A Faster FPTAS for #Knapsack

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Slides by Liran Markin

Counting Knapsack Solutions

capacity C

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Given a set $W = \{w_1, w_2, \ldots, w_n\}$ of n non-negative integers and a capacity C, count the number of subsets of W with total sum of at most C.

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$O(nC)$ time but C is large!

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Deterministic FPTAS $O(n^3 \epsilon^{-1} \log(n\epsilon^{-1}))$ [Štefankovič et al. 2012], [Gopalan et al. 2011].
Best deterministic FPTAS $O(n^3 \epsilon^{-1} \log \epsilon^{-1} / \log n)$ [Rizzi, Tomescu 2014].
Best randomized FPTAS $O(n^{2.5}\sqrt{\log(n\epsilon^{-1})} + \epsilon^{-2} n^2)$ [Dyer 2003].

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This work

A deterministic FPTAS running in $O(n^{2.5}\varepsilon^{-1.5}\log(n\varepsilon^{-1})\log(n\varepsilon))$ time and $O(n^{1.5}\varepsilon^{-1.5})$ space.

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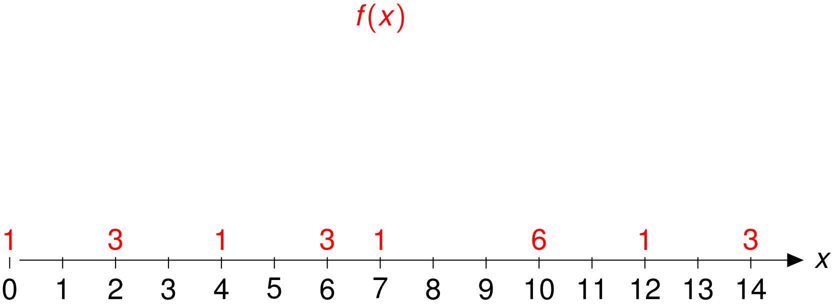
Definition

A function F is a $(1+\epsilon)$-*sum approximation* of f if for every x,

$$f^{\leq}(x) \leq F^{\leq}(x) \leq (1+\epsilon) f^{\leq}(x)$$

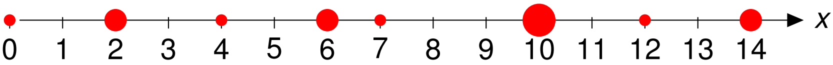
Sum Approximation - Sparsification



Sum Approximation - Sparsification

$f(x)$

wider circle $\rightarrow$ larger value



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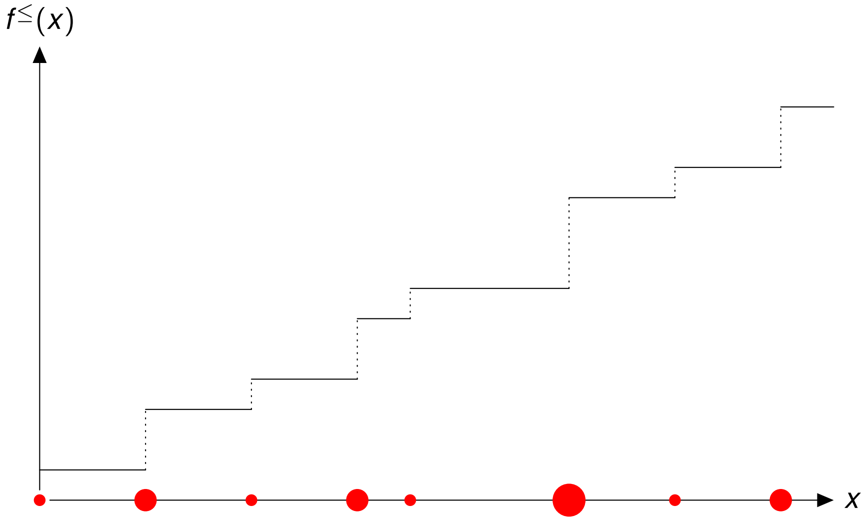

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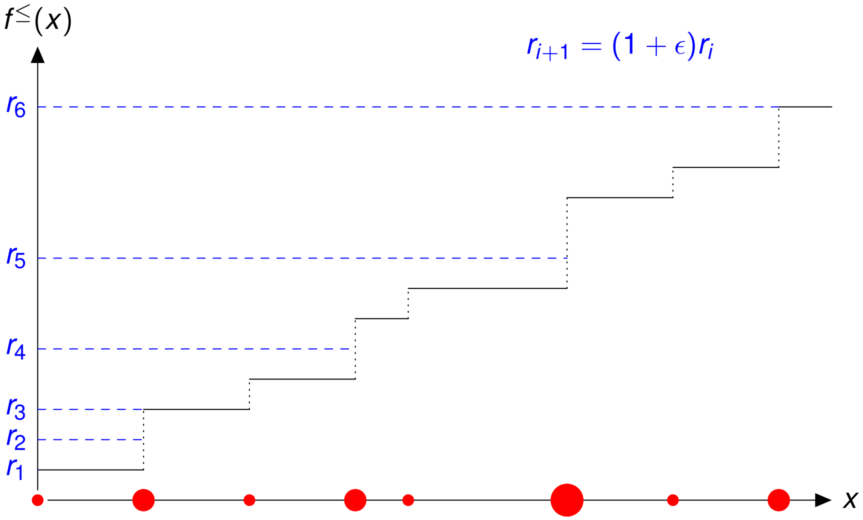

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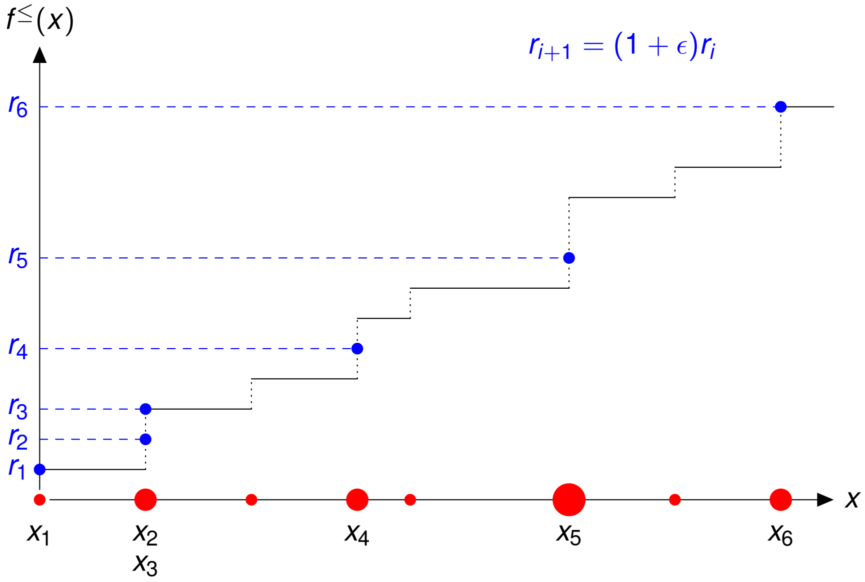

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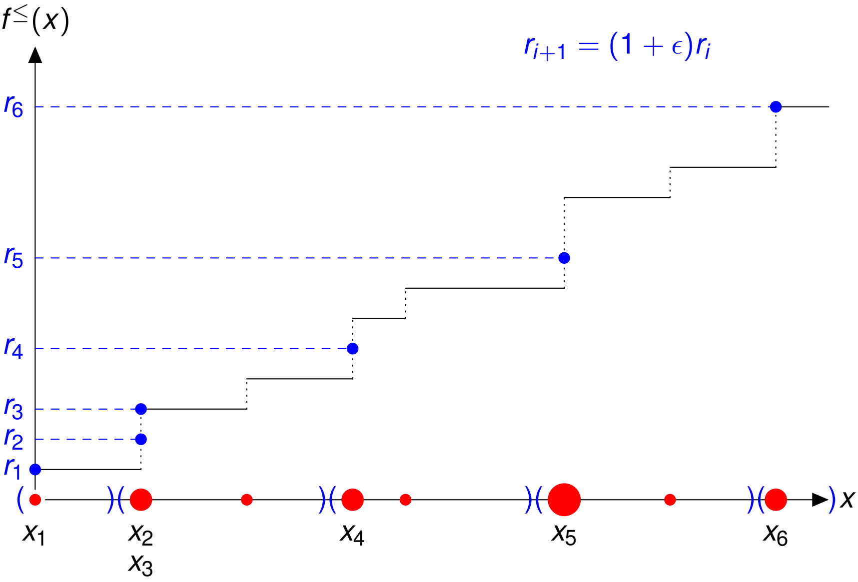

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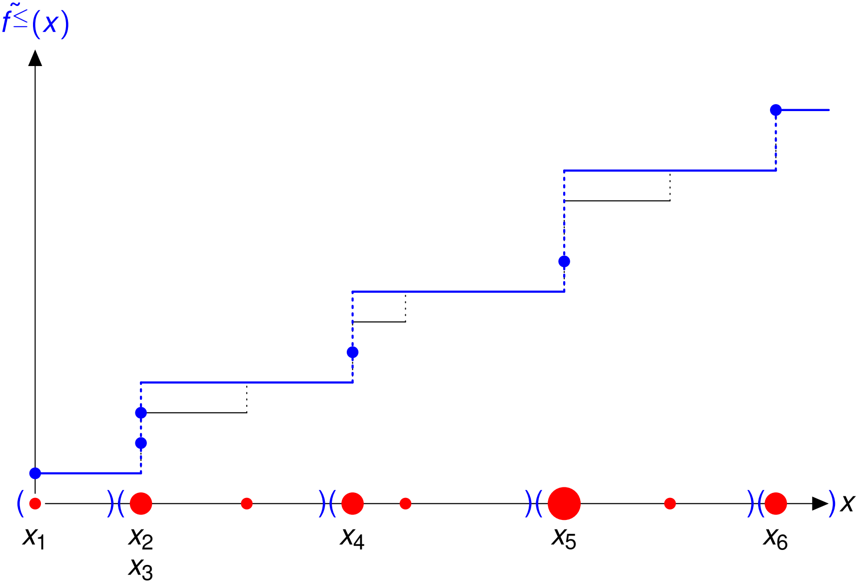

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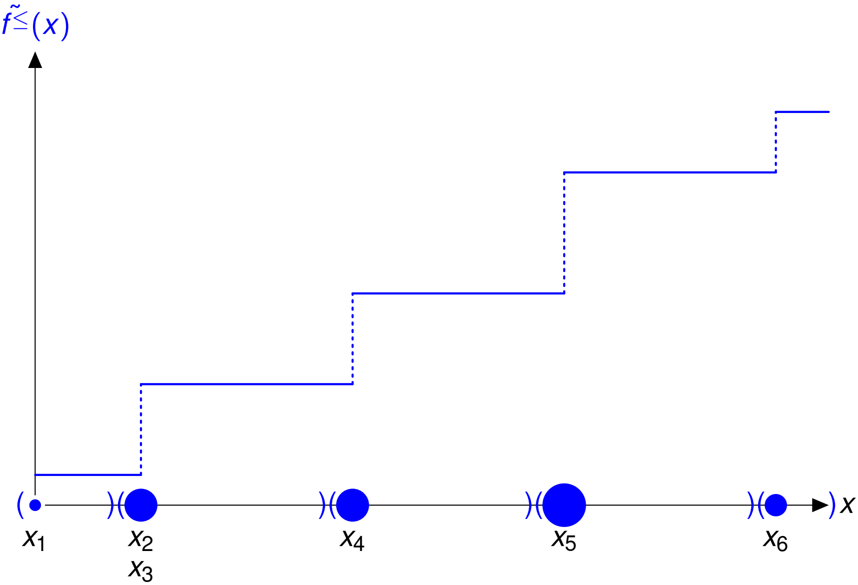

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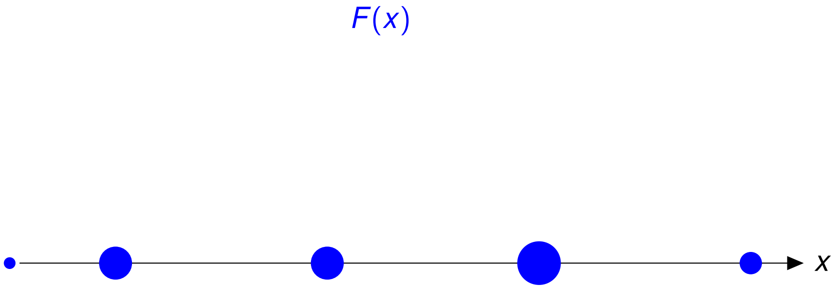
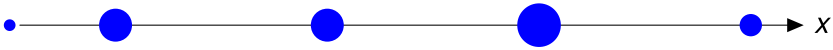


Sum Approximation - Sparsification

$$F^{\leq}(x) = \tilde{f^{\leq}}(x) \leq (1+\epsilon) f^{\leq}(x)$$



Sum Approximation - Properties

Claim

The size of F is $|F| = |r| = \log_{(1+\epsilon)} M$.

Where M is the sum of all values of f.

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For #Knapsack:
$M \leq$ #subsets of $W = 2^n$

Claim

The size of F is $|F| = n/\epsilon$

Sum Approximation - Properties

Lemma

Let F, G be $(1+\epsilon)$-sum approximation of f, g.

Approximation: *A $(1+\epsilon')$-sum approximation of F is a $(1+\epsilon')(1+\epsilon)$-sum approximation of f.*

Summation: *$(F+G)$ is a $(1+\epsilon)$-sum approximation of $(f+g)$.*

Shifting: *$F(x-w)$ is a $(1+\epsilon)$-sum approximation of $f(x-w)$ for any $w>0$.*

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Convolution: *$(F*G)$ is a $(1+\varepsilon)^2$-sum approximation of $(f*g)$.*

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Let $k_S(x)$ be a function that equals to the number of subsets of the set S with a total weight of **exactly** x.

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K_S is the sum-approximation of k_S.

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As in the naive algorithm

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n steps $\cdot |K_S|$ time $\Rightarrow O(n^3/\epsilon)$

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Key observation:

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$O(n^{2.5}\varepsilon^{-1.5}\log(n\varepsilon^{-1})\log(n\varepsilon))$ time and $O(n^{1.5}\varepsilon^{-1.5})$ space.

Sparsification parameter should be adjusted for every level of the recursion. See the paper.

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Open Problems

- Ignore the dependency on ϵ (constant), is there an algorithm with running time of $\tilde{O}(n^{2.5-\alpha})$?
- Deterministic FPTAS with running time of $\tilde{O}(n^{2.5}\epsilon^{-1.5+\alpha})$?

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Thank You!

Questions?