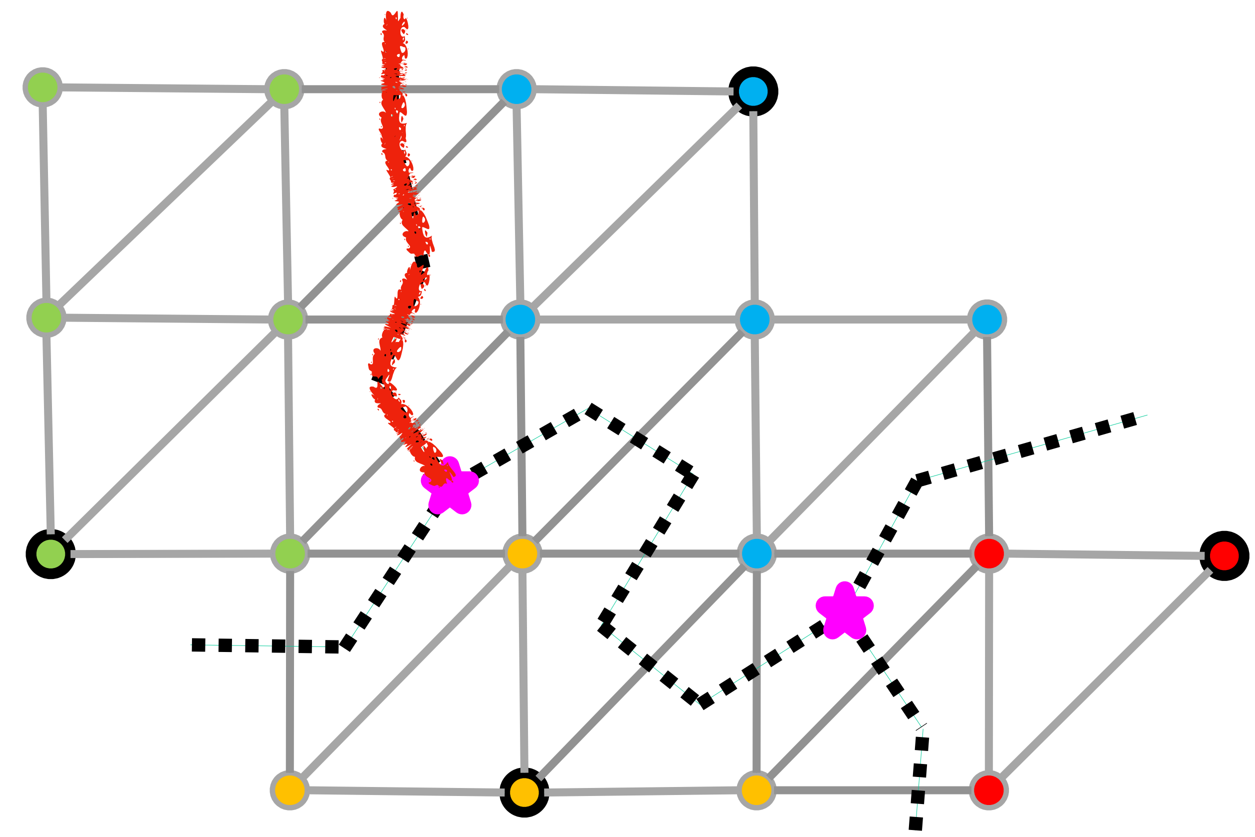

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- Precompute and store all possible bisectors (for all pairs of sites and all possible weights…)

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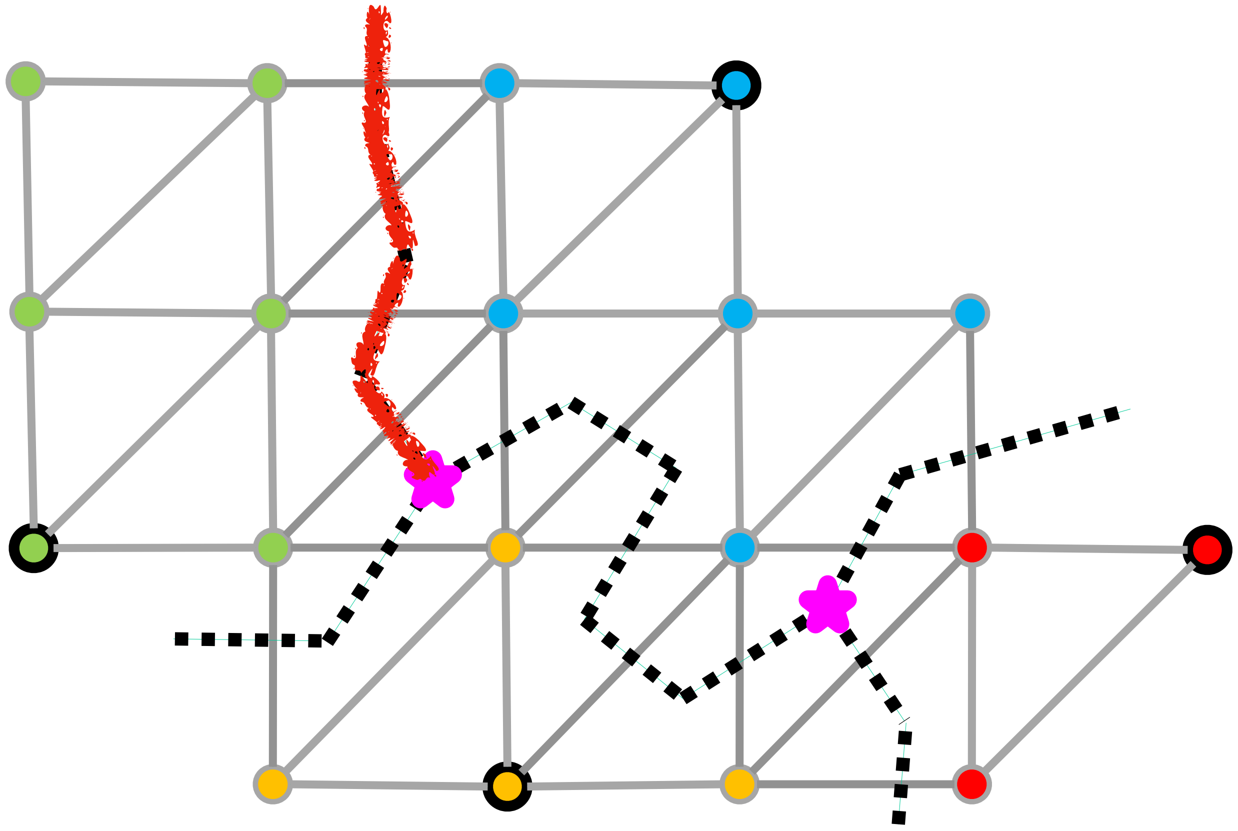

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- Represent Voronoi edges as subpaths of bisectors

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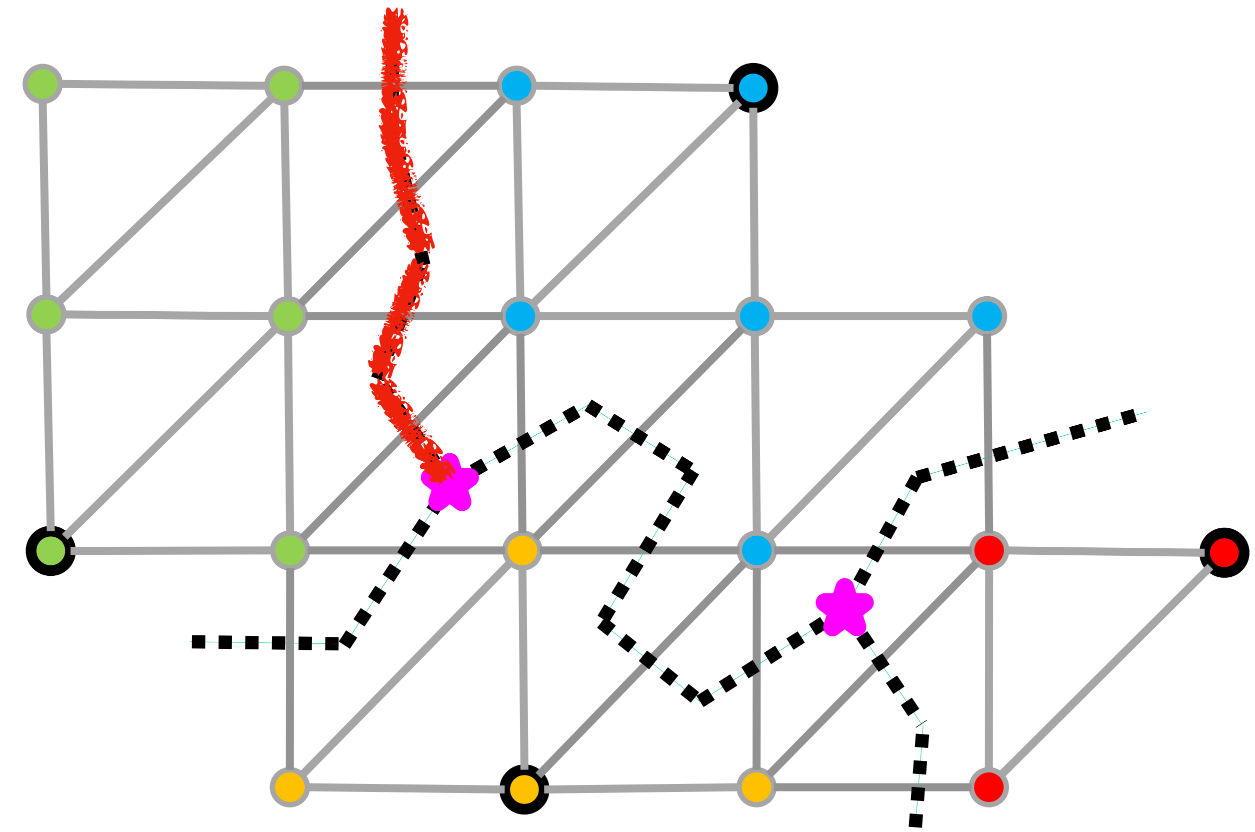

- Every Voronoi edge is a subpath of a bisector
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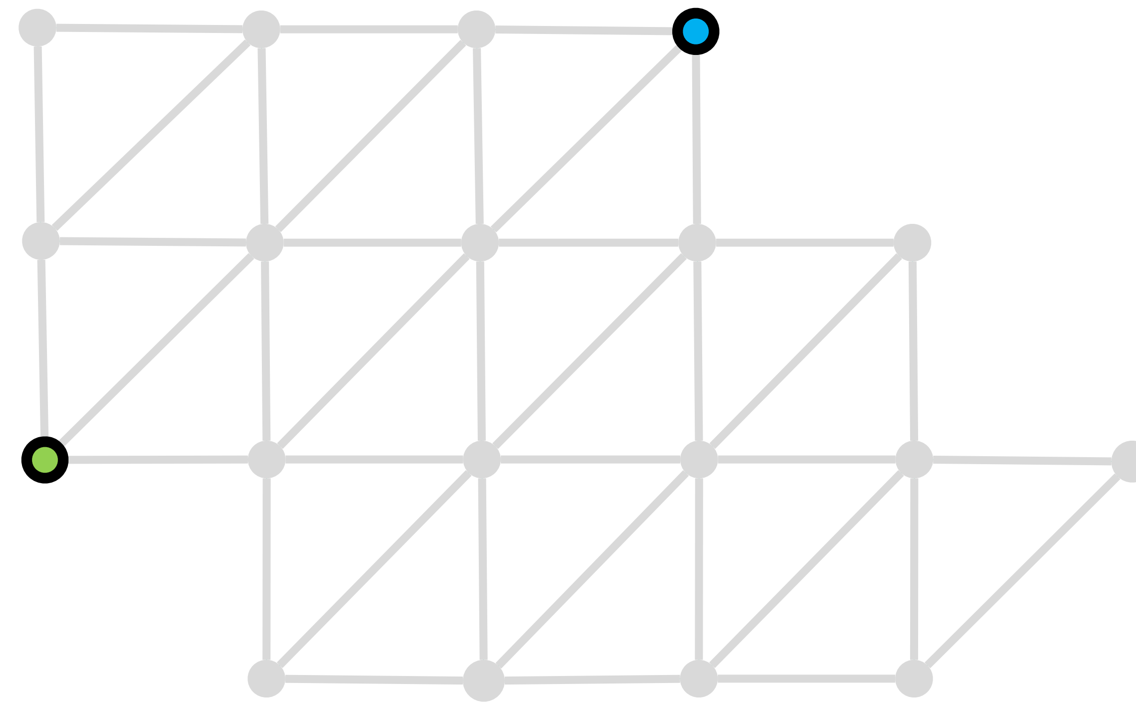
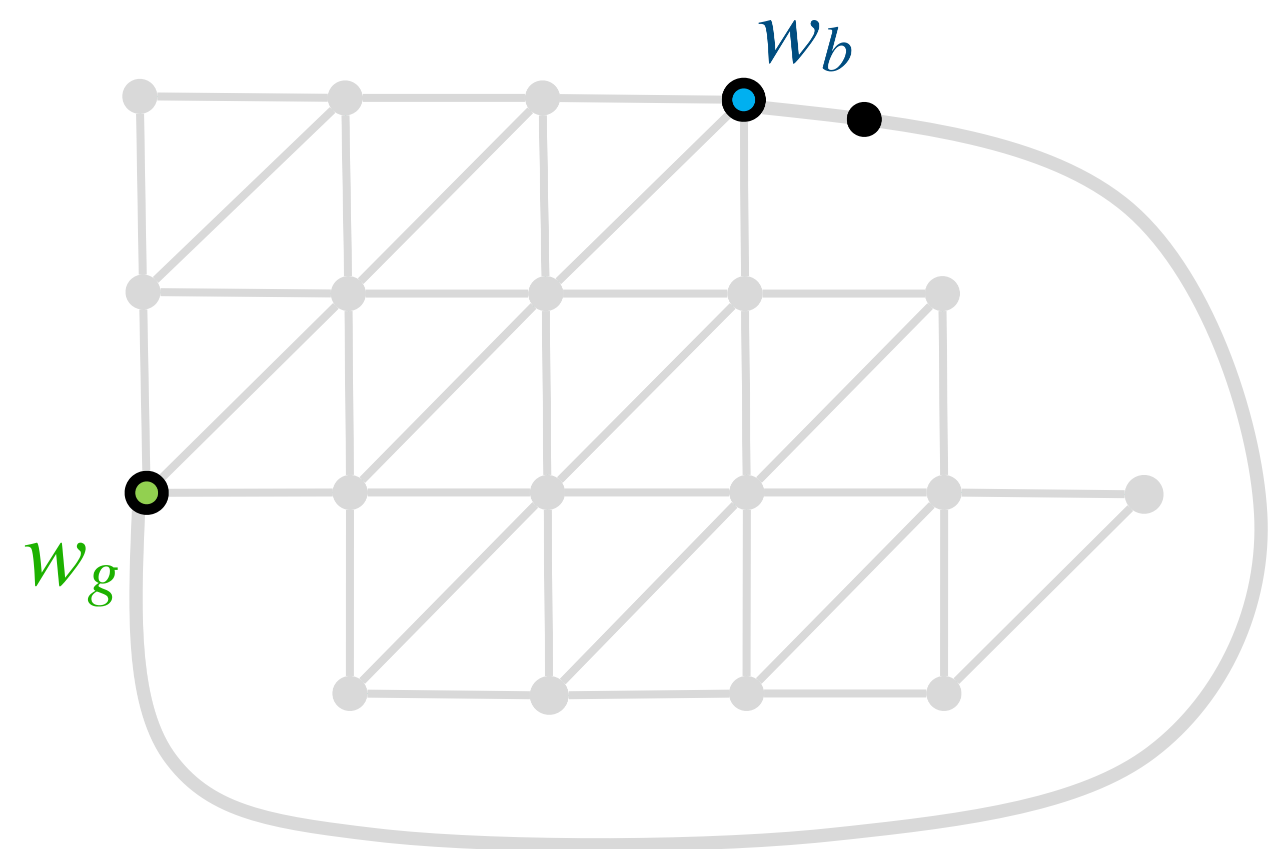
- Precompute and store all possible bisectors (for all pairs of sites and all possible weights…)
- Represent Voronoi edges as subpaths of bisectors
- Construct Voronoi diagram with b sites in $\tilde{O}(b)$ time using divide and conquer by intersecting bisectors

Computing bisectors



Computing bisectors

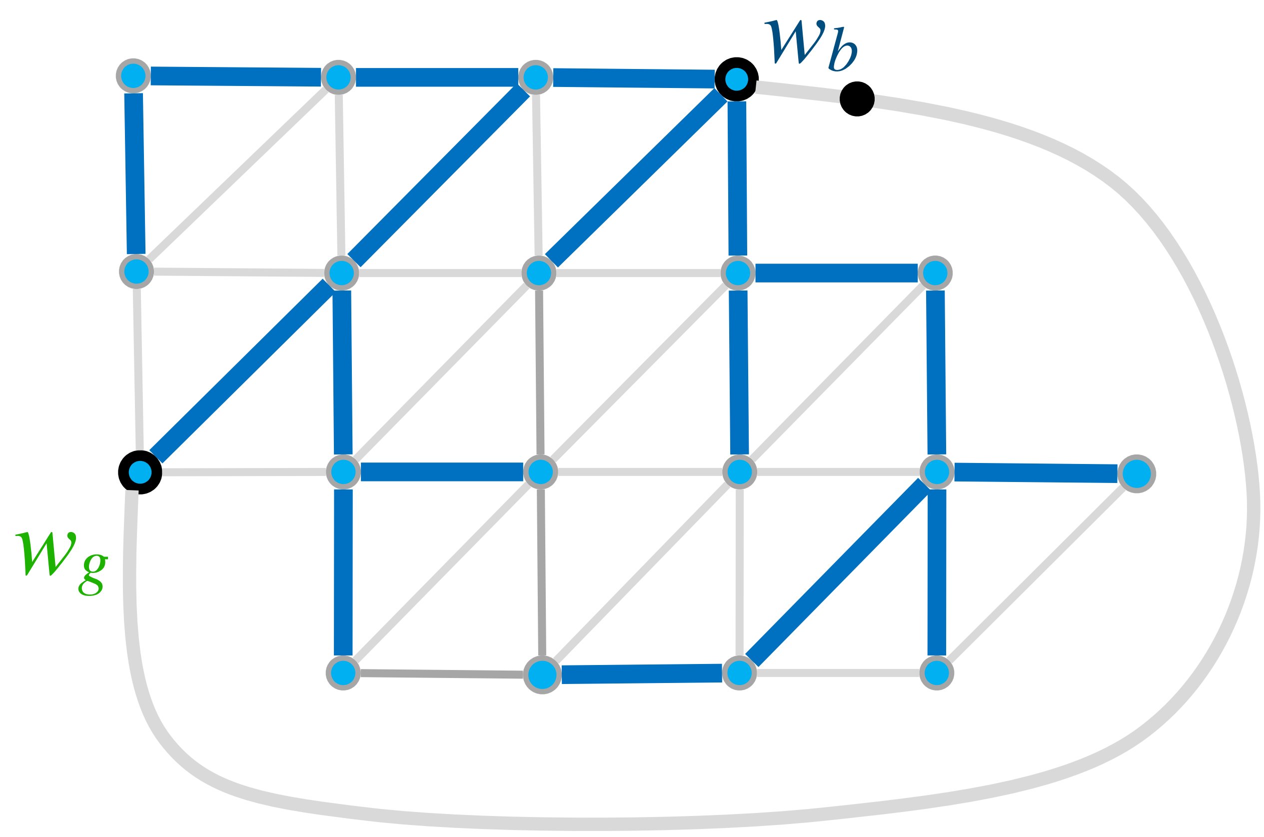
- only depends on the difference w_b-w_g



reminiscent of MSSP
[Cabello, Chambers, Erickson]

Computing bisectors

- only depends on the difference w_b-w_g
- as we increase w_b-w_g the bisector sweeps the graph
- changes occur at discrete critical values, where blue vertices become green



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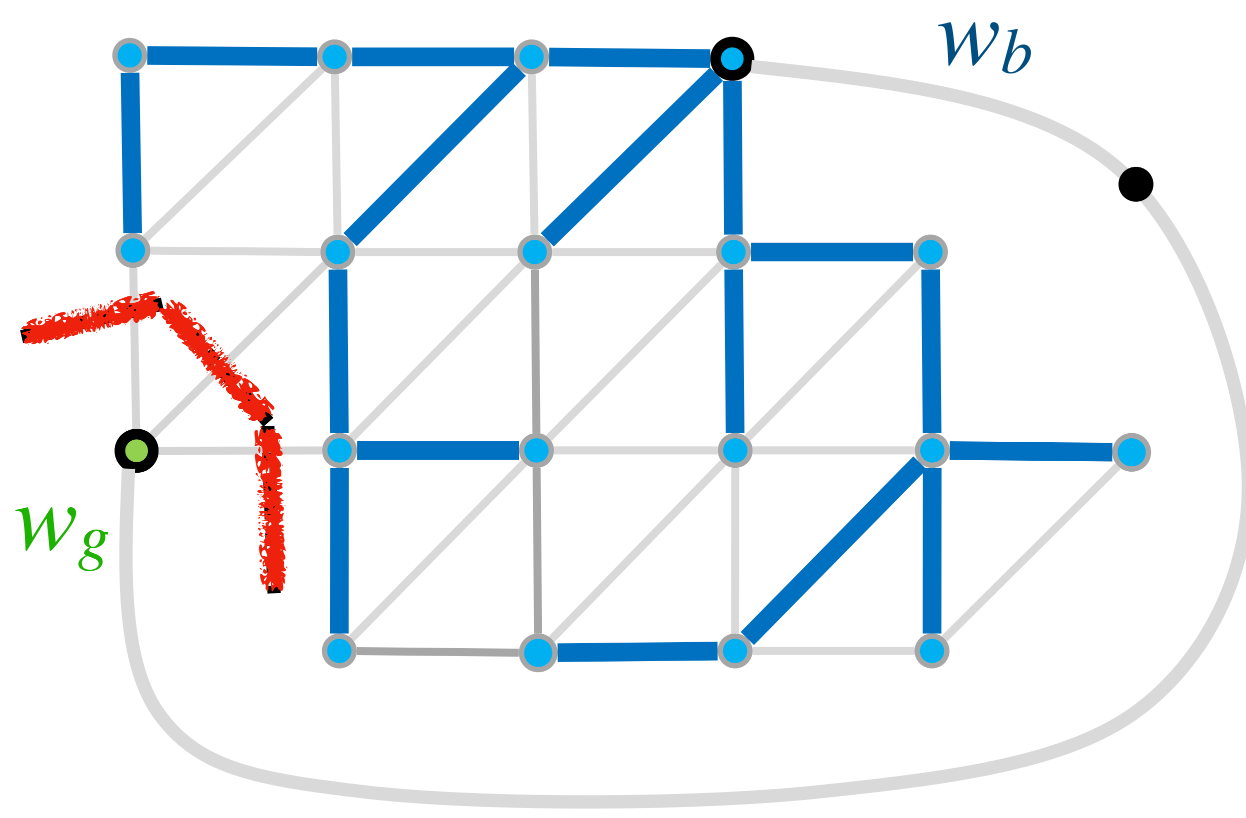

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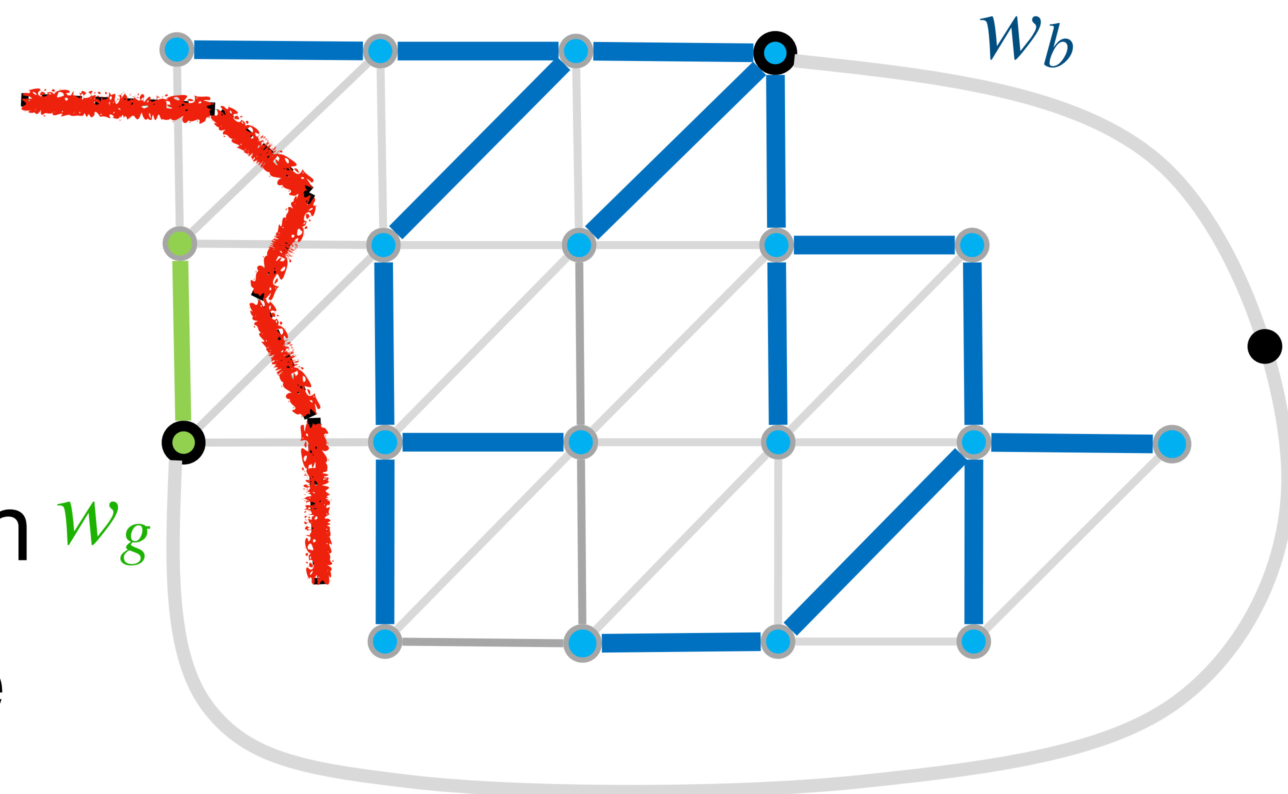

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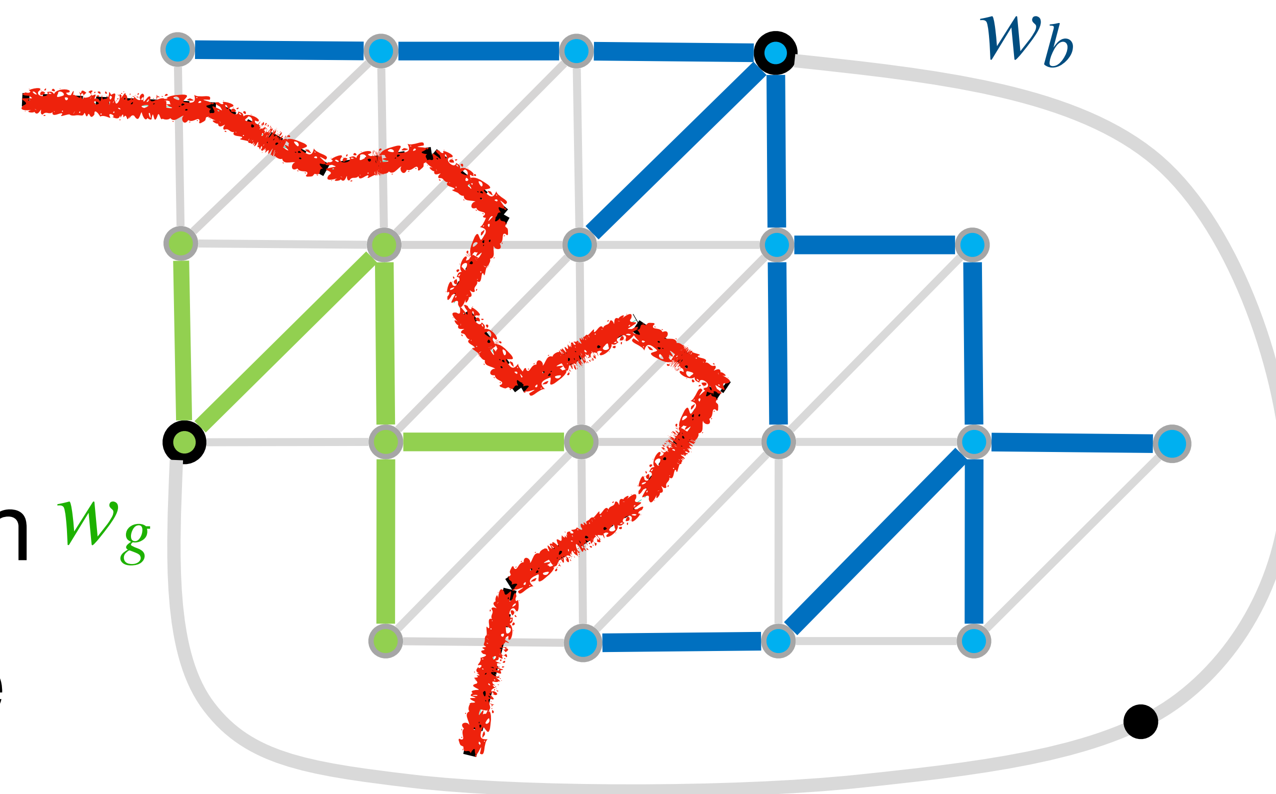

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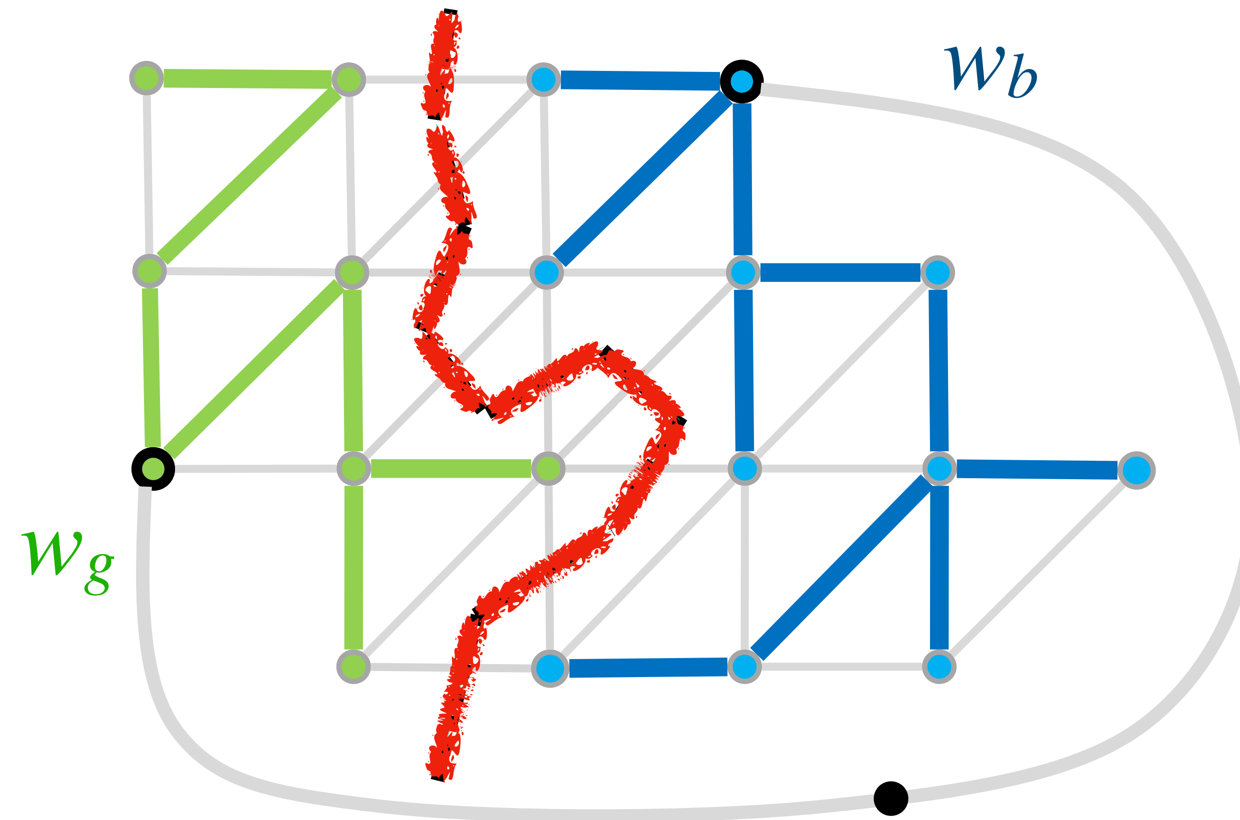

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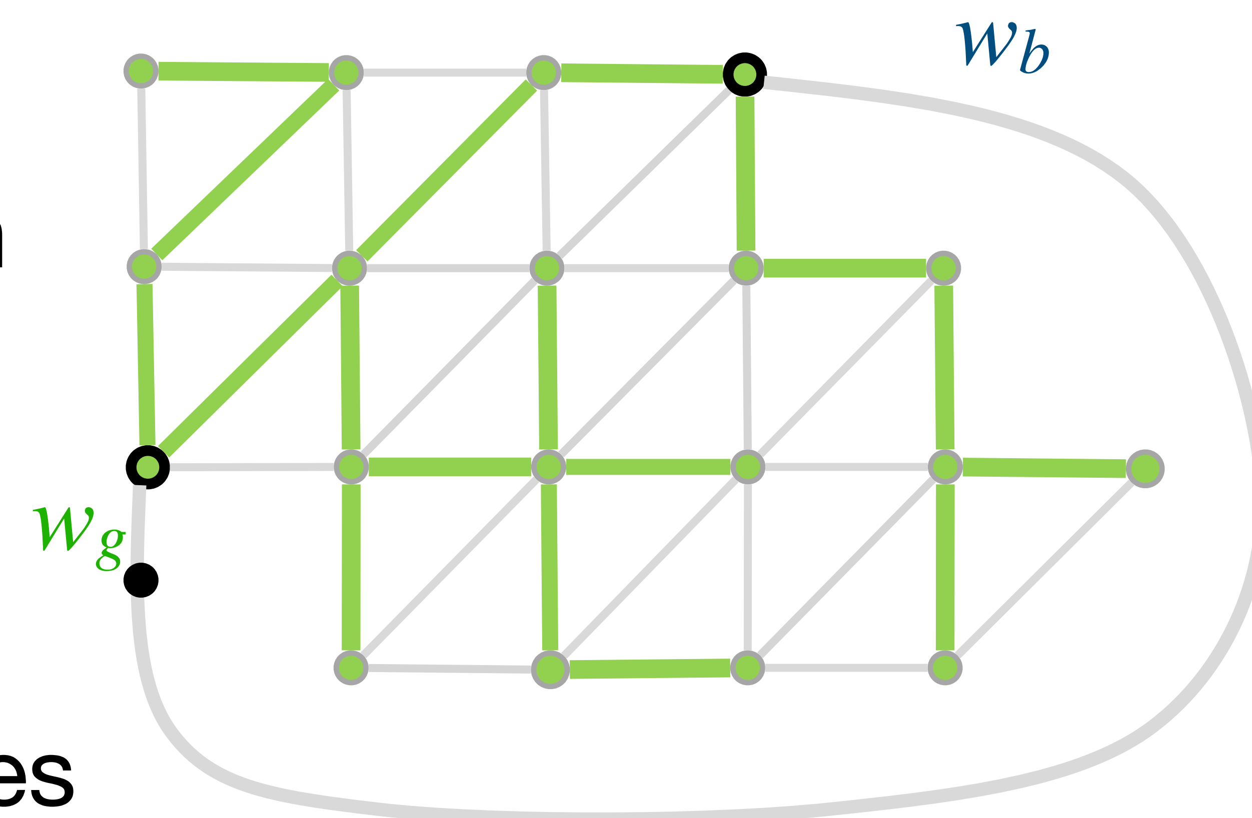

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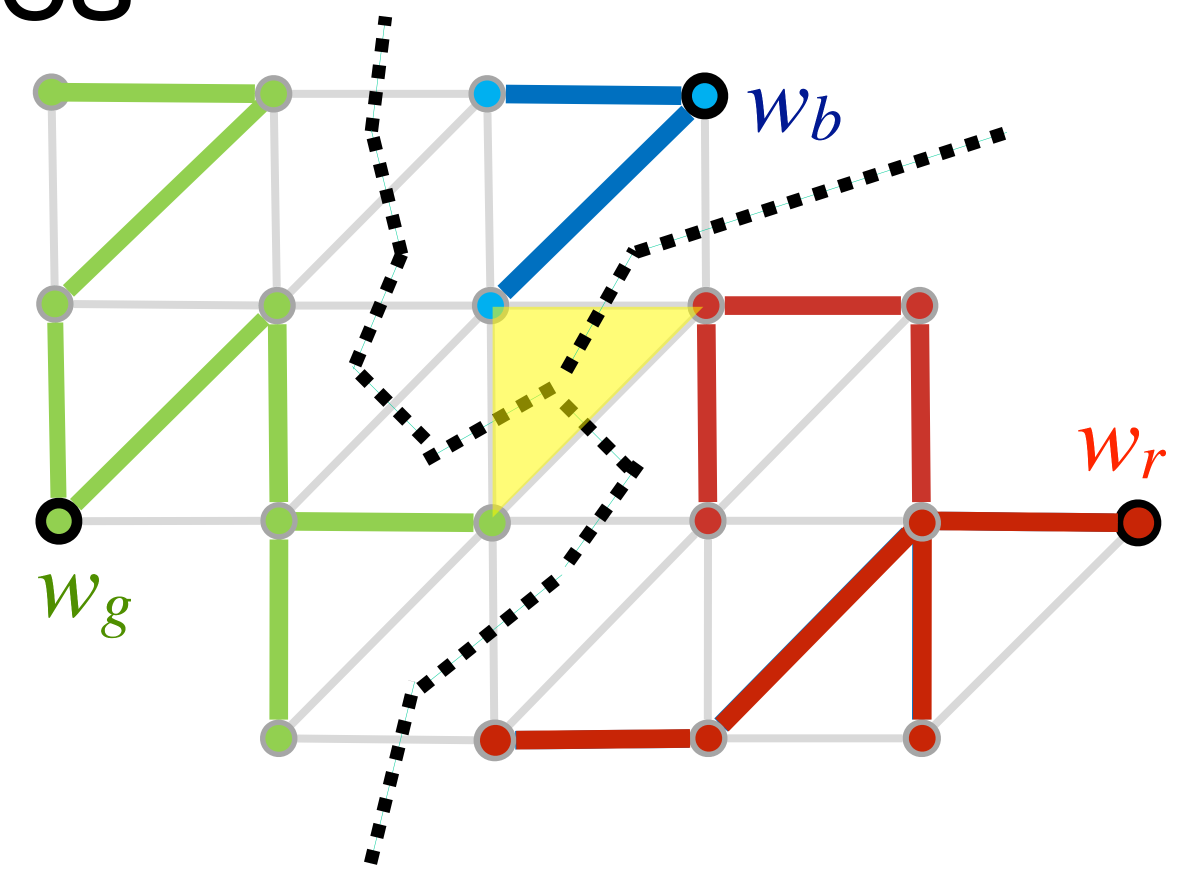
➡ In a graph with $O(r)$ vertices there are only $O(r)$ bisectors (for each pair of sites). can be computed in $\tilde{O}(r)$ time

- for each pair of sites, all bisectors stored in $\tilde{O}(r)$ space and time using persistent binary search trees
- for $r^{1/2}$ sites total preprocessing $\tilde{O}(r^2)$ time and space
- compared to $\tilde{O}(r^3)$ time and space in Cabello's

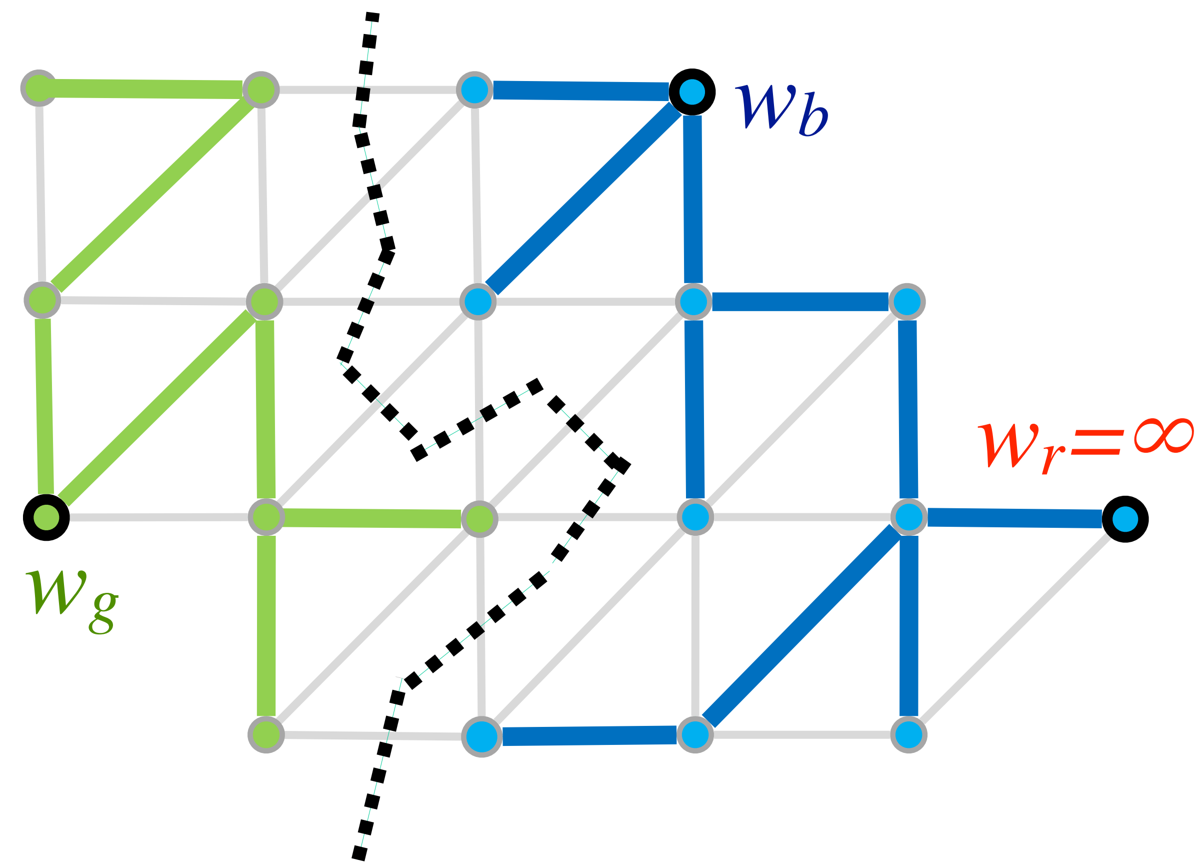
Intersecting bisectors = finding trichromatic faces

- three sites with weights w_b, w_g, w_g.
- want to find a face with one vertex in each of the Voronoi cells.



Dynamics of trichromatic faces

- fix w_b and w_g, and gradually decrease w_g.



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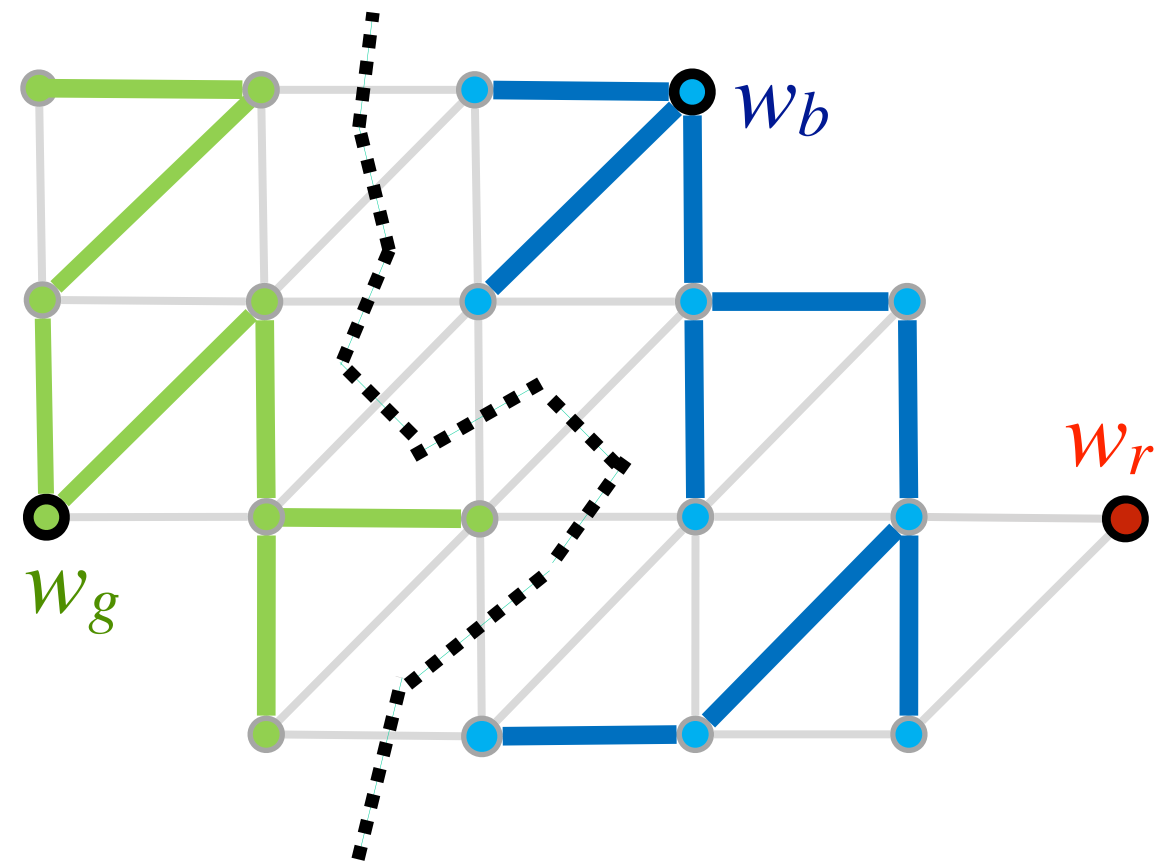
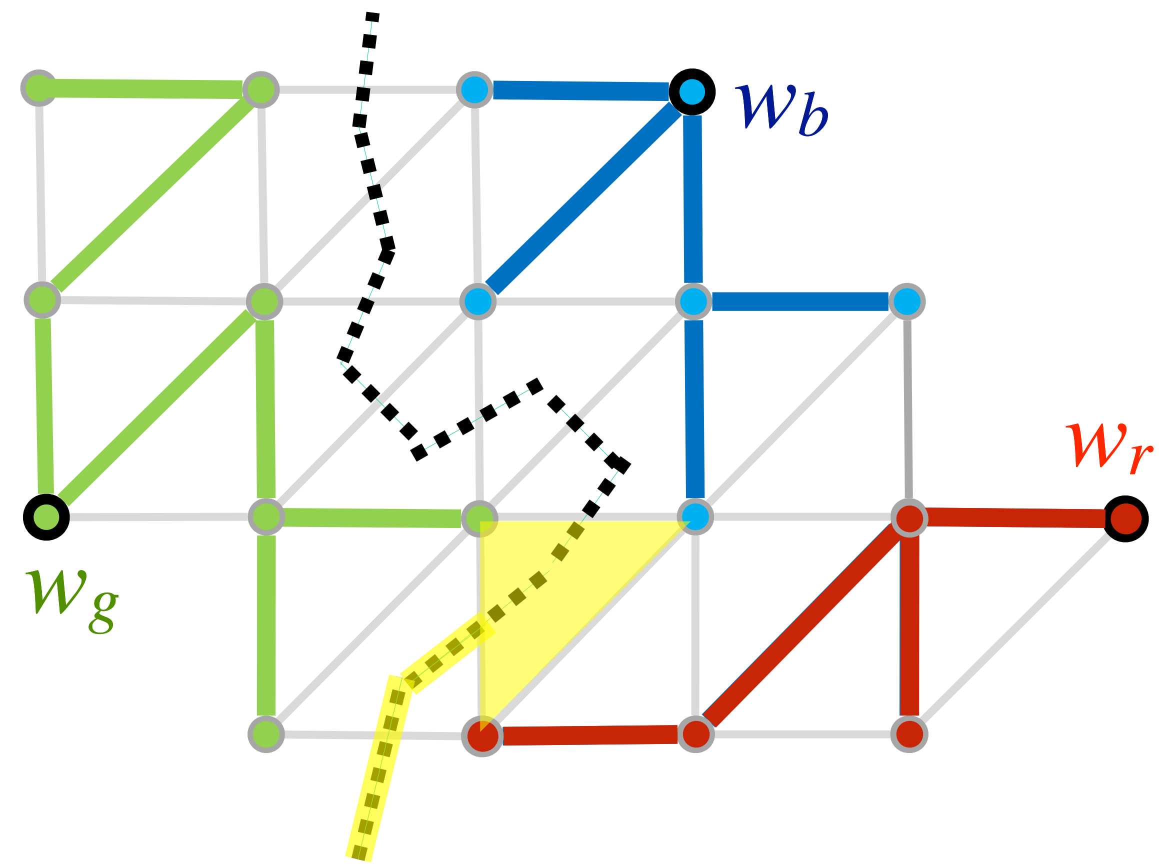


Dynamics of trichromatic faces

- fix w_b and w_g, and gradually decrease w_g.
- as we decrease w_r, the trichromatic face moves monotonically along the (g,b)-bisector.



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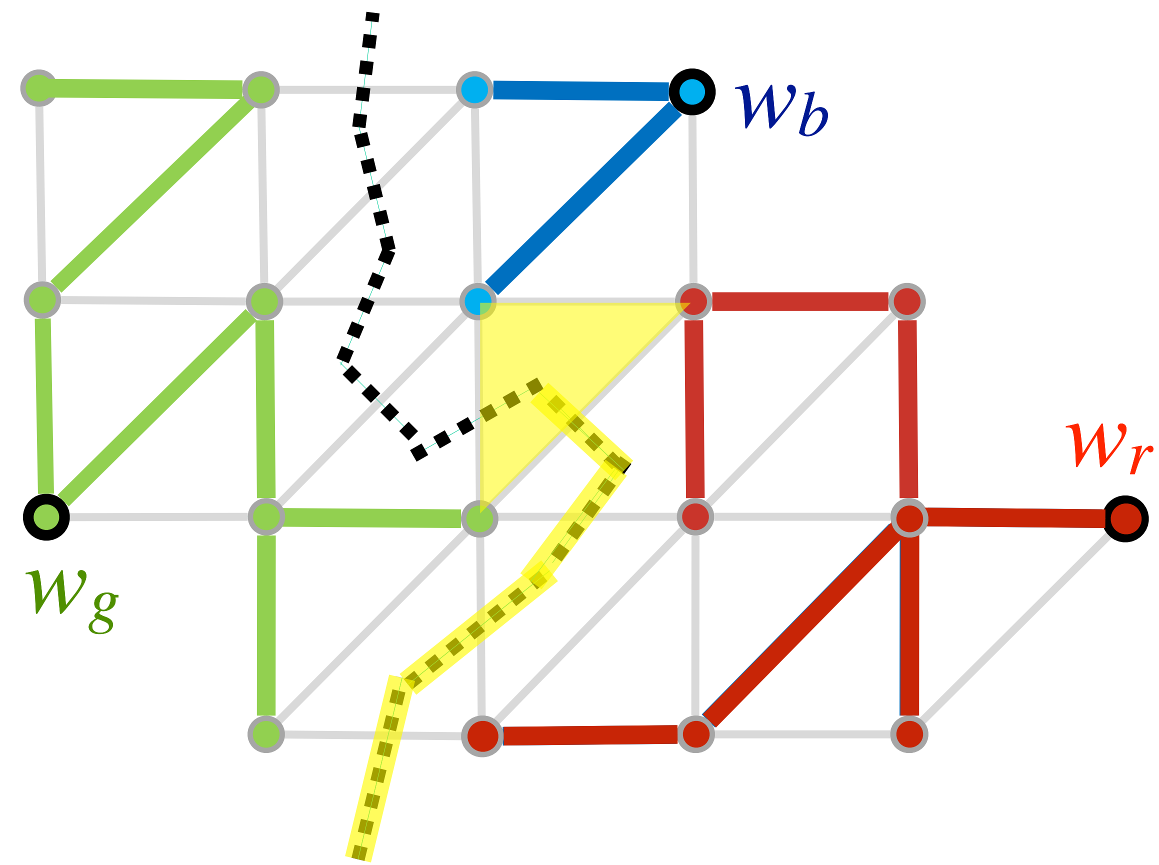

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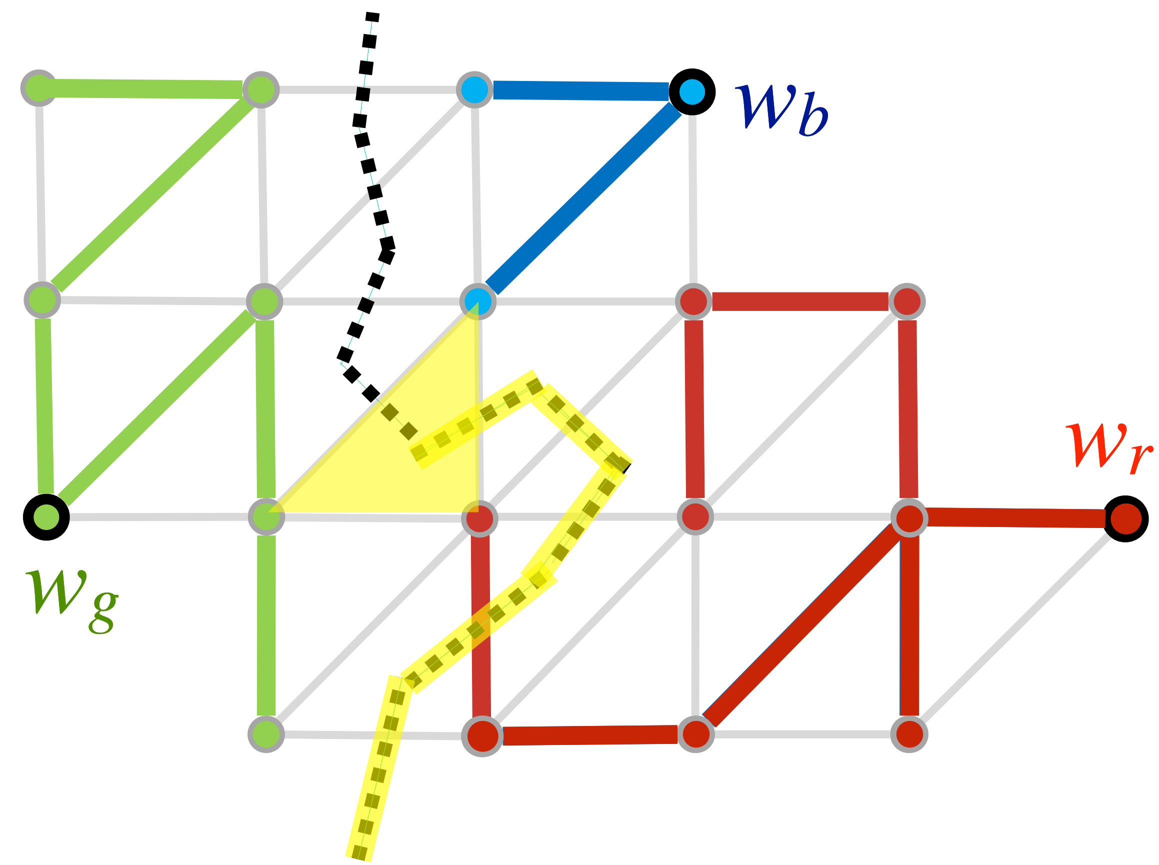
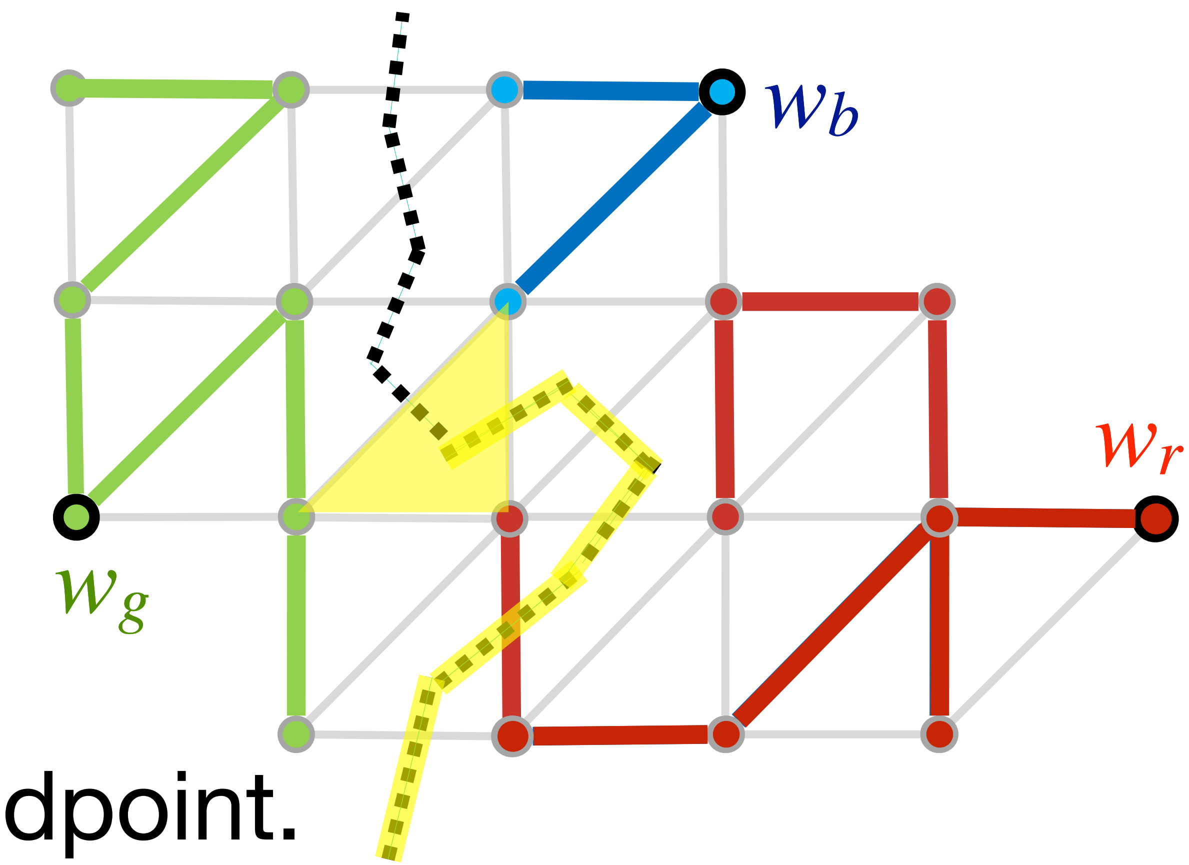


Finding trichromatic faces

- as we decrease w_r, the trichromatic face moves monotonically along the (g,b)-bisector.
- given w_b, w_g, w_r can determine in constant time whether an edge has a red endpoint.

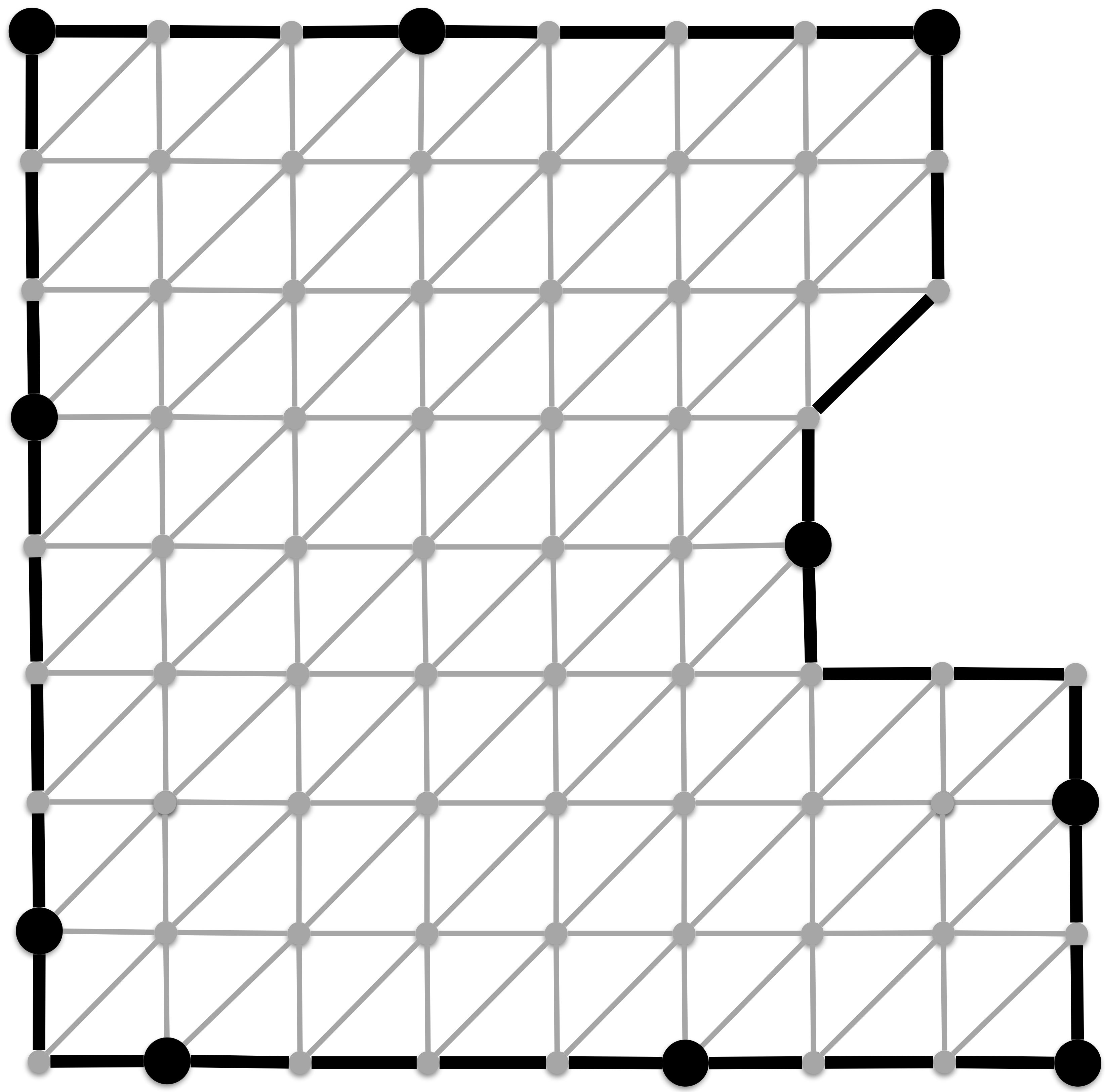


- binary search for the last edge on the (g,b)-bisector that has a red endpoint
- takes $\tilde{O}(1)$ time
- need to extend to groups of sites. Becomes much more complicated

Constructing a Voronoi diagram

- we have precomputed bisectors in $\tilde{O}(r^2)$
- we know how to find a trichromatic face in $\tilde{O}(1)$ time
- we compute the weighted Voronoi diagram of $r^{1/2}$ sites in $\tilde{O}(r^{1/2})$ time using divide and conquer.

Constructing diagram via divide and conquer



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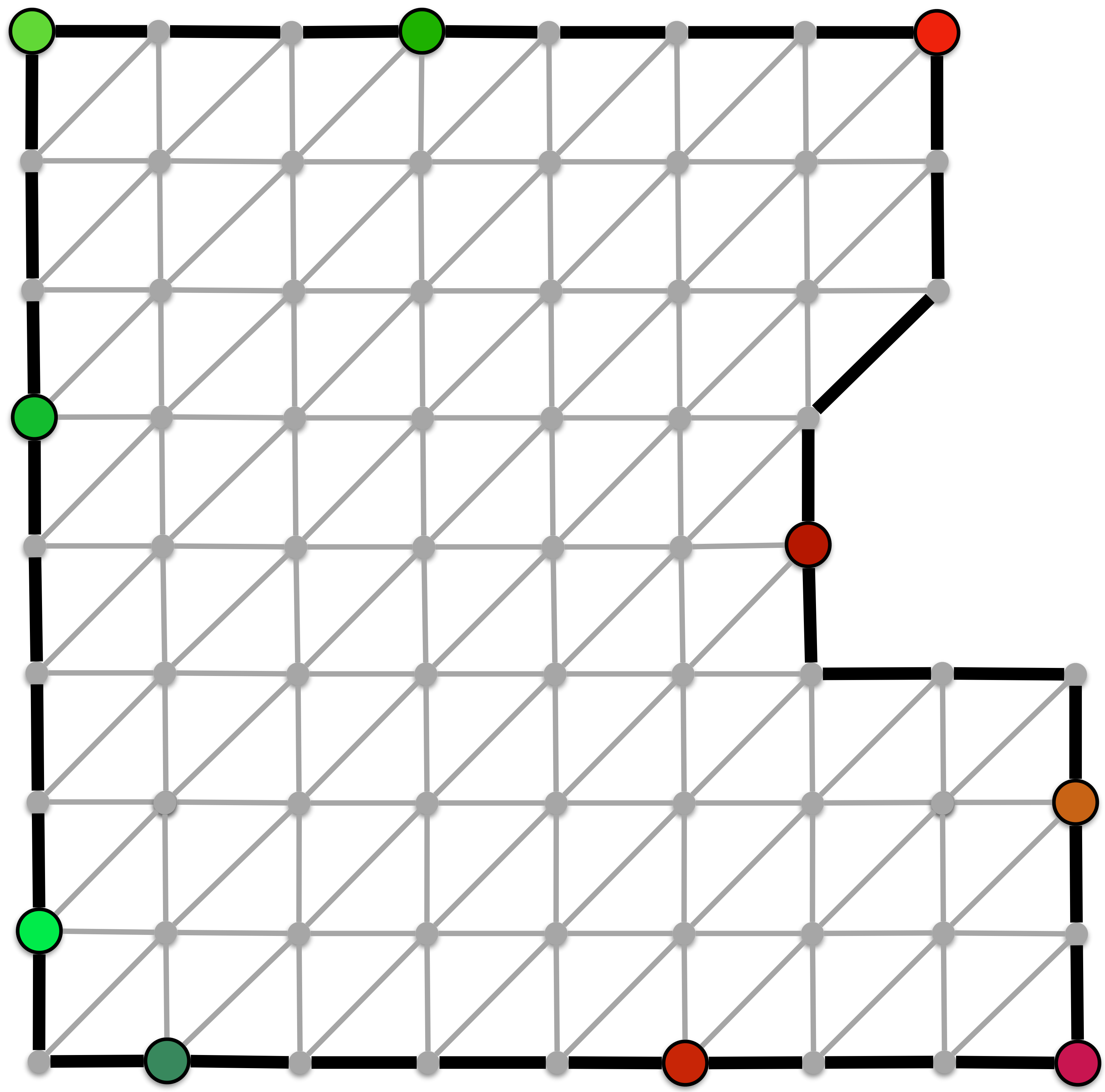

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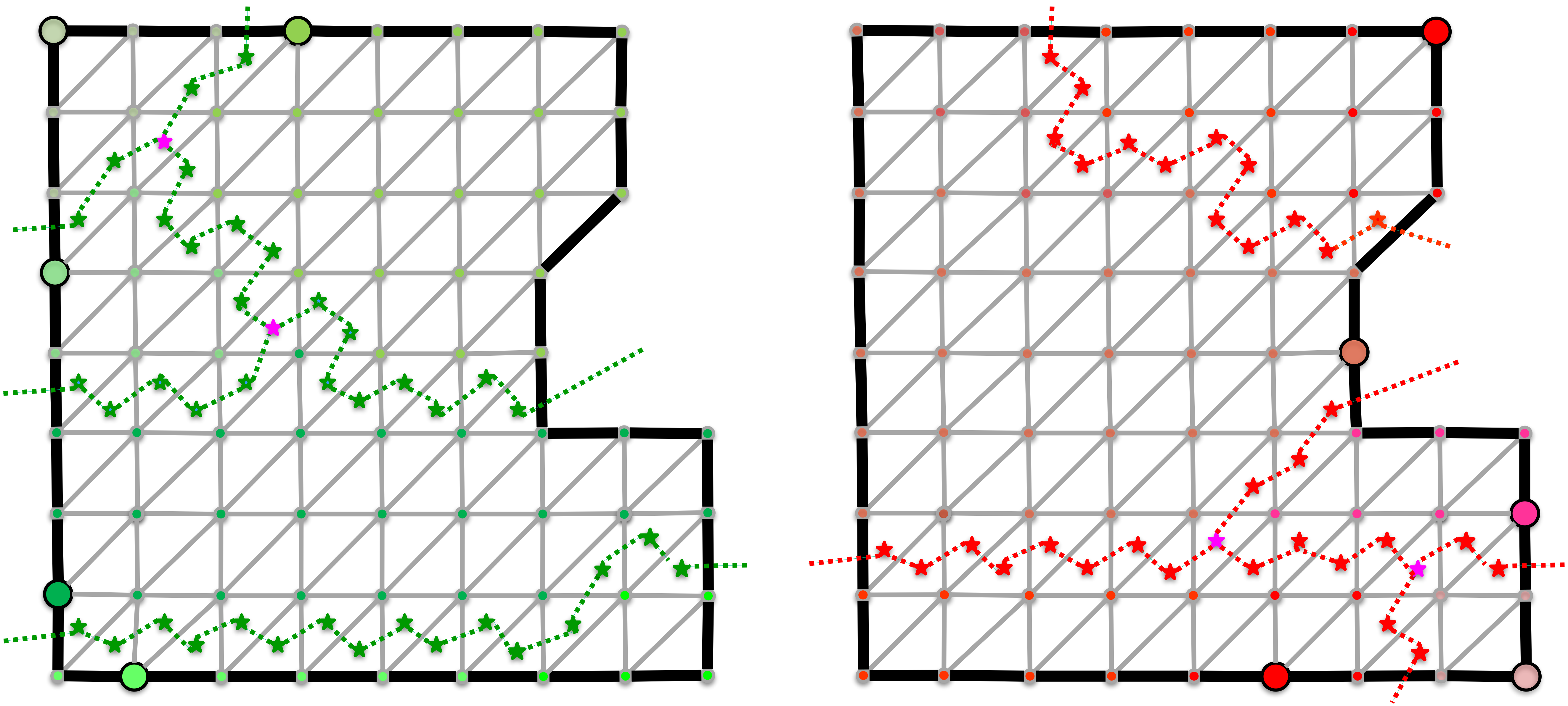
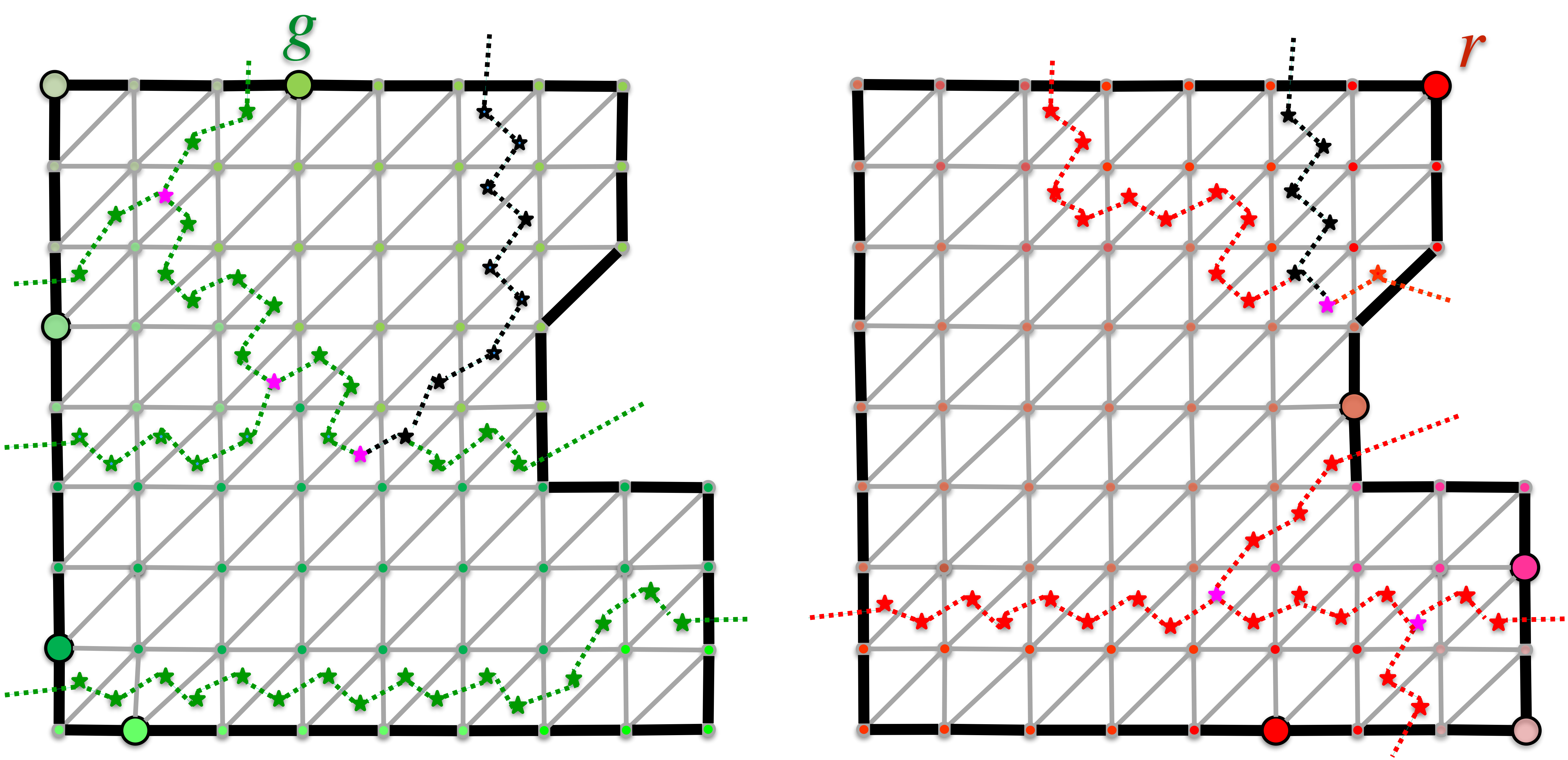
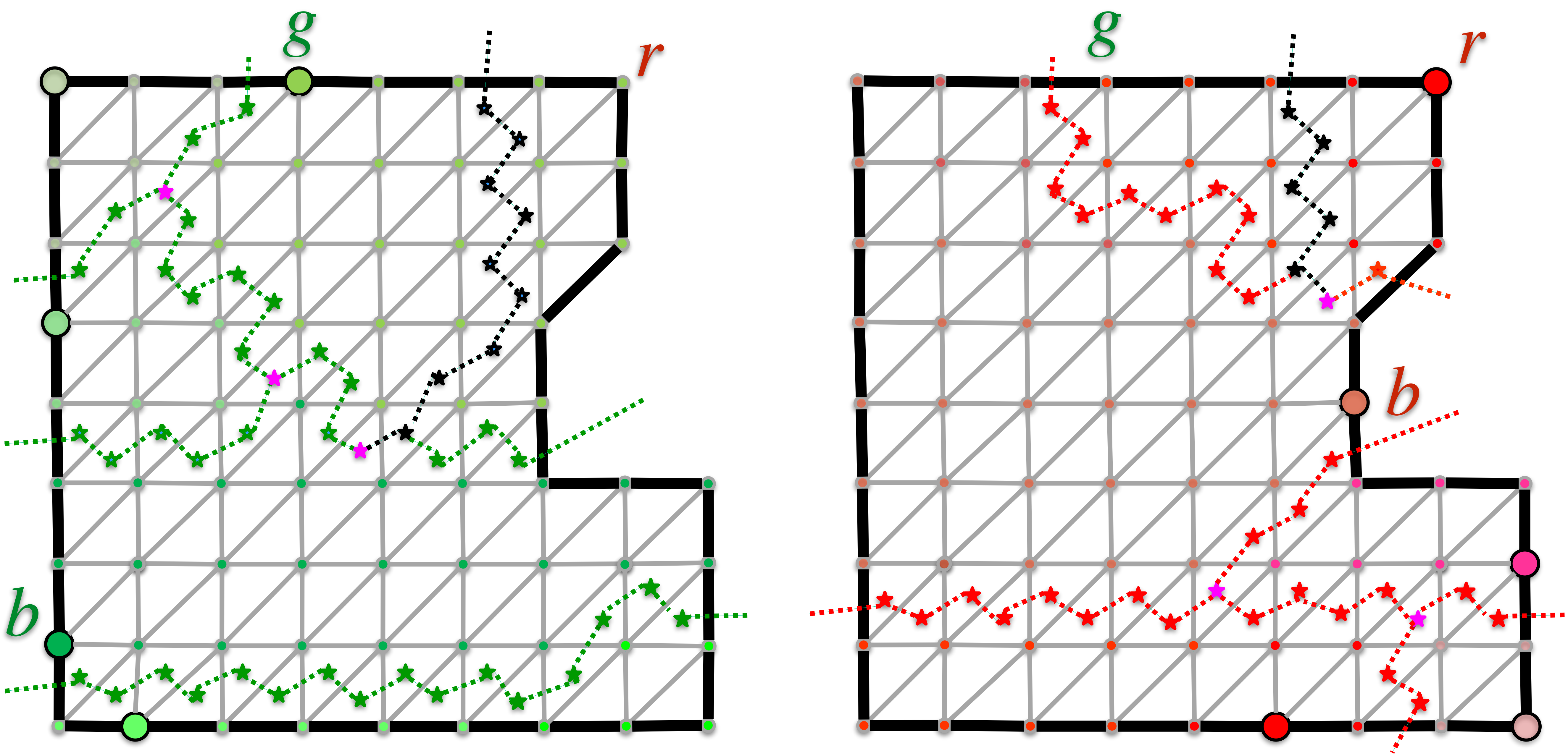


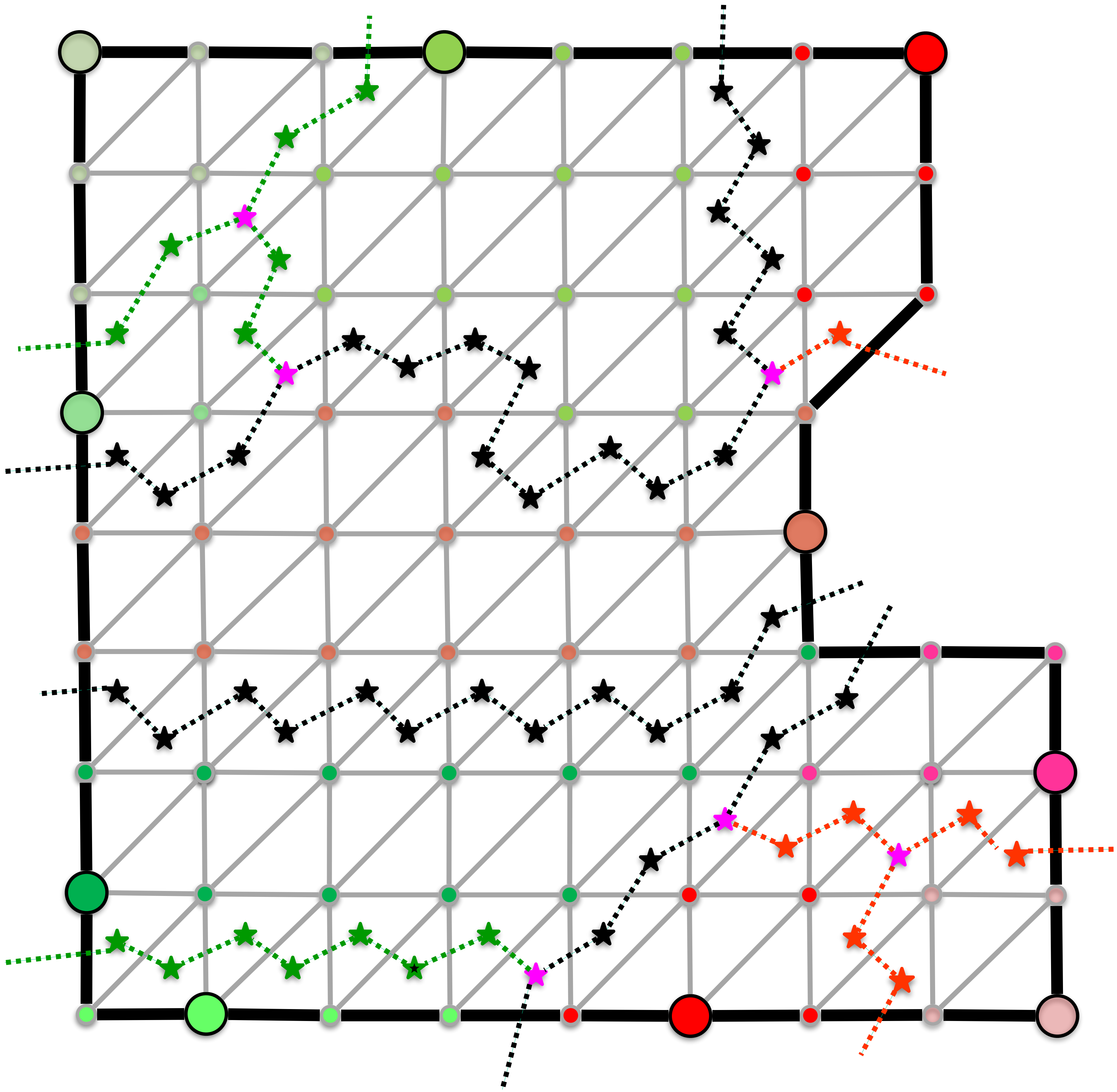


g
r



g
r
b
g
r
b

Combined diagram



Constructing a Voronoi diagram

- since we can compute trichromatic vertices in $\tilde{O}(1)$ time, we can merge two Voronoi diagrams with b_1 and b_2 sites in $\tilde{O}(b_1 + b_2)$ time
- so constructing a Voronoi diagram with $r^{1/2}$ sites takes $\tilde{O}(r^{1/2})$ time

Things I swept under the rug

- handling sites on more than one face (holes)
- mechanism for finding furthest vertex in a Voronoi cell (similar to Cabello's)

Conclusion

- efficient deterministic construction of Voronoi diagrams on planar graphs
- diameter of a planar graph in deterministic $\tilde{O}(n^{5/3})$ time

- can we get below for $\tilde{O}(n^{5/3})$ time diameter?
- nearly linear time / (conditional) lower bounds?
- what other problems can benefit from these techniques?