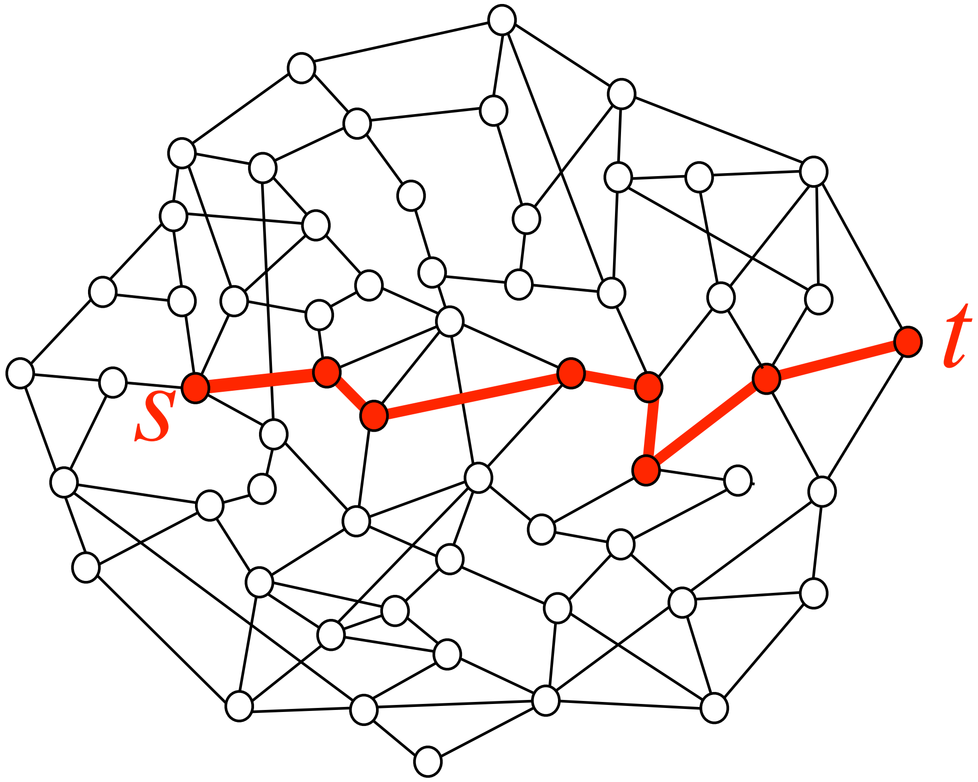
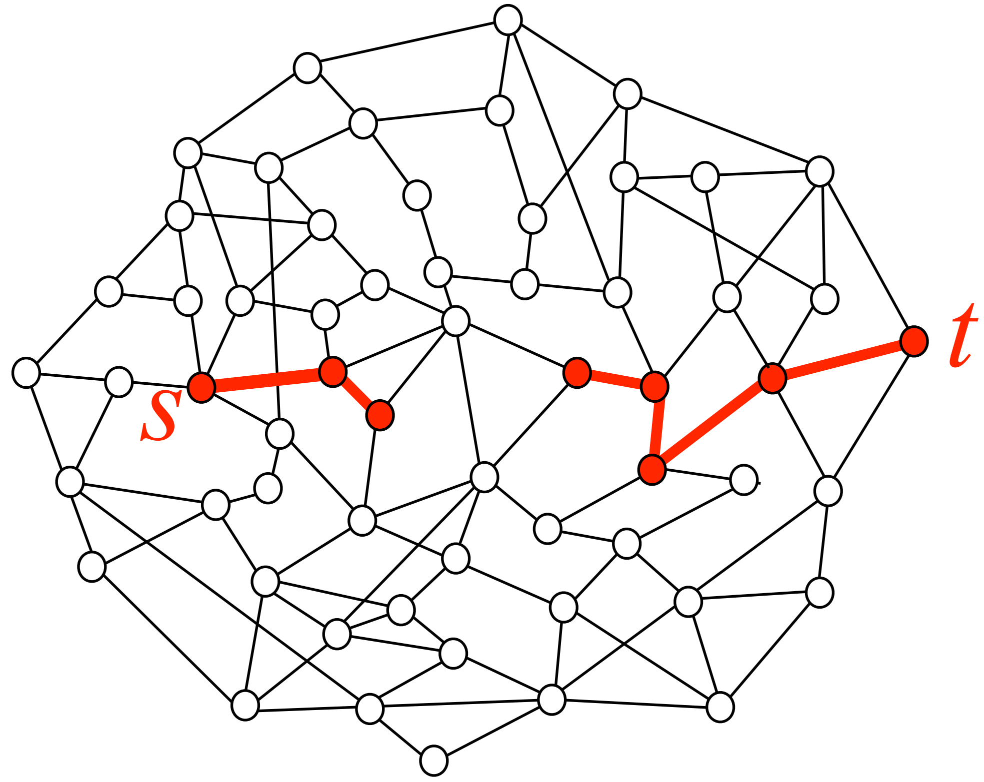
The Replacement Paths Problem



Oren Weimann

Raphy Yuster

The Replacement Paths Problem



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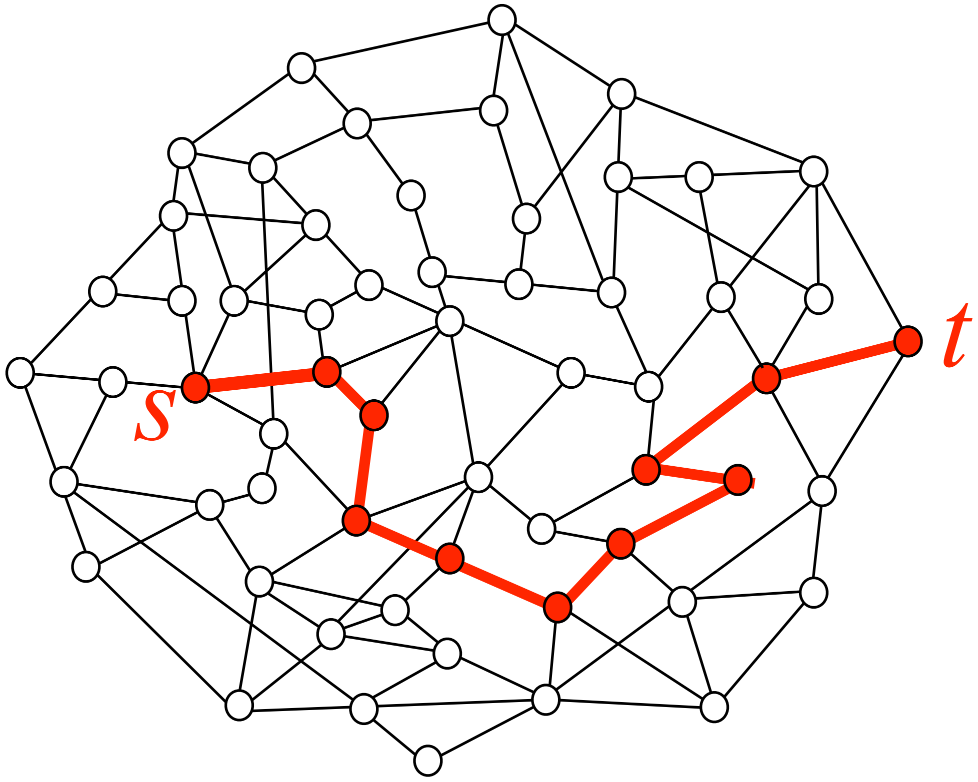

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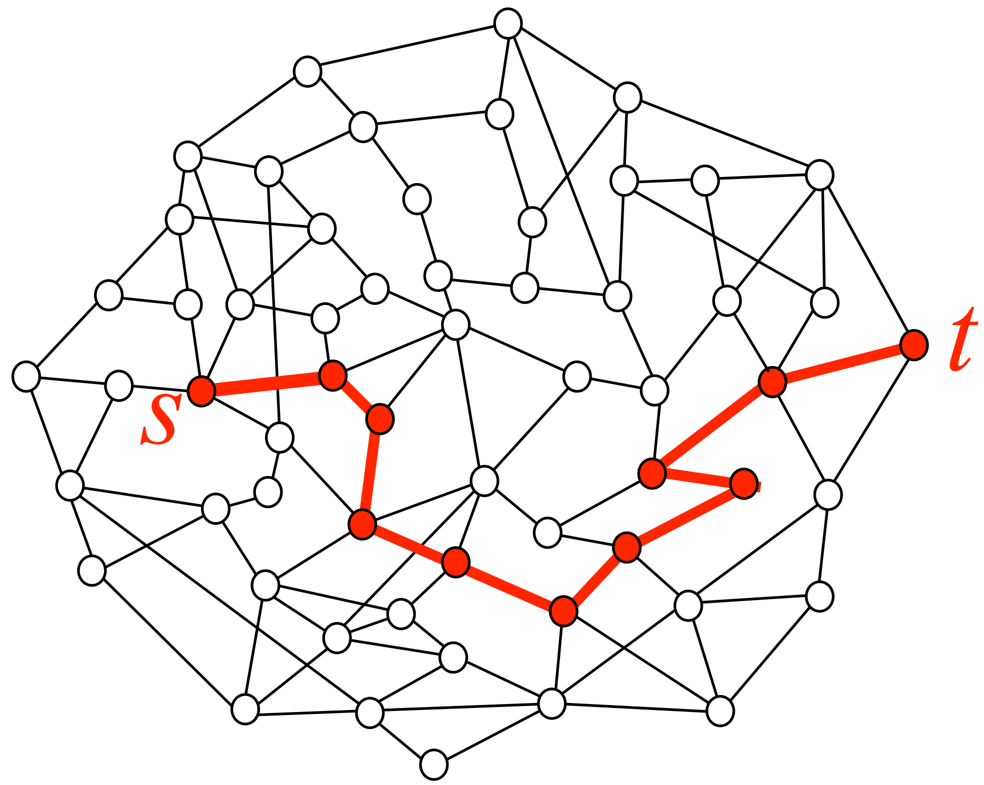
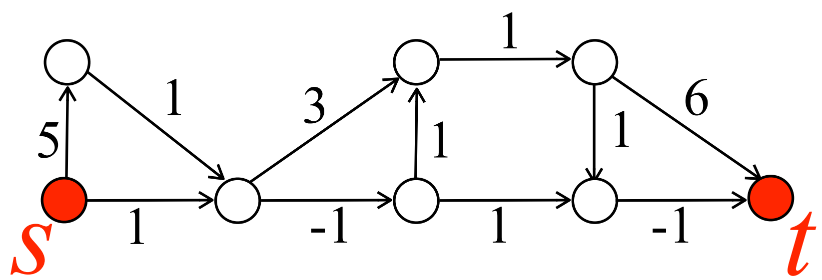


Vickrey pricing of edges owned by selfish agents

The Replacement Paths Problem

- General directed graphs
- Positive and negative lengths



The Replacement Paths Problem

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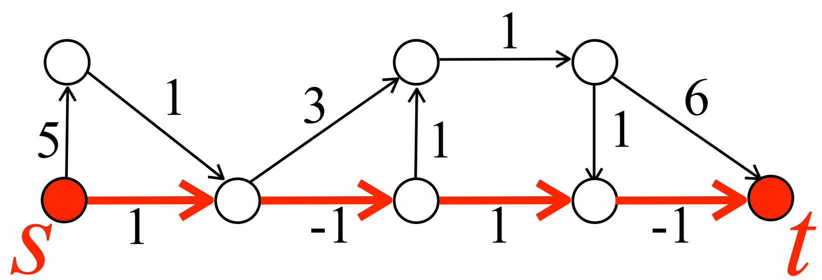

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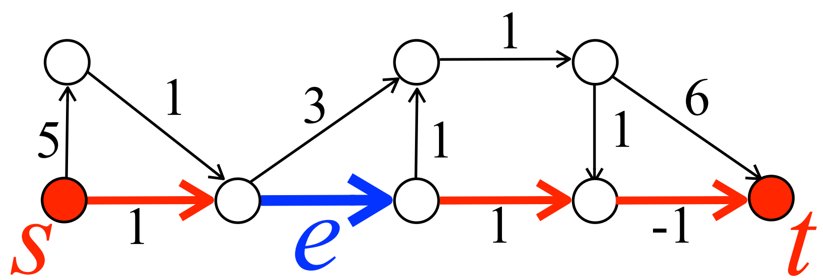

The Replacement Paths Problem

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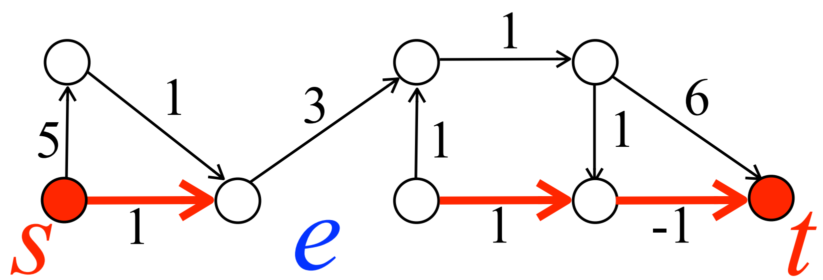

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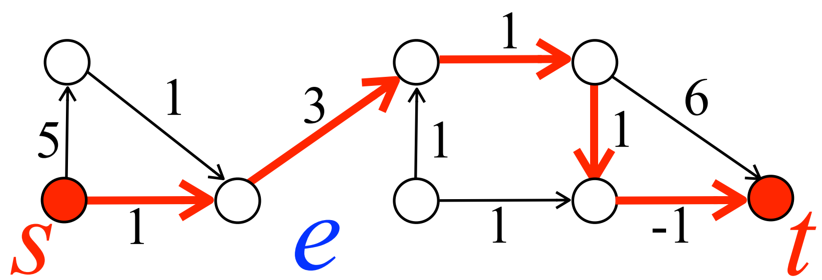
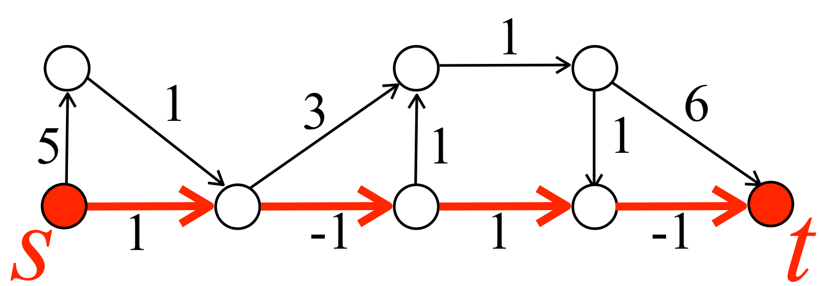


A Naive Solution



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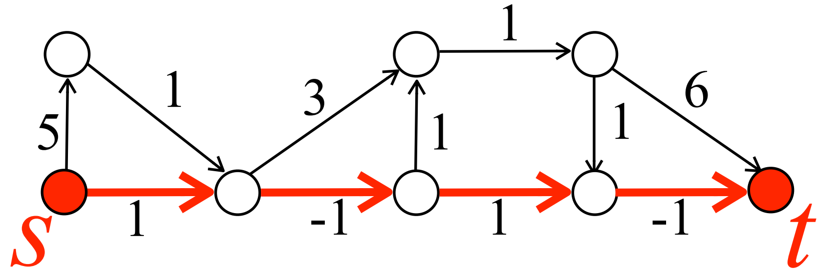

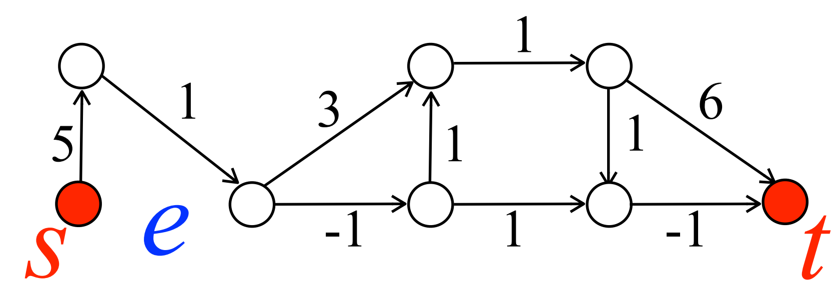

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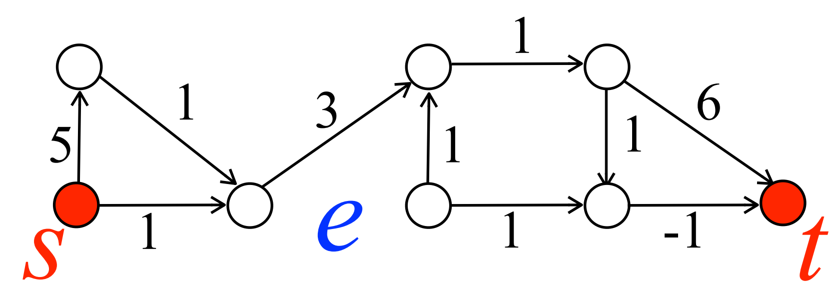

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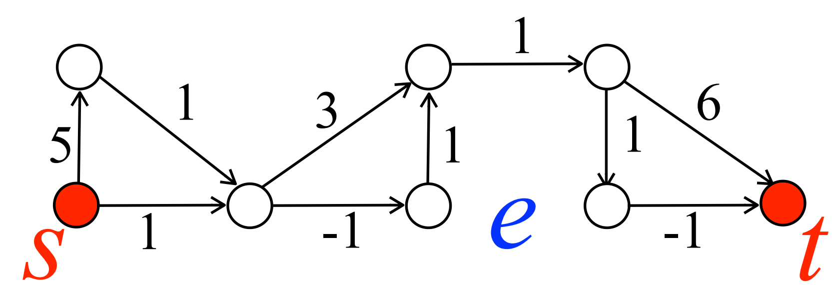

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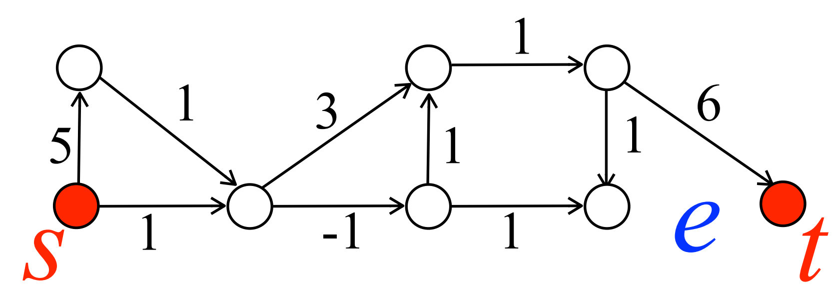

(n)

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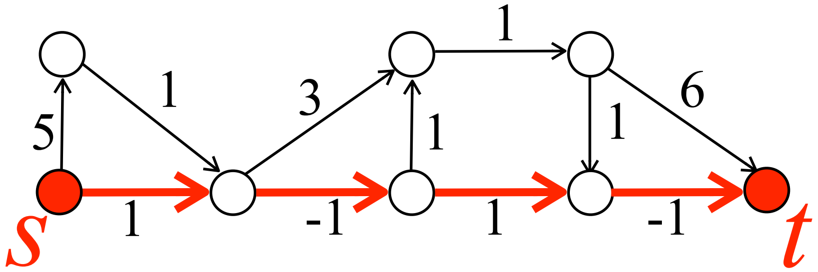
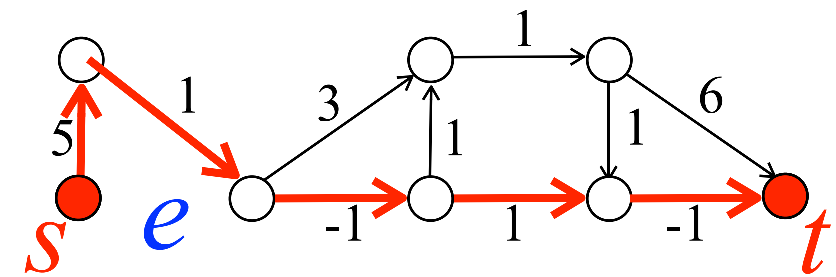
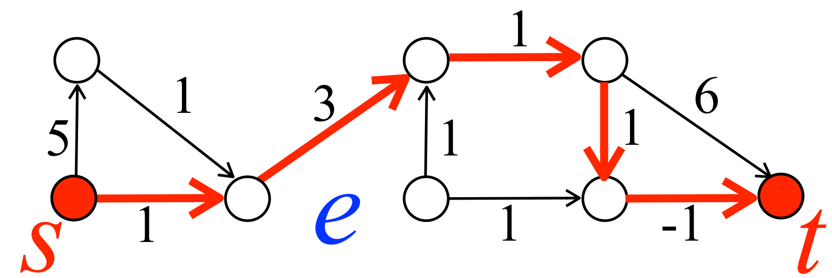




(1)



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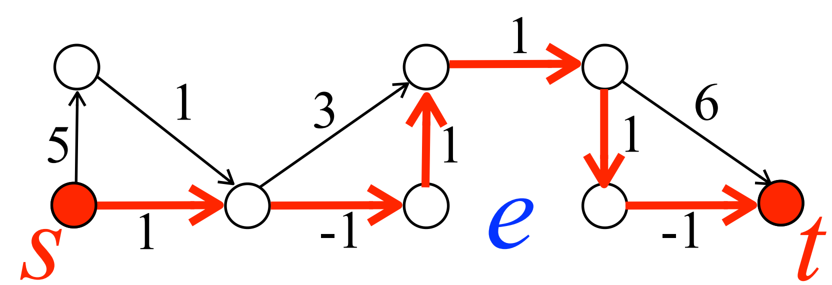

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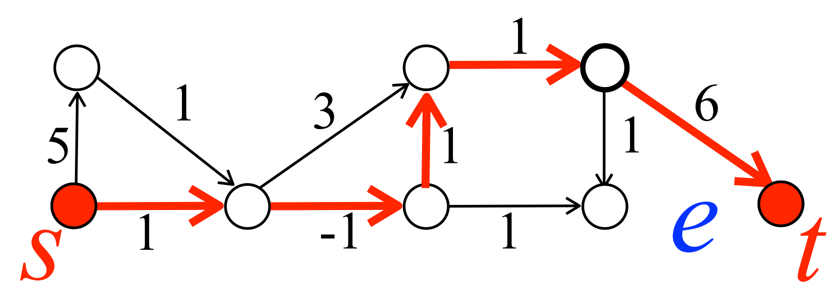
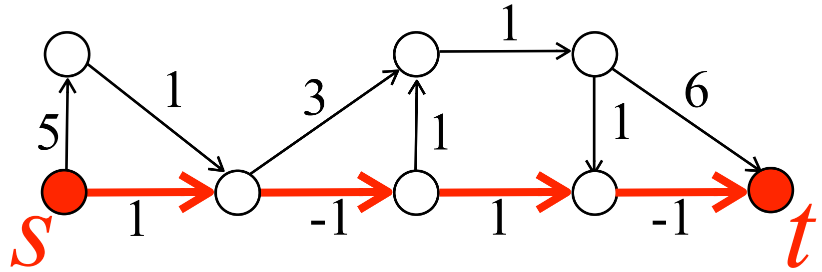
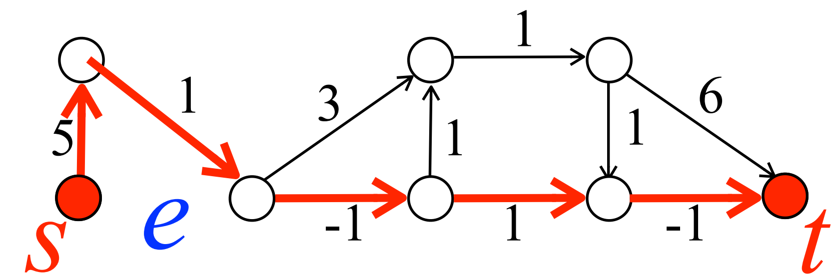


(n)

A Naive Solution

$$O(nm + n^2 \log n) = O(n^3)$$





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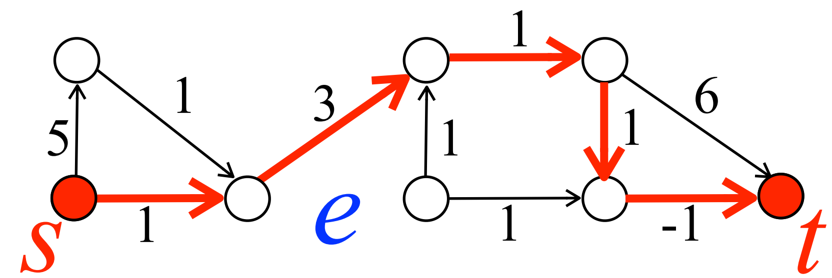
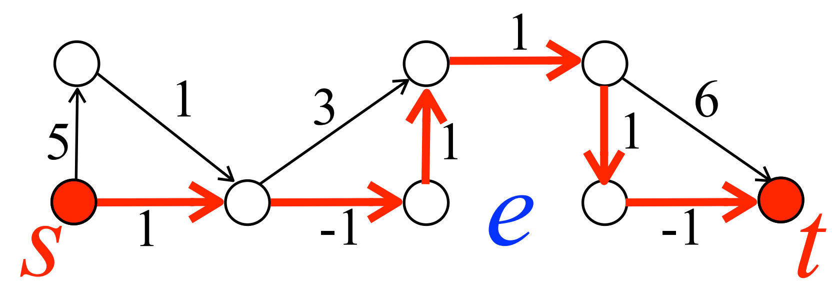


(2)



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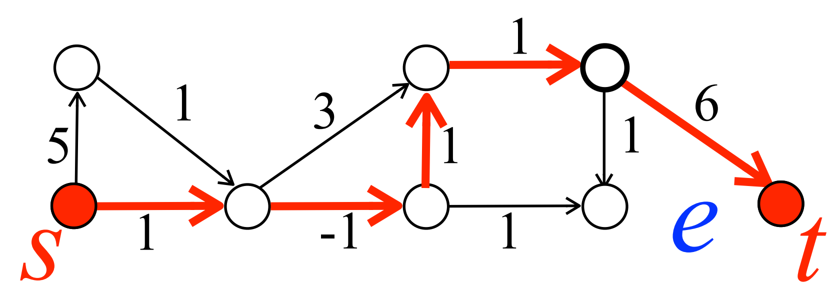

(n)

Related Work

Directed weighted graphs:

- $O(nm + n^2 \log n)$ - naive solution
- $O(nm + n^2 \log\log n)$ - [Gotthilf, Lewenstein 2009]
- $\Omega(m\sqrt{n})$ lower bound - [Hershberger 2003]
- $\tilde{O}(m \log(nM)/\varepsilon)$ for $(1+\varepsilon)$-approximation - [Bernstein 2010]
- $\tilde{O}(Mn^{1+2\omega/3}) = O(Mn^{2.584})$ for weights in $\{-M,\ldots,M\}$ - [Yuster, W.]

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Undirected graphs:

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- $O(m\alpha(n,m))$ - [Nardelli, Proietti, Widmayer 2001]

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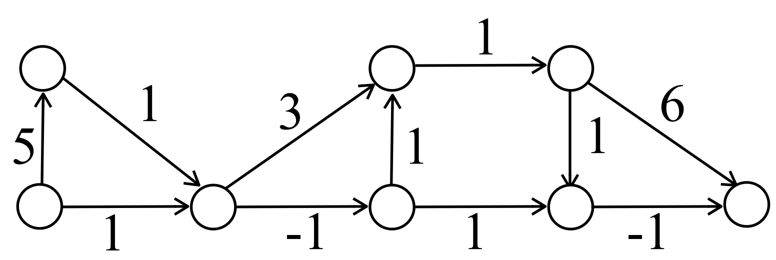
Unweighted graphs:

- $\tilde{O}(m\sqrt{n})$ - [Roditty and Zwick 2005]

The Replacement Paths Problem

All Pairs



The Replacement Paths Problem

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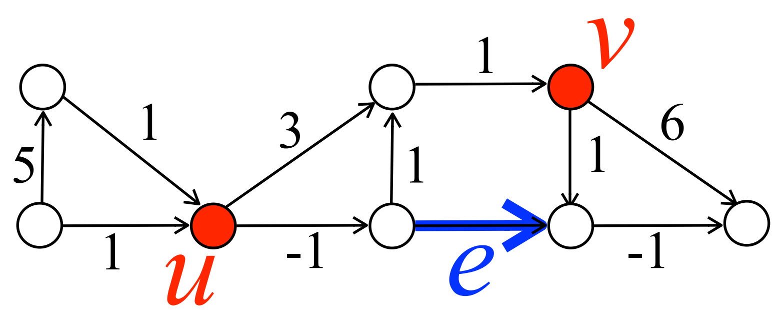
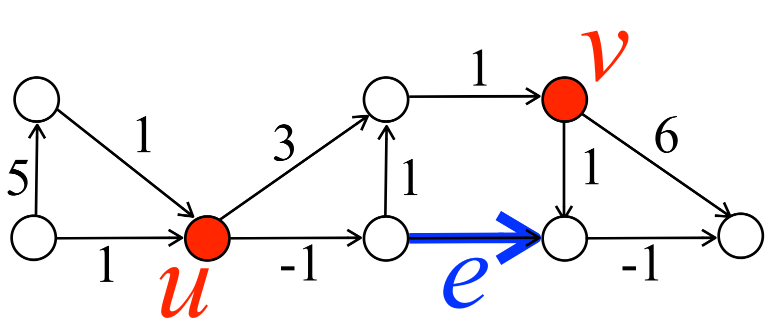


Query: (u,v,e)?

The Replacement Paths Problem

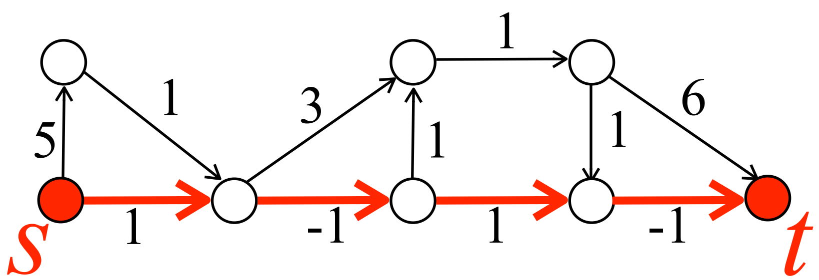
All Pairs

- $\tilde{O}(mn^2)$, constant query time
  [Demetrescu, Thorup, Chowdhury,, Ramachandran 2008]
- $\tilde{O}(mn)$, constant query time
  [Bernstein, Karger 2009]
- Poly., $O(\log n)$ query time, 2 failures
  [Duan, Pettie 2009]
- Poly., $\tilde{O}(f)$ query time, f failures
  constant-approximation
  [Chechick, Langberg, Peleg, Roditty 2009]
- $\tilde{O}(Mn^{1+\omega-\alpha})$, $\tilde{O}(n^{1+\alpha})$ query time, f failures
  [Yuster, W.]



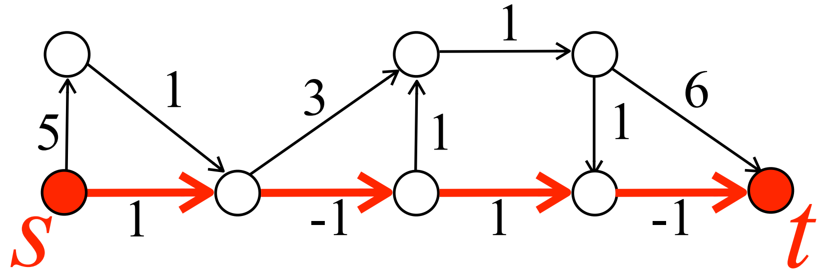
Query: (u,v,e)?

A Replacement Paths Algorithm



A Replacement Paths Algorithm

- Success probability 1-1/n



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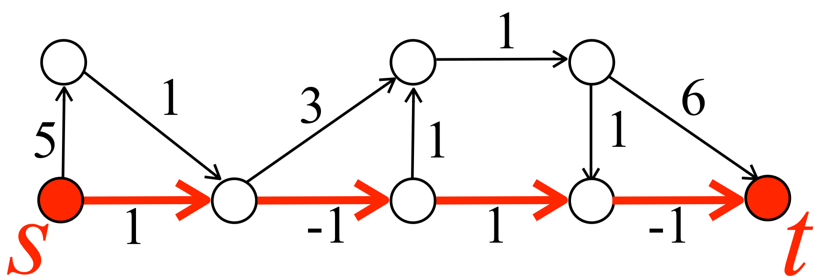
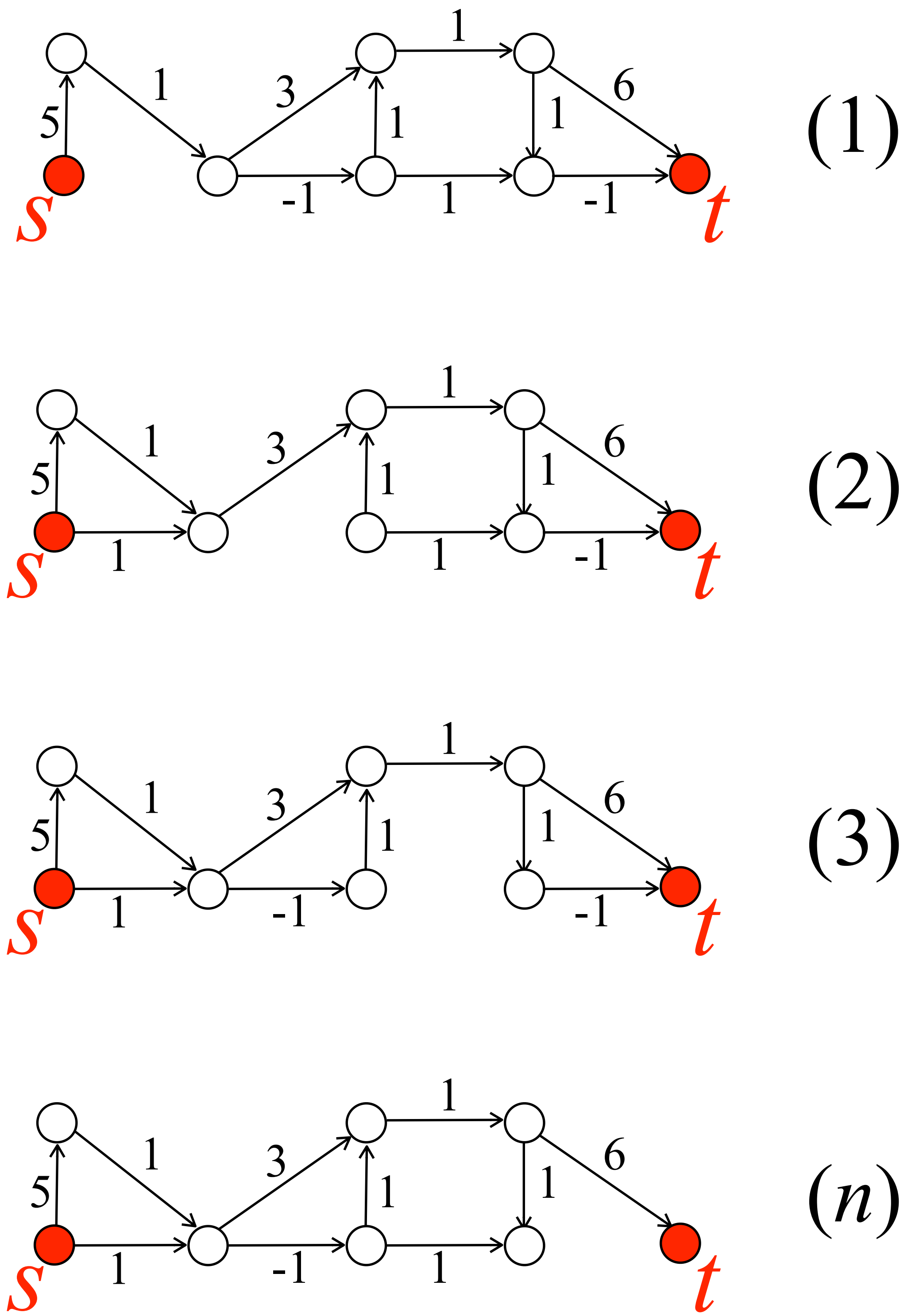
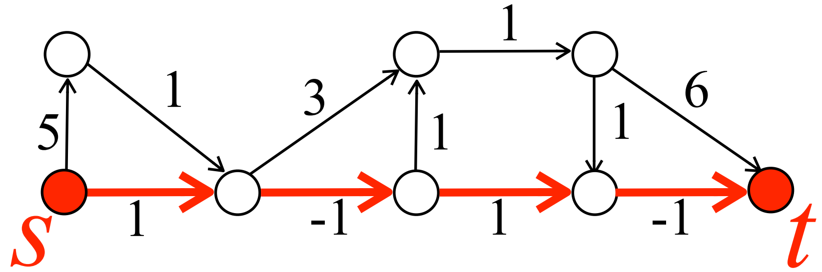




A Replacement Paths Algorithm

- Success probability 1-1/n
- $r = \tilde{O}(n^{1-\alpha})$ for $0 < \alpha < 1$



(1)

(2)

(3)

(r)

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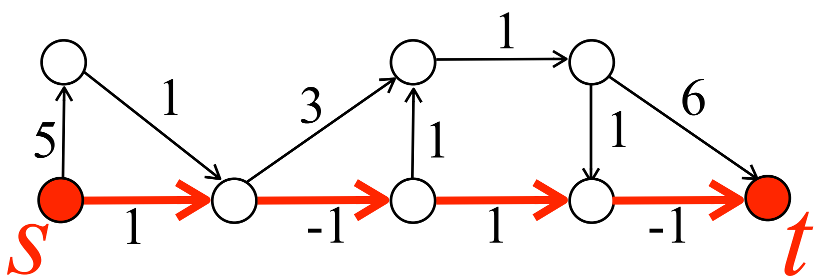

Remove every edge
with probability $n^{\alpha - 1}$

(1)

(2)

(3)

(r)

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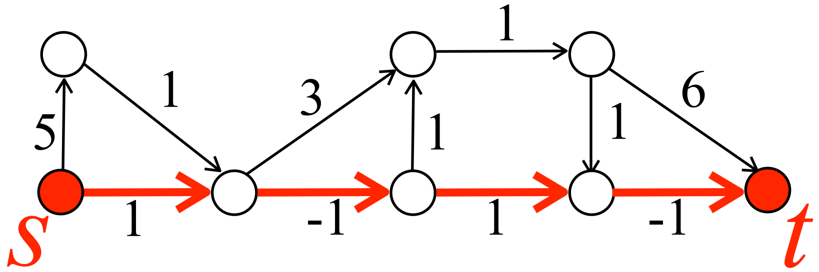

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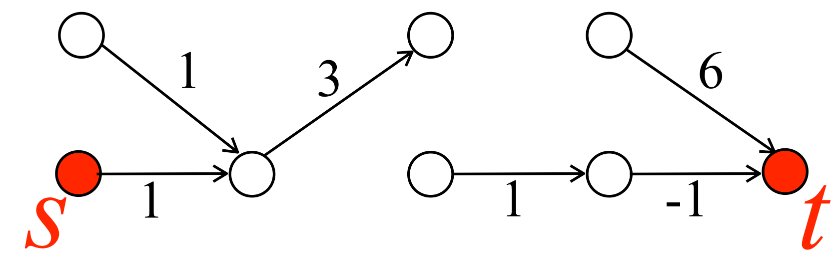

(1)

(2)

(3)

(r)

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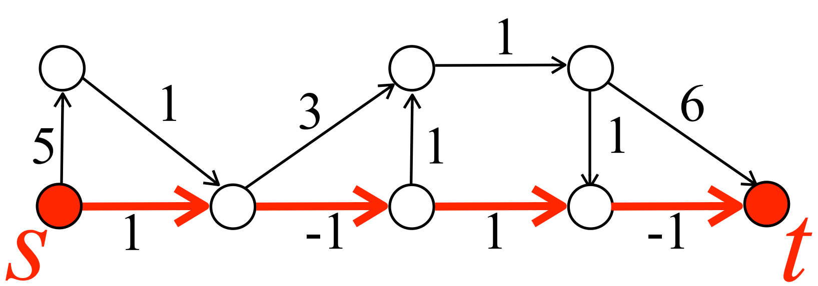

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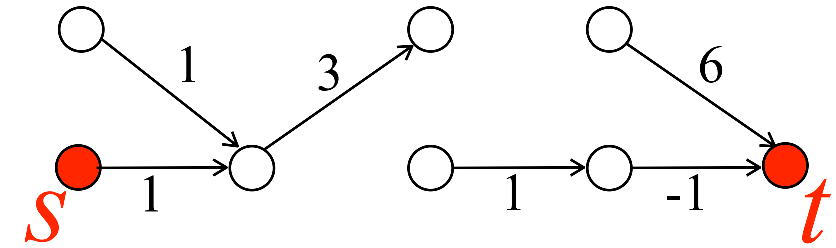
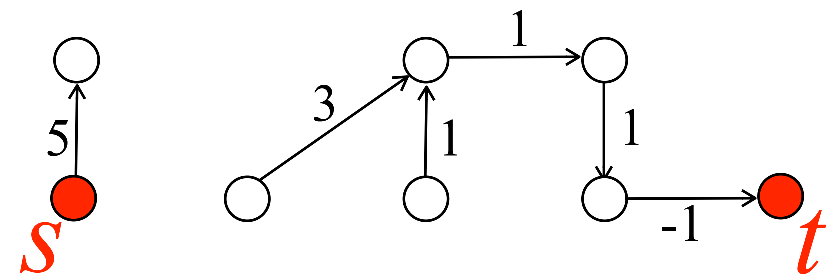


(1)



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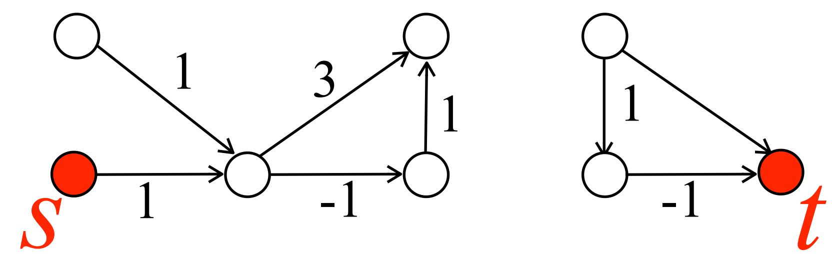

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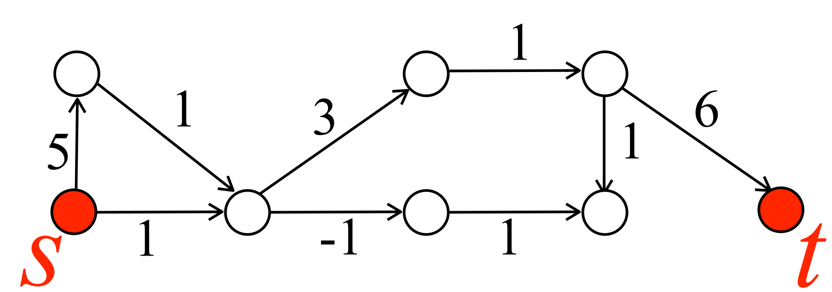

(*r*)

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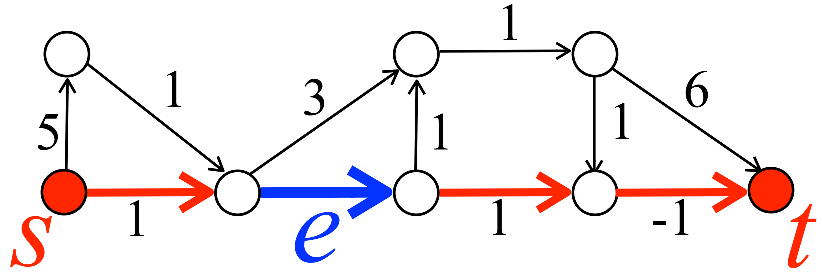

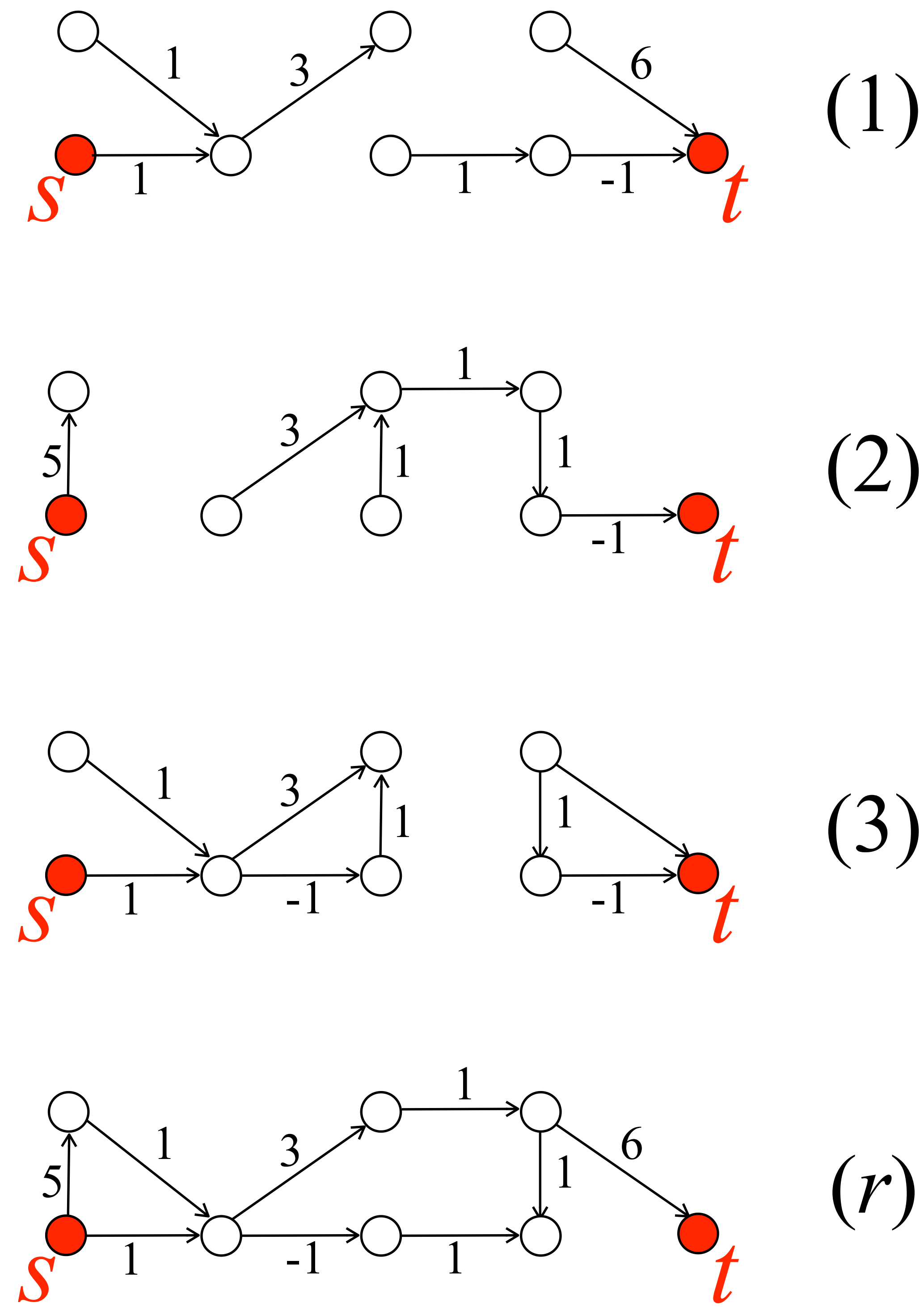

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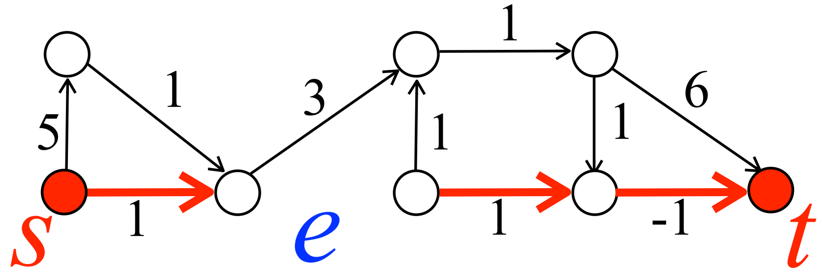

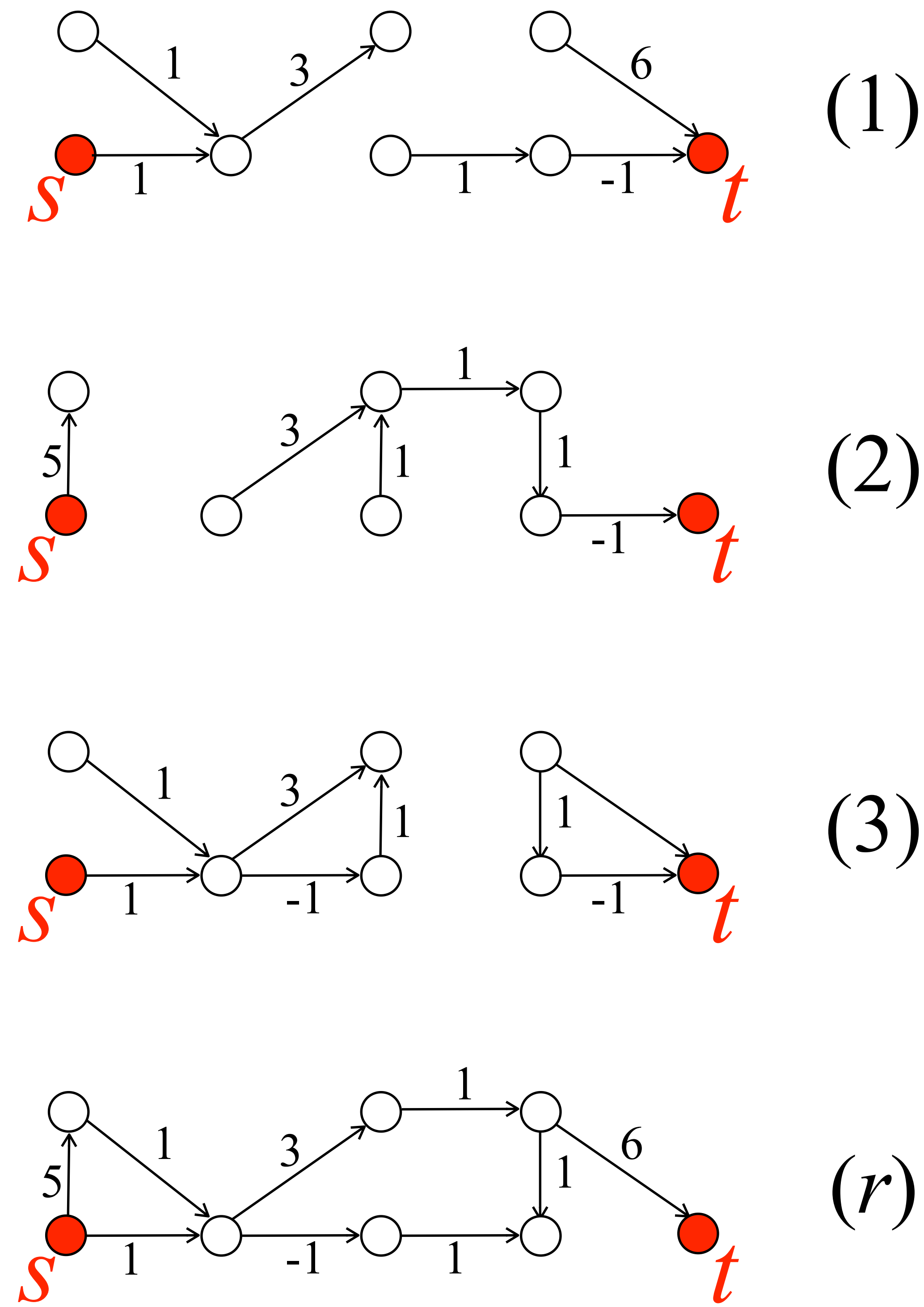

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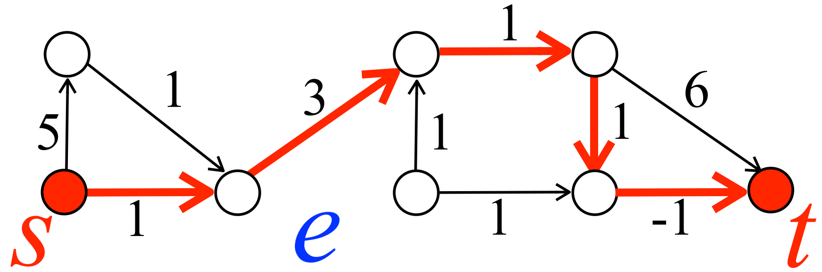

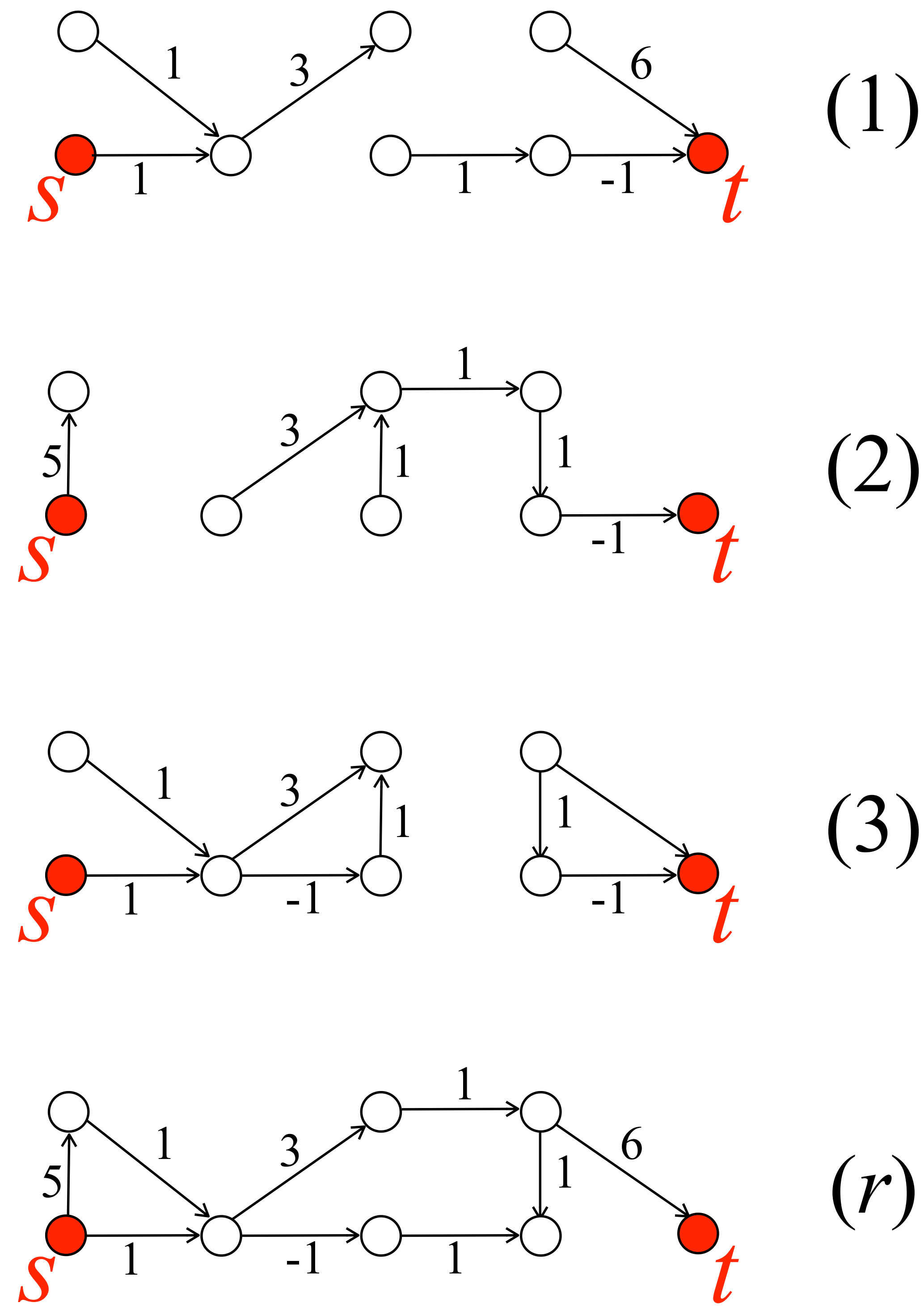

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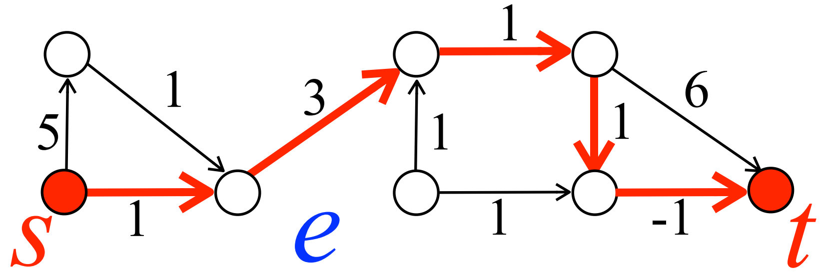

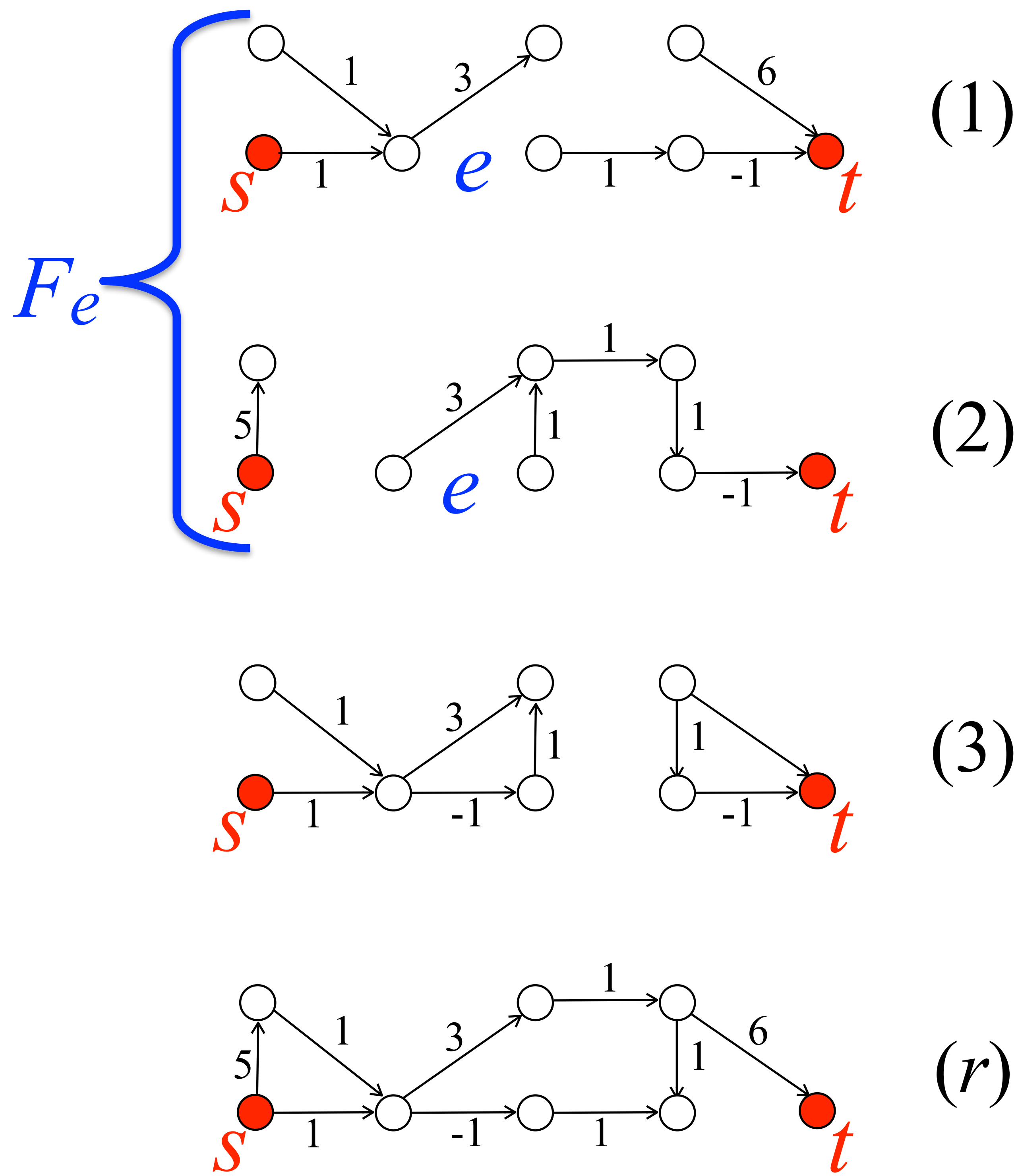

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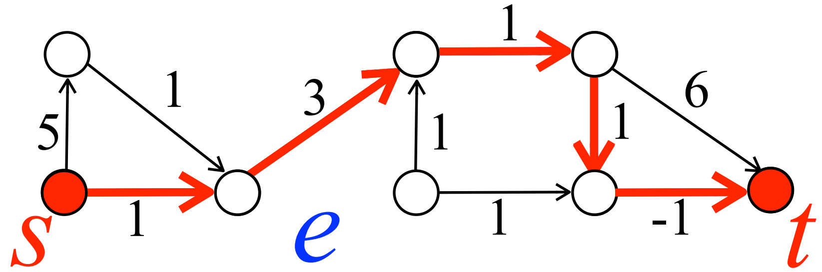
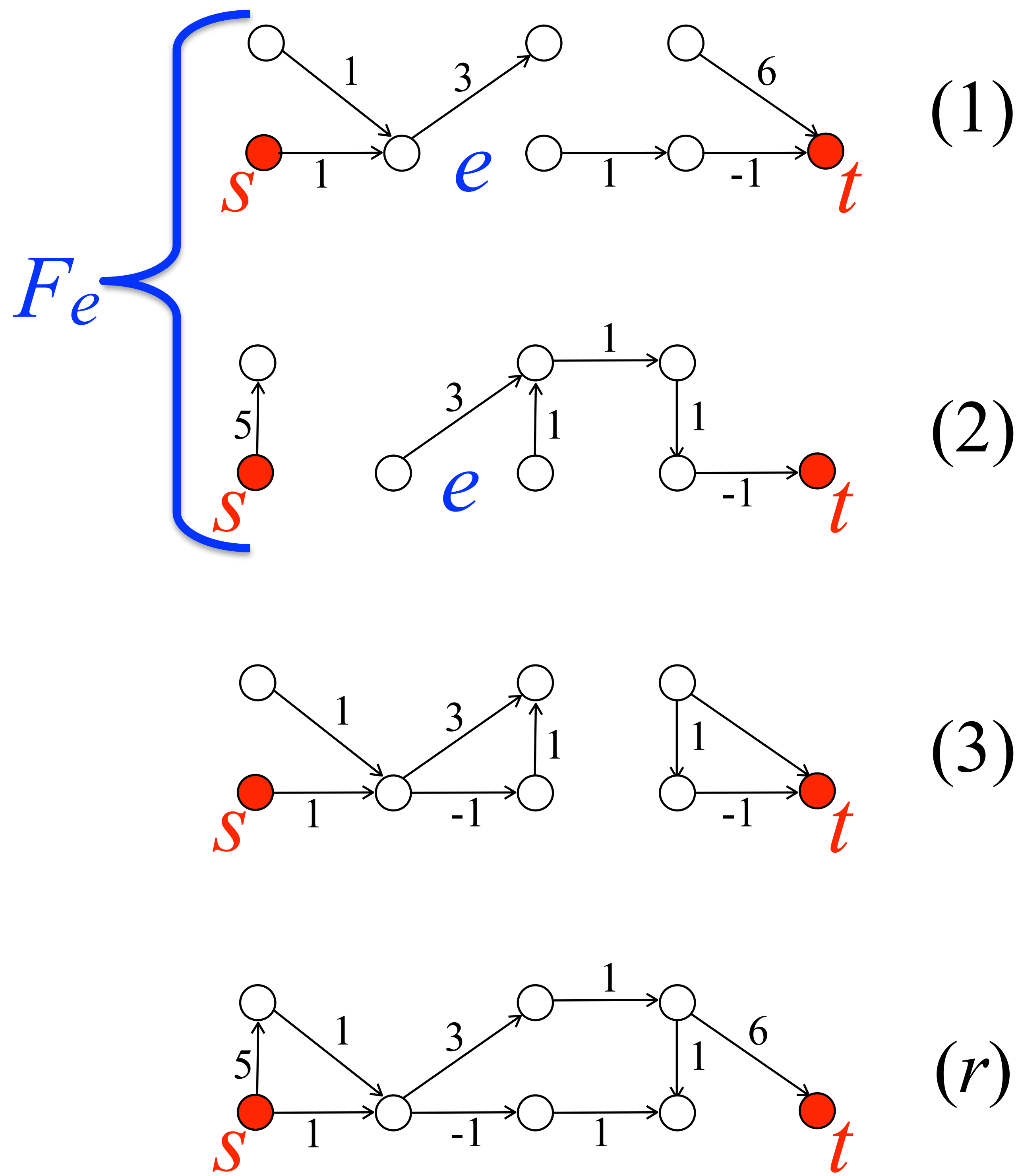


- With high probability
  - short subpaths survive



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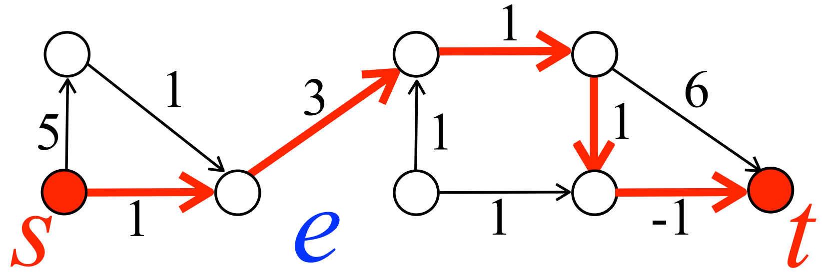

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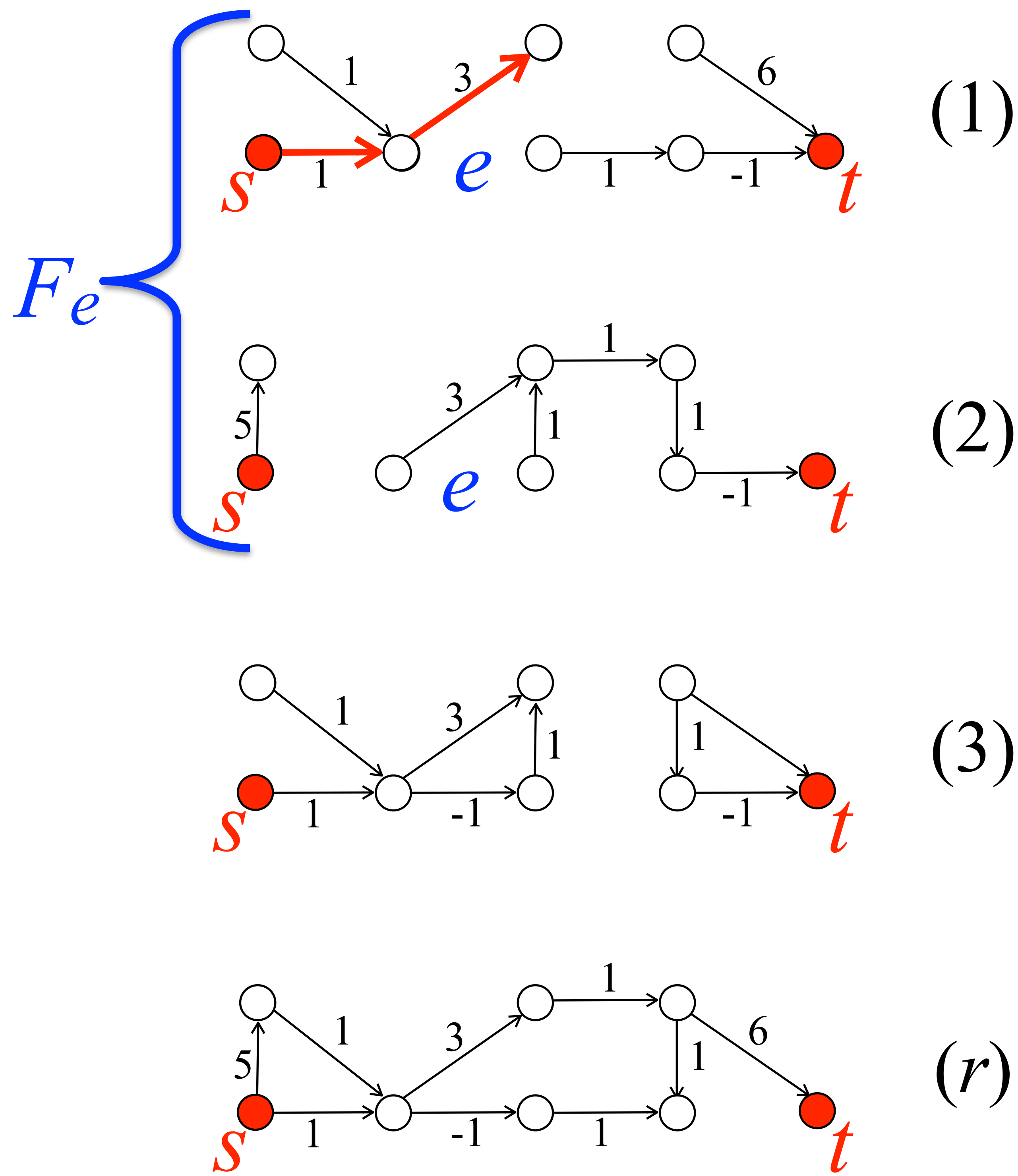

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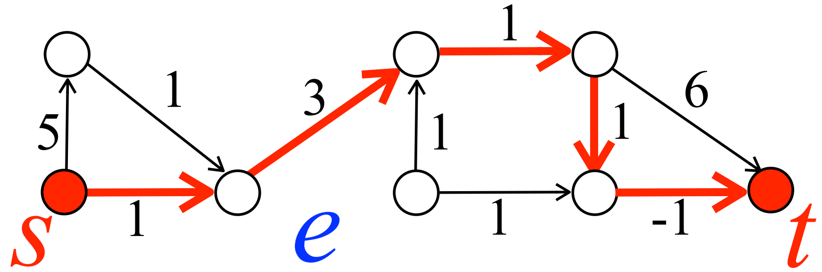

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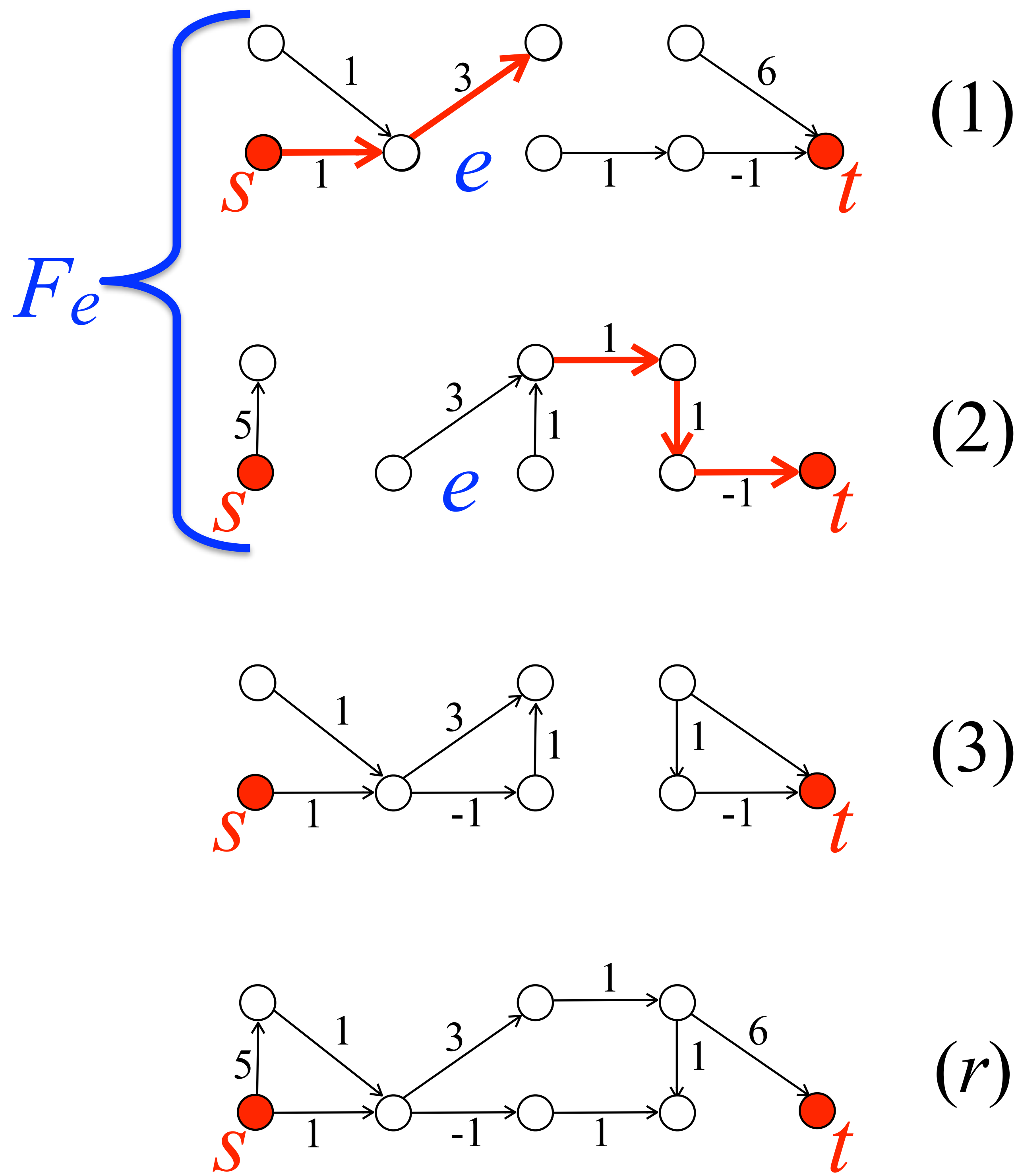

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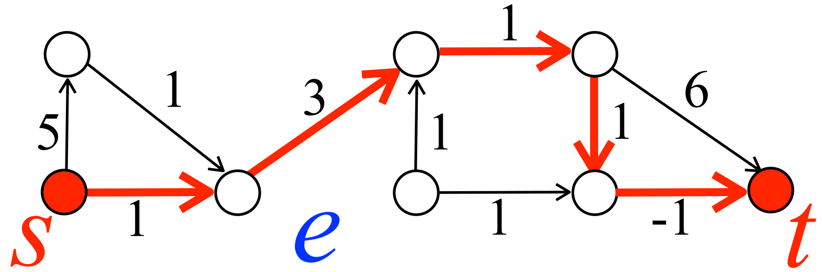

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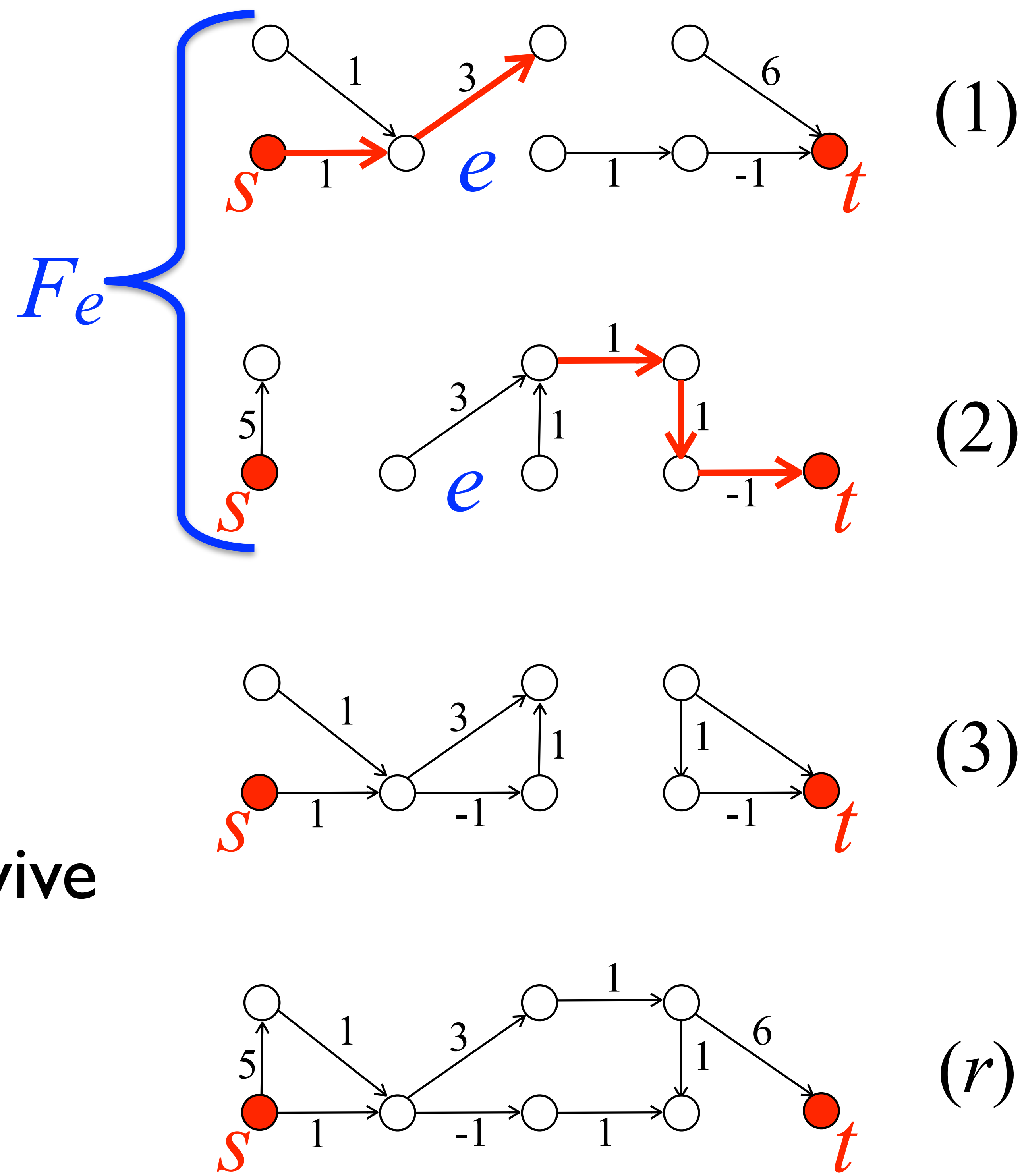
  - short $(n^{1-\alpha})$-subpaths survive



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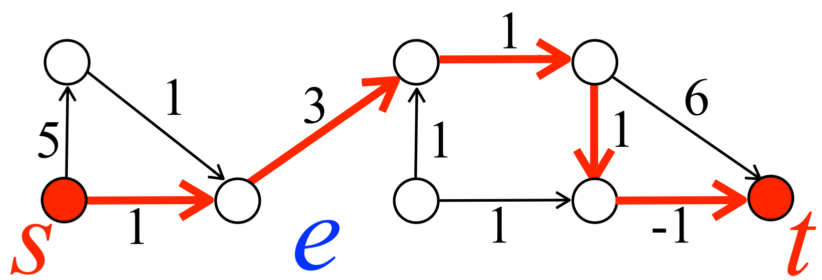

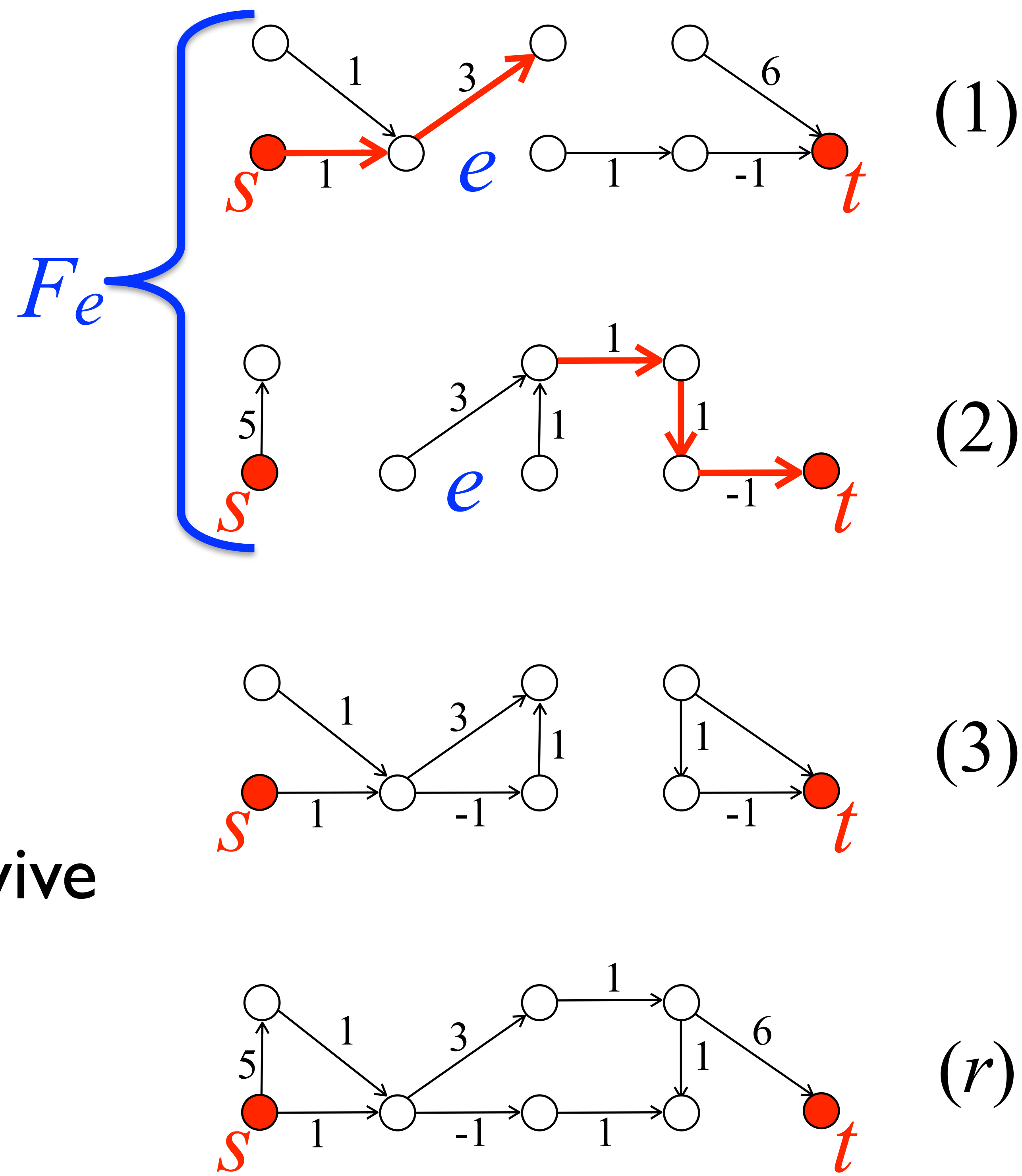

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  - $|F_e| = \Theta(\log n)$

A Replacement Paths Algorithm

- Success probability 1-1/n
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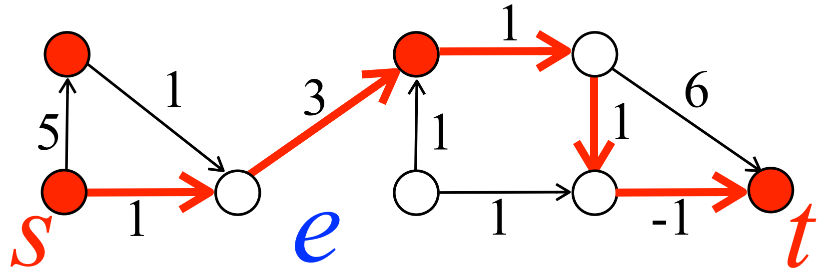
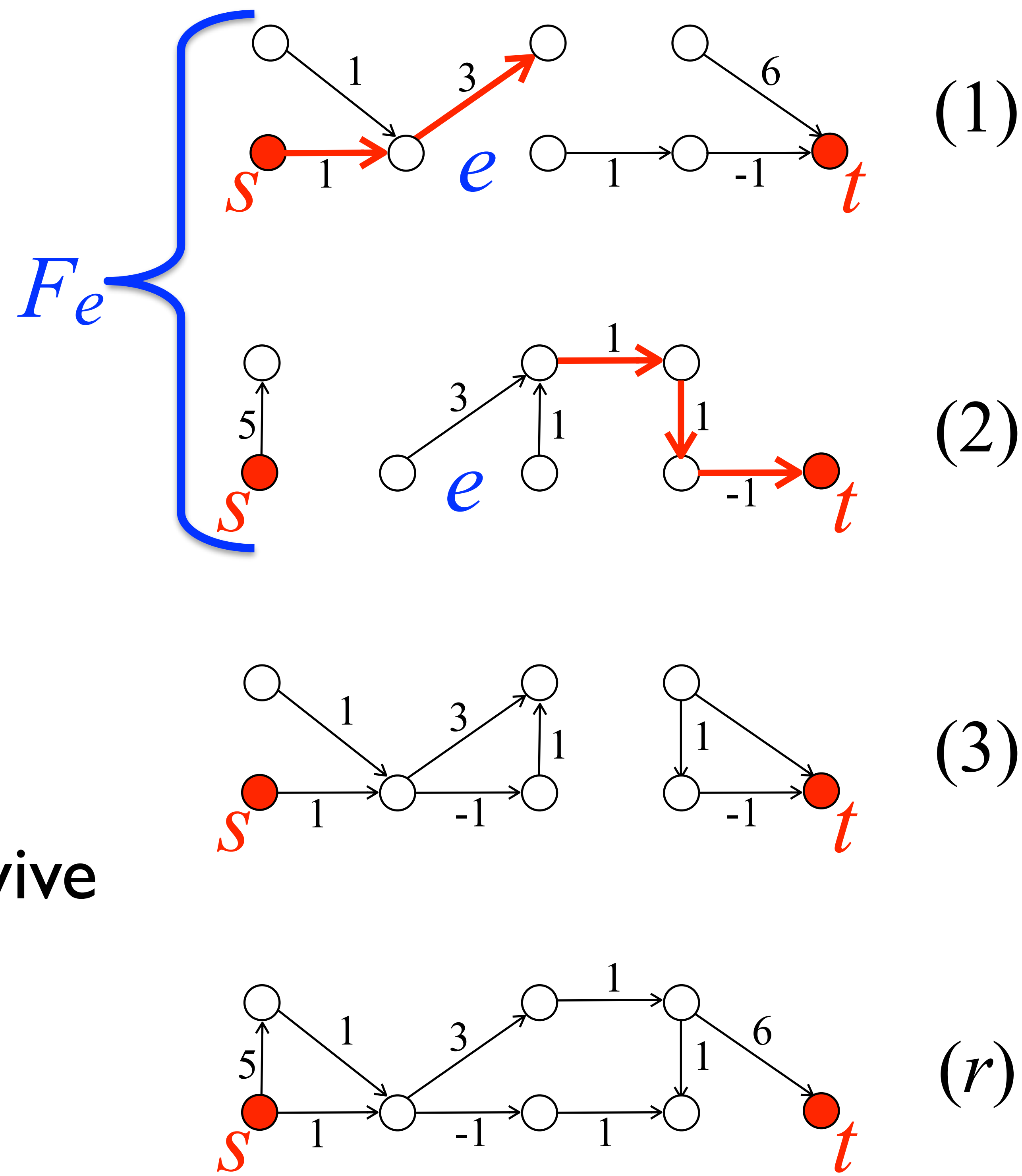


- With high probability
  - short $(n^{1-\alpha})$-subpaths survive
  - $|F_e| = \Theta(\log n)$



A Replacement Paths Algorithm

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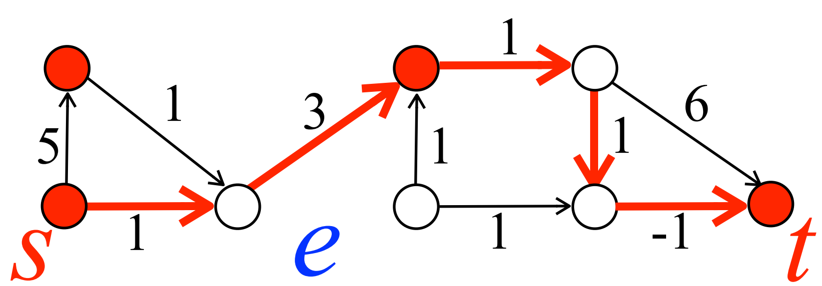

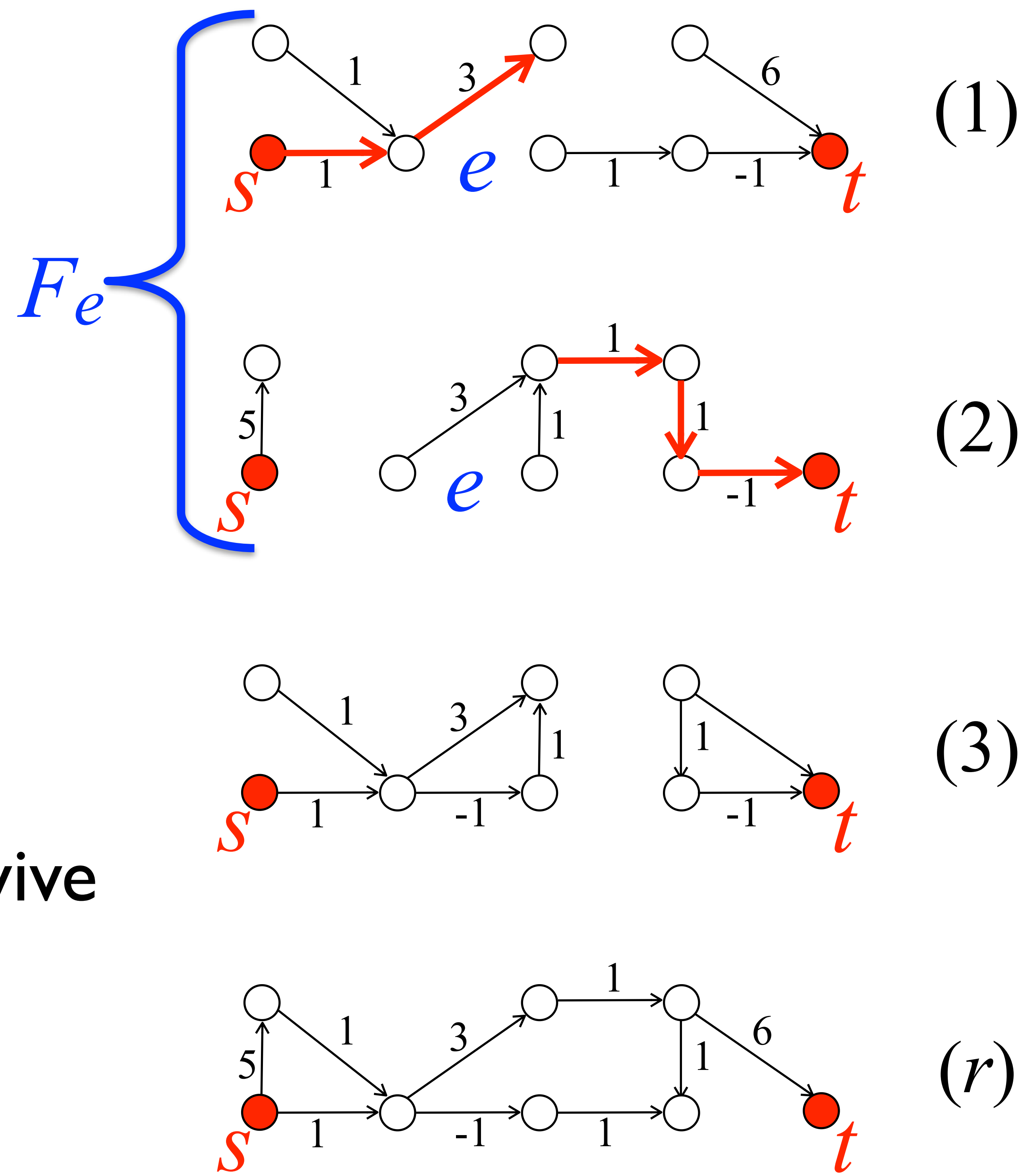

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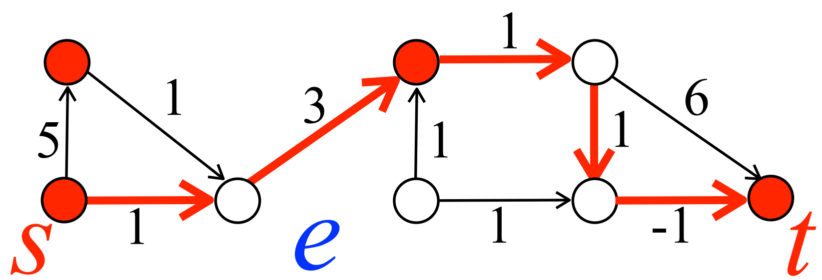

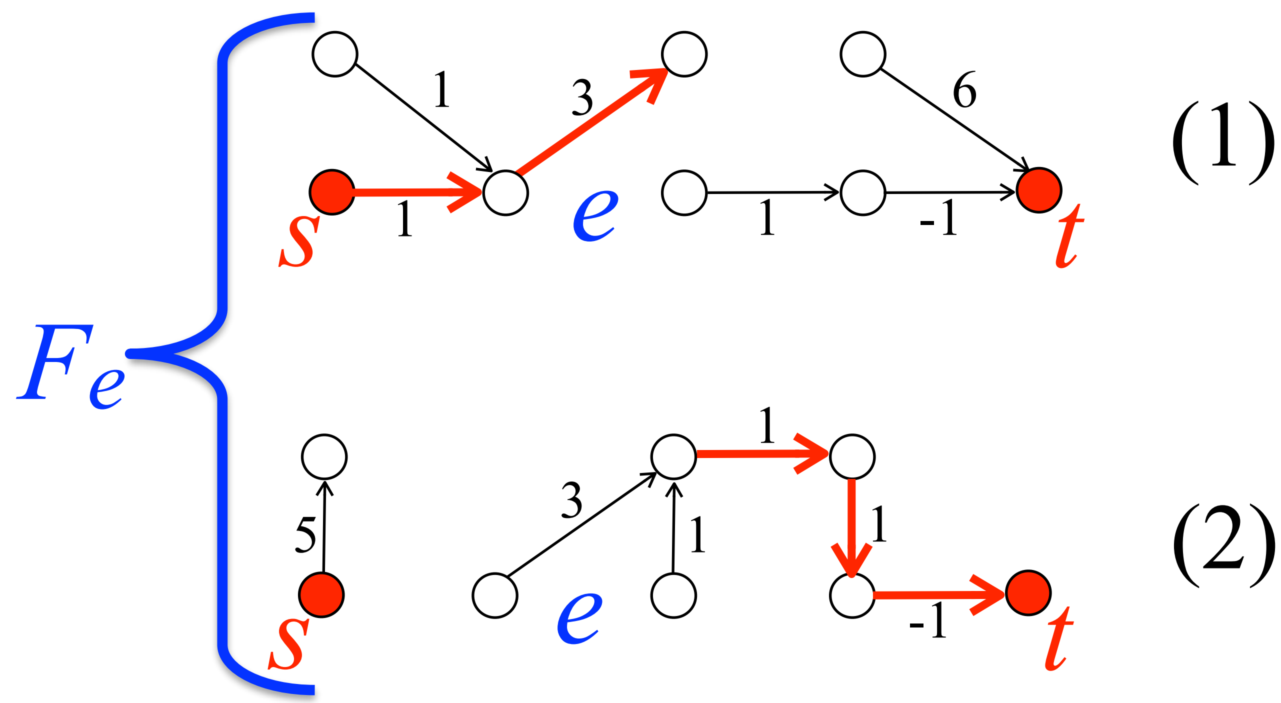

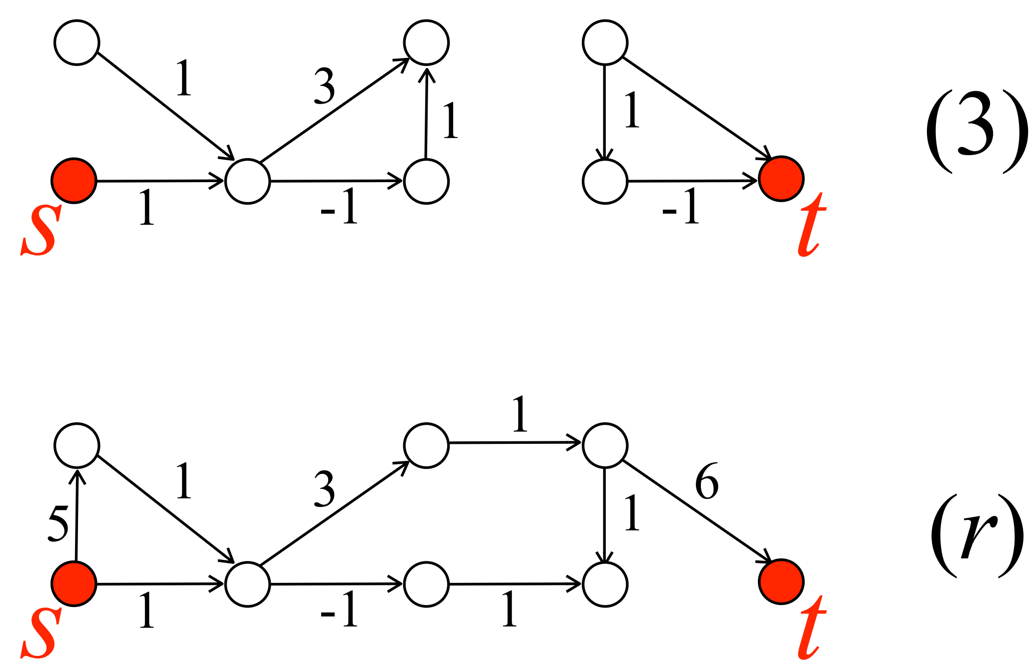

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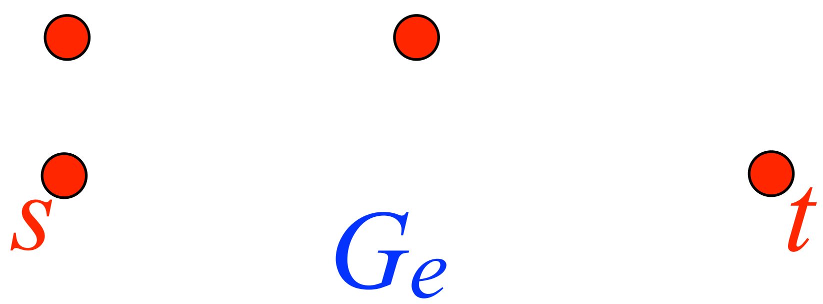

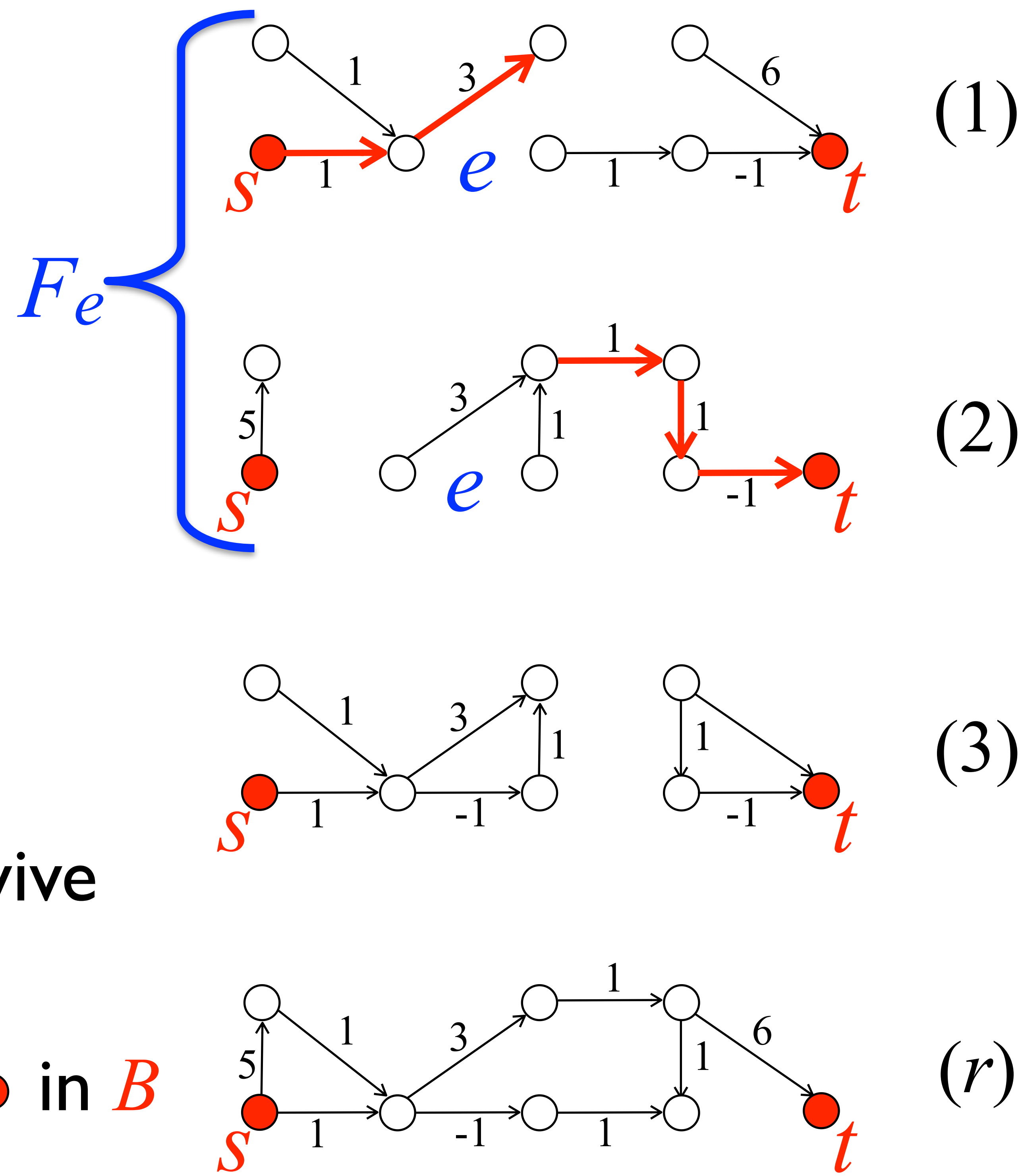

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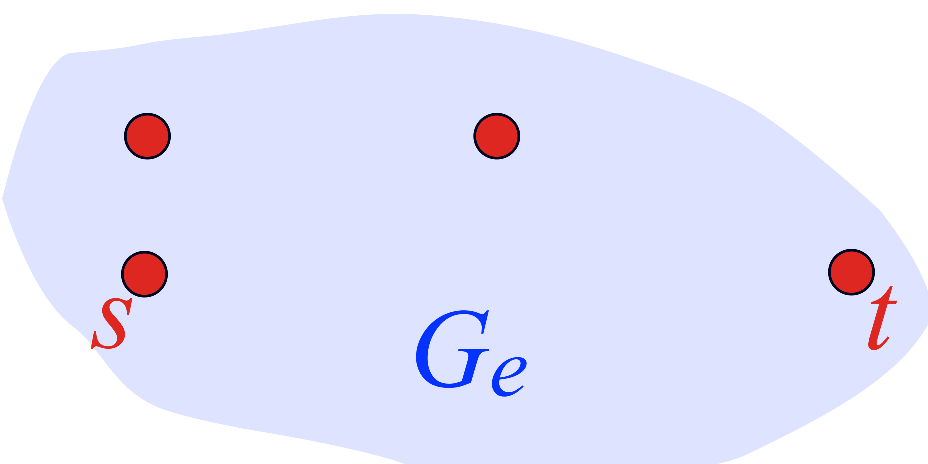

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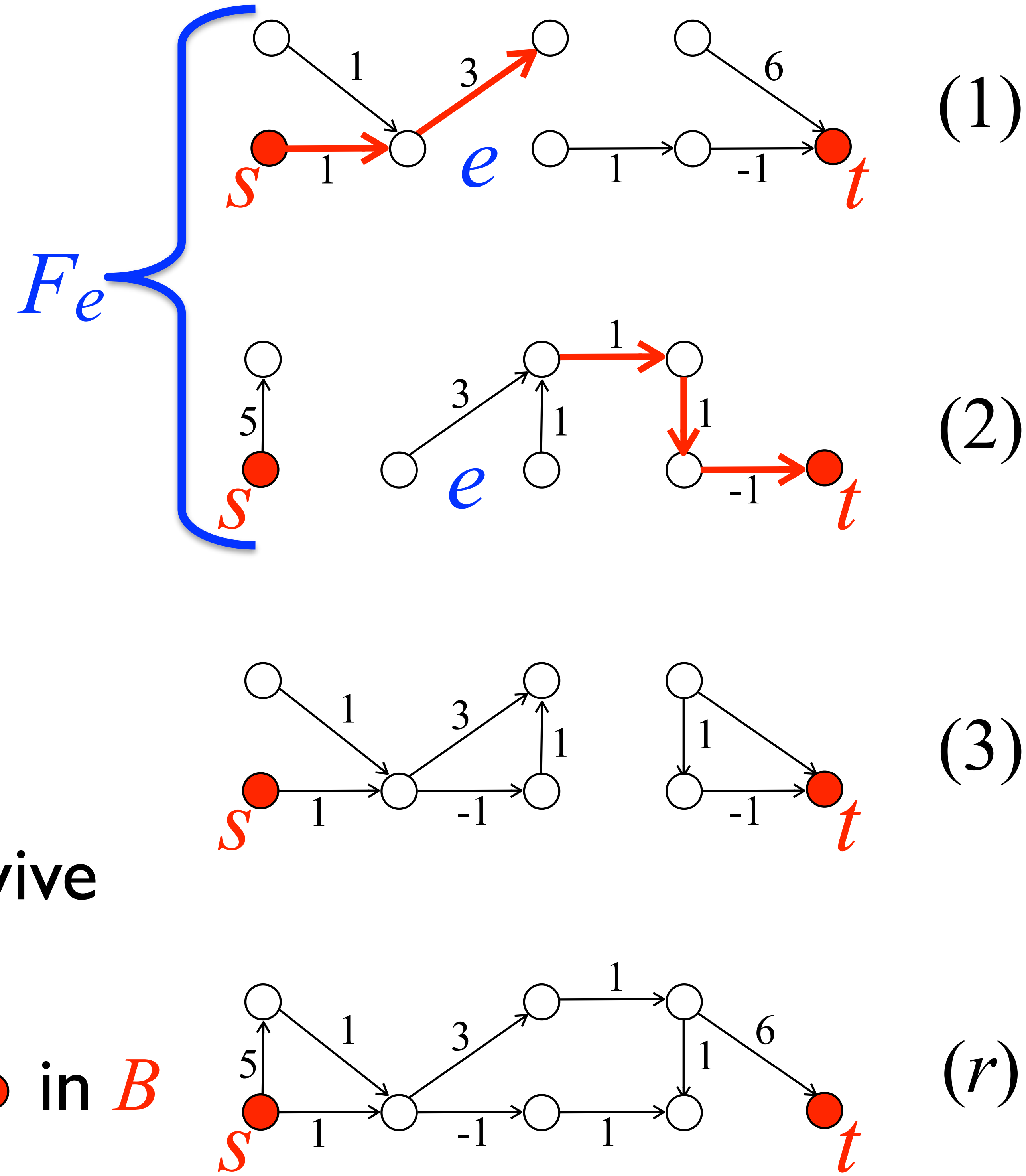

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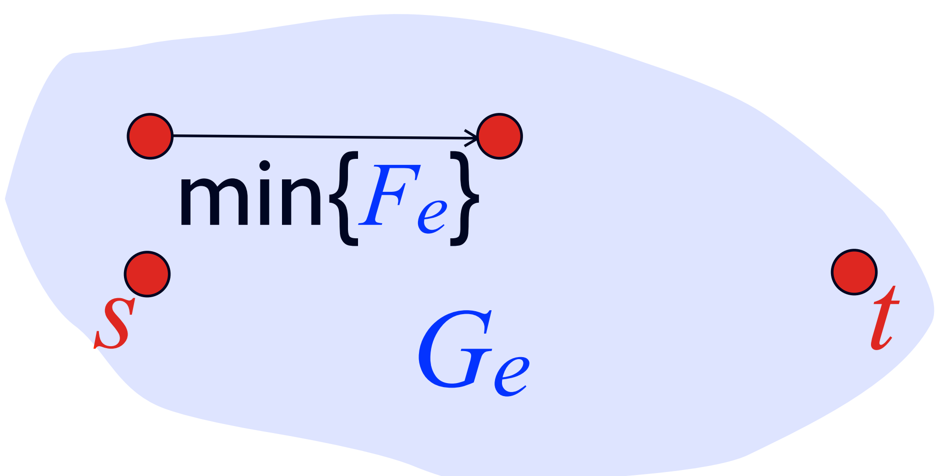

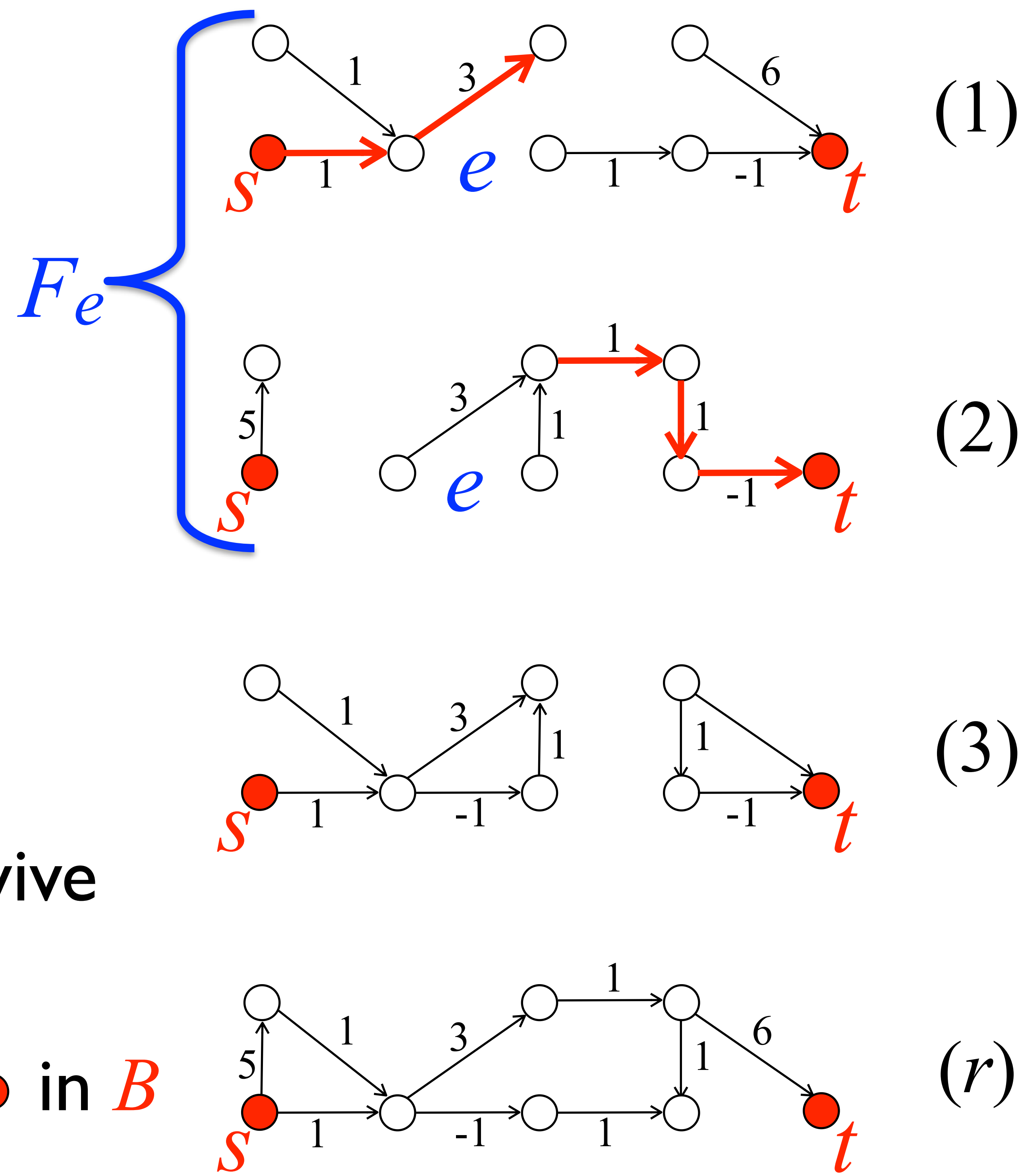

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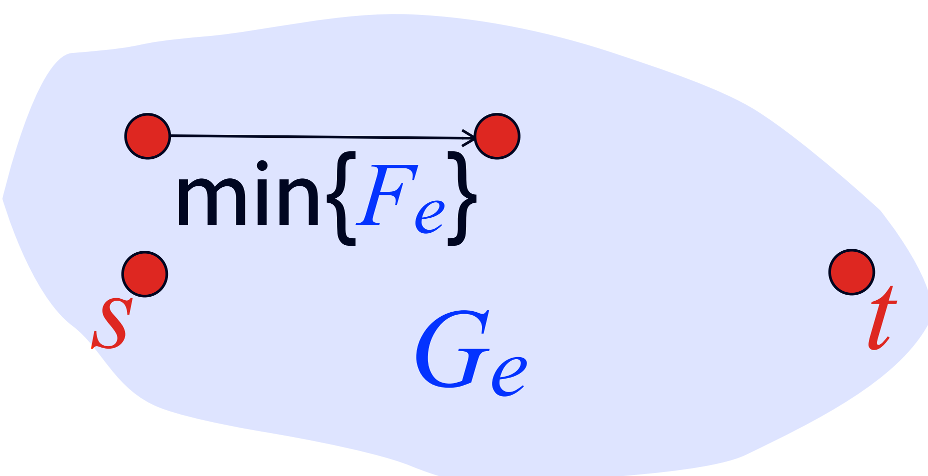
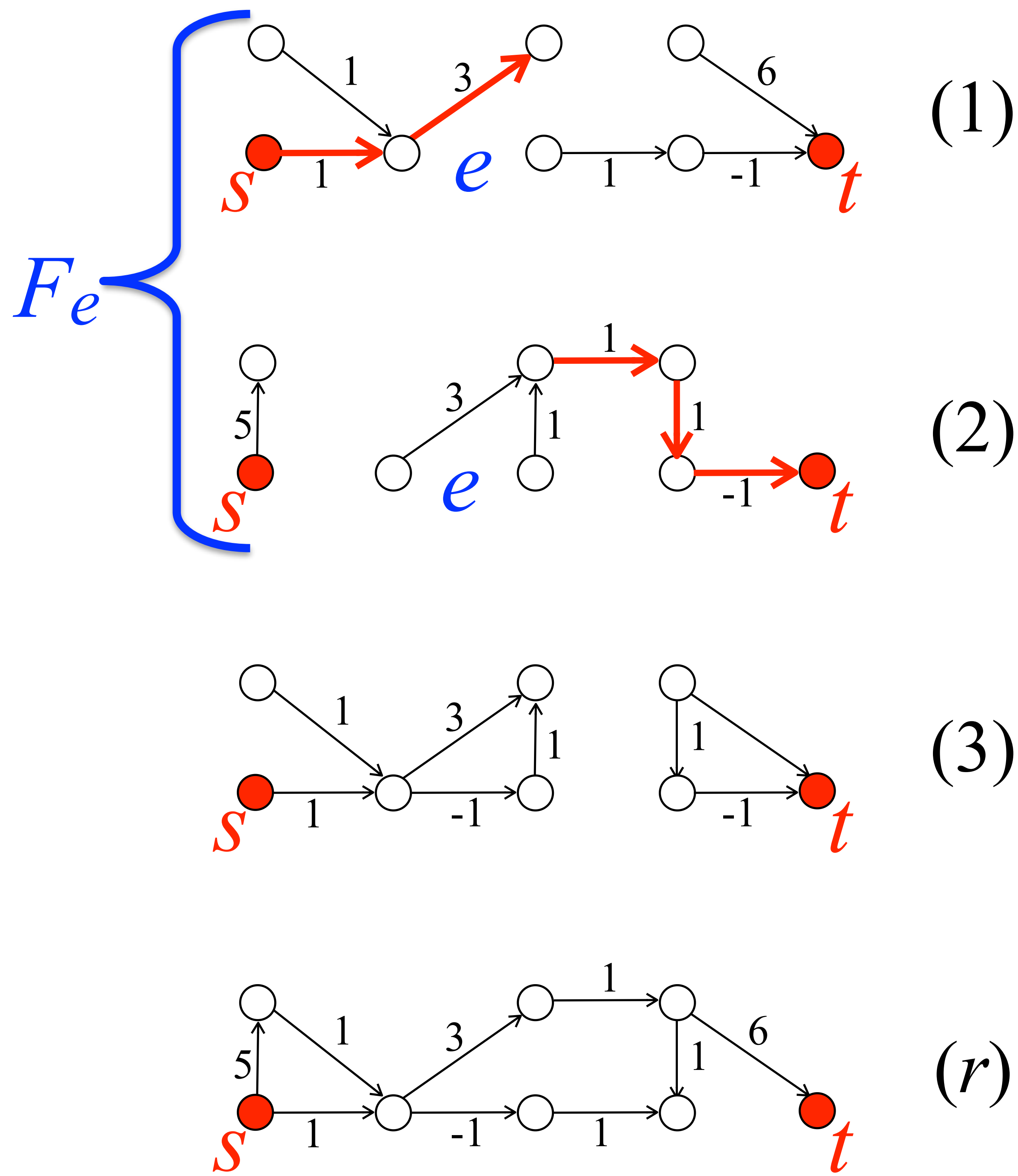
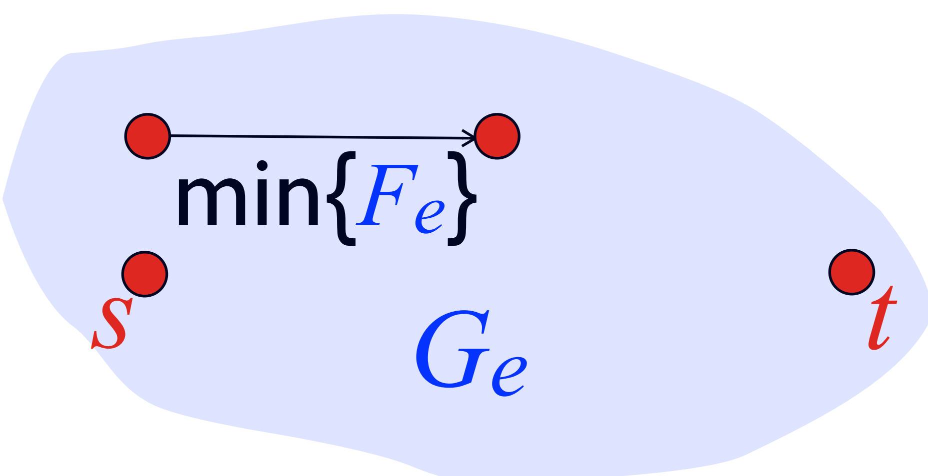
Challenges:

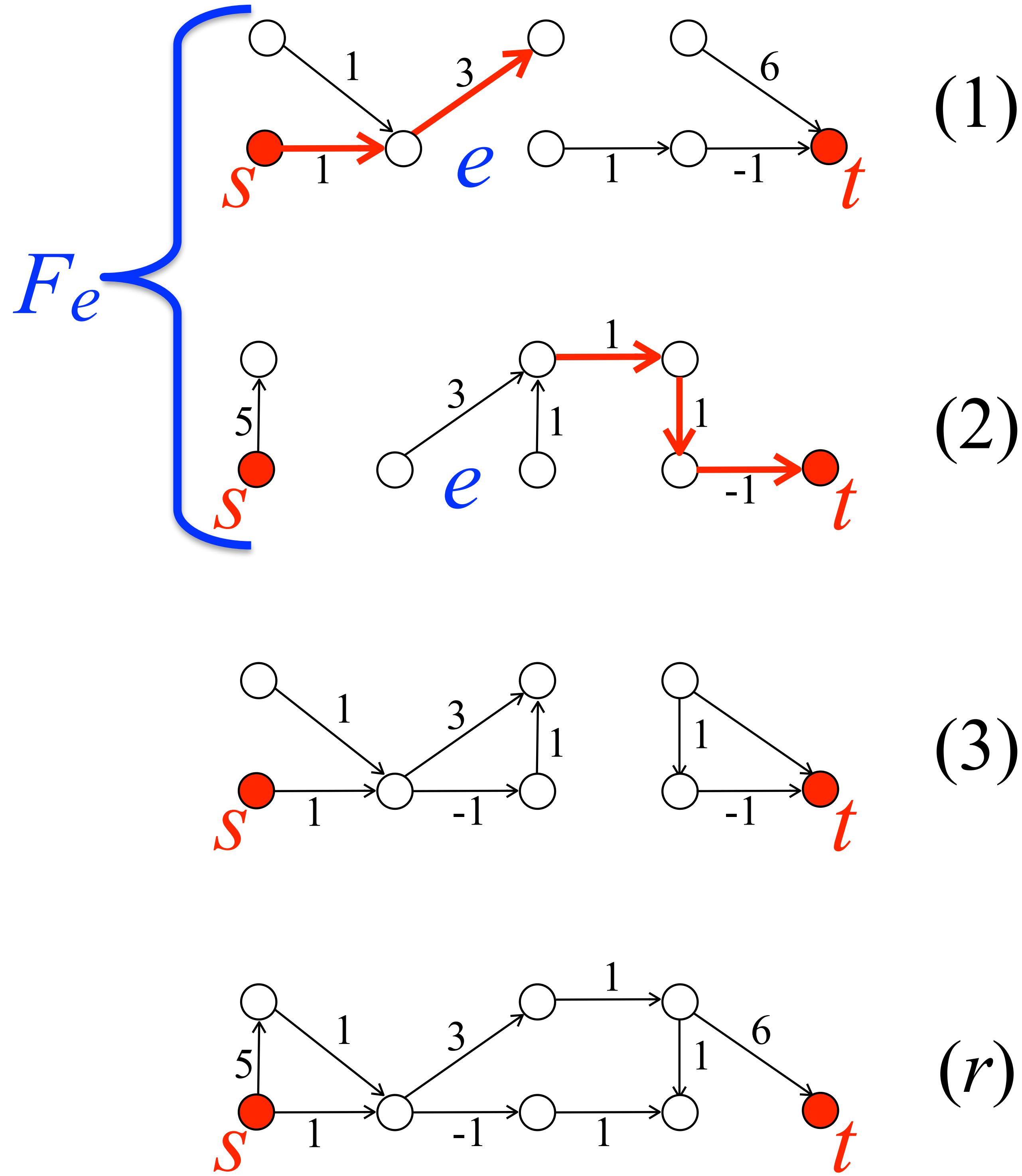
- Construct G_e
- Compute st-paths in all G_e





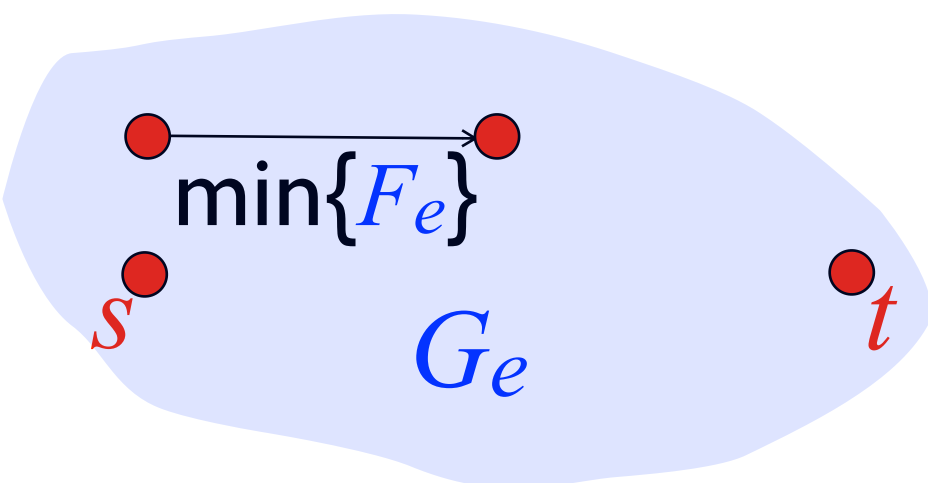
Constructing G_e

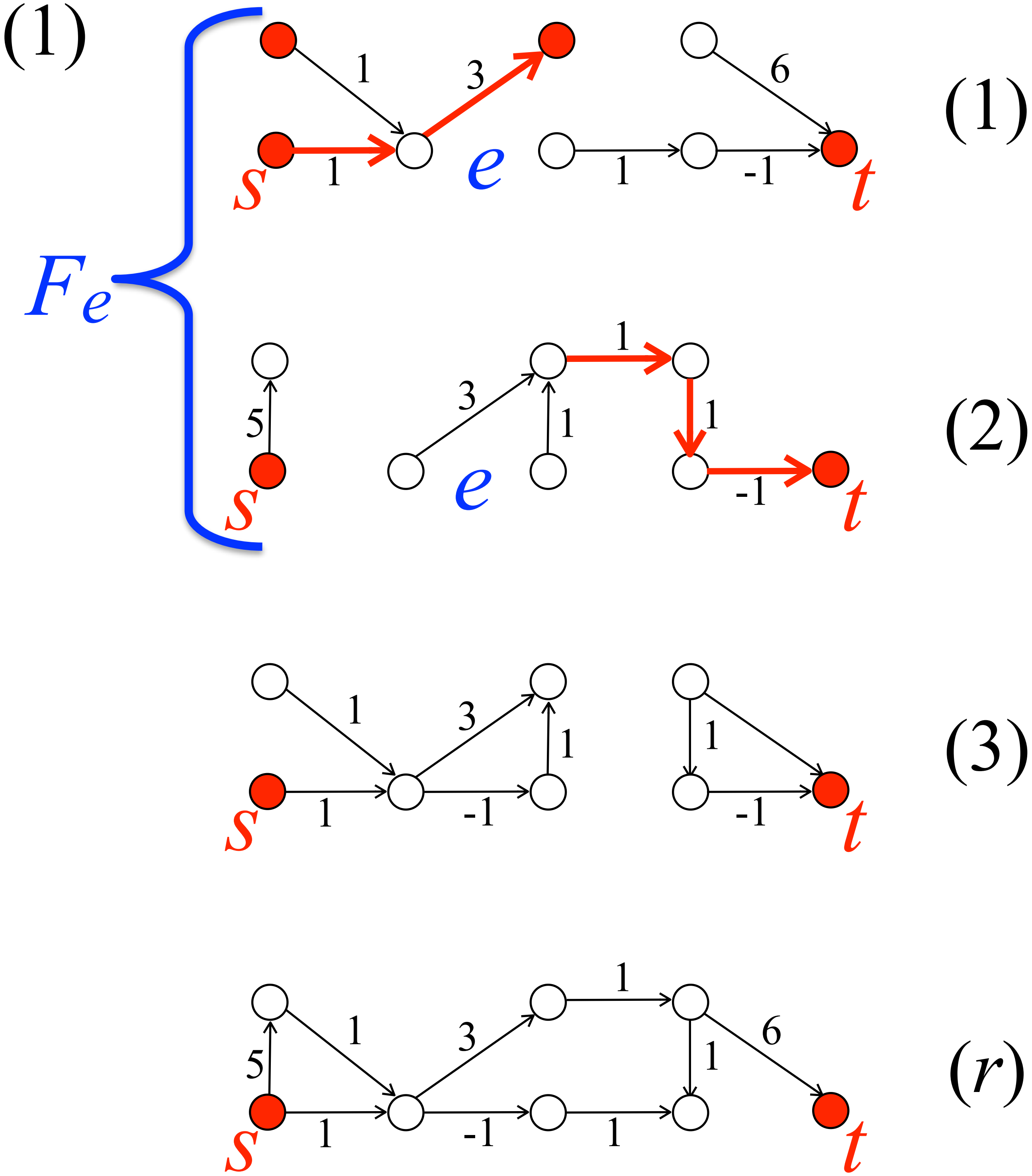




Constructing G_e

- Focus on ●-to-● distances in (1)





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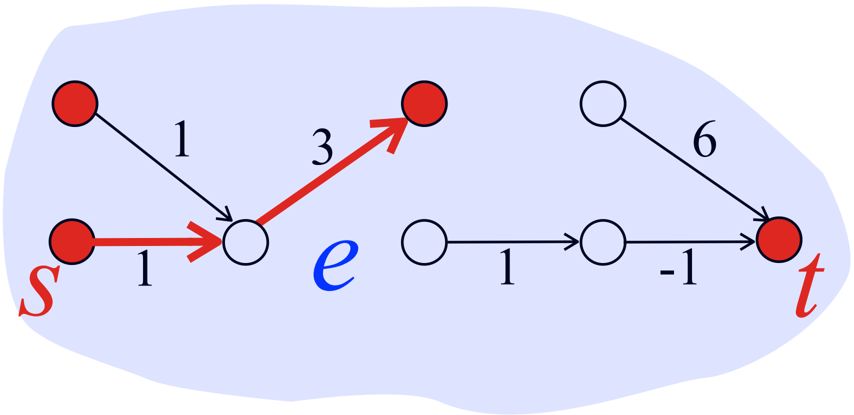
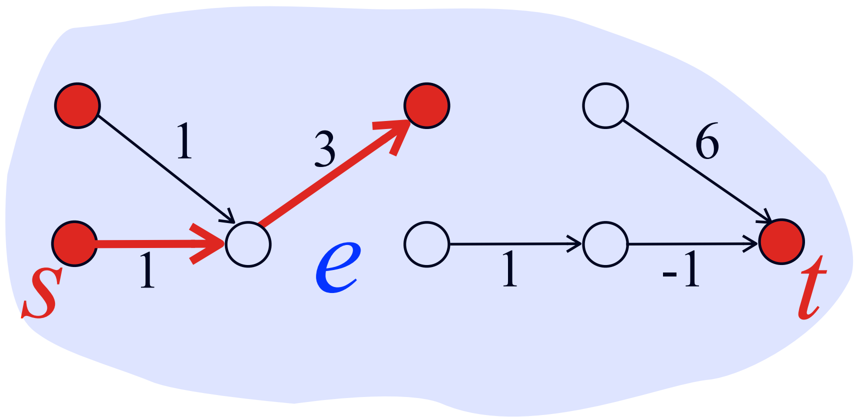


(1)

Constructing G_e

- Focus on ●-to-● distances in (1)
- Lengths in [$-M,\ldots,M$]



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Constructing G_e

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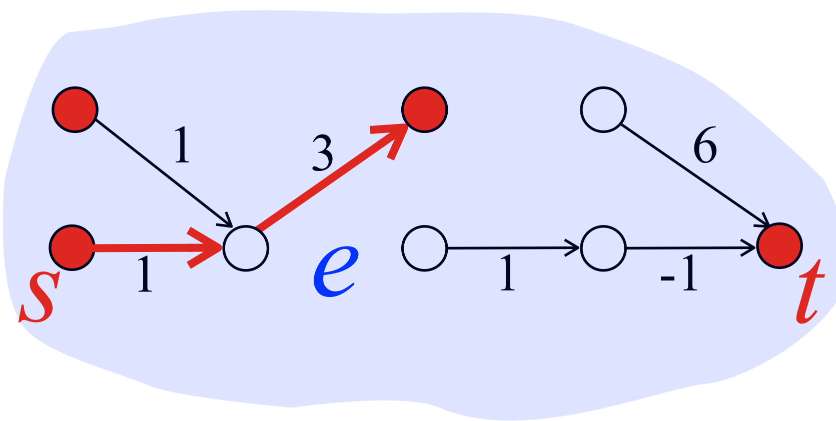

(1)

Constructing G_e

- Focus on ●-to-● distances in (1)
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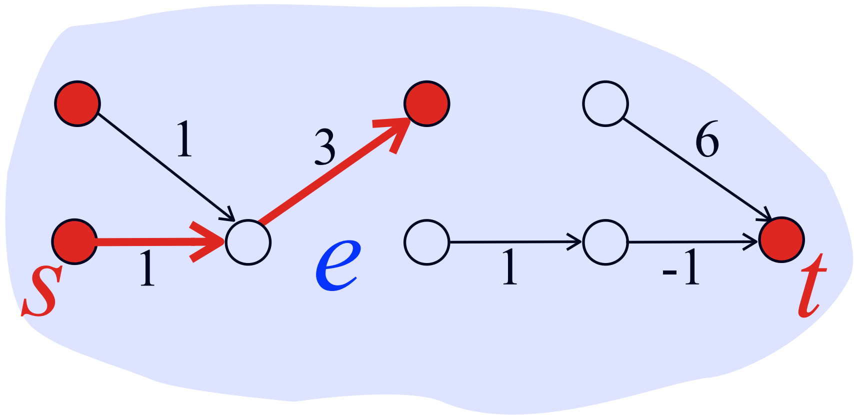
  - Only need B×B subset 



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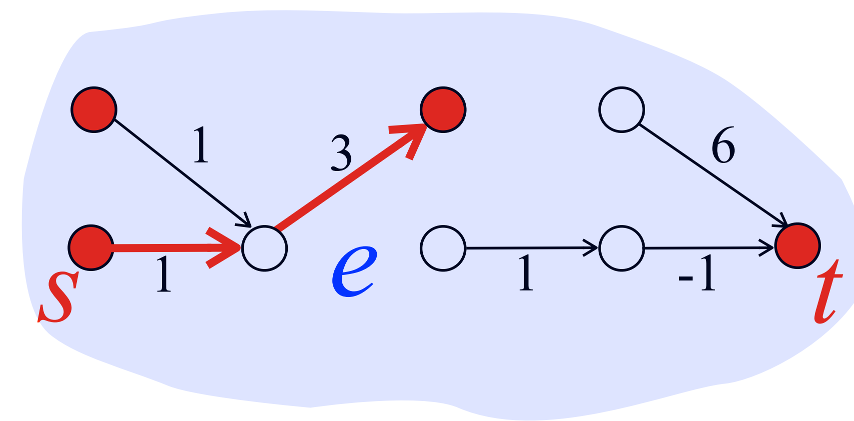

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- $\tilde{O}(Mn^{\omega}$) [Yuster-Zwick 2005]:
  Compute a matrix D s.t $dist(u,v) = \min_k \{D[u,k]+D[k,v]\}$

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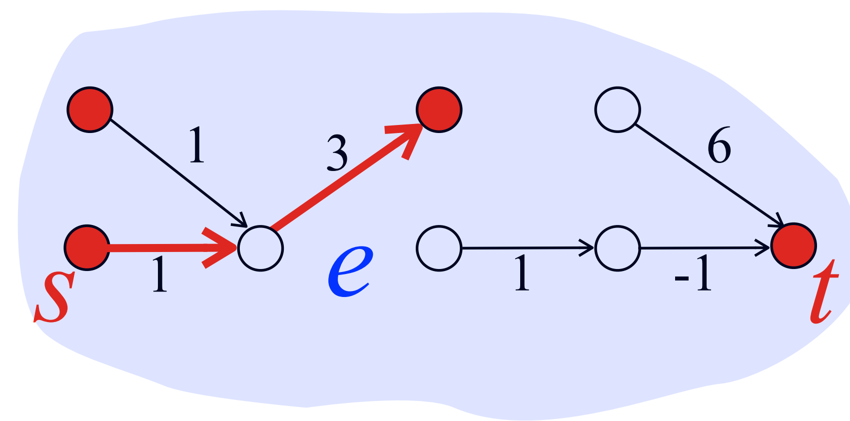

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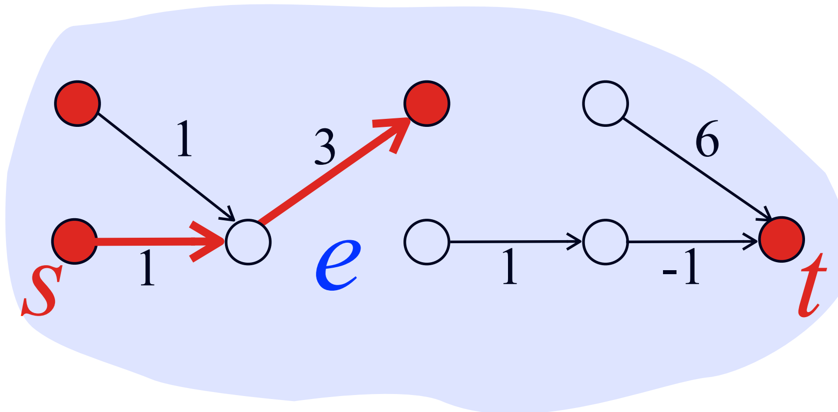

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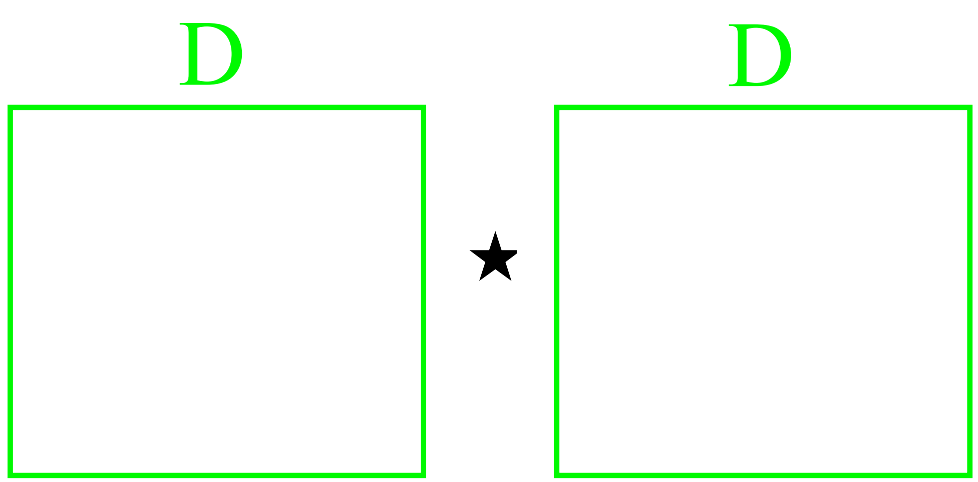

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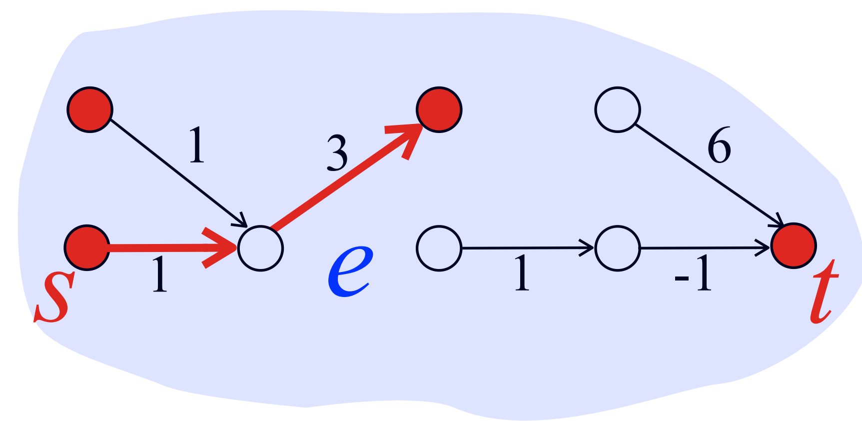

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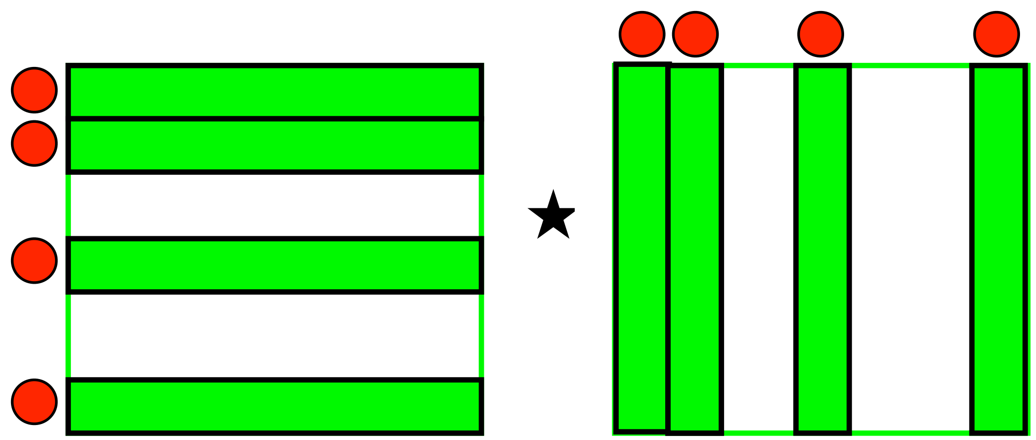

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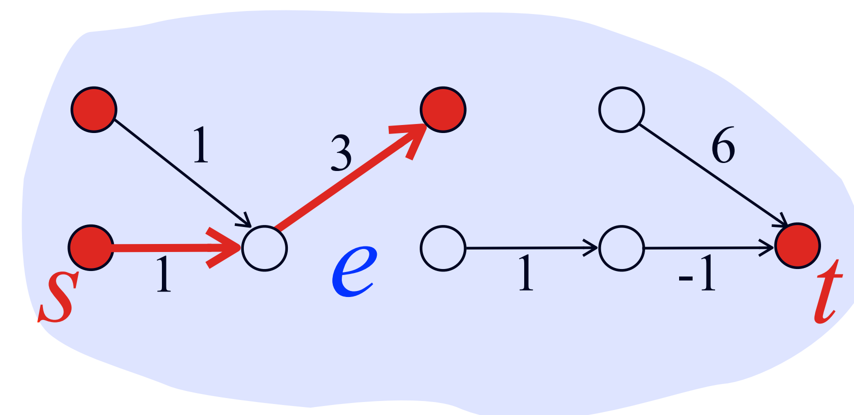

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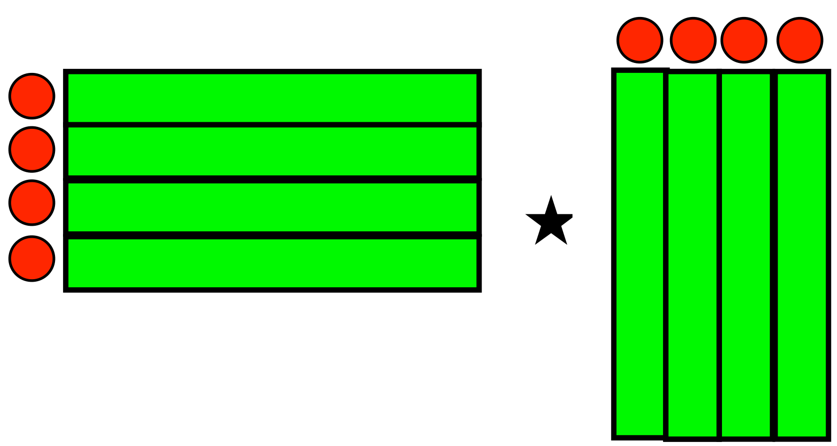

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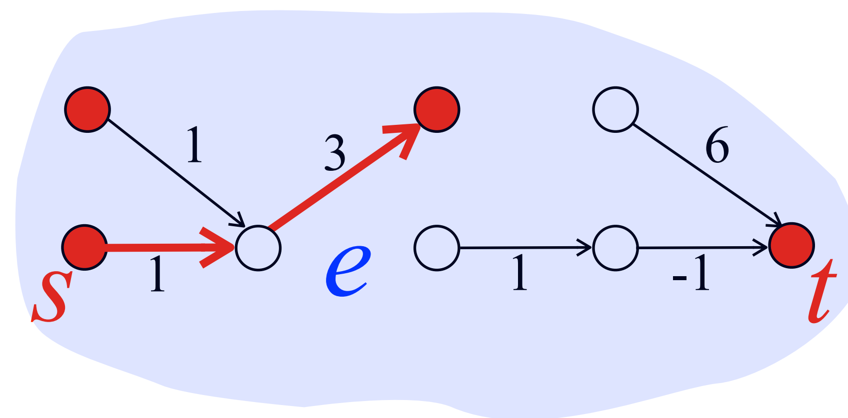

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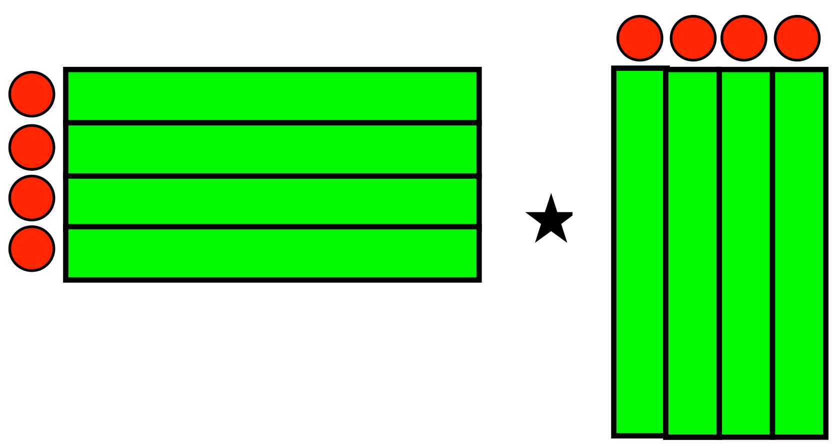

- Rectangular distance product [Alon, Galil, Margalit 1997]

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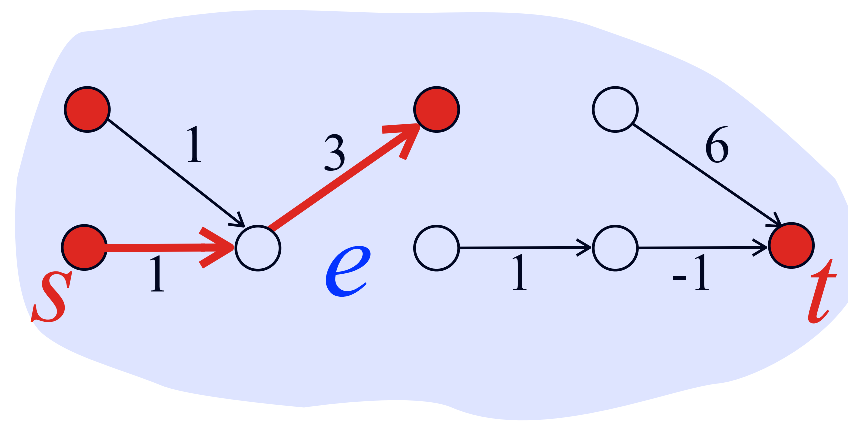

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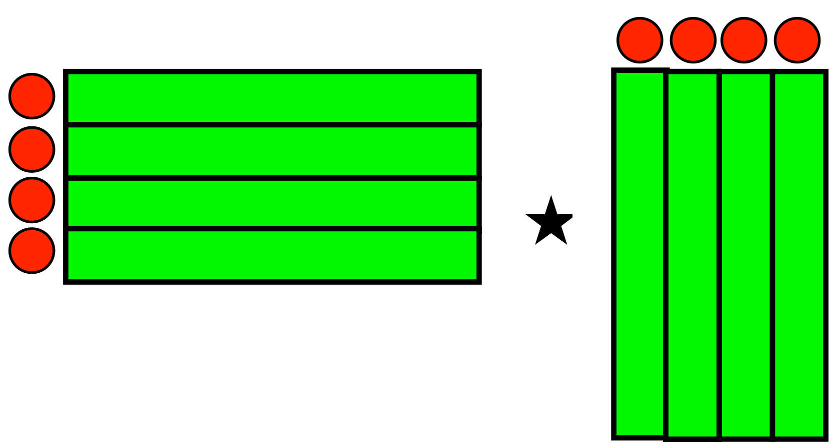

D elements can be as large as nM

- Rectangular distance product [Alon, Galil, Margalit 1997]

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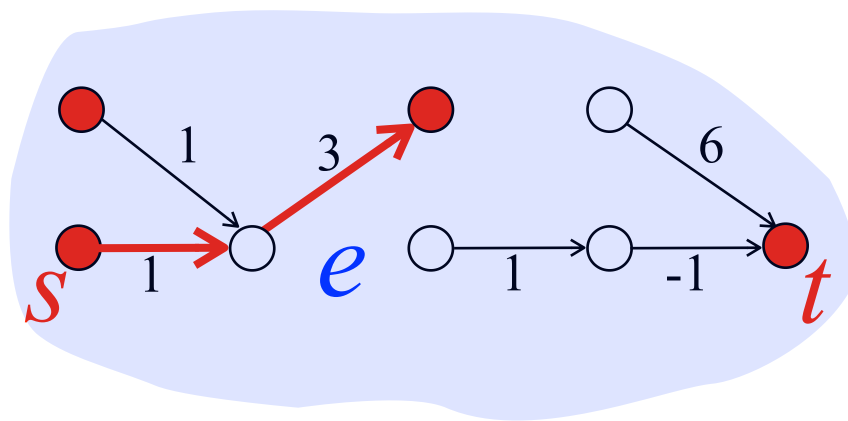

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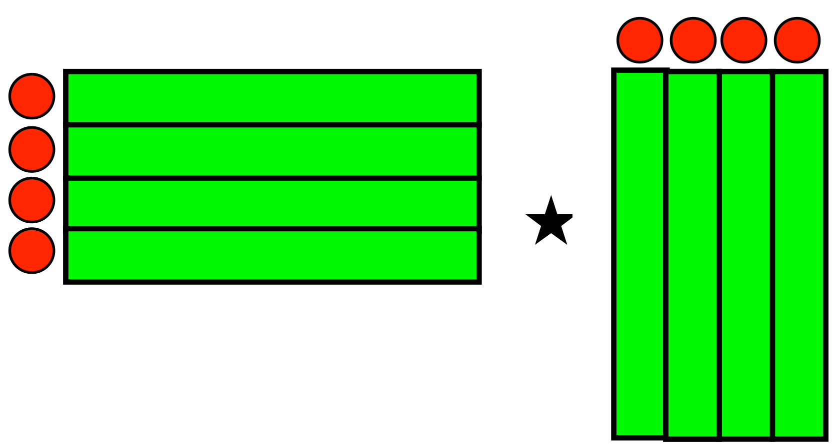
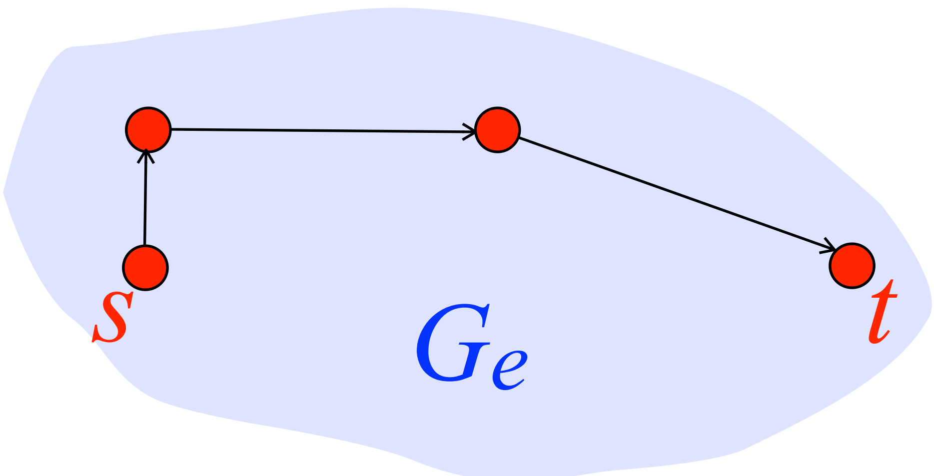


D elements can be as large as nM

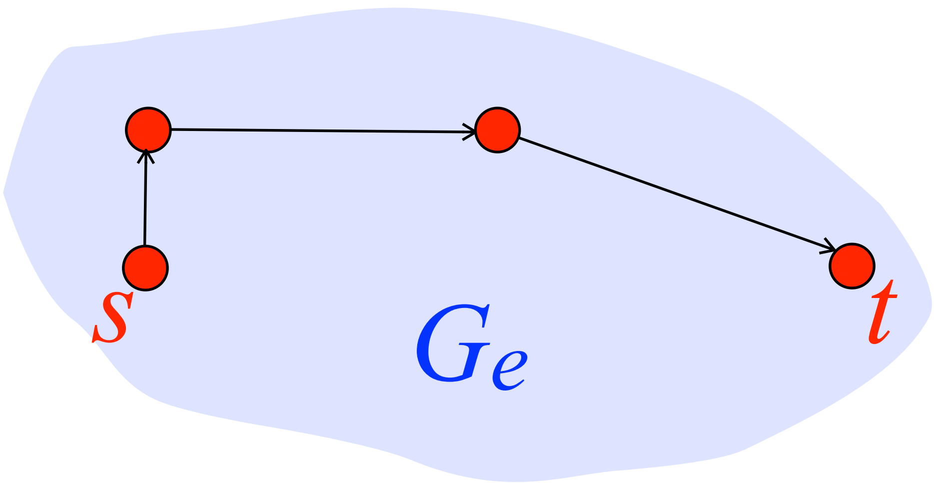
- Rectangular distance product [Alon, Galil, Margalit 1997]
  - Safe to ignore D elements larger than $Mn^{1-\alpha}$

Computing st-paths in all G_e



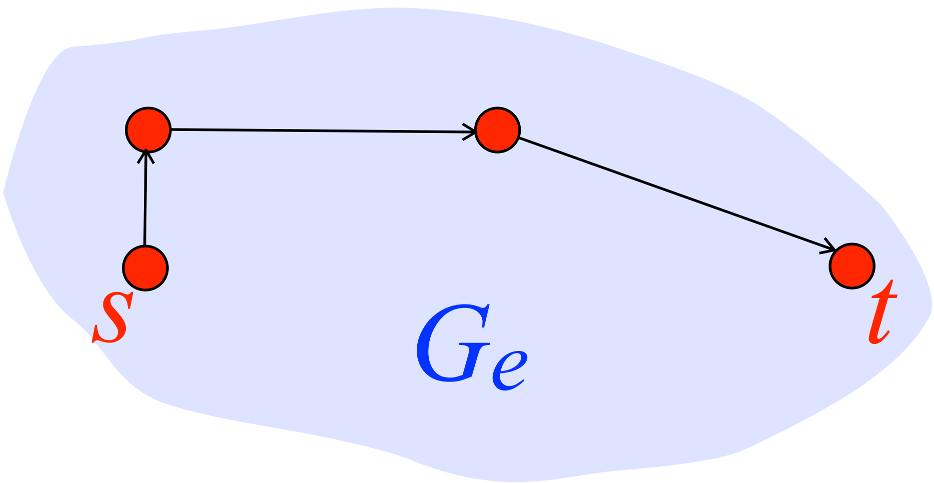
- Unbounded weights.

Computing st-paths in all G_e



- Unbounded weights.
- n executions of Goldberg or Bellman-Ford 

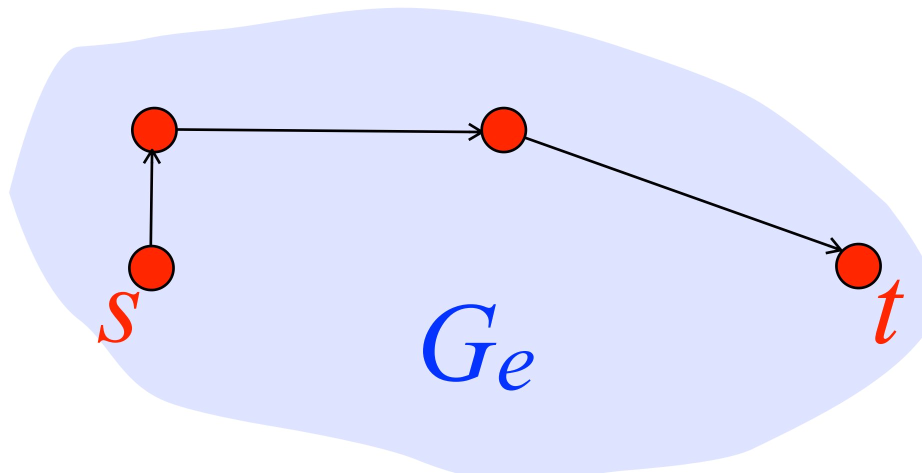
Computing st-paths in all G_e



- Unbounded weights.

- n executions of Goldberg or Bellman-Ford 
- 1 execution of Goldberg, n executions of Dijkstra 

Computing st-paths in all G_e



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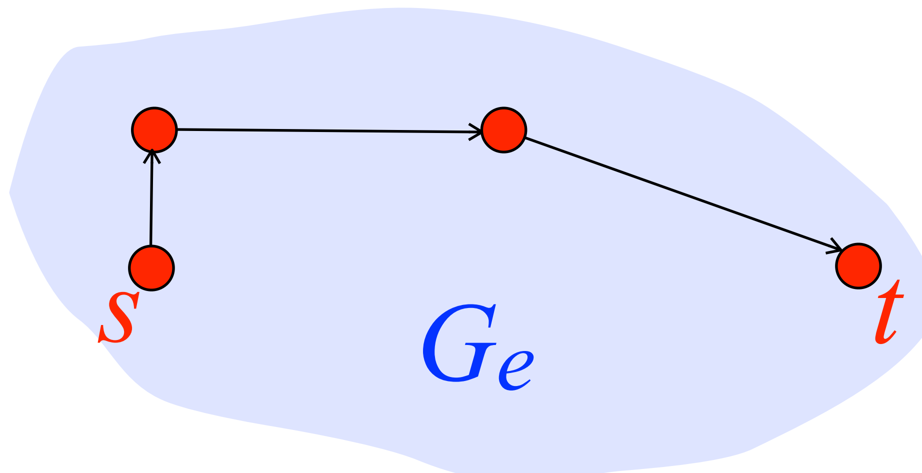
Computing st-paths in all G_e



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- n executions of Goldberg or Bellman-Ford
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  - Compute shortest paths $d(v)$ from s in G
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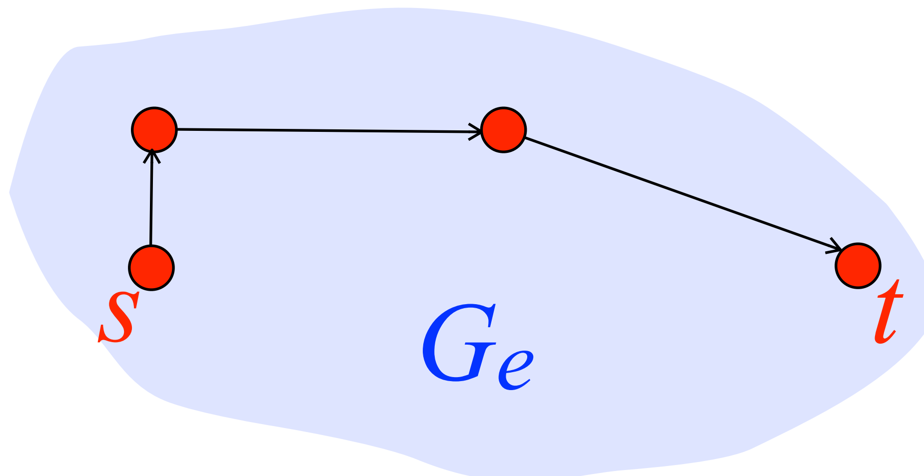
Computing st-paths in all G_e



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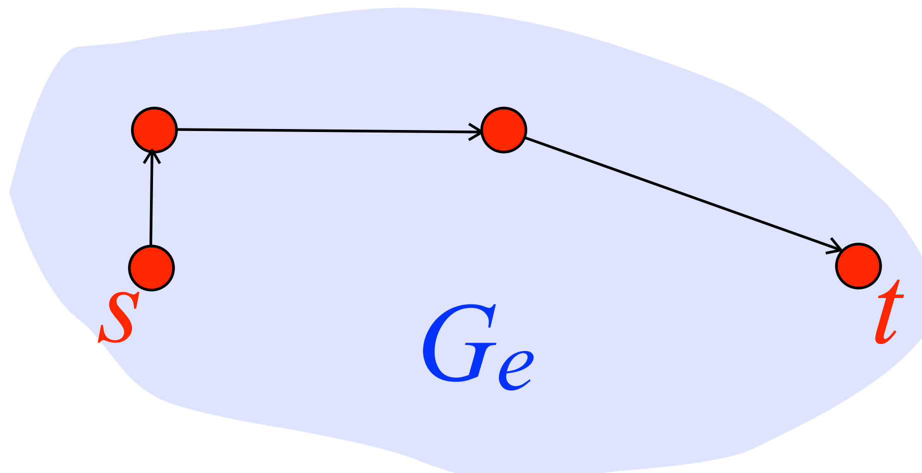

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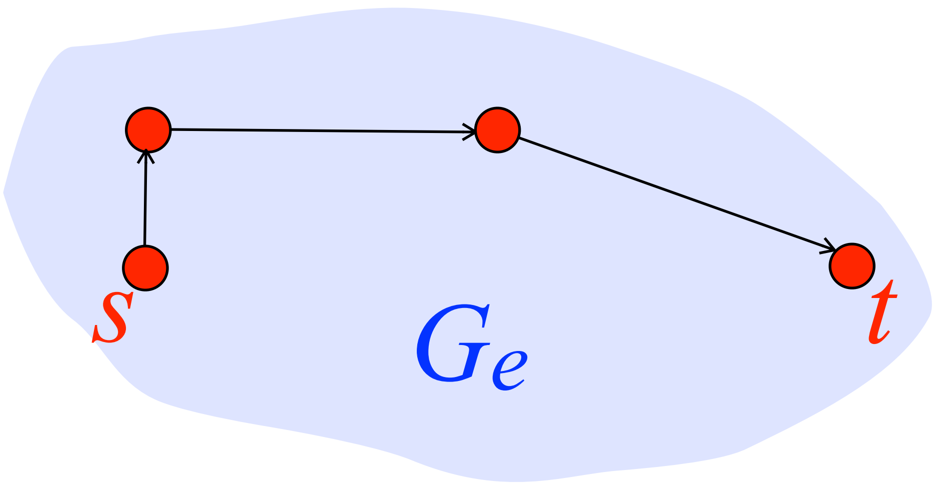
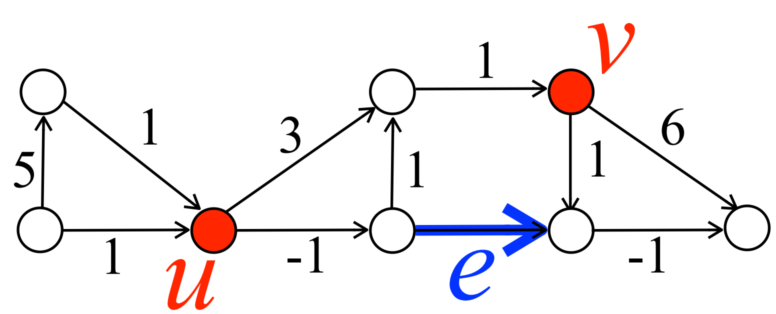
    - Dijkstra on every (positive weighted) G_e
    - $O(n|B|^2)$

Putting it all Together



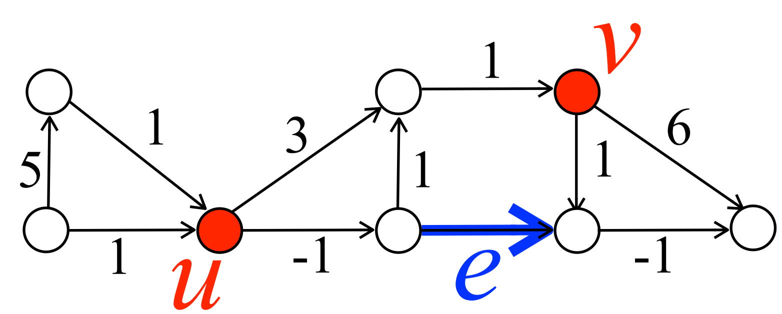
- Replacement paths in $\tilde{O}(Mn^{1+2\omega/3}) = O(Mn^{2.584})$ time.

All-pairs Replacement Paths



Query: (u, v, e)?

All-pairs Replacement Paths

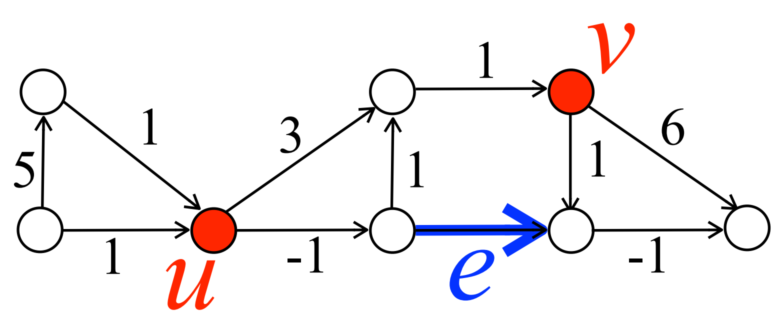


Query: (u,v,e)?

- $\tilde{O}(Mn^{1+\omega-\alpha})$ preprocessing: (not knowing u,v,e)

- $\tilde{O}(n^{1+\alpha})$ query time: (knowing u,v,e)

All-pairs Replacement Paths



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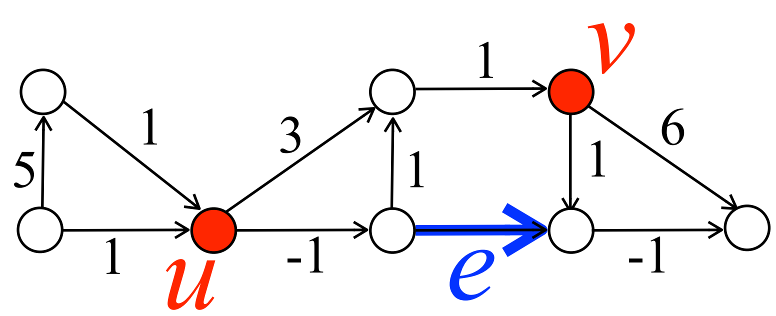
- $\tilde{O}(Mn^{1+\omega-\alpha})$ preprocessing: (not knowing u,v,e)

Construct the random graphs

Pick B, and compute B x B distances

- $\tilde{O}(n^{1+\alpha})$ query time: (knowing u,v,e)

All-pairs Replacement Paths



Query: (u,v,e)?

- $\tilde{O}(Mn^{1+\omega-\alpha})$ preprocessing: (not knowing u,v,e)

Construct the random graphs

Pick B, and compute B x B distances

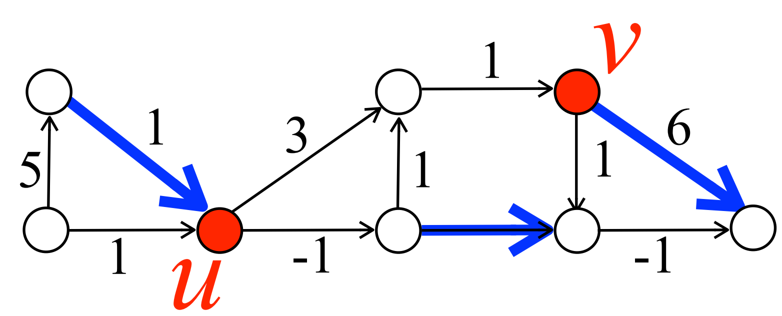
- $\tilde{O}(n^{1+\alpha})$ query time: (knowing u,v,e)

Identify the set F_e

Add u,v to B, and extend B x B distances

Construct G_e and run Dijkstra

Multiple Failures



Query: (u,v,S)?

- $\tilde{O}(Mn^{1+\omega-\alpha})$ preprocessing: (not knowing u,v,e)

Construct the random graphs

Pick B, and compute B x B distances

- $\tilde{O}(n^{1+\alpha})$ query time: (knowing u,v,e)

Identify the set F_e

Add u,v to B, and extend B x B distances

Construct G_e and run Dijkstra

Thank You!