RANDOM YOUNG TABLEAUX AND COMBINATORIAL IDENTITIES

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Abstract. We derive new combinatorial identities which may be viewed as multivariate analogs of summation formulas for hypergeometric series. As in the previous paper [Re], we start with probability distributions on the space of the infinite Young tableaux. Then we calculate the probability that the entry of a random tableau at a given box equals $n = 1, 2, \dots$. Summing these probabilities over n and equating the result to 1 we get a nontrivial identity. Our choice for the initial distributions is motivated by the recent work on harmonic analysis on the infinite symmetric group and related topics.

§0. Introduction

Let Tab be the set of all infinite standard Young tableaux $T = (T(i,j))$. Given a probability measure M on Tab, we may speak about the *random* infinite tableau T. Let $\mathcal{P}_M(T(i,j) = n)$ denote the probability that T has the entry n at the box (i,j). Fix a box (i,j) such that the shape of T contains (i,j) almost surely. Then

$$\sum_{n\geq 0} \mathcal{P}_M(T(i,j) = n) = 1. \tag{0.1}$$

In [Re], it was shown that by specializing M one can get from (0.1) many nontrivial identities. These identities look as summation formulas for multivariate series of hypergeometric type.

For instance, one of the identities is as follows (see [Re, (4.2.1′)]):

$$\begin{aligned}\sum_{\substack{p_1>\dots>p_k\geq 1\\ q_1>\dots>q_l\geq 1}} & \frac{(|p|+|q|+\frac{1}{2}[k+l-(k+l)^2])!\, V^2(p)\, V^2(q)}{\prod\limits_{1\leq r\leq k}(p_r!)^2 \prod\limits_{1\leq s\leq l}(q_s!)^2 \prod\limits_{1\leq r\leq k}\prod\limits_{1\leq s\leq l}(p_r+q_s+1)^2} \\ & \times\left(\prod_{r=1}^{k}\frac{p_r}{p_r+1}\right)\left(\prod_{s=1}^{l}\frac{q_s}{q_s+1}\right) = 1,\end{aligned} \tag{0.2}$$

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where $k, l = 0, 1, \ldots$ are arbitrary, $k + l \geq 1$, and

$$|p| = \sum_{r=1}^{k} p_r \,, \qquad |q| = \sum_{s=1}^{l} q_s \,,$$
$$V^2(p) = \prod_{1 \leq i < j \leq k} (p_i - p_j)^2, \qquad V^2(q) = \prod_{1 \leq i < j \leq l} (q_i - q_j)^2.$$

The identity (0.2) arises from the so–called *Plancherel measure,* the fixed box being $(k+1, l+1)$. From a different point of view, the identity (0.2) (written in an equivalent form) is discussed in [MMW].[1]

In the present paper, which is a continuation of [Re], we derive new identities of the form (0.1). The results are as follows.

Theorem 0.1. *Let $k, l = 0, 1, 2, \ldots$ be arbitrary, $k + l \geq 1$, and $t > 0$ be a parameter. Then*

$$\frac{1}{k!} \sum_{r_1, \ldots, r_l, s_1, \ldots, s_k \geq 0} \frac{(s_1 + \cdots + s_k + r_1 + 2r_2 + \cdots + lr_l + kl + k + l)!}{(s_1 + l + 1) \ldots (s_k + l + 1) 1^{r_1} 2^{r_2} \ldots l^{r_l} r_1! \ldots r_l!} \times \frac{t^{k + r_1 + \cdots + r_l + 1}}{(t)_{s_1 + \cdots + s_k + r_1 + 2r_2 + \cdots + lr_l + kl + k + l + 1}} = 1, \quad (0.3)$$

where $(x)_n = x(x+1) \ldots (x + n - 1)$ is the Pochhammer symbol.

See Theorem 3.3.2 below. We present two proofs of (0.3). One of them follows our general scheme while another is a direct argument, which is largely due to S. Milne [Mi]. Of the general identities derived here and in [Re], so far this is the only case where an independent direct proof was found.

Theorem 0.2. *Let $k = 1, 2, \ldots,$ and let $\theta > 0$ and $z \in \mathbb{C}$ be parameters. Then*

$$\sum_{\mu_1 \geq \cdots \geq \mu_{k+1} = 1} \frac{|\mu|! \cdot \prod_{1 \leq i < j \leq k} ((j - i)\theta + \mu_i - \mu_j)}{\theta \cdot \prod_{1 \leq i \leq k} (\mu_i - \mu_{i+1})! \cdot \prod_{1 \leq i < j \leq k+1} ((j - i)\theta + \mu_i - \mu_{j-1})_{\mu_{j-1} - \mu_{j+1} + 1}} \times \frac{(z - k\theta)(\bar{z} - k\theta) \cdot \prod_{1 \leq i \leq k} [(z - (i-1)\theta)_{\mu_i} (\bar{z} - (i-1)\theta)_{\mu_i}]}{(\theta^{-1} z \bar{z})_{|\mu| + 1}} = 1 \,, \quad (0.4)$$

where $|\mu| = \mu_1 + \cdots + \mu_k$.

See Theorem 2.5.1. For instance, in the simplest case $k = 1$ we get the following special summation formula for the generalized hypergeometric series of type (3,2):

$${}_3F_2(z + 1, \bar{z} + 1, 2; \theta + 2, \theta^{-1} z \bar{z} + 2; 1) = \frac{(\theta + 1)(z \bar{z} + \theta)}{(z - \theta)(\bar{z} - \theta)} \,. \quad (0.5)$$

[1] See Remark 2.8 for a comment to the approach of [MMW].

This can be derived from a certain known formula, see (2.6.1) below.

Specializing $\theta = 1$ and letting $z \to \infty$, we get from (0.4) the following identity:

$$\sum_{\mu_1 \ge \dots \ge \mu_{k+1} = 1} \frac{(\mu_1 + \dots + \mu_k)! \prod_{1 \le i < j \le k} (\mu_i - \mu_j + j - i)}{\prod_{i=1}^{k} (\mu_i - \mu_{i+1})! \cdot \prod_{1 \le i < j \le k+1} (\mu_i - \mu_{j-1} + j - i)_{\mu_{j-1} - \mu_{j+1} + 1}}, \qquad (0.6)$$

which can be transformed (see Proposition 2.7.1) to

$$\sum_{p_1 > \dots > p_k \ge 1} \frac{(p_1 + \dots + p_k - k(k-1)/2)! \cdot \prod_{1 \le i < j \le k} (p_i - p_j)^2}{\prod_{i=1}^{k} [(p_i - 1)!(p_i + 1)!]} = 1\,. \qquad (0.7)$$

The last identity is a particular case of (0.2) (it corresponds to $l = 0$).

Let us explain now the origin of the measures M that lead to the identities (0.3) and (0.4).

Infinite Young tableaux can be identified with infinite paths in a graph $\mathbb{Y}$, called the Young graph [VK] (the vertices of $\mathbb{Y}$ are arbitrary Young diagrams). Let $\mathcal{T}$ stand for the space of all infinite paths in $\mathbb{Y}$. Vershik and Kerov introduced in [VK] the concept of a *central* measure on the space $\mathcal{T}$. Their definition was inspired by the theory of characters of the infinite symmetric group $S(\infty)$. However, it makes sense for certain other graphs as well.

The probability measures M that were considered in [Re] are related to indecomposable characters of $S(\infty)$ [KV1, KV2, T, VK] and to some decomposable characters studied in harmonic analysis on $S(\infty)$ [B2, BO1, BO2, BO3, BO4, KOV, Ro]. Also considered in [Re] were measures related to *projective* characters of $S(\infty)$ [B1, I, N].

In the present paper, we deal with a modified definition of central measures, that of "θ–central measures". Here $\theta \ge 0$ is an additional parameter. This definition, due to Kerov [Ke2, Ke3, Ke5], is related to an additional structure on $\mathbb{Y}$, namely, to certain formal edge multiplicities depending on the parameter θ. The formal edge multiplicities in question come from the Pieri rule for the Jack symmetric functions.[2] When $\theta = 1$, all these multiplicities are equal to 1, which corresponds to the Schur functions and to the Young graph. Thus, the class of θ–central measures includes that of central measures as a particular case.

Another important particular case is that of $\theta = 0$, when the Jack symmetric functions degenerate to the monomial symmetric functions. In that case all the edge multiplicities are natural numbers and we get the so–called Kingman graph. The definition of the Kingman graph, also due to Kerov (see [Ke1]), was initially inspired by Kingman's concept of partition structures, see [Ki1, Ki2].

In Theorem 0.1, we are dealing with the Kingman graph ($\theta = 0$). The identity in question is related to a remarkable family of "0–central measures" depending on a parameter $t > 0$. These measures come from the well–known Poisson–Dirichlet distributions studied by many authors, see, e.g., [Ki3]. The fixed box is $(k+1, l+1)$. Note

[2]About the Jack symmetric functions, see [Ma, S].

that one could write down a generalization of (0.3), which is related to the so–called two–parameter Poisson–Dirichlet distribution, see [Ke4, Pi, PY].

In Theorem 0.2, we are dealing with general Jack edge multiplicities ($\theta > 0$). The corresponding "θ–central measures" depend on the complex parameter z. These measures are studied in [BO3, Ke5]. As the fixed box, we take $(k+1, 1)$. We did not consider an arbitrary box only to simplify the presentation of the identity.

Our formalism has an evident extension to identities of the form

$$\sum_{n_1,\dots,n_k} \mathcal{P}_M(T(i_1,j_1)=n_1,\dots,T(i_k,j_k)=n_k)=1, \tag{0.8}$$

which correspond to several boxes. Some examples are mentioned in [Re, 5.3] and in [MMW]. For simplicity, we do not deal with this case here.

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§1 The general formalism

1.1. The Young graph. Let $\mathbb{Y}_n$ be the set of Young diagrams with n boxes and let $\mathbb{Y} = \mathbb{Y}_0 \sqcup \mathbb{Y}_1 \sqcup \mathbb{Y}_2 \sqcup \dots$ be the set of all Young diagrams. We agree that $\mathbb{Y}_0$ consists of a single element — the empty diagram $\varnothing$.

For a diagram $\lambda \in \mathbb{Y}$ we denote by $|\lambda|$ the number of boxes in λ. Given $\mu, \lambda \in \mathbb{Y}$, we write $\mu \nearrow$ or $\lambda \searrow$ if $\mu \subset \lambda$ and $|\lambda| = |\mu| + 1$. This condition means that λ is obtained from μ by adding a single box. The *Young graph* is the graph whose vertices are arbitrary diagrams $\lambda \in \mathbb{Y}$ and whose edges are arbitrary couples $\mu, \lambda \in \mathbb{Y}$ such that $\mu \nearrow \lambda$. By abuse of notation we denote the Young graph again by the symbol $\mathbb{Y}$.

1.2. Finite tableaux and paths. Recall that a *standard tableau* of a given shape $\lambda \in \mathbb{Y}_n$ is defined by labelling the boxes of λ with the numbers $1, \dots, n$ in such a way that the labels strictly increase from left to right along each row and down each column of λ. Given a standard tableau T we denote by $T(i,j)$ the label it assigns to a box $(i,j) \in \lambda$. The set of all standard tableaux of shape λ will be denoted by $\mathrm{Tab}(\lambda)$.

We will identify a standard tableau $T \in \mathrm{Tab}(\lambda)$ with a *path* $\tau = (\varnothing \nearrow \tau^1 \nearrow \cdots \nearrow \tau^n = \lambda)$. Here, for any $k = 1, \dots, n$, the diagram τ^k consists of those boxes $(i,j) \in \lambda$ for which $T(i,j) \le k$. The correspondence $T \mapsto \tau$ is a bijection between $\mathrm{Tab}(\lambda)$ and the set of all paths in the Young graph starting at $\varnothing$ and ending at λ.

1.3. Infinite tableaux and paths. By an *infinite path* in the graph $\mathbb{Y}$ we mean an infinite sequence of diagrams $\tau = (\varnothing \nearrow \tau^1 \nearrow \tau^2 \nearrow \dots)$. We may view τ as an *infinite standard tableau* T whose shape $D(T)$ is the *infinite Young diagram* $\bigcup\limits_{n \ge 0} \tau^n$.

Let

$$\mathrm{Tab}_n = \bigcup_{\lambda \colon |\lambda| = n} \mathrm{Tab}(\lambda)$$

and let Tab denote the set of all infinite standard tableaux. For any $n \ge 1$ we define the projection $\mathrm{Tab}_n \to \mathrm{Tab}_{n-1}$ as removing from a given tableau $T \in \mathrm{Tab}_n$ the box labelled by n. In terms of paths τ, this means removing the last diagram τ^n. Using these projections we can identify Tab with the projective limit $\varprojlim \mathrm{Tab}_n$.

We equip Tab with the topology of a projective limit of finite sets. In this topology, Tab is a compact metrizable totally disconnected topological space.

Given $\tau \in \mathrm{Tab}_n$, we denote by $\mathrm{Cyl}(\tau) \subset \mathrm{Tab}$ the pull–back of $\{\tau\}$ under the natural projection $\mathrm{Tab} \to \mathrm{Tab}_n$. This is a *cylindrical subset.* Such subsets are both open and closed, and they form a base of the topology of Tab.

1.4. Multiplicity function. Assume we are given a strictly positive function $\varkappa(\mu, \lambda)$ defined on the set of edges of $\mathbb{Y}$. Such a function is called a *multiplicity function,* and its value at (μ, λ) is called the *formal multiplicity* of the edge (μ, ν). Given $\tau \in \mathrm{Tab}_n$, we set

$$\varkappa(\tau) = \varkappa(\varnothing, \tau^1)\varkappa(\tau^1, \tau^2) \dots \varkappa(\tau^{n-1}, \tau^n) ,$$

where $\varnothing \nearrow \tau^1 \nearrow \cdots \nearrow \tau^n$ are the vertices of τ. We agree that the value of $\varkappa$ at the single element of Tab_0 is equal to 1.

For a diagram λ we set

$$\dim_\varkappa(\lambda) = \sum_{\tau \in \mathrm{Tab}(\lambda)} \varkappa(\tau) . \tag{1.4.1}$$

We call $\dim_\varkappa(\cdot)$ the *$\varkappa$-dimension function.* If $\varkappa(\cdot, \cdot) \equiv 1$ then $\dim_\varkappa(\lambda)$ coincides with $|\mathrm{Tab}(\lambda)|$, i.e. it turns into the conventional combinatorial dimension.

The $\varkappa$-dimension function satisfies the recurrence relation

$$\dim_\varkappa(\lambda) = \sum_{\mu : \mu \nearrow \lambda} \dim_\varkappa(\mu)\varkappa(\mu, \lambda) . \tag{1.4.2}$$

Together with the initial condition $\dim_\varkappa(\varnothing) = 1$ this defines the function $\dim_\varkappa(\cdot)$ uniquely.

1.5. $\varkappa$-central measures. A probability measure M on Tab is called *$\varkappa$-central* if for any diagram λ, the masses of the cylindrical sets $\mathrm{Cyl}(\tau)$ with $\tau \in \mathrm{Tab}(\lambda)$ are proportional to the numbers $\varkappa(\tau)$, i.e.,

$$M(\mathrm{Cyl}(\tau)) = \varkappa(\tau)\varphi(\lambda) , \tag{1.5.1}$$

where $\varphi(\lambda)$ is a certain (positive real valued) function on the Young diagrams. It is easy to see that φ is nonnegative, $\varphi(\varnothing) = 1$ (since $\varkappa(\varnothing) = 1 = M(\mathrm{Cyl}(\varnothing))$) and for all μ,

$$\varphi(\mu) = \sum_{\lambda : \lambda \searrow \mu} \varkappa(\mu, \lambda)\varphi(\lambda). \tag{1.5.2}$$

Indeed, to see this, pick $\tau \in \mathrm{Tab}(\mu)$ and multiply both sides by $\varkappa(\tau)$. Then in the left–hand side we get $M(\mathrm{Cyl}(\tau))$, and in the right–hand side we get $\sum M(\mathrm{Cyl}(\tau'))$, where the summation is taken over all paths τ' obtained from τ by adding one extra edge. Since $\mathrm{Cyl}(\tau)$ is the disjoint union of the sets $\mathrm{Cyl}(\tau')$, we get $M(\mathrm{Cyl}(\tau)) = \sum M(\mathrm{Cyl}(\tau'))$, which is equivalent to (1.5.2).

The relation (1.5.2) is called the *$\varkappa$-harmonicity condition.* The correspondence $M \mapsto \varphi$ is a bijection between the *$\varkappa$-central* probability measures on Tab and the nonnegative, $\varkappa$-harmonic functions on $\mathbb{Y}$, normalized at $\varnothing \in \mathbb{Y}$.

In the particular case $\varkappa(\cdot,\cdot) \equiv 1$, $\varkappa$-central measures and $\varkappa$-harmonic functions turn into central measures and harmonic functions, respectively, as defined in [VK].

1.6. Transition and cotransition probabilities. Let M be a $\varkappa$-central measure on Tab. We view (Tab, M) as a probability space, which makes it possible to speak about the random element of Tab, i.e., the *random infinite tableau* or, equivalently, the *random infinite path.*

Given $\lambda \in \mathbb{Y}$, the probability that the random path passes through λ equals

$$\dim_\varkappa \lambda \cdot \varphi(\lambda).$$

Let (μ, λ) be an edge. The probability that the random path passes through μ, conditional that it passes through λ, equals

$$q(\mu,\lambda) = \frac{\dim_\varkappa \mu \cdot \varkappa(\mu,\lambda)}{\dim_\varkappa \lambda}. \tag{1.6.1}$$

Note that $q(\mu,\lambda)$ does not depend on M. This is the so-called *cotransition function.* Thus, all $\varkappa$-central measures with a fixed $\varkappa$ have one and the same cotransition function.

The *transition function* $p_M(\mu,\lambda)$ is, by definition, the probability that the random path passes through λ, conditional that it passes through μ. We have

$$p_M(\mu,\lambda) = \frac{\varkappa(\mu,\lambda)\varphi(\lambda)}{\varphi(\mu)}\ . \tag{1.6.2}$$

This function is well defined, provided that $\varphi(\mu) > 0$, and it depends on M.

1.7. Reachable boxes. Given an infinite tableau $T \in \mathrm{Tab}$, we denote by $D(T)$ its shape, which is an infinite diagram. A box (i,j) is called *completely reachable* (with respect to M) if it is contained in $D(T)$ for almost all $T \in \mathrm{Tab}$ (with respect to M). Call (i,j) *partially reachable* if the same event holds with a nonzero probability. Note that this terminology differs from that of [Re, section 2.5].

Let $D(M)$ and $\tilde{D}(M)$ stand for the set of all completely reachable and partially reachable boxes, respectively. Then $D(M) \subseteq \tilde{D}(M)$, and both sets are infinite diagrams. The set $\tilde{D}(M)$ of partially reachable boxes is easily described in terms of φ:

$$\tilde{D}(M) = \bigcup_{\lambda:\varphi(\lambda)>0} \lambda\ .$$

As for the set $D(M)$, which is more interesting for us, its structure is not so evident. It may well happen that $D(M) \neq \tilde{D}(M)$. For instance, take $M = \frac{1}{2}(M_1 + M_2)$, where M_1 is the delta-measure concentrated on the infinite path going along the first row $i = 1$, while M_2 is defined similarly by replacing the first row by the first column $j = 1$. Both M_1 and M_2 are $\varkappa$-central for any choice of $\varkappa$. Consequently, M is $\varkappa$-central, too. Then $\tilde{D}(M)$ is the hook formed by the first row and the first column, while $D(M)$ consists of a single box $(1, 1)$.

1.8. The identity — general form. Given a box (i, j), we denote by $H'(i, j)$ the set of finite diagrams μ such that μ does not contain (i, j) and the shape $\mu \cup (i, j)$ is a diagram. Here we prove

Theorem 1.8.1. *Let M be a $\varkappa$-central measure, φ the corresponding $\varkappa$-harmonic function and let (i, j) be a completely reachable box.*

Then

$$\sum_{\mu \in H'(i,j)} \dim_\varkappa \mu \cdot \varkappa(\mu, \mu \cup (i, j)) \cdot \varphi(\mu \cup (i, j)) = 1 \ . \tag{1.8.1}$$

In fact, Theorem 1.8.1 is a consequence of the slightly more general Theorem 1.8.3, which is given below.

Let $\mathcal{T}(i, j)$ be the set of the (infinite) paths $\tau = (\varnothing \nearrow \tau^1 \nearrow \ldots)$ such that the diagram of τ contains (i, j). For any $\tau \in \mathcal{T}(i, j)$ there exists a unique n such that (i, j) is contained in τ^n (hence in $\tau^{n+1}, \tau^{n+2}, \ldots$), but not in τ^{n-1}. Denote this by $\tau(i, j) = n$. Given a probability measure M on Tab, we consider the M–probability that a random path τ satisfies $\tau(i, j) = n$, and we denote it by $\mathcal{P}_M(\tau(i, j) = n)$.

Let $\mathcal{T}(i, j; n) \subseteq$ Tab denote the set of the infinite tableaux (paths) τ such that $\tau^{n-1} \in H'(i, j)$, $\tau^n = \tau^{n-1} \cup (i, j)$ (i.e such that $\tau(i, j) = n$):

$$\mathcal{T}(i, j; n) = \{\tau \in \text{Tab} \mid \tau^{n-1} \in H'(i, j) \text{ and } \tau^n = \tau^{n-1} \cup (i, j)\} \ .$$

Clearly, $\mathcal{P}_M(\tau(i, j) = n) = M(\mathcal{T}(i, j; n))$.

Lemma 1.8.2. *We have*

$$\mathcal{P}_M(\tau(i, j) = n) = M(\mathcal{T}(i, j; n)) = \sum_{\substack{\mu \in H'(i,j) \\ |\mu| = n-1}} \dim_\varkappa \mu \cdot \varkappa(\mu, \mu \cup (i, j)) \cdot \varphi(\mu \cup (i, j)) \ . \tag{1.8.2}$$

Proof. Given $\mu \in H'(i, j)$ with $|\mu| = n - 1$, let

$$\mathcal{T}(i, j; \mu) = \{\tau \in \text{Tab} \mid \tau^{n-1} = \mu \text{ and } \tau^n = \tau^{n-1} \cup (i, j)\} \,. \tag{1.8.3}$$

Clearly,

$$\mathcal{T}(i, j; n) = \bigsqcup_{\substack{\mu \in H'(i,j) \\ |\mu| = n-1}} \mathcal{T}(i, j; \mu) \ ,$$

where $\bigsqcup$ indicates a disjoint union, hence

$$M(\mathcal{T}(i,j;n)) = \sum_{\substack{\mu \in H'(i,j) \\ |\mu|=n-1}} M(\mathcal{T}(i,j;\mu)) \ .$$

Now the set $\mathcal{T}(i,j;\mu)$ is the disjoint union of (d_μ) cylindrical sets $\mathrm{Cyl}(\sigma)$, where σ is an arbitrary path $\varnothing \nearrow \sigma' \nearrow \cdots \nearrow \sigma^{n-1} \nearrow \sigma^n$ such that $\sigma^{n-1} = \mu$, $\sigma^n = \mu \cup (i,j)$. Here $d_\mu = \dim \mu$ is the number of standard tableaux of shape μ.

By the very definition, $M(\mathrm{Cyl}(\sigma)) = \prod_{i=1}^n \varkappa(\sigma^{i-1}, \sigma^i) \cdot \varphi(\sigma^n)$. This can be written as the product of two expressions

$$M(\mathrm{Cyl}(\sigma)) = \left[\prod_{i=1}^{n-1} \varkappa(\sigma^{i-1}, \sigma^i)\right] \cdot [\varkappa(\mu, \mu \cup (i,j)) \cdot \varphi(\mu \cup (i,j))] \,. \tag{1.8.4}$$

The second expression does not depend on σ, while summing the first expression over all possible (d_μ) σ gives $\dim_\varkappa(\mu)$. This concludes the proof. □

Recall that $\mathcal{T}(i,j) = \{\tau \in \mathrm{Tab} \mid (i,j) \in D(\tau)\}$. Thus $M(\mathcal{T}(i,j))$ is the probability of a random tableau τ to have $(i,j) \in D(\tau)$. Denote that probability by $\mathcal{P}_M(i,j)$.

Theorem 1.8.3. *We have*

$$\mathcal{P}_M(i,j) = \sum_{\mu \in H'(i,j)} \dim_\varkappa \mu \cdot \varkappa(\mu, \mu \cup (i,j)) \cdot \varphi(\mu \cup (i,j)) \,. \tag{1.8.5}$$

Proof. Clearly, $\mathcal{T}(i,j) = \bigsqcup_{n \geq 1} \mathcal{T}(i,j;n)$, a disjoint union, hence

$$\mathcal{P}_M(i,j) = M(\mathcal{T}(i,j)) = \sum_{n=1}^{\infty} M(\mathcal{T}(i,j;n))$$

and by the above lemma,

$$\mathcal{P}_M(i,j) = \sum_{\mu \in H'(i,j)} \dim_\varkappa \mu \cdot \varkappa(\mu, \mu \cup (i,j)) \cdot \varphi(\mu \cup (i,j)) \,. \tag{1.8.6}$$

□

If $(i,j) \in D(M)$ (i.e. (i,j) is completely reachable), $\mathcal{P}_M(i,j) = 1$, and we have

$$\sum_{\mu \in H'(i,j)} \dim_\varkappa \mu \cdot \varkappa(\mu, \mu \cup (i,j)) \cdot \varphi(\mu \cup (i,j)) = 1 \ .$$

This completes the proof of Theorem 1.8.1.

§2 The Jack graph case

2.1. Jack edge multiplicities. Fix a positive parameter θ. Let P_μ denote the Jack symmetric function with parameter θ and index μ. Here μ is a Young diagram, and the normalization of P_μ is that of [Ma, VI.10]. Note that Macdonald uses as the parameter $\alpha = \theta^{-1}$.

The simplest case of Pieri's formula for the Jack symmetric functions has the form

$$P_\mu P_{(1)} = \sum_{\lambda:\lambda\searrow\mu} \varkappa_\theta(\mu,\lambda) P_\lambda \ , \tag{2.1.1}$$

where $\varkappa_\theta(\mu,\lambda)$ are certain strictly positive numbers. An explicit expression for $\varkappa_\theta(\mu,\lambda)$ is as follows (see [Ma, VI.10, VI.6]). Let (i,j) be the box $\lambda\backslash\mu$. Then

$$\varkappa_\theta(\mu,\lambda) = \prod_{k=1}^{i-1} \frac{(a(k,j)+(\ell(k,j)+2)\theta)\cdot(a(k,j)+1+\ell(k,j)\theta)}{(a(k,j)+(\ell(k,j)+1)\theta)\cdot(a(k,j)+1+(\ell(k,j)+1)\theta)} \tag{2.1.2}$$

where

$$a(k,j) = \mu_k - j \ , \quad \ell(k,j) = \mu'_j - k$$

are the arm–length and the leg–length of the box (k,j) in μ.

We will interprete the numbers $\varkappa_\theta(\mu,\lambda)$ as formal multiplicities of the edges of the Young graph. By the *Jack graph* we mean the Young graph together with this additional structure. Of course, the Jack graph is not a graph in the conventional sense.

In the particular case $\theta = 1$ the Jack function P_μ turns into the Schur function S_μ, and all the edge multiplicities are equal to 1, so that the Jack graph reduces to the ordinary Young graph.

The dimension function $\dim_\varkappa \mu$ corresponding to $\varkappa(\cdot,\cdot) = \varkappa_\theta(\cdot,\cdot)$ will be denoted as $\dim_\theta \mu$. The following formula is a generalization of the classical hook formula:

$$\dim_\theta \mu = \frac{|\mu|!}{\mathcal{H}_\theta(\mu)} \ , \tag{2.1.3}$$

where

$$\mathcal{H}_\theta(\mu) = \prod_{b\in\mu}(a_\mu(b) + \theta\ell_\mu(b) + 1) = \prod_{b\in\mu} h_\mu(b) \tag{2.1.4}$$

and where $h_\mu(b) = a_\mu(b) + \theta\ell_\mu(b) + 1$. Here $b \in \mu$ means a box $b = (i,j)$ of μ.

A proof of this formula can be obtained from [S, Theorem 5.4] or [Ma, VI.10]. A different proof is given in [Ke5, Corollary (6.10)]. We shall need another expression, which is similar to $\mathcal{H}_\theta(\mu)$ but different from it:

$$\mathcal{H}'_\theta(\mu) = \prod_{b\in\mu}(a_\mu(b) + \theta\ell_\mu(\theta) + \theta) = \prod_{b\in\mu} h'_\mu(b) \tag{2.1.5}$$

where $h'_\mu(b) = a_\mu(b) + \theta\ell_\mu(\theta) + \theta$. In the particular case $\theta = 1$, $\mathcal{H}_\theta(\mu)$ and $\mathcal{H}'_\theta(\mu)$ coincide.

Finally, for a box $b = (i,j)$, we denote

$$c_\theta(b) = (j-1) - (i-1)\theta \; .$$

This is the "θ-version" of the conventional content $c(b) = j - i$ of a box $b = (i,j)$.

Alternative expressions for $\mathcal{H}_\theta(\mu)$ and $\mathcal{H}'_\theta(\mu)$:

$$\mathcal{H}_\theta(\mu) = \prod_{1\le i<j\le \ell(\mu)} \frac{(1+(j-i-1)\theta)_{\mu_i-\mu_j}}{(1+(j-i)\theta)_{\mu_i-\mu_j}} \cdot \prod_{i=1}^{\ell(\mu)} (1+(\ell(\mu)-i)\theta)_{\mu_i} \tag{2.1.6}$$

and

$$\mathcal{H}'_\theta(\mu) = \prod_{1\le i<j\le \ell(\mu)} \frac{((j-i)\theta)_{\mu_i-\mu_j}}{((j-i+1)\theta)_{\mu_i-\mu_j}} \cdot \prod_{i=1}^{\ell(\mu)} ((\ell(\mu)+1-i)\theta)_{\mu_i} \, . \tag{2.1.7}$$

2.2. The z-measures and the Plancherel measures. We fix $\theta > 0$ and deal with the multiplicity function $\varkappa(\cdot,\cdot) = \varkappa_\theta(\cdot,\cdot)$ as defined above.

Theorem 2.2.1. *For any $z \in \mathbb{C}$ there exists a $\varkappa_\theta$-central measure M_z with the corresponding $\varkappa_\theta$-harmonic function*

$$\varphi_z(\lambda) = \frac{1}{(\theta^{-1}|z|^2)_n} \cdot \frac{\prod_{b\in\lambda} |z + c_\theta(b)|^2}{\mathcal{H}'_\theta(\lambda)} \; , \tag{2.2.1}$$

where $n = |\lambda|$, and $(x)_n = x(x+1)\dots(x+n-1)$ stands for the Pochhammer symbol.

We call M_z the *z-measure.* About the proof of this claim see [Ke5] and [BO3, §3]. In the particular case $\theta = 1$ these measures are discussed in [B2, BO1, BO2, BO4, KOV].

Theorem 2.2.2. *There exists a $\varkappa_\theta$-central measure M_∞ such that the corresponding $\varkappa_\theta$-harmonic function φ_∞ is the pointwise limit of φ_z as $|z| \to \infty$. We have*

$$\varphi_\infty(\lambda) = \frac{\theta^n}{\mathcal{H}'_\theta(\lambda)} \; , \quad n = |\lambda| \; . \tag{2.2.2}$$

The measure M_∞ is called the *Plancherel measure.* See [KV1, KV2, VK] for the case of the Young graph and [Ke5] for the general case.

2.3. Reachability. Here we use the definitions introduced in section 1.7.

Theorem 2.3.1. *Let $M = M_\infty$ or $M = M_z$, where $z \in \mathbb{C}$, $z \notin \mathbb{Z} + \mathbb{Z}\theta$. Then all boxes are completely reachable.*

To prove the theorem we need the following two lemmas.

Lemma 2.3.2. *Fix a box (i,j), let $\mu \in H'(i,j)$ be arbitrary, and let $\lambda = \mu \cup (i,j)$. Let $p_\infty(\cdot,\cdot)$ denote the transition probability for the Plancherel measure M_∞. Then there exists $\epsilon = \epsilon(i,j,\theta) > 0$, not depending on μ, such that*

$$p_\infty(\mu, \lambda) \geq \epsilon \ .$$

Proof. Recall the general formula

$$p(\mu,\lambda) = \frac{\varkappa(\mu,\lambda)\varphi(\lambda)}{\varphi(\mu)} \ . \tag{2.3.1}$$

When $\varkappa(\mu,\lambda) = \varkappa_\theta(\mu,\lambda)$, it follows from the explicit expression for $\varkappa_\theta(\mu,\lambda)$ that this is the product of $i-1$ factors, each of the type

$$\frac{(a+\ell\theta+2\theta)(a+\ell\theta+1)}{(a+\ell\theta+\theta)(a+\ell\theta+\theta+1)} \ .$$

Since $\theta > 0$ is fixed and $a + \ell\theta \geq 0$, this is bounded from below by a positive constant, uniformly on $a+\ell\theta$. It remains to examine $\varphi(\lambda)/\varphi(\mu) = \varphi_\infty(\lambda)/\varphi_\infty(\mu)$. By (2.2.2),

$$\frac{\varphi_\infty(\lambda)}{\varphi_\infty(\mu)} = \frac{\theta \cdot \mathcal{H}'_\theta(\mu)}{\mathcal{H}'_\theta(\lambda)} \ . \tag{2.3.2}$$

Recall that

$$\mathcal{H}'_\theta(\lambda) = \prod_{b\in\lambda}(a_\lambda(b) + \theta\ell_\lambda(b) + \theta)$$

$$\mathcal{H}'_\theta(\mu) = \prod_{b\in\mu}(a_\mu(b) + \theta\ell_\mu(b) + \theta) \ ,$$

where the subscript indicates that the arm-length and the leg-length are taken in the corresponding diagram.

Since $\lambda = \mu \cup (i,j)$, we have $a_\lambda(b) = a_\mu(b)$, $\ell_\lambda(b) = \ell_\mu(b)$ whenever b does not lie in the same column or in the same row that (i,j). Note also that for $b = (i,j) \in \lambda$ we have $a_\lambda(b) = \ell_\lambda(b) = 0$, so that $a_\lambda(i,j) + \theta\ell_\lambda(i,j) + \theta = \theta$, which cancels out with θ in the numerator. Thus, we get

$$\frac{\varphi_\infty(\lambda)}{\varphi_\infty(\mu)} = \prod_{k=1}^{i-1} \frac{a_\mu(k,j) + \theta\ell_\mu(k,j) + \theta}{a_\mu(k,j) + \theta\ell_\mu(k,j) + 2\theta} \times \prod_{k=1}^{j-1} \frac{a_\mu(i,k) + \theta\ell_\mu(i,k) + \theta}{a_\mu(i,k) + \theta\ell_\mu(i,k) + \theta + 1} \ . \tag{2.3.3}$$

Clearly, this expression is also bounded from below by a positive constant not depending on μ. $\square$

Lemma 2.3.3. *Fix a box (i,j), let $\mu \in H'(i,j)$ be arbitrary, and let $\lambda = \mu \cup (i,j)$. Let $p_z(\mu,\lambda)$ denote the transition probability for the z-measure M_z. Assume that $z \notin \mathbb{Z}+\mathbb{Z}\theta$. Then there exists $\epsilon = \epsilon(i,j,\theta,z) > 0$ not depending on μ, such that*

$$p_z(\mu,\lambda) \geq \frac{\epsilon}{n}\ , \quad n = |\lambda|\ .$$

Proof. Comparing the formulas for φ_z and φ_∞ we see that

$$p_z(\mu,\lambda) = \frac{|z+(j-1)-(i-1)\theta|^2}{|z|^2+\theta(n-1)} \cdot p_\infty(\mu,\lambda)\ . \tag{2.3.4}$$

By the assumptions on z, we have $|z+(j-1)-(i-1)\theta| > 0$. So, the claim follows from Lemma 2.3.2. □

Proof of Theorem 2.3.1. Given an infinite Young diagram D, let $\mathcal{T}(D)$ denote the set of all paths $\tau = (\varnothing \nearrow \tau^1 \nearrow \dots)$ such that $D(\tau) = D$. We call D *proper* if D is distinct from the set of all the boxes. Then the claim of the theorem will follow once we prove that for any proper D, $M(\mathcal{T}(D)) = 0$. So, fix a proper infinite diagram D. There exists a box (i,j) adjacent to D, i.e. (i,j) does not lie in D but $D \cup (i,j)$ is a diagram. Given $\mu \in H'(i,j)$, let $\mathcal{T}(D,\mu)$ denote the subset of those $\tau \in \mathcal{T}(D)$ which pass through μ, i.e. $\tau^{|\mu|} = \mu$. Clearly, $\mathcal{T}(D) = \bigcup_{\mu \in H'(i,j)} \mathcal{T}(D,\mu)$, a countable union. Thus, it suffices to prove that $M(\mathcal{T}(D,\mu)) = 0$ for any $\mu \in H'(i,j)$. Set $m = |\mu|$. For any $n \geq m$ let $\mathcal{T}_n(i,j,\mu)$ denote the set of all $\tau \in \mathcal{T}$ such that $\tau^m = \mu$ and $\tau^n \in H'(i,j)$. Clearly, the sets $\mathcal{T}_n(i,j,\mu)$ decrease as $n \to \infty$, and for $n \geq m$, $\mathcal{T}_n(i,j,\mu)$ contains $\mathcal{T}(D,\mu)$. Thus, it suffices to prove that

$$\lim_{n\to\infty} M(\mathcal{T}_n(i,j,\mu)) = 0\ .$$

We have

$$M(\mathcal{T}_{m+1}(i,j,\mu)) = M(\mathcal{T}_m(i,j,\mu)) \cdot (1 - \mathcal{P}_M(\mu, \mu \cup (i,j)))\ .$$

By Lemmas 2.3.2, 2.3.3, we have an estimate

$$\mathcal{P}_M(\mu, \mu \cup (i,j)) \geq \epsilon_{|\mu|+1}\ ,$$

where

$$\epsilon_n = \begin{cases} \epsilon\ , & \text{case of } M = M_\infty \\ \frac{\epsilon}{n}\ , & \text{case of } M = M_z\ . \end{cases}$$

Therefore

$$M(\mathcal{T}_{m+1}(i,j,\mu)) \leq (1 - \epsilon_{|\mu|+1}) M(\mathcal{T}_m(i,j,\mu))\ .$$

Repeating the same argument for any $n > m$, we get

$$M(\mathcal{T}_n(i,j,\mu)) \leq (1-\epsilon_n) M(\mathcal{T}_{n-1}(i,j,\mu))\ .$$

Since the series $\sum \epsilon_n$ is divergent, it follows that $M(\mathcal{T}_n(i,j,\mu)) \to 0$. □

Theorem 2.3.1 justifies the application of the general Theorem 1.8.1 in Theorems 2.4.1 and 2.5.1 below. The argument of Theorem 2.3.1 is also useful to check complete reachability for the examples considered in [Re].

2.4. The θ-Plancherel identity for $(k+1,1)$. Here we specialize $(i,j)=(k+1,1)$, and Theorem 1.8.1 to the case of the Jack graph, and we deduce

Theorem 2.4.1. *Let $\mu=(a)=(a_1,\dots,a_k)$, $\quad a_1\geq\cdots\geq a_k\geq 1=a_{k+1}$, $|\mu|=|a|=a_1+\cdots+a_k=n-1$, $\quad(\lambda=(a_1,\dots,a_k,1)$, $\quad|\lambda|=n)$. As usual, $a_r=0$ if $r>k+1$.*

Then the θ–Plancherel identity for the box $(k+1,1)$ is

$$\sum_{a_1\geq\cdots\geq a_{k+1}=1}\frac{|a|!\cdot\theta^{|a|}\cdot\prod_{1\leq i<j\leq k}((j-i)\theta+a_i-a_j)}{\prod_{1\leq i\leq k}(a_i-a_{i+1})!\cdot\prod_{1\leq i<j\leq k+1}((j-i)\theta+a_i-a_{j-1})_{a_{j-1}-a_{j+1}+1}}=1. \tag{2.4.1}$$

Proof. We apply Theorem 1.8.1 (the complete reachability is ensured by Theorem 2.3). Let $\mu=(\mu_1,\dots,\mu_k)\vdash n-1$, $\mu_k\geq 1$, let $\lambda=(\mu_1,\dots,\mu_k,1)\vdash n$, and denote (by abuse of notation) $\mu_{k+1}=1$. Also $\mu_r=0$ if $r\geq k+2$. Recall that

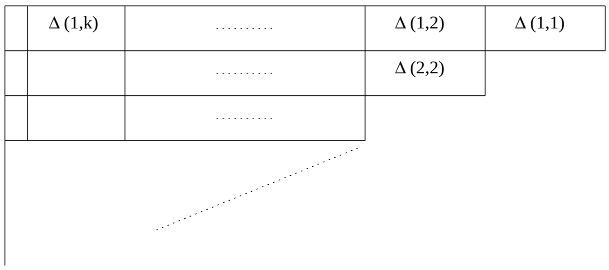
$$\mathcal{H}_\theta(\mu)=\prod_{b\in\mu}(a_\mu(b)+\theta\ell_\mu(b)+1),\quad \dim_\theta\mu=\frac{|\mu|!}{\mathcal{H}_\theta(\mu)},\quad \mathcal{H}'_\theta(\lambda)=\prod_{b\in\lambda}(a_\lambda(\ell)+\theta\ell_\lambda(b)+\theta)$$

and $\varphi_\infty(\lambda)=\varphi_\theta(\lambda)=\frac{\theta^{|\lambda|}}{\mathcal{H}'_\theta(\mu)}$. Thus, by Theorem 1.8.1, for the box $(k+1,1)$ we deduce the identity

$$\sum_{\mu_1\geq\cdots\geq\mu_k\geq 1}\frac{|\mu|!}{\mathcal{H}_\theta(\mu)}\cdot\varkappa_\theta(\mu,\mu\cup(k+1,1))\cdot\frac{\theta^{|\mu|+1}}{\mathcal{H}'_\theta(\lambda)}=1\ . \tag{2.4.2}$$

To calculate the left hand side of (2.4.2) explicitly, split the boxes in λ – hence in μ – into the following disjoint subsets

$$\Delta(i,r)=\{(i,j)\,|\,\mu_{r+1}+1\leq j\leq\mu_r\}\ ,\quad 1\leq i\leq r\leq k+1\ .$$



Thus, for $1\leq i\leq k$, the i-th row in either μ or λ is the disjoint union $\bigsqcup_{r=1}^{k+1}\Delta(i,r)$. Note that $\Delta(i,k+1)=\{(i,1)\}$. Also, $\Delta(k+1,k+1)=\{(k+1,1)\}$, $\quad(k+1,1)\in\lambda$ and $(k+1,1)\notin\mu$.

Denote

$$\lambda_*=\bigsqcup_{1\leq i\leq r\leq k-1}\Delta(i,r)\ ,\quad \mu_*=\bigsqcup_{1\leq i\leq r\leq k-1}\Delta(i,r+1)\ .$$

Clearly,

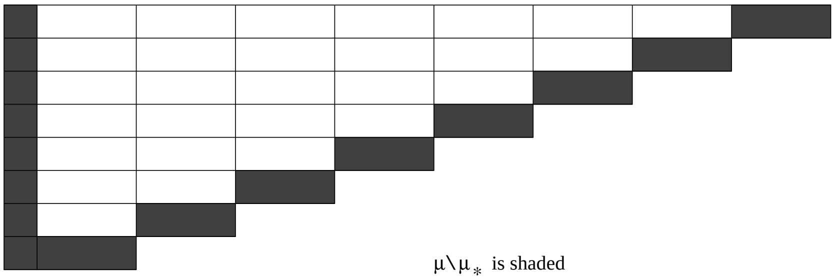
$$\begin{aligned}\lambda\backslash\lambda_* = \left(\bigsqcup_{i=1}^{k}\Delta(i,k)\right)\sqcup\left(\bigsqcup_{i=1}^{k+1}\Delta(i,k+1)\right)\\ = \Delta(k+1,k+1)\sqcup\left(\bigsqcup_{i=1}^{k}(\Delta(i,k)\sqcup\Delta(i,k+1))\right)\\ = \{(k+1,1)\}\cup\{(i,j)|1\le i\le k, 1\le j\le \mu_k\},\end{aligned} \tag{2.4.3}$$

and

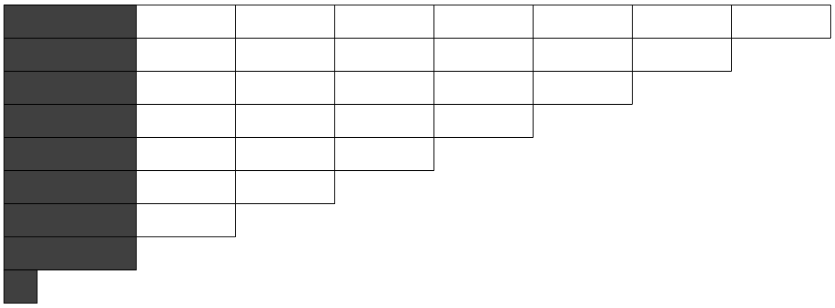
$$\mu\backslash\mu_* = \left(\bigsqcup_{i=1}^{k}\Delta(i,i)\right)\sqcup\{(i,1)|1\le i\le k\}\ , \tag{2.4.4}$$

disjoint unions.

k = 8



$\mu\backslash\mu_*$ is shaded



$\lambda\backslash\lambda_*$ is shaded

Denote (and abbreviate)

$$\mathcal{H}_\theta(\mu) = \prod_{b\in\mu} h_\mu(b) = P(b\in\mu) = P(b\in\mu_*)\cdot P(b\in\mu\backslash\mu_*)\ ,$$

where $P(b \in \mu)$ is the product - of the numbers $h_\mu(b)$ - over the boxes $b \in \mu$, while $P(b \in \mu_*)$ is such a product - over the boxes $b \in \mu_*$, etc.

Similarly, let $P'(b \in \lambda)$ denote the product - of the numbers $h'_\lambda(b)$ - over the boxes $b \in \lambda$, etc., so that

$$\mathcal{H}'_\theta(\lambda) = \prod_{b\in\lambda} h'_\lambda(b) = P'(b \in \lambda) = P'(b \in \lambda_*) \cdot P'(b \in \lambda\backslash\lambda_*)\,.$$

Thus,

$$\mathcal{H}_\theta(\mu)\mathcal{H}'_\theta(\lambda) = (P(b \in \mu_*) \cdot P'(b \in \lambda_*)) \cdot P(b \in \mu\backslash\mu_*) \cdot P'(b \in \lambda\backslash\lambda_*)\,. \tag{2.4.5}$$

We establish below that

$$\begin{aligned} P(b \in \mu_*) \cdot P'(b \in \lambda_*) &\overset{\text{def}}{=} \prod_{b\in\mu_*} h_\mu(b) \cdot \prod_{b\in\lambda_*} h'_\mu(b) \\ &= \prod_{1\le i\le r\le k-1} \frac{(\mu_i - \mu_r + (r+1-i)\theta)_{\mu_r-\mu_{r+2}+1}}{\mu_i - \mu_{r+1} + (r+1-i)\theta}\,, \end{aligned} \tag{2.4.6}$$

$$P(b \in \mu\backslash\mu_*) \overset{\text{def}}{=} \prod_{b\in\mu\backslash\mu_*} h_\mu(b) = \prod_{i=1}^{k} (\mu_i - \mu_{i+1})! \cdot \prod_{i=1}^{k} (\mu_i + \theta(k-i)), \tag{2.4.7}$$

and that

$$\begin{aligned} P'(b \in \lambda\backslash\lambda_*) &\overset{\text{def}}{=} \prod_{b\in\lambda\backslash\lambda_*} h'_\lambda(b) \\ &= \theta \cdot \prod_{i=1}^{k} \frac{(\mu_i - \mu_k + (k+1-i)\theta)_{\mu_k-\mu_{k+2}+1}}{\mu_i - \mu_{k+1} + (k+1-i)\theta} \cdot \frac{\mu_i - 1 + (k+2-i)\theta}{\mu_i + (k-i+1)\theta}\,. \end{aligned} \tag{2.4.8}$$

Thus, (2.4.6), (2.4.7) and (2.4.8) clearly imply that

$$\begin{aligned} &\mathcal{H}_\theta(\mu) \cdot \mathcal{H}'_\theta(\lambda) \\ &\quad= \frac{\theta \cdot \prod_{1\le i\le r\le k} (\mu_i - \mu_r + (r+1-i)\cdot\theta)_{\mu_r-\mu_{r+2}+1} \cdot \prod_{i=1}^{k} (\mu_i - \mu_{i+1})!}{\prod_{1\le i\le r\le k} (\mu_i - \mu_{r+1} + (r+1-i)\theta)} \times \\ &\qquad\qquad \times \prod_{i=1}^{k} \frac{(\mu_i + (k-i)\theta)(\mu_i - 1 + (k-i+2)\theta)}{\mu_i + (k-i+1)\theta}\,. \end{aligned} \tag{2.4.9}$$

Substitute $s = r+1$ so that

$$\begin{aligned} &\prod_{1\le i\le r\le k} (\mu_i - \mu_{r+1} + (r+1-i)\theta) = \\ &\qquad= \prod_{1\le i<s\le k+1} (\mu_i - \mu_s + (s-i)\theta) \\ &\qquad= \prod_{1\le i<s\le k} (\mu_i - \mu_s + (s-i)\theta) \cdot \prod_{i=1}^{k} (\mu_i - 1 + (k+1-i)\theta)\,. \end{aligned} \tag{2.4.10}$$

Thus (2.4.9) can be rewritten as $\mathcal{H}_\theta(\mu)\cdot\mathcal{H}'_\theta(\lambda) = A\cdot B$, where

$$A = \frac{\theta\cdot\prod_{i=1}^{k}(\mu_i-\mu_{i+1})!\cdot\prod_{1\le i<s\le k+1}((s-i)\theta+\mu_i-\mu_{s-1})_{\mu_{s-1}-\mu_{s+1}+1}}{\prod_{1\le i<s\le k}((s-i)\theta+\mu_i-\mu_s)}$$

and

$$B = \prod_{i=1}^{k}\frac{(\mu_i-1+(k-i+2)\theta)\cdot(\mu_i+(k-i)\theta)}{(\mu_i+(k-i+1)\theta)\cdot(\mu_i-1+(k-i+1)\theta)}\ .$$

Note that by (2.1.2), $B=\varkappa_\theta(\mu,\lambda)$, hence

$$\begin{aligned}\frac{|\mu|!\cdot\theta^{|\mu|+1}}{\mathcal{H}_\theta(\mu)\cdot\mathcal{H}'_\theta(\lambda)}\varkappa_\theta(\mu,\lambda) &= \frac{|\mu|!\cdot\theta^{|\mu|+1}}{A}\\ &= \frac{|\mu|!\cdot\theta^{|\mu|}\cdot\prod_{1\le i<s\le k}((s-i)\theta+\mu_i-\mu_s)}{\prod_{i=1}^{k}(\mu_i-\mu_{i+1})!\cdot\prod_{1\le i<s\le k+1}((s-i)\theta+\mu_i-\mu_{s-1})_{\mu_{s-1}-\mu_{s+1}+1}}\ ,\end{aligned}\tag{2.4.11}$$

and formula (2.4.1) follows.

It remains to verify (2.4.6), (2.4.7) and (2.4.8).

Proof of (2.4.6). Let $1\le i\le r\le k-1$ (hence $\Delta(i,r+1)\subseteq\mu_*$ and $\Delta(i,r)\subseteq\lambda_*$) and denote

$$P_{i,r}\stackrel{\text{def}}{=}\prod_{b\in\Delta(i,r)}h'_\lambda(b)\cdot\prod_{b\in\Delta(i,r+1)}h_\mu(b)\ ,\tag{2.4.12}$$

then it suffices to show that

$$P_{i,r}=\frac{(\mu_i-\mu_r+(r+1-i)\theta)_{\mu_r-\mu_{r+2}+1}}{\mu_i-\mu_{r+1}+(r+1-i)\theta}\ .\tag{2.4.13}$$

So, let $b=(i,j)\in\Delta(i,r)$, then $a_\mu(b)=\mu_i-j$ and $\ell_\mu(b)=r-i$, therefore $h_\theta(b)=\mu_i-j+(r-i)\theta+1$. Replacing r by $r+1$ we get

$$\begin{aligned}\prod_{b\in\Delta(i,r+1)}h_\theta(b) &= \prod_{j=\mu_{r+2}+1}^{\mu_{r+1}}(\mu_i-j+(r+1-i)\theta+1)\\ &= (\mu_i-\mu_{r+1}+(r+1-i)\theta+1)_{\mu_{r+1}-\mu_{r+2}} =_{\left((z+1)_n=\frac{1}{z}\cdot(z)_{n+1}\right)}\\ &= \frac{(\mu_i-\mu_{r+1}+(r+1-i)\theta)_{\mu_{r+1}-\mu_{r+2}+1}}{\mu_i-\mu_{r+1}+(r+1-i)\theta}\ .\end{aligned}\tag{2.4.14}$$

Similarly, let $b=(i,j)\in\Delta(i,r)$ $(r\le k-1)$, then $h'_\lambda(b)=a_\lambda(b)+\theta\cdot\ell_\lambda(b)+\theta$. Trivially, $a_\lambda(b)=a_\mu(b)=\mu_i-j$. Since $r\le k-1$, hence $r+1\le k$, so $1\le\mu_k\le\mu_{r+1}$ and $2\le\mu_{r+1}+1\le j$. Therefore $\ell_\lambda(b)=\ell_\mu(b)=r-i$, so $h'_\theta(b)=\mu_i-j+(r+1-i)\theta$. Thus

$$\prod_{b\in\Delta(i,r)}h'_\lambda(b)=\prod_{j=\mu_{r+1}+1}^{\mu_r}(\mu_i-j+(r+1-i)\theta)=(\mu_i-\mu_r+(r+1-i)\theta)_{\mu_r-\mu_{r+1}}\ .\tag{2.4.15}$$

Conclude that for $1 \leq i \leq r \leq k-1$, by (2.4.14) and (2.4.15),

$$\prod_{b\in\Delta(i,r)} h'_\lambda(b) \cdot \prod_{b\in\Delta(i,r+1)} h_\mu(b) =$$
$$= \frac{(\mu_i - \mu_r + (r+1-i)\theta)_{\mu_r-\mu_{r+1}} \cdot (\mu_i - \mu_{r+1} + (r+1-i)\theta)_{\mu_{r+1}-\mu_{r+2}+1}}{\mu_i - \mu_{r+1} + (r+1-i)\theta}$$
$$= \frac{(\mu_i - \mu_r + (r+1-i)\theta)_{\mu_r-\mu_{r+2}+1}}{\mu_i - \mu_{r+1} + (r+1-i)\theta} \ ,$$

which proves (2.4.6).

Proof of (2.4.7). Obviously,

$$\prod_{b\in\mu\setminus\mu_*} h_\mu(b) = \left[\prod_{i=1}^{k} \prod_{b\in\Delta(i,i)} h_\mu(b)\right] \cdot \left[\prod_{i=1}^{k} h_\mu(i,1)\right]$$

(note that $(k,1) \notin \Delta(k,k)$, but $(k,1)$ belongs to the left column of μ). If $b \in \Delta(i,i)$ then $\ell_\mu(b) = 0$, therefore

$$\prod_{b\in\Delta(i,i)} h_\mu(b) = \prod_{j=\mu_{i+1}+1}^{\mu_i} (\mu_i - j + 1) = (\mu_i - \mu_{i+1})! \ . \tag{2.4.17}$$

Also

$$h_\mu(i,1) = a_\mu(i,1) + \theta\ell_\mu(i,1) + 1 = \mu_i - 1 + (k-i)\theta + 1 = \mu_i + (k-i)\theta \ .$$

Deduce that

$$\prod_{b\in\mu\setminus\mu_*} h_\mu(b) = \prod_{i=1}^{k}(\mu_i - \mu_{i+1})! \cdot \prod_{i=1}^{k}(\mu_i + (k-i)\theta) \ , \tag{2.4.18}$$

which proves (2.4.7).

Proof of (2.4.8). Recall that $\Delta(k+1,k+1) = \{(k+1,1)\}$ and $h'_\lambda(k+1,1) = \theta$ since $a_\lambda(k+1,1) = \ell_\lambda(k+1,1) = 0$. Note also that $\Delta(i,k) \sqcup \Delta(i,k+1) = \{(i,j) \in \lambda | 1 \leq j \leq \mu_k\}$. Let $1 \leq i \leq k$, $1 \leq j \leq \mu_k$, then $h'_\lambda(i,j) = (\mu_i - j) + \theta(k-i) + \theta$, hence

$$\prod_{j=2}^{\mu_k} h'_\lambda(i,j) = \prod_{j=2}^{\mu_k}(\mu_i - j + (k+1-i)\theta) = (\mu_i - \mu_k + (k+1-i)\theta)_{\mu_k-1}.$$

If $b = (i,1) \in \lambda$ with $1 \leq i < k$, then, with respect to λ,

$$h'_\lambda(i,1) = (\mu_i - 1) + (k+1-i)\theta + \theta = \mu_i - 1 + (k-i+2)\theta.$$

Thus

$$\prod_{j=1}^{\mu_k} h'_\lambda(i,j) = (\mu_i - \mu_k + (k+1-i)\theta)_{\mu_k - 1} \cdot (\mu_i - 1 + (k-i+2)\theta) =$$

$$= \frac{(\mu_i - \mu_k + (k+1-i)\theta)_{\mu_k - \mu_{k+2}+1}}{\mu_i - \mu_{k+1} + (k+1-i)\theta} \cdot \frac{\mu_i - 1 + (k-i+2)\theta}{\mu_i + (k-i+1)\theta} \qquad (2.4.19)$$

(since $\mu_{k+1} = 1$, $\mu_{k+2} = 0$ and $(u)_n = \frac{(u)_{n+2}}{(u+n)(u+n+1)}$). Therefore

$$\prod_{b \in \lambda \setminus \lambda_*} h'_\lambda(b) = h'_\lambda(k+1,1) \cdot \prod_{i=1}^{k} \prod_{j=1}^{\mu_k} h'_\lambda(i,j)$$

$$= \theta \cdot \prod_{i=1}^{k} \frac{(\mu_i - \mu_k + (k+1-i)\theta)_{\mu_k - \mu_{k+2}+1}}{\mu_i - \mu_{k+1} + (k+1-i)\theta} \cdot \frac{\mu_i - 1 + (k-i+2)\theta}{\mu_i + (k-i+1)\theta} \ ,$$

which verifies (2.4.8).

This completes the proof of Theorem 2.4.1.

2.5. The z-measures identity for $(i,j) = (k+1,1)$. The z-measures and $(i,j) = (k+1,1)$ imply:

Theorem 2.5.1. *For the box $(i,j) = (k+1,1)$, the z-measures on the Jack graph imply the following identity:*

$$\sum_{\mu_1 \geq \cdots \geq \mu_{k+1} = 1} \frac{|\mu|! \cdot \prod_{1 \leq i < j \leq k} ((j-i)\theta + \mu_i - \mu_j)}{\theta \cdot \prod_{1 \leq i \leq k} (\mu_i - \mu_{i+1})! \cdot \prod_{1 \leq i < j \leq k+1} ((j-i)\theta + \mu_i - \mu_{j-1})_{\mu_{j-1} - \mu_{j+1} + 1}}$$

$$\times \frac{(z - k\theta)(\bar{z} - k\theta) \cdot \prod_{1 \leq i \leq k} \left[(z - (i-1)\theta)_{\mu_i} (\bar{z} - (i-1)\theta)_{\mu_i} \right]}{(\theta^{-1} z \bar{z})_{|\mu|+1}} = 1 \ , \qquad (2.5.1)$$

where $|\mu| = \mu_1 + \cdots + \mu_k \quad (= n-1)$.

Proof. The proof follows from Theorem 1.8.1., i.e., from the identity

$$\sum_{\mu} \dim_\theta(\mu) \cdot \varkappa_\theta(\mu, \lambda) \cdot \varphi_{z\bar{z}}(\lambda) = 1$$

by arguments similar to those of the previous theorem. Note that here the function $\varphi_{z\bar{z}}(\lambda)$ in the $(k+1,1)$ case equals

$$\varphi_{z\bar{z}}(\lambda) = \frac{(z - k\theta)(\bar{z} - k\theta) \cdot \prod_{1 \leq i \leq k} \left[(z - (i-1)\theta)_{\mu_i} (\bar{z} - (i-1)\theta)_{\mu_i} \right]}{(\theta^{-1} z \bar{z})_{|\mu|+1}} \ . \qquad (2.5.2)$$

2.6. Some special cases. When $(k+1,1)=(2,1)$, simple manipulations in the last theorem yield the amazing identity:

$$ {}_3F_2(z+1,\bar{z}+1,2;\theta+2,\theta^{-1}z\bar{z}+2;1)=\frac{(\theta+1)(z\bar{z}+\theta)}{(z-\theta)(\bar{z}-\theta)}\,. \tag{2.6.1} $$

A long list of cases when the hypergeometric series ${}_3F_2(a_1,a_2,a_3;b_1,b_2;1)$ can be summed in a closed form is given in the handbook [PBW], subsection 7.4.4. At a first glance, (2.6.1) does not appear in that list. However, it is hidden in formula No. 27, which is as follows:

$$ {}_3F_2(1,a,b;c,d;1)+\frac{ab(2+a+b-c-d)}{cd[ab-(c-1)(d-1)]}\cdot{}_3F_2(2,a+1,b+1;c+1,d+1;1)= $$
$$ \frac{(1-c)(d-1)}{ab-(c-1)(d-1)}\,, \tag{2.6.2} $$

where $Re(a+b-c-d)<-2$.

To get (2.6.1) from (2.6.2), multiply both sides of (2.6.2) by $ab-(c-1)(d-1)$ so it cancels from the denominators, then substitute $a=z$, $b=\bar{z}$, $c=\theta+1$, $d=\theta^{-1}z\bar{z}+1$. Under this substitution $ab-(c-1)(d-1)=0$, hence the first summand on the left vanishes, and the identity (2.6.1) clearly follows.

For $k\geq 2$, no direct proofs of the identities given by Theorems 2.4.1 and 2.5.1 are known at the moment.

We now list some special cases of Theorem 2.4.1. Also included is the θ-Plancherel identity for $(i,j)=(2,2)$, which was obtained directly from Theorem 1.8.1.

- The box $(i,j)=(2,2)$ implies the identity

$$ \sum_{r,s=0}^{\infty}\frac{(r+s+3)!\cdot\theta^{r-s+2}}{[r+2+(s+1)\theta](r+1)!(\theta^{-1})_{s+1}[r+1+(s+2)\theta][r+2\theta][1+(s+1)\theta](\theta)_r s!}\times $$
$$ \frac{(r+2\theta)(r+1)}{(r+1+\theta)(r+\theta)}=1\,. \tag{2.6.3} $$

- The box $(i,j)=(2,1)$ implies

$$ \sum_{r\geq 0}\frac{(r+1)\cdot\theta^{r+1}}{(\theta)_{r+2}}=\sum_{r\geq 0}\frac{(r+1)!\cdot\theta^{r+1}}{r!\cdot(\theta)_{r+2}}=1. \tag{2.6.4} $$

It is easy to give (2.6.4) a direct proof.

- The box $(i,j)=(3,1)$ implies

$$ \sum_{s\geq r\geq 0}\frac{(s+r+2)!\cdot\theta^{s+r+2}\cdot(\theta+s-r)}{r!\cdot(s-r)!\cdot(\theta)_{r+2}\cdot(\theta)_{s+1}(2\theta+s-r)_{r+2}}=1. \tag{2.6.5} $$

- The box $(i,j)=(4,1)$ implies that $\sum_{u\geq s\geq r\geq 0} f_4(r,s,u;\theta)=1$, where

$$f_4(r,s,u;\theta)=\frac{(r+s+u+3)!\cdot\theta^{r+s+u+3}(\theta+s-r)(\theta+u-s)(2\theta+u-r)}{r!\cdot(s-r)!\cdot(u-s)!(t)_{r+2}(\theta)_{s+1}(\theta)_{u-r+1}}\times \frac{1}{(2\theta+s-r)_{r+2}(2\theta+u-s)_{s+1}(3\theta+u-r)_{r+2}}. \quad (2.6.7)$$

- The box $(i,j)=(5,1)$ implies that $\sum_{v\geq u\geq s\geq r\geq 0} f_5(r,s,u,v;t)=1$, where

$$f_5(r,s,u,v;\theta)=\frac{(r+s+u+v+4)!\cdot\theta^{r+s+u+v+4}(\theta+s-r)(t+u-s)(2\theta+u-r)}{r!\cdot(s-r)!\cdot(u-s)!(\theta)_{r+2}(\theta)_{s+1}(\theta)_{u-r+1}}\times \frac{1}{(2\theta+s-r)_{r+2}(2\theta+u-s)_{s+1}(3\theta+u-r)_{r+2}}\times \frac{(\theta+v-u)(2\theta+v-s)(3\theta+v-r)}{(v-u)!(\theta)_{v-s+1}(2\theta+v-u)_{u-r+1}(3\theta+v-s)_{s+1}(4\theta+v-r)_{r+2}}. \quad (2.6.8)$$

2.7. The case $\theta=1$. In this case the measure M_∞ becomes the Plancherel measure for the Young graph $\mathbb{Y}$, and the identity (2.4.1), which corresponds to the box $(k+1,1)$, takes a simpler form given in (0.6). On the other hand, the identity for the Plancherel measure on $\mathbb{Y}$ and a general box $(k+1,l+1)$ was given in [Re], see (0.2) above. The specialization of (0.2) for $l=1$ takes the form (0.7). Here we verify directly that the left hand sides of these two identities, (0.6) and (0.7), coincide, thus proving they indeed are the same identity.

Proposition 2.7.1. *Let*

$$f_k(p_1,\dots,p_k)=\frac{(p_1+\cdots+p_k-k(k-1)/2)!\cdot\prod_{1\leq i<j\leq k}(p_i-p_j)^2}{\prod_{i=1}^k[(p_i-1)!(p_i+1)!]} \quad (2.7.1)$$

so that (0.6) is the identity

$$\sum_{p_1>\cdots>p_k\geq 1} f_k(p_1,\dots,p_k)=1.$$

Similarly, let $\mu_{k+1}=1$ and

$$g_k(\mu_1,\dots,\mu_k)=\frac{(\mu_1+\cdots+\mu_k)!\prod_{1\leq i<j\leq k}(\mu_i-\mu_j+j-i)}{\prod_{i=1}^k(\mu_i-\mu_{i+1})!\cdot\prod_{1\leq i<j\leq k+1}(\mu_i-\mu_{j-1}+j-i)_{\mu_{j-1}-\mu_{j+1}+1}} \quad (2.7.2)$$

so that (0.7) is the identity

$$\sum_{\mu_1 \geq \cdots \geq \mu_{k+1}=1} g_k(\mu_1, \ldots, \mu_k) = 1.$$

Finally, let $p_j = \mu_j + k - j, \quad 1 \leq j \leq k+1$ *(so* $p_{k+1} = 0$*),* $p_r = 0$ *if* $r \geq k+1$*.*

Then the expressions (2.7.1) and (2.7.2) are equal. Hence (0.6) and (0.7) indeed are the same identity.

Proof. Define $f_k^*(p)$ via

$$g_k(\mu_1, \ldots, \mu_k) = g_k(p_1 - (k-1), \ldots, p_k) = f_k^*(p_1, \ldots, p_k). \tag{2.7.3}$$

The statement of the above proposition clearly follows from

Claim 1. $f_k(p_1, \ldots, p_k) = f_k^*(p_1, \ldots, p_k)$.

Proof of Claim 1. First calculate $f_k^*(p_1, \ldots, p_k)$. We have

$$\mu_1 + \cdots + \mu_k = p_1 + \cdots + p_k - k(k-1)/2, \qquad \mu_i - \mu_j + j - i = p_i - p_j,$$
$$\mu_i - \mu_{i+1} = p_i - p_{i+1} - 1$$

and

$$\prod_{1 \leq i < j \leq k+1} (\mu_i - \mu_{j-1} + j - i)_{\mu_{j-1} - \mu_{j+1} + 1}$$
$$= \prod_{1 \leq i < j \leq k} (p_i - p_{j-1} + 1)_{p_{j-1} - p_{j+1} - 1} \prod_{i=1}^{k} (p_i - p_k + 1)_{p_k + 1}. \tag{2.7.4}$$

Thus,

$$f_k^*(p) = \frac{(p_1 + \cdots + p_k - k(k-1)/2)! \prod_{1 \leq i < j \leq k}(p_i - p_j)}{\prod_{i=1}^{k}(p_i - p_{i+1} - 1)! \prod_{1 \leq i < j \leq k}(p_i - p_{j-1} + 1)_{p_{j-1} - p_{j+1} - 1}}$$
$$\times \frac{1}{\prod_{i=1}^{k}(p_i - p_k + 1)_{p_k + 1}}. \tag{2.7.5}$$

Comparing $f_k(p)$ with $f_k^*(p)$ and cancelling $(p_1 + \cdots + p_k - k(k-1)/2)! \prod_{1 \leq i < j \leq k}(p_i - p_j)$, we see that Claim 1 is equivalent to

Claim 2.

$$\prod_{1 \leq i < j \leq k} [(p_i - p_j) \cdot (p_i - p_{j-1} + 1)_{p_{j-1} - p_{j+1} - 1}] \prod_{i=1}^{k} (p_i - p_{i+1} - 1)!(p_i - p_k + 1)_{p_k + 1} =$$
$$= \prod_{i=1}^{k} (p_i - 1)!(p_i + 1)! \tag{2.7.6}$$

Note that in the second factor on the left, when $i = k$, the corresponding term in that product becomes $(p_k - 1)!(p_k + 1)!$. This same term also appears on the right hand side — again when $i = k$. Cancelling it on both sides we see that Claim 2 is equivalent to

Claim 3.

$$\prod_{1\le i<j\le k}[(p_i-p_j)\cdot(p_i-p_{j-1}+1)_{p_{j-1}-p_{j+1}-1}]\prod_{i=1}^{k-1}(p_i-p_{i+1}-1)!(p_i-p_k+1)_{p_k+1}=$$

$$=\prod_{i=1}^{k-1}(p_i-1)!(p_i+1)! \quad (2.7.7)$$

On the left–hand side of (2.7.7) the terms involving the index $i=1$ are

$$R_1=\left[\prod_{j=2}^{k}(p_1-p_j)(p_1-p_{j-1}+1)_{p_{j-1}-p_{j+1}-1}\right]\cdot(p_1-p_2-1)!(p_1-p_k+1)_{p_k+1}. \quad (2.7.8)$$

By induction on k, the proof of Claim 3 will follow once we prove

Claim 4.

$$R_1=(p_1-1)!(p_1+1)!$$

We need

Claim 5. *Let $2\le r\le k-1$, then*

$$(p_1-p_2-1)!\cdot\left[\prod_{j=2}^{r}(p_1-p_j)(p_1-p_{j-1}+1)_{p_{j-1}-p_{j+1}-1}\right]=(p_1-p_r)!(p_1-p_{r+1}-1)! \quad (2.7.9)$$

Proof of Claim 5. Induction on r, where $2\le r\le k-1$.

$r=2:\quad (p_1-p_2)(p_1-p_1+1)_{p_1-p_3-1}(p_1-p_2-1)!=(p_1-p_2)!(p_1-p_3-1)!$

$r\Rightarrow r+1$:

$$\begin{aligned}\prod_{j=2}^{r+1}\cdots=(\prod_{j=2}^{r}\dots)\times(p_1-p_{r+1})(p_1-p_r+1)_{p_r-p_{r+2}-1}\quad&\text{by induction}\\ =(p_1-p_r)!(p_1-p_{r+1}-1)!\times(p_1-p_{r+1})(p_1-p_r+1)_{p_r-p_{r+2}-1}&\\ =(p_1-p_{r+1})!(p_1-p_{r+2}-1)!&\quad (2.7.10)\end{aligned}$$

which proves Claim 5.

We can now prove Claim 4, thus completing the proof of proposition 2.7.1.
Rearrange terms in R_1 and apply Claim 5 with $r=k-1$:

$$\begin{aligned}R_1&=\{\prod_{j=2}^{k-1}[(p_1-p_j)(p_1-p_{j-1}+1)_{p_{j-1}-p_{j+1}-1}](p_1-p_2-1)!\}\\ &\qquad\times(p_1-p_k)(p_1-p_{k-1}+1)_{p_{k-1}-1}(p_1-p_k+1)_{p_k+1}\\ &=\{(p_1-p_{k-1})!(p_1-p_k-1)!\}(p_1-p_k)(p_1-p_{k-1}+1)_{p_{k-1}-1}(p_1-p_k+1)_{p_k+1}\\ &=[(p_1-p_{k-1})!(p_1-p_{k-1}+1)_{p_{k-1}-1}][(p_1-p_k-1)!(p_1-p_k)(p_1-p_k+1)_{p_k+1}]\\ &=(p_1-1)![(p_1-p_k)!(p_1-p_k+1)_{p_k+1}]\\ &=(p_1-1)!(p_1+1)! \quad (2.7.11)\end{aligned}$$

This completes the proof of Proposition 2.7.1. □

2.8. Remark on [MMW]**.** Recall that the space Tab can be viewed as the projective limit space $\varprojlim \mathrm{Tab}_n$. Let M be the Plancherel measure ($\theta = 1$) on Tab, let M_n be the pushforward of M under the projection $\mathrm{Tab} \to \mathrm{Tab}_n$, and let M'_n be the uniform probability measure on Tab_n. Note that the measures M_n and M'_n are quite different. Indeed,

$$M_n(\mathrm{Tab}(\lambda)) = \mathrm{const}_1 \cdot (\dim \lambda)^2, \qquad M'_n(\mathrm{Tab}(\lambda)) = \mathrm{const}_2 \cdot \dim \lambda.$$

Nevertheless, as is shown by McKay, Morse, and Wilf,

$$\lim_{n\to\infty} M'_n = M, \tag{2.8.1}$$

see [MMW, Section 4]. From this fact McKay, Morse, and Wilf deduce the following result. Let $T_n \in \mathrm{Tab}_n$ be the random (with respect to M'_n) finite tableau and let $p(n; i, j, k)$ stand for the probability that T_n has the entry k at the box (i, j). Then the limit $p_{i,j}(k) = \lim_{n\to\infty} p(n; i, j, k)$ exists and coincides with the probability $\mathcal{P}_M(T(i, j) = k)$ (which can be calculated as is shown in [Re]).

An explanation of the equality (2.8.1) can be extracted from [V].

§3 The Kingman graph case

3.1. Kingman edge multiplicities. As the Jack parameter $\theta > 0$ goes to 0 (denoted $\theta \downarrow 0$), the Jack symmetric functions P_μ degenerate to the monomial symmetric functions m_μ [Ma]. The simplest case of Pieri's formula for the monomial symmetric functions has the form

$$m_\mu m_{(1)} = \sum_{\lambda:\lambda\searrow\mu} \varkappa_0(\mu, \lambda) m_\lambda \ , \tag{3.1.1}$$

where $\varkappa_0(\mu, \lambda)$ is a strictly positive integer. Specifically, if k stands for the length of the row in λ containing the box $\lambda\backslash\mu$, then $\varkappa_0(\mu, \lambda)$ is the multiplicity of k in λ.

One can verify that

$$\varkappa_0(\mu, \lambda) = \lim_{\theta\downarrow 0} \varkappa_\theta(\mu, \lambda) \ , \quad \mu \nearrow \lambda \ . \tag{3.1.2}$$

We take the numbers $\varkappa_0(\mu, \lambda)$ as edge multiplicities. The graph $\mathbb{Y}$ together with these edge multiplicities is called the *Kingman graph,* see [Ke1, KOO, BO3]. The dimension function of the Kingman graph will be denoted as $\dim_0 \mu$. It is given by a simple formula

$$\dim_0 \mu = \frac{|\mu|!}{\mu_1!\mu_2!\dots\mu_\ell!} \ , \tag{3.1.3}$$

where $\ell = \ell(\mu)$ is the number of nonzero rows in μ.

Again, one can verify that

$$\dim_0 \mu = \lim_{\theta\downarrow 0} \dim_\theta \mu \ .$$

3.2. The t-measures. We fix a parameter $t > 0$.

Theorem 3.2.1. *For any $t > 0$ there exists a $\varkappa_0$-central measure M_t such that the corresponding $\varkappa_0$-harmonic function has the form*

$$\psi_t(\lambda) = \frac{(\lambda_1 - 1)! \dots (\lambda_\ell - 1)!}{r_1(\lambda)! r_2(\lambda)! \dots} \cdot \frac{t^\ell}{(t)_n}, \tag{3.2.1}$$

where $\ell = \ell(\lambda)$ and $r_k(\lambda)$ is the multiplicity of k in λ.

Proof. See [BO3]. □

3.3. Reachability for t-measures.

Lemma 3.3.1. *Let $M = M_t$. Then all the boxes are completely reachable.*

Proof. It is readily verified that Lemma 2.3.3 holds for the measure M. Then we apply the same argument as in the proof of Theorem 2.3.1. □

Theorem 3.3.2. *Let $k = 0, 1, \dots,$ $l = 1, 2, \dots,$ and $t > 0$. Then*

$$\frac{1}{k!} \sum_{r_1, \dots, r_l, s_1, \dots, s_k \ge 0} \frac{(s_1 + \dots + s_k + r_1 + 2r_2 + \dots + lr_l + kl + k + l)!}{(s_1 + l + 1) \dots (s_k + l + 1) 1^{r_1} 2^{r_2} \dots l^{r_l} r_1! \dots r_l!} \\ \times \frac{t^{k + r_1 + \dots + r_l + 1}}{(t)_{s_1 + \dots + s_k + r_1 + 2r_2 + \dots + lr_l + kl + k + l + 1}} = 1. \tag{3.3.1}$$

We prove this in two ways: the first proof is based on Theorem 1.8.1, while the second one is a direct derivation. A crucial step in that direct proof was shown to us by S. Milne.

3.4. First proof. Fix the box $(k + 1, l + 1)$ and apply Theorem 1.8.1 (the complete reachability is ensured by Lemma 3.3.1). Since the box $(k + 1, l + 1)$ can be added to μ (yielding λ), we have

$$\begin{gathered} \mu_1 \ge \dots \ge \mu_k \ge l + 1, \quad \mu_{k+1} = l, \quad r_l(\mu) \ge 1, \\ r_l(\lambda) = r_l(\mu) - 1, \quad r_{l+1}(\lambda) = r_{l+1}(\mu) + 1, \\ r_j(\lambda) = r_j(\mu) \quad \text{if } j \ne l, l + 1, \\ \ell(\lambda) = k + r_1(\mu) + \dots + r_l(\mu). \end{gathered}$$

Let us abbreviate

$$\begin{gathered} r_1 = r_1(\mu), \ r_2 = r_2(\mu), \ \dots, \\ \ell = \ell(\lambda) = k + r_1 + \dots + r_l \end{gathered}$$

and remark that

$$\varkappa(\mu, \lambda) = r_{l+1}(\lambda) = r_{l+1} + 1.$$

In this notation, the identity

$$\sum_{\mu} \dim \mu \cdot \varkappa(\mu, \lambda) \cdot \psi_t(\lambda) = 1$$

provided by Theorem 1.8.1 becomes

$$\sum_{r_1,\dots,r_{l-1}\geq 0} \sum_{r_l \geq 1} \sum_{\mu_1\geq\cdots\geq\mu_k\geq l+1} \frac{|\mu|!}{\mu_1!\mu_2!\dots}(r_{l+1}+1)\frac{(\lambda_1-1)!(\lambda_2-1)!\dots}{r_1(\lambda)!r_2(\lambda)!\dots}\frac{t^\ell}{(t)_{|\mu|+1}} = 1. \tag{3.4.1}$$

Now $\lambda_{k+1} = \mu_{k+1} + 1 = l + 1$ and $\lambda_j = \mu_j$ if $j \neq k+1$. Therefore

$$\frac{(\lambda_1-1)!(\lambda_2-1)!\dots}{\mu_1!\mu_2!\dots} = \frac{1}{\mu_1\dots\mu_k 1^{r_1}2^{r_2}\dots(l-1)^{r_{l-1}}l^{r_l-1}}\,. \tag{3.4.2}$$

Also,

$$\frac{r_{l+1}+1}{r_1(\lambda)!r_2(\lambda)!\dots} = \frac{1}{r_1!\dots r_{l-1}!(r_l-1)!r_{l+1}!r_{l+1}!\dots}\,. \tag{3.4.3}$$

Substituting (3.4.2) and (3.4.3) into (3.4.1) and replacing $r_l - 1$ by r_l, we obtain

$$\sum_{r_1,\dots,r_l\geq 0} \sum_{\mu_1\geq\cdots\geq\mu_k\geq l+1} \frac{|\mu|!}{\mu_1\dots\mu_k 1^{r_1}\dots l^{r_l} r_1!\dots r_l!}\frac{1}{r_{l+1}!r_{l+2}!\dots}\frac{t^\ell}{(t)_{|\mu|+1}} = 1\,, \tag{3.4.4}$$

where

$$|\mu| = \mu_1 + \cdots + \mu_k + r_1 + 2r_2 + \cdots + lr_l + l$$

and

$$\ell = \ell(\lambda) = k + r_1 + \cdots + r_l + 1$$

(because we have changed the definition of r_l).

Note that $r_{l+1} + r_{l+2} + \cdots = k$. Multiply and divide (3.4.4) by $k!$ and note that $\dfrac{k!}{r_{l+1}!r_{l+2}!\dots}$ equals the number of permutations of $\mu_1, \dots, \mu_k$. We can therefore cancel the factor $\dfrac{k!}{r_{l+1}!r_{l+2}!\dots}$ by replacing '$\mu_1 \geq \cdots \geq \mu_k$' by '$\mu_1, \cdots, \mu_k$' under the summation sign, and (3.4.4) becomes

$$\frac{1}{k!}\sum_{r_1,\dots,r_l\geq 0} \sum_{\mu_1,\dots,\mu_k\geq l+1} \frac{|\mu|!}{\mu_1\dots\mu_k 1^{r_1}\dots l^{r_l} r_1!\dots r_l!}\frac{t^\ell}{(t)_{|\mu|+1}} = 1. \tag{3.4.5}$$

Finally, replace μ_j by $s_j + l + 1$, $s_j \geq 0$ and observe that

$$|\mu| = s_1 + \cdots + s_k + r_1 + 2r_2 + \cdots + lr_l + kl + k + l.$$

The first proof of Proposition 3.3.2 clearly follows. $\square$

3.5. Second proof (direct). We first transform the left hand side of (3.3.1).

Lemma 3.5.1. *Let L denote the left hand side of (3.3.1). We have*

$$L = \frac{t^{k+1}}{k!} \int_0^1 (1-v)^{t-1} \cdot v^l \cdot \exp\left[t\left(v + \frac{v^2}{2} + \cdots + \frac{v^l}{l}\right)\right] \times \\ \times \left[-\ln(1-v) - \left(v + \frac{v^2}{2} + \cdots + \frac{v^l}{l}\right)\right]^k dv. \quad (3.5.1)$$

Corollary 3.5.2. *The identity $L = 1$, i.e, the identity* (3.3.1)*, is equivalent to the following integral identity:*

$$\int_0^1 (1-v)^{t-1} \cdot v^l \cdot \exp\left[t\left(v + \frac{v^2}{2} + \cdots + \frac{v^l}{l}\right)\right] \times \\ \times \left[-\ln(1-v) - \left(v + \frac{v^2}{2} + \cdots + \frac{v^l}{l}\right)\right]^k dv = \frac{k!}{t^{k+1}}. \quad (3.5.2)$$

Proof of Lemma 3.5.1. Note first that the Taylor expansion for $\ln(1-v)$ implies that

$$\sum_{\mu=l+1}^{\infty} \frac{v^\mu}{\mu} = -\ln(1-v) - \left(v + \frac{v^2}{2} + \cdots + \frac{v^l}{l}\right). \quad (3.5.3)$$

Return now to (3.3.1) with $\mu_i = s_i + l + 1$ and write $n = n(\mu, r) = \mu_1 + \cdots + \mu_k + r_1 + 2r_2 + \cdots + lr_l + l$. Then

$$L = \frac{1}{k!} \sum_{r_1,\ldots,r_l=0}^{\infty} \sum_{\mu_1,\ldots,\mu_k=l+1}^{\infty} \frac{n!}{(t)_{n+1}} \cdot \frac{t^{k+1} \cdot t^{r_1+\cdots+r_l}}{\mu_1 \ldots \mu_k \cdot 1^{r_1} \cdot r_1! \cdot 2^{r_2} \cdot r_2! \ldots l^{r_l} \cdot r_l!}. \quad (3.5.4)$$

By Euler's beta integral,

$$\frac{n!}{(t)_{n+1}} = \int_0^1 (1-v)^{t-1} \cdot v^n \, dv, \quad (3.5.5)$$

hence

$$L = \frac{t^{k+1}}{k!} \int_0^1 (1-v)^{t-1} \cdot v^l \cdot \left[\sum_{r_1=0}^{\infty} \frac{t^{r_1} v^{r_1}}{1^{r_1} r_1!}\right] \ldots \left[\sum_{r_l=0}^{\infty} \frac{t^{r_l} v^{lr_l}}{l^{r_l} r_l!}\right] \cdot \left[\sum_{\mu=l+1}^{\infty} \frac{v^\mu}{\mu}\right]^k dv. \quad (3.5.6)$$

Now

$$\sum_{r_j=0}^{\infty} \frac{t^{r_j} v^{jr_j}}{j^{r_j} r_j!} = \exp\left[\frac{t \cdot v^j}{j}\right],$$

hence by (3.5.3),

$$L = \frac{t^{k+1}}{k!} \int_0^1 (1-v)^{t-1} \cdot v^l \cdot \exp\left[t\left(v + \frac{v^2}{2} + \cdots + \frac{v^l}{l}\right)\right] \times$$
$$\times \left[-\ln(1-v) - \left(v + \cdots + \frac{v^l}{l}\right)\right]^k dv. \quad (3.5.7)$$

This proves (3.5.1).

We proceed to the proof of (3.5.2), which, by Corollary 3.5.2, is equivalent to the initial identity.

Denote $y = v + v/2 + \cdots + v^l/l$, then $\dfrac{dy}{dv} = \dfrac{1-v^l}{1-v}$. Therefore

$$\frac{d}{dv}\exp(t\cdot y) = \exp(t\cdot y)\cdot t \cdot \frac{1-v^l}{1-v}. \quad (3.5.8)$$

It follows that

$$\frac{d}{dv}\big((1-v)^t \cdot \exp(t\cdot y)\big) = -t\cdot(1-v)^{t-1}\cdot v^l \cdot \exp(t\cdot y). \quad (3.5.9)$$

Since

$$\frac{d}{dv}[-\ln(1-v) - y] = \frac{v^l}{1-v},$$

we get

$$\frac{d}{dv}[-\ln(1-v) - y]^k = k\cdot[-\ln(1-v) - y]^{k-1}\cdot\frac{v^l}{1-v}. \quad (3.5.10)$$

Thus, for $k \geq 0$

$$\begin{aligned}\frac{d}{dv}\big((1-v)^t \cdot \exp(t\cdot y)\cdot[-\ln(1-v)-y]^k\big) = \\ - t\cdot(1-v)^{t-1}\cdot v^l\cdot\exp(t\cdot y)\cdot[-\ln(1-v)-y]^k + \\ k\cdot(1-v)^{t-1}\cdot v^l\cdot\exp(t\cdot y)\cdot[-\ln(1-v)-y]^{k-1}. \quad (3.5.11)\end{aligned}$$

Since $t > 0$ by the assumption, we have, for any j,

$$\lim_{v\to 1}(1-v)^t\cdot(\ln(1-v))^j = 0.$$

Thus,

$$\big(1-v\big)^t\cdot\exp(t\cdot y)\cdot[-\ln(1-v)-y]^k\Big|_0^1 \;=\; \begin{cases} -1, & k = 0 \\ 0 - 0 = 0, & k > 0 \end{cases} \quad (3.5.12)$$

Denote by a_k the integral in the left–hand side of (3.5.2). When $k = 0$, the integration of (3.5.9) easily implies that

$$a_0 = \int_0^1 (1-v)^{t-1} \cdot v^l \cdot \exp(t \cdot y)\, dv = \frac{1}{t}\,, \tag{3.5.13}$$

as required.

For arbitrary $k = 1, 2, \dots$, integrating the expression (3.5.11) and using (3.5.12) we get

$$a_k = \int_0^1 (1-v)^{t-1} \cdot v^l \cdot \exp(t \cdot y) \cdot [-\ln(1-v) - y]^k \, dv = \\ \frac{k}{t} \int_0^1 (1-v)^{t-1} \cdot v^l \cdot \exp(t \cdot y) \cdot [-\ln(1-v) - y]^{k-1} \, dv, \tag{3.5.14}$$

i.e,

$$a_k = \frac{k}{t} \cdot a_{k-1}\,.$$

Thus, by induction, $a_k = \dfrac{k!}{t^{k+1}}$. □

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