DOUBLE CENTRALIZING THEOREMS FOR THE ALTERNATING GROUPS

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Abstract.

Let $V^{\otimes n}$ be the n–fold tensor product of a vector space V. Following I. Schur we consider the action of the symmetric group S_n on $V^{\otimes n}$ by permuting coordinates. In the 'super' ($\mathbb{Z}_2$ graded) case $V = V_0 \oplus V_1$, a $\pm$ sign is added [BR]. These actions give rise to the corresponding Schur algebras $\mathrm{S}(S_n, V)$. Here $\mathrm{S}(S_n, V)$ is compared with $\mathrm{S}(A_n, V)$, the Schur algebra corresponding to the alternating subgroup $A_n \subset S_n$. While in the 'classical' (signless) case these two Schur algebras are the same for n large enough, it is proved that in the 'super' case where $\dim V_0 = \dim V_1$, $\mathrm{S}(A_n, V)$ is isomorphic to the crossed–product algebra $\mathrm{S}(A_n, V) \cong \mathrm{S}(S_n, V) \times \mathbb{Z}_2$.

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§0. Introduction.

Let V be a finite dimensional vector space over the field $F = \mathbb{C}$ of the complex numbers (in fact, we shall only need the fact that $\sqrt{-1} \in F$), and let $V^{\otimes n} = V \otimes \cdots \otimes V$ n times. The symmetric group S_n acts on $V^{\otimes n}$ (say, from the left) by permuting coordinates. This makes $V^{\otimes n}$ a left FS_n module with the corresponding Schur algebra $End_{FS_n}(V^{\otimes n})$. Here FG denotes the group algebra of a group G. Formally, that action is given by a multiplicative homomorphism $\varphi : \; S_n \longrightarrow End_F(V^{\otimes n})$ which extends linearly to an algebra homomorphism $\varphi : \; FS_n \longrightarrow End_F(V^{\otimes n})$.

Let $A_n \subset S_n$ denote the alternating group, with the corresponding Schur algebra $End_{FA_n}(V^{\otimes n})$. The main purpose of this paper is to study and compare the pair of Schur algebras

$$End_{FS_n}(V^{\otimes n}) \subseteq End_{FA_n}(V^{\otimes n}).$$

We mention first the following phenomena:

Theorem 1 (see Remark 1.9). Let $\dim V = k$ and consider the above (signless!) action of S_n on $V^{\otimes n}$. If $k^2 \lneqq n$ then

$$\varphi(FA_n) = \varphi(FS_n), \qquad \text{hence also} \qquad End_{FA_n}(V^{\otimes n}) = End_{FS_n}(V^{\otimes n}). \tag{0.1}$$

In a sense, Theorem 1 shows an anomaly: even though $|S_n| = 2|A_n|$, nevertheless $\varphi(FS_n) = \varphi(FA_n)$, provided $k^2 \lneqq n$. As indicated below, the incorporation of a $\pm$ sign to the above permutation action of S_n seems to be natural and to remove that anomaly. Such a sign permutation action is related to the representation theory of Lie superalgebras [BR] [Se]. We now briefly explain that S_n action, and this will allow us to formulate the main result of this paper, which is Theorem 2 below.

Let $V = V_0 \oplus V_1$, $\dim V_0 = k$, $\dim V_1 = l$, and let S_n act on $V^{\otimes n}$ by permuting coordinates as before, but now, together with a $\pm$ sign; that sign is obtained by considering the elements of V_0 as being central, and the non-zero elements of V_1 as anti-commuting among themself. This is the sign-permutation action $*$ of S_n on $V^{\otimes n}$ [BR §1], [Se]. This action determines the algebra homomorphism $\varphi^* : \; FS_n \longrightarrow End_F(V^{\otimes n})$. It endows $V^{\otimes n}$ with a new FS_n module structure, which yields the corresponding (new!) Schur algebras

$$End_{FS_n}(V^{\otimes n}) = End_{\varphi^*(FS_n)}(V^{\otimes n}) \stackrel{\text{def}}{=} \mathcal{B}_n\,, \qquad \text{and}$$

$$End_{FA_n}(V^{\otimes n}) = End_{\varphi^*(FA_n)}(V^{\otimes n}) \stackrel{\text{def}}{=} \mathcal{A}_n\,,$$

and clearly $\mathcal{B}_n \subseteq \mathcal{A}_n$. The main result of this paper is the following *crossed product theorem.*

Theorem 2 (The Cross Product Theorem. See Theorem 1.1). Let $V = V_0 \oplus V_1$ with $\dim V_0 = \dim V_1$ and consider the above $*$-action of S_n on $V^{\otimes n}$. Then

$$\dim \varphi^*(FS_n) = 2 \dim \varphi^*(FA_n) \quad \text{and} \quad \dim \mathcal{A}_n = 2 \dim \mathcal{B}_n\,. \tag{0.6}$$

Moreover, there exists an algebra automorphism ω of order two, $\omega : \mathcal{B}_n \to \mathcal{B}_n$, such that $\mathcal{A}_n$ is isomorphic to the crossed product $\mathcal{A}_n \cong \mathcal{B}_n \times F[H] \cong \mathcal{B}_n \times F[\mathbb{Z}_2]$, where $H = \{1, \omega\}$.

The case $\dim V_0 \neq \dim V_1$ is considered in Section 3, and we prove

Theorem 3 (Theorem 3.2). Let $\dim V_0 \neq \dim V_1$, then each of the above subalgebras of $End_F(V^{\otimes n})$ splits into a direct sum of two subalgebras as shown below, and the following relations hold between these summands.

1)
$$\varphi^*(FS_n) = P \oplus Q \quad \text{and} \quad \varphi^*(FA_n) = P' \oplus Q', \tag{0.7}$$

satisfying $Q' = Q, \quad P' \subset P \quad \text{and} \quad \dim P = 2 \dim P'$.

2)

$$End_{FS_n}(V^{\otimes n}) = \mathcal{B}_n = \mathcal{B}_P \oplus \mathcal{B}_Q, \quad End_{FA_n}(V^{\otimes n}) = \mathcal{A}_n = \mathcal{A}_P \oplus \mathcal{A}_Q\,, \tag{0.8}$$

satisfying $\mathcal{B}_Q = \mathcal{A}_Q$ and $\mathcal{B}_P \subset \mathcal{A}_P$.

At the moment, when $\dim V_0 \neq \dim V_1$, the relations between $\mathcal{B}_P$ and $\mathcal{A}_P$ are not clear.

Section 2 contains some necessary preliminaries from the representation theory of S_n and A_n – which are applied in Section 3.

In Section 4 we first consider dimensions growth rates, as $n \to \infty$, of the factors in (0.7) and (0.8). We show (Remark 4.1) that when $\dim V_0 \neq \dim V_1$, the above P-summands have much smaller dimensions than the corresponding Q-summands. In particular, this implies that in that case,

$$\lim_{n \longrightarrow \infty} \frac{\dim \varphi^*(FA_n)}{\dim \varphi^*(FS_n)} = \lim_{n \longrightarrow \infty} \frac{\dim \mathcal{A}_n}{\dim \mathcal{B}_n} = \lim_{n \longrightarrow \infty} \frac{\dim End_{FA_n}(V^{\otimes n})}{\dim End_{FS_n}(V^{\otimes n})} = 1\,. \tag{0.9}$$

In the rest of Section 4 the case $\dim V_0 = \dim V_1$ is revisited. There, we study dimensions growth rates, as $n \to \infty$, of certain factors that appear in theorem 2. These factors are related to the self–conjugate and to the non–self–conjugate partitions in the (k, k) hook $H(k, k; n)$. It is shown that the asymptotics for the dimensions of these factors do exist, and can be computed explicitly – by Selberg integrals.

Remark. An important step in the proof of Theorem 2 (i.e. Theorem 1.1) is Proposition 1.2. In its original proof, a considerable amount of the representation theory of S_n and of A_n was applied. Its present elegant proof is due to G. Olshanski.

§1 The case $\dim V_0 = \dim V_1$.

Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$ graded vector space over a field F of characteristic zero, and consider the sign action $*$ of S_n on $V^{\otimes n}$ [BR] [Se]. Denote that action by φ^* . For example, let $n = 2$, $\sigma = (1,2) \in S_2$ and let $y_1, y_2 \in V$ be homogeneous, then

$$\varphi^*_{(1,2)}(y_1 \otimes y_2) = (-1)^{(\deg y_1)(\deg y_2)}(y_2 \otimes y_1)\,.$$

This yields the algebra homomorphism

$$\varphi^* :\ FS_n \longrightarrow End_{_F}(V^{\otimes n})$$

and defines the centralizer $\mathcal{B}_n = End_{_{FS_n}}(V^{\otimes n})$ of $\varphi^*(FS_n)$ in $End_{_F}(V^{\otimes n})$. Obviously, for the alternating group $A_n \subset S_n$ we have

$$\varphi^*(FA_n) \subseteq \varphi^*(FS_n) \qquad \text{and} \qquad End_{_{FS_n}}(V^{\otimes n}) \subseteq End_{_{FA_n}}(V^{\otimes n}),$$

and we denote $\mathcal{A}_n = End_{_{FA_n}}(V^{\otimes n})$. Thus, $\mathcal{A}_n$ is the centralizer of $\varphi^*(FA_n)$ in $End_{_F}(V^{\otimes n})$. We are interested in the comparison between $\mathcal{B}_n$ and $\mathcal{A}_n$ and also between the images $\varphi^*(FA_n)$ and $\varphi^*(FS_n)$. In this section we study the case when $\dim V_0 = \dim V_1$. The main result of this paper is the following theorem.

Theorem 1.1. Let $V = V_0 \oplus V_1$, $\dim V_0 = \dim V_1$ and consider the above sign permutation action $*$ of S_n on $V^{\otimes n}$. Then there is a subgroup H of the automorphisms of $\mathcal{B}_n = End_{FS_n}(V^{\otimes n})$, $H \cong \mathbb{Z}_2$, such that the algebra $\mathcal{A}_n = End_{FA_n}(V^{\otimes n})$ is isomorphic to a crossed product of the algebra $\mathcal{B}_n = End_{FS_n}(V^{\otimes n})$ with the group H . In particular,

$$\dim\left(End_{_{FA_n}}(V^{\otimes n})\right) = 2\dim\left(End_{_{FS_n}}(V^{\otimes n})\right) \quad \text{i.e.} \quad \dim \mathcal{A}_n = 2\dim \mathcal{B}_n\,. \qquad (1.1.1)$$

Remark. By applying the representation theory of A_n we show in the next section that in the situation of Theorem 1.1, also

$$\dim \varphi^*(FS_n) = 2\dim \varphi^*(FA_n). \qquad (1.1.2)$$

The Key for proving Theorem 1.1 is

Proposition 1.2. Let $V = V_0 \oplus V_1$, $\dim V_0 = \dim V_1$, len $n \geq 2$ and let S_n act with the sign permutation action $*$ on $V^{\otimes n}$ and let $\mathcal{A}_n = End_{FA_n}(V^{\otimes n})$ and $\mathcal{B}_n = End_{FS_n}(V^{\otimes n})$, then

$$\dim \mathcal{A}_n = 2\dim \mathcal{B}_n\,. \qquad (1.2.1)$$

The following proof of Proposition 1.2 was suggested to us by G. Olshanski.

Consider first the general (k,l) case: let $\dim V_0 = k$, $\dim V_1 = l$, $V = V_0 \oplus V_1$, and consider the action $*$ of S_n on $V^{\otimes n}$ [BR] [Se].

Definition 1.3. Denote by $I_n^+(V)$ the subspace of $V^{\otimes n}$ of the symmetric tensors, and by $I_n^-(V)$ the subspace of $V^{\otimes n}$ of the anti-symmetric tensors. Thus

$$I_n^+(V) = \{w \in V^{\otimes n} \mid \varphi_\sigma^*(w) = w \quad \text{for all} \quad \sigma \in S_n\} \tag{1.3.1}$$

and

$$I_n^-(V) = \{w \in V^{\otimes n} \mid \varphi_\sigma^*(w) = \operatorname{sgn}(\sigma) w \quad \text{for all} \quad \sigma \in S_n\}\,. \tag{1.3.2}$$

Finally, denote by $\mathcal{L}_n(V)$ the subspace of $V^{\otimes n}$ of the A_n – invariant vectors:

$$\mathcal{L}_n(V) = \{w \in V^{\otimes n} \mid \varphi_\sigma^*(w) = w \quad \text{for all} \quad \sigma \in A_n\}\,. \tag{1.3.3}$$

Lemma 1.4. We have

$$\mathcal{L}_n(V) = I_n^+(V) \oplus I_n^-(V)\,. \tag{1.4.1}$$

Proof. The inclusion $\supseteq$ is obvious. The opposite inclusion is obtained from

$$w = \frac{1}{2}(w + \varphi_{(1,2)}^* w) + \frac{1}{2}(w - \varphi_{(1,2)}^* w), \tag{1.4.2}$$

where $w \in \mathcal{L}_n(V)$.

Remark 1.5. It is well known that if $l = \dim V_1 = 0$ and $k = \dim V_0 \lneqq n$ then $I_n^-(V) = 0$.

We show next that when $\dim V_0 = \dim V_1$, $I_n^+(V) \cong I_n^-(V)$ (Lemma 1.8). In fact, we produce an explicit such isomorphism T. To do so, denote

$$E = End_F(V)$$

and note it is $\mathbb{Z}_2$ graded.

The identification $E^{\otimes n} \equiv End_F(V^{\otimes n})$. We use standard sign convention for the canonical isomorphism $X \otimes Y \to Y \otimes X$, that is, for homogeneous x, y,

$$x \otimes y \to (-1)^{\deg x \deg y} y \otimes x\,.$$

This induces a corresponding identification $E^{\otimes n} \equiv End_F(V^{\otimes n})$.
For example, let $n = 2$, f_1, $f_2 \in E$ homogeneous and v_1, $v_2 \in V$ homogeneous, then

$$(f_1 \otimes f_2)(v_1 \otimes v_2) = (-1)^{(\deg(f_2))(\deg(v_1))}(f_1(v_1) \otimes f_2(v_2))\,.$$

When V is replaced by E, the action φ^* of S_n on $V^{\otimes n}$ is replaced by the corresponding action η^* of S_n on $E^{\otimes n}$. These two actions are related by formula (1.6.1) below.

Lemma 1.6. [Se, Lemma 1]. Let $L \in End_F(V^{\otimes n})$ and $\sigma \in S_n$, then

$$\eta^*_\sigma L = \varphi^*_\sigma \circ L \circ \varphi^*_{\sigma^{-1}} . \tag{1.6.1}$$

This obviously implies

Corollary 1.7. Let $E = End_F(V)$, then

$$I^+_n(E) = End_{FS_n}(V^{\otimes n}) = \mathcal{B}_n \tag{1.7.1}$$

and

$$\mathcal{L}_n(E) = End_{FA_n}(V^{\otimes n}) = \mathcal{A}_n . \tag{1.7.2}$$

Lemma 1.8. Let $\dim V_0 = \dim V_1$. Then there exist $T \in \mathcal{A}_n$ which induces an isomorphism between $I^+_n(V)$ and $I^-_n(V)$: $T(I^+_n(V)) = I^-_n(V)$ and $T(I^-_n(V)) = I^+_n(V)$. Moreover,

$$T^2 = (-1)^{\binom{n}{2}} \cdot I \quad (I \text{ is the identity map}), \tag{1.8.1}$$

and

$$T \circ \varphi^*_{(1,2)} = -\varphi^*_{(1,2)} \circ T . \tag{1.8.2}$$

Proof. First, construct τ : Let $t_1, \cdots, t_k \in V_0$, $u_1, \cdots, u_k \in V_1$ be bases, and let $\tau \in E$ be given by $\tau t_i = u_i$ and $\tau u_i = t_i$, $i = 1, \cdots, k$. Note that $\tau \in E_1$, i.e. $\deg \tau = 1$, where $E = End_F(V) = E_0 \oplus E_1$.

Now define $T = \tau^{\otimes n}$. For example, if $n = 2$ and $y_1, y_2 \in V$ are homogeneous,

$$T(y_1 \otimes y_2) = (\tau \otimes \tau)(y_1 \otimes y_2) = (-1)^{\deg y_1} \cdot \tau y_1 \otimes \tau y_2 .$$

Since $\tau \in E_1$, it follows from the definition of η^*_σ that for any $\sigma \in S_n$, $\eta^*_\sigma T = \mathrm{sgn}(\sigma) T$. By (1.6.1) it follows that

$$\varphi^*_\sigma \circ T = \mathrm{sgn}(\sigma) T \circ \varphi^*_\sigma , \qquad \sigma \in S_n ,$$

which clearly implies that $T \in \mathcal{A}_n$ and is an isomorphism between $I^+_n(V)$ and $I^-_n(V)$. Property (1.8.2) is now obvious, while (1.8.1) follows from the definition of T and from the fact that if $y \in V$ is homogeneous, then

$$\deg(y) + \deg(\tau(y)) = 1 .$$

□

We can now complete

The proof of Proposition 1.2. Apply (1.7.2), then apply (1.4.1) with E replacing V, to deduce that $\mathcal{A}_n = I_n^+(E) \oplus I_n^-(E)$. By (1.7.1) $\mathcal{B}_n = I_n^+(E)$, and the proof follows from Lemma 1.8, again with E replacing V. $\square$

Remark 1.9. Let $\dim V_1 = 0$, so that the sign permutation action φ^* is the classical permutation action φ of S_n on $V^{\otimes n}$, as considered by Schur. Let $\dim V = k$, hence $\dim E = k^2$, and let $k^2 < n$. By Remark 1.5 and by (1.4.1), with E replacing V, $\mathcal{L}_n(E) = I_n(E)$, and by (1.7.1) and (1.7.2), it follows that

$$\mathcal{A}_n = \mathcal{B}_n\,. \tag{1.9.1}$$

Also, by double centralizing, this implies that in that case,

$$\varphi(FS_n) = \varphi(FA_n)\,. \tag{1.9.2}$$

As was remarked in the Introduction, both (1.9.1) and (1.9.2) may be considered as a sort of anomaly, since one would expect $\dim \varphi(FS_n)$ to be twice as large as $\dim \varphi(FA_n)$, and similarly for $\dim \mathcal{A}_n$ versus $\dim \mathcal{B}_n$. Note that this anomaly disappears when the ordinary case φ is replaced by the super case φ^*, where $\dim V_0 = \dim V_1$.

We now complete the proof of Theorem 1.1.

Lemma 1.10. Let $\dim V_0 = \dim V_1$ and $T \in \mathcal{A}_n$ as in Lemma 1.8, with T satisfying (1.8.1) and (1.8.2). Then

$$\mathcal{A}_n = \mathcal{B}_n \oplus T\mathcal{B}_n\,, \tag{1.10.1}$$

$T\mathcal{B}_nT = \mathcal{B}_n$, and the map

$$\omega : \mathcal{B}_n \longrightarrow \mathcal{B}_n\,, \qquad \omega(b) = \varepsilon \cdot TbT = (-1)^{\binom{n}{2}} \cdot TbT, \quad (\varepsilon = (-1)^{\binom{n}{2}}) \tag{1.10.2}$$

is an automorphism of order two of $\mathcal{B}_n$.
We denote $H = \{1, \omega\}$, which is clearly a group isomorphic to $\mathbb{Z}_2$.

Proof. It easily follows from (1.8.1) and (1.8.2) that $\mathcal{B}_n \cap T\mathcal{B}_n = 0$, therefore $\mathcal{B}_n \oplus T\mathcal{B}_n \cong \mathcal{B}_n + T\mathcal{B}_n \subseteq \mathcal{A}_n$, and equality follows by the equality of the dimensions, as implied by Proposition 1.2 and the obvious equality $\dim \mathcal{B}_n = \dim T\mathcal{B}_n$. This proves (1.10.1). Also, (1.10.2) (i.e. the fact that ω is of order two) easily follows from (1.8.1).

1.11 . Crossed products. Let G be a finite group of automorphisms of an F-algebra R: $g \in G\,; r \in R,\ g : r \to g(r)$. We denote the crossed product of R and G by $R \times FG$, where FG is the group algebra. Recall that as an F vector space, $R \times FG = R \otimes_F FG$. Multiplication is defined by linearity and by $(r_1 \otimes g_1)(r_2 \otimes g_2) = r_1 \cdot g_1(r_2) \otimes g_1g_2$.

Let now ω be given by (1.10.2), let $H = \{1, \omega\}$ and let $\mathcal{B}_n \times FH$ be the corresponding crossed product. Thus, $B_n \times FH = B_n \otimes_F FH$, and multiplication is given as follows. Let $b_1,\ b_2 \in B_n,\ s \in FH$, then

$$(b_1 \otimes 1)(b_2 \otimes s) = b_1b_2 \otimes s \text{ and } (b_1 \otimes \omega)(b_2 \otimes s) = b_1\omega(b_2) \otimes \omega s = b_1Tb_2T \otimes \omega s\,.$$

We now reformulate – and prove – Theorem 1.1. This is

Theorem 1.12 (1.1'). Let $\dim V_0 = \dim V_1$ and let $\mathcal{A}_n = End_{FA_n}(V^{\otimes n})$ and $\mathcal{B}_n = End_{FS_n}(V^{\otimes n})$ (with respect to φ^*). Let ω be given by (1.10.2), $H = \{1, \omega\}$, and $\mathcal{B}_n \times FH$ the corresponding crossed product. Then there is an algebra isomorphism

$$\mathcal{A}_n \cong \mathcal{B}_n \times FH \ \ (\cong \mathcal{B}_n \times F\mathbb{Z}_2\,). \tag{1.12.1}$$

Explicitly, the isomorphism is given by

$$\zeta :\ \mathcal{B}_n \times FH \longrightarrow \mathcal{A}_n\,,$$

where ζ is given by linearity and by

$$\zeta(b \otimes 1) = b \qquad \text{and} \qquad \zeta(b \otimes \omega) = \sqrt{\varepsilon} \cdot bT, \tag{1.12.2}$$

where, again, $\varepsilon = (-1)^{\binom{n}{2}}$.

Proof. It easily follows from Lemma 1.10 that the above ζ is a vector space isomorphism. It is easy to see that it is also an algebra homomorphism. For example,

$$\zeta((b_1 \otimes \omega)(b_2 \otimes 1)) = \zeta((b_1\omega(b_2) \otimes \omega)) =$$
$$\zeta((b_1 \varepsilon T b_2 T \otimes \omega)) = \varepsilon\sqrt{\varepsilon} b_1 T b_2 T T = \sqrt{\varepsilon} b_1 T b_2,$$

while, also,

$$\zeta(b_1 \otimes \omega)\zeta(b_2 \otimes 1) = \sqrt{\varepsilon} b_1 T b_2.$$

Similarly,

$$\zeta((b_1 \otimes \omega)(b_2 \otimes \omega)) = \zeta((b_1\omega(b_2) \otimes 1)) =$$
$$\zeta((b_1 \varepsilon T b_2 T \otimes 1)) = \varepsilon b_1 T b_2 T,$$

while, also,

$$\zeta(b_1 \otimes \omega)\zeta(b_2 \otimes \omega) = (\sqrt{\varepsilon})^2 b_1 T b_2 T.$$

Similarly for the other (trivial) two cases $(b_1 \otimes 1)((b_2 \otimes \omega)$ and $(b_1 \otimes 1)((b_2 \otimes 1)$. □

§2. Some S_n and A_n Representations.

We summarize some facts (and notations) about S_n and A_n representations – that will be needed later [B] [J] [JK] [M] [Sa].

For $\lambda = (\lambda_1, \cdots, \lambda_r) \vdash n$ (i.e. λ is a partition of n) denote $l(\lambda) = r$ if $\lambda_r \neq 0$. Also, λ' is the conjugate partition of λ.

2.1. The group algebra FS_n admits the following decomposition:

$$FS_n = \bigoplus_{\lambda \vdash n} I_\lambda, \tag{2.1.1}$$

where each I_λ is a minimal two-sided ideal in FS_n , and is isomorphic to certain matrix algebra $M_r(F)$ ($r \times r$ matrices over F). Moreover, there is an explicit correspondence $\lambda \longleftrightarrow I_\lambda$, due to Frobenius and to A. Young, for which $I_\lambda \cong M_{f^\lambda}(F)$, where f^λ is the number of standard Young tableaux of shape λ. Also, $I_\lambda \cong J_\lambda^{\oplus f^\lambda}$, where J_λ is a left ideal, irreducible as an FS_n left module.
Moreover, if $\lambda \neq \mu \vdash n$ then $J_\lambda \ncong J_\mu$ as FS_n left modules.

2.2. Consider FS_n as a left FA_n module.
2.2.1. If $\lambda \neq \lambda'$ then, as FA_n modules, $J_\lambda \cong J_{\lambda'}$ and both remain irreducible.
2.2.2. If $\lambda = \lambda'$ then, as FA_n modules, $J_\lambda = J_\lambda^+ \oplus J_\lambda^-$, J_λ^+ and J_λ^- are FA_n irreducible, and $J_\lambda^- = (1,2)J_\lambda^+$ for the transposition $(1,2) \in S_n$. In particular, $\dim J_\lambda^+ = \dim J_\lambda^- = \frac{1}{2}\dim J_\lambda$.

2.3. Note that 2.1 is the decomposition of the S_n regular representation into *isotypic* components: each I_λ is the sum of all the irreducible FS_n submodules of FS_n which are isomorphic to a given irreducible FS_n module. Thus, if $M \subseteq FS_n$ is an FS_n left submodule and $M \cong J_\lambda$ then $M \subseteq I_\lambda$ (J_λ, I_λ as in 2.1).

2.4. Let '$\mu < \lambda$' denote the left lexicographic order on the partitions of n. It follows from 2.2 that the isotypic decomposition of FS_n as a left FA_n module is given by:

$$FS_n = \left[\bigoplus_{\lambda \vdash n;\lambda>\lambda'} (I_\lambda \oplus I_{\lambda'})\right] \oplus \left[\bigoplus_{\lambda \vdash n;\lambda=\lambda'} (I_\lambda^+ \oplus I_\lambda^-)\right], \tag{2.4.1}$$

where for $\lambda \neq \lambda'$, $I_\lambda \oplus I_{\lambda'} \cong J_\lambda^{\oplus 2f^\lambda}$, and for $\lambda = \lambda'$, $I_\lambda = I_\lambda^+ \oplus I_\lambda^-$, $I_\lambda^+ \cong \left(J_\lambda^+\right)^{\oplus f^\lambda}$ and $I_\lambda^- \cong \left(J_\lambda^-\right)^{\oplus f^\lambda}$.
Similarly, it follows that the isotypic decomposition of the regular representation of A_n is

$$FA_n = \left[\bigoplus_{\lambda \vdash n;\lambda>\lambda'} \bar I_\lambda\right] \oplus \left[\bigoplus_{\lambda \vdash n;\lambda=\lambda'} (\bar I_\lambda^+ \oplus \bar I_\lambda^-)\right]. \tag{2.4.2}$$

Here, for $\lambda \neq \lambda'$, $\bar I_\lambda$ is the sum of all the irreducible left FA_n submodules of FA_n which are isomorphic to J_λ . Similarly for $\bar I_\lambda^+$ and for $\bar I_\lambda^-$. Recall that for any group G, the decomposition of the regular representation into isotypic components – coincide with the decomposition of FG into simple two-sided ideals. Comparing the isotypic decompositions of FS_n and of FA_n as FA_n modules, we deduce

Lemma 2.5 The above (2.4.1) gives the decomposition of FS_n into simple two-sided ideals, while (2.4.2) gives that of FA_n into simple two-sided ideals. These decompositions satisfy

a) If $\lambda \neq \lambda'$ then $\bar{I}_\lambda \subseteq I_\lambda \oplus I_{\lambda'}$ and as FA_n modules, $\bar{I}_\lambda \cong I_\lambda \cong I_{\lambda'}$. In particular, $\dim \bar{I}_\lambda = \dim I_\lambda = \dim I_{\lambda'}$.

b) If $\lambda = \lambda'$ then $\bar{I}_\lambda^+ \oplus \bar{I}_\lambda^- \subseteq I_\lambda$ and $\dim \bar{I}_\lambda^+ = \dim \bar{I}_\lambda^- = \frac{1}{4} \dim I_\lambda$.

2.6. The 'hook' theorem for φ^*. The following theorem is a generalization of a classical theorem of Schur and Weyl [W].

Theorem [BR, 3.20]. Let

$$H(k,l;n) = \{(\lambda_1, \lambda_2, \cdots) \vdash n \mid \lambda_{k+1} \le l\} \tag{2.6.1}$$

denote the partitions of n whose diagrams are contained in the k,l hook. As in the previous section, let $V = V_0 \oplus V_1$, $\dim V_0 = k$ and $\dim V_1 = l$, and let S_n act on $V^{\otimes n}$ by the sign permutation action, denoted by φ^*. Then

$$\varphi^*(I_\lambda) \cong \begin{cases} I_\lambda & \text{if} \quad \lambda \in H(k,l;n) \\ 0 & \text{otherwise} \end{cases} \tag{2.6.2}$$

and, moreover,

$$\varphi^*(FS_n) = \bigoplus_{\lambda \in H(k,l;n)} \varphi^*(I_\lambda) \cong \bigoplus_{\lambda \in H(k,l;n)} I_\lambda \, . \tag{2.6.3}$$

2.7. The decomposition of $W = V^{\otimes n}$ and of $\mathcal{B}_n$. The above S_n action φ^* on $W = V^{\otimes n}$ makes W into a left FS_n module, and the decomposition (2.6.3) implies a corresponding decomposition of W :

$$V^{\otimes n} = \bigoplus_{\lambda \in H(k,l;n)} W_\lambda, \tag{2.7.1}$$

where W_λ is the FS_n isotypic component of $V^{\otimes n}$ corresponding to I_λ. Thus

$$I_\lambda W_\lambda \overset{\text{def}}{=} \varphi^*(I_\lambda) W_\lambda = W_\lambda \tag{2.7.2}$$

for $\lambda \in H(k,l;n)$. Also,

$$I_\lambda W_\mu = 0 \quad \text{if} \quad \lambda \neq \mu \vdash n \, . \tag{2.7.3}$$

Similarly,

$$\mathcal{B}_n = End_{FS_n}(V^{\otimes n}) = \bigoplus_{\lambda \in H(k,l;n)} \mathcal{B}_\lambda \, , \tag{2.7.4}$$

where

$$\mathcal{B}_\lambda = End_{I_\lambda}(W_\lambda) \overset{\text{def}}{=} End_{\varphi^*(I_\lambda)}(W_\lambda) \tag{2.7.5}$$

(and it is well known that $End_F(W_\lambda) \cong \varphi^*(I_\lambda) \otimes_F \mathcal{B}_\lambda$).

§**3. The case** $\dim V_0 \neq \dim V_1$ **.**

Let $V = V_0 \oplus V_1$. In this section we compare $\varphi^*(FS_n)$ with $\varphi^*(FA_n)$. If $\dim V_0 = \dim V_1$, we show that $\dim \varphi^*(FS_n) = 2 \dim \varphi^*(FA_n)$. Recall also that in that case, $\dim \mathcal{A}_n = 2 \dim \mathcal{B}_n$ (Proposition 1.2). In the general case, when $\dim V_0 \neq \dim V_1$, the picture is more involved, and depends on the following decomposition of $H(k,l;n)$.

Definition 3.1. Denote $H_0(k,l;n) = \{\lambda \mid \lambda, \lambda' \in H(k,l;n)\}$ and let $H_1(k,l;n) = H(k,l;n) \backslash H_0(k,l;n)$, so that

$$H(k,l;n) = H_0(k,l;n) \cup H_1(k,l;n)\,, \quad \text{a disjoint union.} \tag{3.1.1}$$

If $\lambda \in H_1(k,l;n)$ then, by definition, $\lambda' \notin H(k,l;n)$. Assume, without loss of generality, that $l \leq k$. If $l = k$ then $H(k,k;n) = H_0(k,k;n)$ and $H_1(k,k;n) = \varnothing$. If $l < k$, it is easy to see that $H_0(k,l;n) = \{\lambda \in H(k,l;n) \mid \lambda_{l+1} \leq k\}$ (and $H_0(k,l;n)$ is close to $H(l,l;n)$ in an obvious sense).

Recall that $\mathcal{B}_n = End_{FS_n}(V^{\otimes n})$ and $\mathcal{A}_n = End_{FA_n}(V^{\otimes n})$.

The disjoint union decomposition (3.1.1) implies the following "$P - Q$" direct sum decompositions.

Theorem 3.2. Let $\dim V_0 \neq \dim V_1$. The decomposition (3.1.1) implies that

(a) $\varphi^*(FS_n) = P \oplus Q$, $\varphi^*(FA_n) = P' \oplus Q'$, satisfying $Q' = Q$, $P' \subset P$, and $\dim P = 2 \dim P'$.

(b) Also corresponding to (3.1.1) we have $\mathcal{A}_n = \mathcal{A}_P \oplus \mathcal{A}_Q$ and $\mathcal{B}_n = \mathcal{B}_P \oplus \mathcal{B}_Q$, satisfying $\mathcal{B}_P \subset \mathcal{A}_P$ and $\mathcal{B}_Q = \mathcal{A}_Q$.

We remark that beside the fact that $\dim \mathcal{B}_P < \dim \mathcal{A}_P$, at the moment, the relations between $\mathcal{A}_P$ and $\mathcal{B}_P$ are not clear.

We remark also that in Section 4 we show that if $\dim V_0 \neq \dim V_1$ then, asymptotically, both $\dim \varphi^*(FS_n) \cong \dim Q' = \dim Q \cong \dim \varphi^*(FA_n)$ and $\dim \mathcal{A}_n \cong \dim \mathcal{A}_Q = \dim \mathcal{B}_Q \cong \dim \mathcal{B}_n$.

We proceed with the details.

By (2.6.3) and (3.1.1),

$$\varphi^*(FS_n) = P \oplus Q\,, \tag{3.2.1}$$

where $P = \bigoplus_{\lambda \in H_0(k,l;n)} \varphi^*(I_\lambda) \cong \bigoplus_{\lambda \in H_0(k,l;n)} I_\lambda$,

and $Q = \bigoplus_{\lambda \in H_1(k,l;n)} \varphi^*(I_\lambda) \cong \bigoplus_{\lambda \in H_1(k,l;n)} I_\lambda$.

By pairing together $\lambda', \lambda \in H_0(k,l;n)$ (and choosing the notation such that $\lambda' \le \lambda$ in the left lexicographic order), the term P can be rewritten as

$$P \cong \bigoplus_{\lambda \in H_0(k,l;n)} I_\lambda = \left[\bigoplus_{\lambda' < \lambda \in H_0(k,l;n)} (I_\lambda \oplus I_{\lambda'}) \right] \oplus \left[\bigoplus_{\lambda' = \lambda \in H_0(k,l;n)} I_\lambda \right] . \tag{3.2.2}$$

Similarly,

$$\varphi^*(FA_n) = P' \oplus Q' , \tag{3.2.3}$$

with P' corresponding to $H_0(k,l;n)$ and Q' – to $H_1(k,l;n)$. By Lemma 2.5,

$$P' \cong \left[\bigoplus_{\lambda' < \lambda \in H_0(k,l;n)} I_\lambda \right] \oplus \left[\bigoplus_{\lambda' = \lambda \in H_0(k,l;n)} (\bar{I}_\lambda^+ \oplus \bar{I}_\lambda^-) \right]$$

$$\text{and} \qquad Q' \cong \bigoplus_{\lambda \in H_1(k,l;n)} \bar{I}_\lambda . \tag{3.2.4}$$

Similarly, (2.7.1) and (3.1.1) imply that

$$V^{\otimes n} = W_P \oplus W_Q , \tag{3.2.5}$$

$$\text{where} \qquad W_P = \bigoplus_{\lambda \in H_0(k,l;n)} W_\lambda, \qquad \text{and} \qquad W_Q = \bigoplus_{\lambda \in H_1(k,l;n)} W_\lambda .$$

Also here we can rearrange terms and rewrite

$$W_P = \bigoplus_{\lambda \in H_0(k,l;n)} W_\lambda = \left[\bigoplus_{\lambda' < \lambda \in H_0(k,l;n)} (W_\lambda \oplus W_{\lambda'}) \right] \oplus \left[\bigoplus_{\lambda' = \lambda \in H_0(k,l;n)} W_\lambda \right] . \tag{3.2.6}$$

Lemma 3.3. Let $\lambda \in H_1(k,l;n)$, and refer to (2.4.1) and (2.4.2). Then $\varphi^*(I_{\lambda'}) = 0$ and $\varphi^*(\bar{I}_\lambda) = \varphi^*(I_\lambda)$. Thus, by (3.2.1) and (3.2.3), $Q = Q'$.

Proof. Since $\lambda' \notin H(k,l;n)$, $\varphi^*(I_{\lambda'}) = 0$. Also, $\lambda \neq \lambda'$ therefore $\bar{I}_\lambda \cong I_\lambda$ and $\bar{I}_\lambda \subset I_\lambda \oplus I_{\lambda'}$, and hence $\varphi^*(\bar{I}_\lambda) \subseteq \varphi^*(I_\lambda)$. To prove equality, it clearly suffices to show that $\varphi^*(\bar{I}_\lambda) \neq 0$.

Assume $\varphi^*(\bar{I}_\lambda) = 0$, so $\varphi^*(\bar{I}_\lambda) W_\lambda = 0$. If $\mu \neq \lambda$, then

(1) if $\mu \neq \mu'$, $\varphi^*(\bar{I}_\mu) W_\lambda = 0$ since $\varphi^*(\bar{I}_\mu) \subseteq \varphi^*(I_\mu \oplus I_{\mu'})$; (the case $\mu' = \lambda'$ also follows, since $\varphi^*(I_{\lambda'}) = 0$)

(2) if $\mu = \mu'$, $\varphi^*(\bar{I}_\mu^+ \oplus \bar{I}_\mu^-) W_\lambda = 0$ since $\varphi^*(\bar{I}_\mu^+ \oplus \bar{I}_\mu^-) \subset \varphi^*(I_\mu)$.

We deduce that the assumption that $\varphi^*(\bar{I}_\lambda) = 0$ implies that $\varphi^*(FA_n) W_\lambda = 0$, which is a contradiction, since $1 \in FA_n$ and $\varphi^*(1) W_\lambda = W_\lambda$ for any $\lambda \in H(k,l;n)$. $\square$

Proposition 3.4. Let P, P' as in (3.2.2) and (3.2.3). Then $P' \subset P$ and $\dim P = 2\dim P'$.

Proof. The inclusion $P' \subset P$ follows easily from Lemma 2.5.
Let $\lambda \in H_0(k,l;n)$. If $\lambda \neq \lambda'$ then P contains the summand $\cong I_\lambda \oplus I_{\lambda'}$ $(\dim I_{\lambda'} = \dim I_\lambda)$ while P' contains the summand $\cong \bar{I}_\lambda \cong I_\lambda$.
If $\lambda = \lambda'$ then P contains the summand $\cong I_\lambda$ while P' contains the summand $\cong \bar{I}_\lambda^+ \oplus I_\lambda^-$ and by Lemma 2.5.b, $\dim \bar{I}_\lambda^+ \oplus I_\lambda^- = \frac{1}{2}\dim I_\lambda$. This obviously implies the proof. □

Note that Lemma 3.3 and Proposition 3.4 prove part (a) of Theorem 3.2

Corollary 3.5. Let $\mathcal{B}_n = \mathcal{B}_P \oplus \mathcal{B}_Q$,

$$\text{where}\quad \mathcal{B}_P = End_{FS_n}(W_P) = \bigoplus_{\lambda\in H_0(k,l;n)} \mathcal{B}_\lambda,$$

$$\text{and}\quad \mathcal{B}_Q = End_{FS_n}(W_Q) = \bigoplus_{\lambda\in H_1(k,l;n)} \mathcal{B}_\lambda$$

(and where $\mathcal{B}_\lambda = End_{FS_n}(W_\lambda)$),

Similarly,

$$\mathcal{A}_n = \mathcal{A}_P \oplus \mathcal{A}_Q,$$

$$\text{where}\quad \mathcal{A}_P = End_{FA_n}(W_P) \text{ and } \mathcal{A}_Q = End_{FA_n}(W_Q)$$

Then $\mathcal{A}_Q = \mathcal{B}_Q$.

Proof. Recall from (3.2.1) that $\varphi^*(FS_n) = P \oplus Q$ and $PW_Q = 0$. Therefore

$$\mathcal{B}_Q = End_{P\oplus Q}(W_Q) = End_Q(W_Q).$$

Similarly, $\varphi^*(FA_n) = P' \oplus Q'$ and $P'W_Q = 0$, while by Lemma 3.4, $Q' = Q$. Thus also

$$\mathcal{A}_Q = End_{P'\oplus Q'}(W_Q) = End_{Q'}(W_Q) = End_Q(W_Q)\ .$$

Note that Corollary 3.5 proves part (b) of Theorem 3.2.

§4 Asymptotics, as $n \to \infty$.

We examine first the case $\dim V_0 \neq \dim V_1$. Let

$$P,\ Q,\ P',\ Q',\ \mathcal{A}_n,\ \mathcal{A}_P,\ \mathcal{A}_Q,\ \mathcal{B}_n,\ \mathcal{B}_P,\ \mathcal{B}_Q,\ W_P \ \text{ and }\ W_Q$$

as in Section 3.

Remark 4.1. Let $\dim V_0 \neq \dim V_1$, say, $\dim V_1 < \dim V_0$. Then

$$\lim_{n\longrightarrow\infty} \frac{\dim P}{\dim \varphi^*(FS_n)} = 0 \qquad \text{and} \qquad \lim_{n\longrightarrow\infty} \frac{\dim Q}{\dim \varphi^*(FS_n)} = 1\,, \tag{4.1.1}$$

and similarly

$$\lim_{n\longrightarrow\infty} \frac{\dim P'}{\dim \varphi^*(FA_n)} = 0 \qquad \text{and} \qquad \lim_{n\longrightarrow\infty} \frac{\dim Q'}{\dim \varphi^*(FA_n)} = 1\,. \tag{4.1.2}$$

Together with $Q = Q'$, this clearly implies

$$\lim_{n\longrightarrow\infty} \frac{\dim \varphi^*(FA_n)}{\dim \varphi^*(FS_n)} = 1\,. \tag{4.1.3}$$

The proof follows since, as $n \to \infty$, it can be shown (by arguments similar to those in [BR §7]) that the growth of $\dim P$ is close to $(2\dim V_1)^n$, while that of $\dim Q$ is close to $(\dim V_0 + \dim V_1)^n$. This implies (4.1.1). The proof of (4.1.2) follows by similar arguments. We skip the details.
It is also possible to show that here,

$$\lim_{n\longrightarrow\infty} \frac{\dim \mathcal{B}_n}{\dim \mathcal{A}_n} = 1 \tag{4.1.4}$$

and also

$$\lim_{n\longrightarrow\infty} \frac{\dim W_P}{\dim V^{\otimes n}} = 0 \qquad \text{and} \qquad \lim_{n\longrightarrow\infty} \frac{\dim W_Q}{\dim V^{\otimes n}} = 1\,. \tag{4.1.5}$$

Again, we skip the details.

4.2. In the rest of this section we study the case $\dim V_0 = \dim V_1 = k$.

It follows from (2.4.1) and (2.6.3) that

$$\varphi^*(FS_n) = M_1 \oplus M_2, \qquad \text{where}$$

$$M_1 \cong \left[\bigoplus_{\lambda'<\lambda\in H(k,k;n)} (I_\lambda \oplus I_{\lambda'})\right] \quad \text{and} \quad M_2 \cong \left[\bigoplus_{\lambda'=\lambda\in H(k,k;n)} I_\lambda\right]. \tag{4.2.1}$$

Denote

$$H_{sc}(k,k;n) = \{\lambda \in H(k,k;n) \mid \lambda = \lambda'\} \quad (\text{‘}sc\text{’ for ‘self-conjugate’}),$$

$$\text{and} \quad H_{nsc}(k,k;n) = H(k,k;n) \backslash H_s(k,k;n) \quad (\text{‘}nsc\text{’ for ‘non-self-conjugate’}).$$

Thus, (4.2.1) can be rewritten with

$$M_1 \cong \left[\bigoplus_{\lambda \in H_{nsc}(k,k;n)} I_\lambda \right] \quad \text{and} \quad M_2 \cong \left[\bigoplus_{\lambda \in H_{sc}(k,k;n)} I_\lambda \right].$$

We compare $\dim M_1$ with $\dim M_2$, as well as the cardinalities of $H_{sc}(k,k;n)$ and of $H_{nsc}(k,k,;n)$, as $n \to \infty$. Here we have

Proposition 4.3. As $n \longrightarrow \infty$,

$$\frac{|H_{sc}(k,k;n)|}{|H_{nsc}(k,k;n)|} \approx \frac{|H_{sc}(k,k;n)|}{|H(k,k;n)|} \approx c_1 \cdot \left(\frac{1}{n}\right)^k \quad \text{where} \quad c_1 = \frac{k \cdot (2k-1)!}{2^{k-1}}, \qquad (4.3.1))$$

and

$$\frac{\dim M_2}{\dim M_1} \approx \frac{\dim M_2}{\dim \varphi^*(FS_n)} \approx c_2 \cdot \left(\frac{1}{\sqrt{n}}\right)^k, \qquad (4.3.2)$$

where c_2 is a constant that can be calculated explicitly by Selberg integrals.
In particular,

$$\frac{\dim M_2}{\dim \varphi^*(FS_n)} \approx 0 \quad \text{and} \quad \frac{\dim M_1}{\dim \varphi^*(FS_n)} \approx 1. \qquad (4.3.3)$$

Proof. First, let $H'_{sc}(k,k;n) = \{(\lambda_1, \lambda_2, \cdots) \in H_{sc}(k,k;n) \mid \lambda_k \geq k\}$ (i.e $k \times k \subseteq \lambda$). It can be shown, for example by [BR 7.3] and by induction on k, that $|H_{sc}(k,k;n)| \approx |H'_{sc}(k,k;n)|$. It is easy to see, for example by the arguments below, that $|H(k,k;n)| \approx |H_{nsc}(k,k;n)|$. Therefore, to prove (4.3.1) it suffices to show that

$$\frac{|H'_{sc}(k,k;n)|}{|H(k,k;n)|} \approx c_1 \cdot \left(\frac{1}{n}\right)^k. \qquad (4.3.1')$$

It follows from [BR 7.3] that

$$|H(k,k;n)| \approx \frac{n^{2k-1}}{k!k!(2k-1)!}.$$

Let $Par_k(m)$ denote the number of the partitions of m to at most k parts. It is easy to see that

$$|H'_{sc}(k,k;n)| = Par_k((n-k^2)/2).$$

Let $m = \frac{n-k^2}{2}$, then $m \approx \frac{n}{2}$. Again by [BR 7.3] (with $l = 0$), $Par_k(m) \approx \frac{m^{k-1}}{k!(k-1)!}$. Thus

$$H'_{sc}(k,k;n)| \approx \frac{n^{k-1}}{k!(k-1)!2^{k-1}},$$

and the proof of (4.3.1') follows.

To prove (4.3.2), follow the rest of Section 7 in [BR]. Let $\lambda' = \lambda \in H(k,k;n)$. Since $\lambda' = \lambda$, in the notations of [BR 7.9 – 7.21] , $a_i = b_i$ and $\alpha_i = \beta_i,\ i = 1, \cdots, k$. By 7.16.1 there,

$$d_\lambda \approx c(k,k) \cdot D_k^2(\alpha_1, \cdots, \alpha_k) \cdot e^{-2k(\sum \alpha_i^2)} \cdot \left(\frac{1}{n}\right)^{\theta(k,k)} \cdot (2k)^n, \tag{4.3.4}$$

where $\theta(k,k) = \frac{1}{2}(k(k+1)-1)$.
As in the proof of [BR, Thm 7.18], we follow [R,§2] to deduce that

$$\dim M_2 = \sum_{\lambda'=\lambda \in H(k,k;n)} d_\lambda^2 \approx \left[c(k,k) \cdot \left(\frac{1}{n}\right)^{\theta(k,k)} \cdot (2k)^n\right]^2 \cdot (\sqrt{n})^{k-1} \cdot I(k,k,2), \tag{4.3.5}$$

where

$$I(k,k,2) = \int_{P(k)} \left[D_k^2(x_1, \cdots, x_k) \cdot e^{-2k(\sum x_i^2)}\right]^2 d^{(k-1)}(x) \tag{4.3.6}$$

with

$$P(k) = \{(x_1, \cdots, x_k) \,|\, x_1 \geq \cdots \geq x_k \text{ and } \sum x_i = 0\}$$

(correction: the term $\sum \alpha_i^2 + \sum \beta_j^2$ just below (7.18.1) in [BR] – should have been $\sum x_i^2 + \sum y_j^2$). On the other hand, [BR, Thm 7.18] gives the asymptotics of $S_{k,k}^{(2)}(n) = \dim M$:

as $n \longrightarrow \infty$,

$$\dim M \approx \left[c(k,k) \cdot \left(\frac{1}{n}\right)^{\theta} (k,k) \cdot (2k)^n\right]^2 \cdot (\sqrt{n})^{2k-1} \cdot I'(k,k,2) \tag{4.3.7}$$

for a certain multi-integral $I'(k,k,2)$ (given explicitly in [BR (7.18.1)]).
The proof of (4.3.2) now clearly follows from (4.3.5) and (4.3.7). $\square$

Remark 4.4. Recall that

$$M_1 = M_1(k,n) \cong \left[\bigoplus_{\lambda' < \lambda \in H(k,k;n)} (I_\lambda \oplus I_{\lambda'})\right] \quad \text{and}$$

$$M_2 = M_2(k,n) \cong \left[\bigoplus_{\lambda' = \lambda \in H(k,k;n)} I_\lambda\right].$$

Similarly, by removing the restriction that $\lambda \in H(k,k;n)$, one can define

$$M_1(n) = \left[\bigoplus_{\lambda' < \lambda \vdash n} (I_\lambda \oplus I_{\lambda'}) \right] \quad \text{and} \quad M_2(n) = \left[\bigoplus_{\lambda' = \lambda \vdash n} I_\lambda \right]. \tag{4.4.1}$$

In analogy with Proposition 4.3, it should be interesting to calculate the asymptotics of the ratios

$$\frac{\dim M_2(n)}{\dim M_1(n)} \qquad \text{and} \qquad \frac{\dim M_2(n)}{n!}$$

as $n \longrightarrow \infty$, *if* such asymptotics exists – which is not at all obvious.

References

[BR] Berele A., Regev A., Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras, *Adv. Math.* **64** (1987) 118–175.

[B] Boerner H., Representations of groups, North-Holland, Amsterdam, 1963.

[J] James, G. D., The representations of the symmetric groups, Springer LNM, No. 682, Springer-Verlag, 1978.

[JK] James, G. D. and Kerber A., The representation theory of the symmetric group, Encyclopedia of Mathematics and its Applications, Vol. 16, Addison-Wesley, 1981.

[M] Macdonald I. G., Symmetric functions and Hall polynomials, 2nd edition, Oxford University Press, 1995.

[R] Regev A., Asymptotic values for degrees associated with strips of Young diagrams, *Adv. Math.* **41** (1981) 115–136.

[Sa] Sagan B. E., The symmetric group, representations, combinatorial algorithms, and symmetric functions, Wadsworth & Brooks/Cole Mathematics Series, 1991.

[Sc] Schur I., Uber die rationalen Darstellungen der allgemeinen linearen Gruppe (1927), *in* "I.Schur, Gesammetle Abhandlungen III," pp. 68-85, Springer-Verlag, Berlin, 1973.

[Se] Sergeev A., The tensor algebra of the identity representation as a module over the Lie superalgebras $gl(n, m)$ and $Q(n)$, (Russian) Tom 123 (165) (1984). English translation: Math, USSR Sbornik Vol. 51 (1985), No. 2, 419-427.

[W] Weyl H., The Classical groups, Princeton Math. Series, No. 1, Princeton Univ. Press, Princeton, N.J.