Weyl Modules for the Schur Algebra of the Alternating Group

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Abstract

Let F be the field of complex numbers and $V = V_0 \oplus V_1$ a vector space over F with dim $V_0 =$ dim V_1. The symmetric group acts on $V^{\otimes n}$ by the sign-permutation action (see [1]). Let $S^*(V,n) \subset S^*_A(V,n)$ be the corresponding Schur algebras of S_n and of $A_n \subset S_n$ respectively where A_n is the alternating group (see [8]). Following the fundamental work of H. Weyl, the explicit decomposition of $V^{\otimes n}$ as an $S^*(V,n)$-module was given in [1]. By applying the character theory and the representation theory of A_n we give here the explicit decomposition of $V^{\otimes n}$ as an $S^*_A(V,n)$-module.

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1 Introduction

1.1. Let F be the field of complex numbers and assume that all objects (modules, algebras, vector spaces) are finite dimensional. It should be noted that many of the arguments also work for a field of positive characteristic. Consider the vector space $V = V_0 \oplus V_1$ where dim $V_0 =$ dim $V_1 = k$ and the n-fold tensor space $W := V^{\otimes n}$ with $n \geq 2$. The symmetric group acts on W via place permutations with a sign and the corresponding representation is $\varphi^* : FS_n \to \mathrm{End_F(V^{\otimes n})}$. For example, let $n = 2$, $\sigma = (1,2) \in S_2$ and let $y_1, y_2 \in V$ be homogeneous (i.e. $y_1, y_2 \in V_0 \cup V_1$), then

$$\varphi^*_{(1,2)}(y_1 \otimes y_2) = (-1)^{(\deg y_1)(\deg y_2)}(y_2 \otimes y_1).$$

For a precise definition of this sign action see [1]. The Schur super algebra is defined by

$$S^*(V,n) = \mathrm{End}_{\varphi^*(\mathrm{FS_n})}(\mathrm{V^{\otimes n}}).$$

More generally, given a subgroup G of the symmetric group S_n, the Schur super algebra of G is given by

$$S_G(V,n) = \mathrm{End}_{\varphi^*(\mathrm{FG})}(\mathrm{V^{\otimes n}}).$$

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Of course, $S_G(V,n)$ depends on the particular embedding of G in S_n. In the present paper we are interested in $S^*_A(V,n)$, the Schur super algebra of the alternating group $A = A_n$.

1.2. The aim of this paper is to understand the structure of $W = V^{\otimes n}$ as an $S^*_A(V,n)$-module. For $S^*(V,n)$ (i.e for $G = S_n$) this is well understood: it is a superalgebra generalization of the classical work of Schur and Weyl for $V = V_0$ and $V_1 = 0$. It involves the decomposition of W into a direct sum of its isotypic components and into the so-called *Weyl modules*. Moreover, these decompositions are given explicitly in terms of Young tableaux (see [1], Section 3). The Schur-Weyl theory and its super-generalization in [1], both require a detailed description of the group algebra FS_n. In particular, an explicit decomposition of FS_n into a direct sum of minimal one-sided ideals is required, as well as a description in terms of tableaux for the generators of such left ideals.

The analogue decomposition of W as an $S^*_A(V,n)$-module requires an analogue description for FA_n. In Section 2 we review some generalities and some known facts about FA_n. Then in Section 3 we develope the analogue structure theory for FA_n. We describe explicit generators g_{T_λ} for its minimal left ideals, where the T_λ's are certain (half of the) standard tableaux. The elements g_{T_λ} are similar to the well known generators $e_{T_\lambda} = R^+_{T_\lambda} C^-_{T_\lambda}$ of the minimal left ideals of FS_n; here R_{T_λ} (C_{T_λ}) denotes the row (column) stabilizer of T_λ. It follows from the well known von-Neumann Lemma (see Section 4.3) that the e_{T_λ}'s are semi-idempotents, and this fact is of great importance in the structure theory of FS_n. Similar to the S_n-case, most of the above g_{T_λ}'s are semi-idempotents. In particular, if $\lambda \neq \lambda'$ then g_{T_λ} is indeed semi-idempotent. However, there is a subset of the self-conjugate partitions $\lambda = \lambda'$ for which the g_{T_λ}'s are *not* semi-idempotents. We do show that this subset is very small: in every infinite $k \times k$-hook of partitions there are only finitely many such elements λ. The details are given in the Appendix, which was written together with Tamar Seeman. We nevertheless show, in Section 4, that there are infinitely many partitions λ for which g_{T_λ} is not semi-idempotent.

In the rest of the paper we describe the *explicit* structure and decompositions of $W = V^{\otimes n}$ as an $S^*_A(V,n)$ module. We describe three types of (related) decompositions of W:

1) into its isotypic components,

2) into a direct sum of Weyl-modules,

3) into a direct sum of irreducible $S^*_A(V,n)$-modules.

For $S^*(V,n)$, the e_{T_λ}'s define the Weyl-submodules of W both in the classical case and in the super case. Similarly, for $S^*_A(V,n)$ the g_{T_λ}'s define (the analogues of) the Weyl-submodules of W. In Section 5 we describe the decomposition of W into a direct sum of such $S^*_A(V,n)$-Weyl-submodules – in terms of partitions and certain standard tableaux. For $S^*(V,n)$, the Weyl modules are all irreducible. For $S^*_A(V,n)$, some of these Weyl modules are reducible: those corresponding to non-self-conjugate partitions remain irreducible, but those corresponding to self-conjugate partitions split as a direct sum of two non-isomorphic submodules. The details are given in Theorem 5.3. Note that this is closely related to a similar phenomena between the characters of S_n and of A_n.

In Section 6 we examine the various Weyl-modules through the eigenvalues of the actions of permutations on W. We show that these eigenvalues are closely related to – and can be calculated by – the values of S_n and of A_n characters. In particular, for a self-conjugate partition this yields the decomposition of the corresponding Weyl-module into a direct sum of two eigenspaces.

In Section 7 we describe explicit bases for the irreducible $S^*_A(V,n)$-submodules of W in terms of partitions and of semi-standard tableaux. As for the irreducible $S^*(V,n)$-submodules, these bases allow us to attach to each irreducible $S^*_A(V,n)$-module a certain super-symmetric function (see [1], Section 6). However, unlike for $S^*(V,n)$, in the case of a self-conjugate partition these functions do not characterize the corresponding irreducible $S^*_A(V,n)$-modules.

2 Generalities

2.1. *Generalities.* All modules in this paper are left modules. The following is well-known: Let M be a vector space over F and assume that R is a semisimple ring with q (non-zero) simple components R_i and such that $R \subseteq \mathrm{End_F(M)}$. The decomposition $R = \oplus_{i=1}^q R_i$ implies a decomposition of the identity in R, say $1 = \sum_{i=1}^q u_i$ with $u_i \in R_i$. Then $u_i u_j = \delta_{i,j} u_i$. The R_i's are the minimal two-sided ideals in R, each isomorphic to a certain matrix algebra over F, while the u_i's are the minimal central idempotents in R. Let $M_i = u_i M$ then $M = \oplus M_i$ and $R_j M_i = 0$ for $i \neq j$. Moreover, the M_i's are the isotypic components of M also as an R-module. Consider the ring $S := \mathrm{End_R(M)}$ and set $S_i = \mathrm{End_{R_i}(M_i)} = \mathrm{End_R(M_i)}$. Then $S = \oplus_{i=1}^q S_i$ where each S_i is simple with $u_i \in S_i$. Again, the S_i's are the minimal two-sided ideals in S, each isomorphic to a certain matrix algebra over F, while the u_i's are the minimal central idempotents in S, and the M_i's are the isotypic components of M as an S-module. So in particular this shows that M has the same isotypic decomposition as an R and as an S-module. Moreover $R_i = \mathrm{End_{S_i}(M_i)}$ and $\mathrm{End_F(M_i) \simeq R_i \otimes_F S_i}$. So we have $R = \mathrm{End_S(M)}$ and $\mathrm{End_F(M) = \oplus_{i=1}^q (R_i \otimes_F S_i)}$.

Let $1 \leq i \leq q$. As an R_i-module, $M_i \simeq N_i^{\oplus a}$ where N_i is R_i-irreducible and N_i is unique up to isomorphism. Similarly $M_i \simeq K_i^{\oplus b}$ where K_i is S_i-irreducible, and unique up to isomorphism. We apply this in Section 2.2 to $R = \varphi^*(FA_n)$ and $M = W = V^{\otimes n}$.

2.2. *Minimal two-sided ideal decompositions.* In order to apply the situation described in Section 2.1 to $R = \varphi^*(FA_n)$ and $M = W = V^{\otimes n}$ we need to understand the isotypic decomposition of R. Recall the classical and unique decomposition of the group algebra FS_n into minimal two-sided ideals:

$$FS_n = \bigoplus_{\lambda \vdash n} I_\lambda. \tag{1}$$

We say a partition λ is *self-conjugate* if it equals its conjugate partition λ', otherwise we call the partition *non-self-conjugate*. The group algebra FA_n has an analogue decomposition into minimal two-sided ideals as FS_n. To the pair $\lambda \neq \lambda'$ corresponds in FA_n a minimal ideal $\bar{I}_\lambda = \bar{I}_{\lambda'} \subsetneq I_\lambda \oplus I_{\lambda'}$ (by (2.4.2) in [8]) and $\bar{I}_\lambda \simeq I_\lambda \simeq I_{\lambda'}$ as left A_n-modules; and to a self-conjugate partition λ correspond two minimal ideals $\bar{I}^+_\lambda$ and $\bar{I}^-_\lambda$ with $\bar{I}^+_\lambda \oplus \bar{I}^-_\lambda \subsetneq I_\lambda$ and where dim $\bar{I}^+_\lambda =$ dim $\bar{I}^-_\lambda = (1/4)$dim I_λ (see [8], Lemma 2.5). Let $\lambda > \lambda'$ denote the left lexicographic order. Then

$$FA_n \;\; = \;\; [\bigoplus_{\lambda \vdash n, \lambda > \lambda'} \bar{I}_\lambda] \oplus [\bigoplus_{\lambda \vdash n, \lambda = \lambda'} \bar{I}^+_\lambda \oplus \bar{I}^-_\lambda]. \tag{2}$$

Denote the set of partitions which lie inside the (k, l)- hook by $H(k, l; n)$, that is $H(k, l; n) = \{(\lambda_1, \lambda_2, \dots) \vdash n \mid \lambda_{k+1} \leq l\}$. By [1], Theorem 3.20 (since here $l = k$),

$$\varphi^*(FS_n) = \bigoplus_{\lambda \in H(k,k;n)} \varphi^*(I_\lambda)$$

and φ^* restricted to I_λ for a partition λ lying in the (k, k)-hook is an isomorphism. Hence by 2.1 we have $W = \oplus_{\lambda \in H(k,k;n)} W_\lambda$ where $W_\lambda = \varphi^*(I_\lambda)W$ are the isotypic components of W as $\varphi^*(FS_n)$-module. An analogue of this decomposition also holds for $\varphi^*(FA_n)$. Let $H_1(k, k; n)$ be the set of all non-self-conjugate partitions λ lying in the (k, k)-hook such that λ is bigger than λ' in the left lexicographic order. Let $H_2(k, k; n)$ denote the set of all self-conjugate partitions lying in the (k, k)-hook. Then

$$R = \varphi^*(FA_n) \;\; = \;\; [\bigoplus_{\lambda \in H_1(k,k;n)} \varphi^*(\bar{I}_\lambda)] \oplus [\bigoplus_{\lambda \in H_2(k,k;n)} \varphi^*(\bar{I}^+_\lambda \oplus \bar{I}^-_\lambda)].$$

Moreover, since a minimal ideal is a simple algebra, φ^* applied to a minimal ideal is either zero or isomorphic to that same minimal ideal. Since here $\varphi^*(I_\lambda) \simeq I_\lambda$ we have $\varphi^*(\bar{I}_\lambda) \simeq \bar{I}_\lambda$ and $\varphi^*(\bar{I}^\epsilon_\lambda) \simeq \bar{I}^\epsilon_\lambda$ for $\epsilon = \pm$ which says that

$$\varphi^*(FA_n) \simeq [\bigoplus_{\lambda \in H_1(k,k;n)} \bar{I}_\lambda] \oplus [\bigoplus_{\lambda \in H_2(k,k;n)} \bar{I}^+_\lambda \oplus \bar{I}^-_\lambda].$$

2.3. *Conjugacy classes and characters of* A_n. Let μ be a partition of n and C_μ be its corresponding conjugacy class in S_n consisting of permutations of cycle type μ. We denote by c_μ the class sum of C_μ. The subgroup A_n of S_n is given by the even permutations of S_n. We call a partition μ of n *even* if any permutation in C_μ is even. Recall that the A_n-conjugacy classes are determined by the conjugacy classes C_μ of S_n as follows (see [7] Section 1.2):

1. If $\mu = (\mu_1, \mu_2, \ldots)$ is a partition of n where all μ_i are distinct and odd then the S_n-conjugacy class C_μ splits into two A_n-conjugacy classes which we denote as C_{μ^+} and C_{μ^-}. Denote the corresponding A_n-class sums as c_{μ^+} and c_{μ^-}.

2. If μ is an even partition of n but is not of the type described in 1. then C_μ is also the conjugacy class in A_n.

The irreducible characters of S_n, denoted χ^λ are parametrised by the partitions λ of n. Write $\chi^\lambda(\mu)$ for the character value of χ^λ on an element of the conjugacy class C_μ. Suppose λ is non-self-conjugate then the S_n-character χ^λ restricted to A_n is irreducible (and we denote it again by χ^λ) whereas, for λ self-conjugate it splits into two irreducible characters, $\chi^\lambda = \chi^{\lambda+} + \chi^{\lambda-}$ (see [7], Section 2.5). Let μ be an even partition and denote by $h(\lambda) := (h_{11}(\lambda), \ldots, h_{kk}(\lambda))$ the partition recording the hook lengths $h_{i,i}(\lambda)$ of λ. Then $\chi^\lambda(\mu)$ is a character value for A_n if λ is non-self-conjugate and C_μ non-split. If C_μ splits (but not χ^λ)) or χ^λ splits with $\mu \neq h(\lambda)$ then

$$\chi^\lambda(\mu^\pm) = \chi^\lambda(\mu)/2 \quad \text{and} \quad \chi^{\lambda\pm}(\mu) = \chi^\lambda(\mu)/2 \quad \text{and} \quad \chi^{\lambda\pm}(\mu^\pm) = \chi^\lambda(\mu)/2,$$

where the left-hand side denotes the A_n-character and the right-hand side the S_n-character. For λ self-conjugate and $\mu = h(\lambda)$ the conjugacy class C_μ splits with (see [7], 2.5.13)

$$\begin{aligned} \chi^{\lambda\pm}(h(\lambda)^+) &= \tfrac{1}{2}(\chi^\lambda(h(\lambda)) \pm \sqrt{\chi^\lambda(h(\lambda)) \textstyle\prod_i h^\lambda_{ii}}, \\ \chi^{\lambda\pm}(h(\lambda)^-) &= \tfrac{1}{2}(\chi^\lambda(h(\lambda)) \mp \sqrt{\chi^\lambda(h(\lambda)) \textstyle\prod_i h^\lambda_{ii}}. \end{aligned}$$

Note that since $\chi^{\lambda'} = \chi^\lambda \otimes \chi^{(1^n)}$ we have $\chi^{\lambda'}(\mu) = \chi^\lambda(\mu)$ for μ even. Hence in particular $\chi^{\lambda'}(1) = \chi^\lambda(1)$.

2.4. *Minimal central idempotents.* We describe next the minimal central idempotents of FS_n and FA_n. For S_n, the minimal central idempotent lying in I_λ is denoted by e_λ or $e_\lambda^{S_n}$. An explicit expression for $e_\lambda^{S_n}$ is given for example in [4], Lemma 3. For A_n, the minimal central idempotent lying in $\bar{I}_\lambda = \bar{I}_{\lambda'}$ for λ non-self-conjugate is denoted by $e_\lambda^{A_n} = e_{\lambda'}^{A_n}$; the minimal central idempotent of FA_n lying in $\bar{I}_\lambda^\epsilon$ for λ self-conjugate is denoted by e_λ^ϵ where $\epsilon = +$ or $\epsilon = -$. Note that $e_\lambda^+ + e_\lambda^- = e_\lambda^{S_n}$.

In describing explicit expressions for these idempotents we follow [4], Theorem 4 and [3]Theorem 5. For a non-self-conjugate partition λ we have $e_\lambda^{A_n} = e_\lambda^{S_n} + e_{\lambda'}^{S_n}$; for a self-conjugate partition λ the central minimal idempotents $e_\lambda^\pm$ are given as follows: Let d_λ be the number of standard tableaux of shape λ (see [2, 9, 11]) and define

$$p_\lambda = (-1)^{(n-k)/2} \prod_{i=1}^{k} h_{ii}(\lambda).$$

[3]The equation in the last line of page 473 (respectively the first line of page 474) in the proof of Theorem 5 in [4] is not correct. However, the authors of [4] pointed out that the weaker equation needed for the proof can be obtained by using Equation (33.11) in [3].

Then the minimal central idempotents $e_\lambda^\pm$ are given by

$$\begin{aligned} e_\lambda^+ &= \frac{1}{2}e_\lambda^{S_n}(1+\frac{d_\lambda}{2\sqrt{p_\lambda}\mid C_{h(\lambda)^+}\mid}(c_{h(\lambda)^+}-c_{h(\lambda)^-})), \\ e_\lambda^- &= \frac{1}{2}e_\lambda^{S_n}(1-\frac{d_\lambda}{2\sqrt{p_\lambda}\mid C_{h(\lambda)^+}\mid}(c_{h(\lambda)^+}-c_{h(\lambda)^-})). \end{aligned}$$

2.5. *Further notation.* Consider the algebra automorphism $f : FS_n \to FS_n$ given by linearity and by $f(\sigma) = \mathrm{sgn}(\sigma)\sigma$ for a permutation $\sigma \in S_n$. It is well known that $f(I_\lambda) = I_{\lambda'}$. Also, $x + f(x) \in FA_n$ for any $x \in FS_n$. Let G be a subgroup of S_n. We define elements G^+ and G^- in the group algebra FG by

$$G^+ = \sum_{g\in G} g \qquad \text{and} \qquad G^- = f(G^+) = \sum_{g\in G} sgn(g)g.$$

Given a tableau T_λ of shape λ where λ is a partition of n, we write R_{T_λ} for the group of row permutations and C_{T_λ} for the group of column permutations determined by the tableau T_λ. Moreover, corresponding to the tableau T_λ we have the elements $e_{T_\lambda} \in I_\lambda$ and $e'_{T_\lambda} \in I_{\lambda'}$ given by

$$e_{T_\lambda} = R^+_{T_\lambda} \cdot C^-_{T_\lambda} \quad \text{and} \quad e'_{T_\lambda} = f(e_{T_\lambda}) = R^-_{T_\lambda} \cdot C^+_{T_\lambda} \tag{3}$$

For the rest of this paper we fix the following notation: $V = V_0 \oplus V_1$ with $\dim V_0 = \dim V_1 = k$, and $W = V^{\otimes n}$. We shall delete the φ^* from the action of FS_n on $V^{\otimes n}$. For example, for the Schur super algebras and the super Weyl modules (see later sections) we write

$$\begin{aligned} &S^*(V,n) = \mathrm{End}_{\varphi^*(\mathrm{FS_n})}(\mathrm{V^{\otimes n}}) = \mathrm{End}_{\mathrm{FS_n}}(\mathrm{V^{\otimes n}}), \\ &S^*_A(V,n) = \mathrm{End}_{\varphi^*(\mathrm{FA_n})}(\mathrm{V^{\otimes n}}) = \mathrm{End}_{\mathrm{FA_n}}(\mathrm{V^{\otimes n}}), \\ &e_{T_\lambda}W := \varphi^*(e_{T_\lambda})W. \end{aligned}$$

3 The structure of FA_n

3.1. *Description of results.* Let T_λ be a tableau of shape λ. We are interested in the elements e_{T_λ} introduced in Section 2.5. They are the fundamental objects for analyzing the structure of the group algebra FS_n: The elements e_{T_λ} (and e'_{T_λ}) are semi-idempotents, that is $e^2_{T_\lambda} = \alpha e_{T_\lambda}$ for some non-zero α. Here α equals the product of the hook numbers of the diagram λ. A complete but in general not unique decomposition of FS_n into minimal left ideals is given by equation (1) and by the decomposition

$$I_\lambda = \bigoplus_{T_\lambda \text{ standard}} FS_n e_{T_\lambda}.$$

The set of semi-idempotents $\{e_{T_\lambda} \mid T_\lambda \text{ is standard }\}$ consists in general of mutually non-orthogonal elements. A replacement of this set by a set of orthogonal idempotents is given explicitly for example in [4], Lemma 3.

In this section we introduce elements g_{T_λ} in FA_n which share most of the above properties of the e_{T_λ}. Theorem 3.2 shows how the decomposition of the minimal two-sided ideals of FA_n (given by Equation (2)) into a direct sum of minimal left ideals is given explicitly by these elements g_{T_λ}. The g_{T_λ} in general are not semi-idempotents but, however, Theorem 4.1 and Propositions 4.2 indicate that for most λ these elements are indeed semi-idempotents.

Definition 3.1 *We define* $g_{T_\lambda} := e_{T_\lambda} + e'_{T_\lambda} = e_{T_\lambda} + f(e_{T_\lambda}) = R^+_{T_\lambda} C^-_{T_\lambda} + R^-_{T_\lambda} C^+_{T_\lambda}$.

Let $\lambda = \lambda'$ and let $T_1, \ldots, T_h$ be the standard tableaux of shape λ with one and two in their first column. Define T'_i to be the tableau obtained from T_i by conjugation. The conjugate tableau T'_λ of a standard tableau of shape λ is a standard tableau of shape λ' which means in this special situation of shape λ. Moreover T'_i is a tableau whose first row starts with one and two. This implies that the set of all T_i and all T'_i is the set of all standard tableaux of shape λ and hence $d_\lambda = 2h$. The main result of this section is:

Theorem 3.2 *Let λ be a partition of n. With the notation from above we have:*

1. *Let λ be non-self-conjugate. Then $\bar{I}_\lambda \subset I_\lambda \oplus I_{\lambda'}$ and $\bar{I}_\lambda$ decomposes into a direct sum of minimal left ideals of FA_n as follows:*

$$\bar{I}_\lambda = \bigoplus_{T_\lambda \ \textit{standard}} FA_n g_{T_\lambda}.$$

2. *Let λ be self-conjugate and let $\epsilon = +$ or $\epsilon = -$. Then $\bar{I}^+_\lambda \oplus \bar{I}^-_\lambda \subset I_\lambda$ and $\bar{I}^\epsilon_\lambda$ decomposes into a direct sum of left ideals of FA_n as follows:*

$$\bar{I}^\epsilon_\lambda = \bigoplus_{i=1}^{h} FA_n e^\epsilon_\lambda g_{T_i},$$

3.2. *Proof of main result.* In this section we prove Theorem 3.2. We assume the notation introduced in Section 3.1 and we start with the following simple Lemma which is frequently applied throughout the paper.

Lemma 3.3 *Let T be a tableau on $1, \ldots, n$, let $1 \leq a \neq b \leq n$ and assume a and b appear in the same column in T (a similar statement can be formulated for the rows). Then*

$$\begin{array}{lcl} (1+(a,b))C^+_T = C^+_T(1+(a,b)) = 2C^+_T & & (1+(a,b))C^-_T = C^-_T(1+(a,b)) = 0 \\ (1-(a,b))C^-_T = C^-_T(1-(a,b)) = 2C^-_T & \textit{and} & (1-(a,b))C^+_T = C^+_T(1-(a,b)) = 0. \end{array}$$

Proof. This follows since the transposition (a,b) is in the column stabilizer C_{T_i}, so we can write $C^+_T = (1+(a,b))q^+ = q^+(1+(a,b))$ and $C^-_T = (1-(a,b))q^- = q^-(1-(a,b))$ for some $q^+, q^- \in FS_n$. ∎

Lemma 3.4 *Let λ be a self-conjugate partition of n and T_λ any tableau of shape λ. Then*

$$e_\lambda^\pm g_{T_\lambda} = e_\lambda^\pm (e_{T_\lambda} + e'_{T_\lambda}) \neq 0.$$

Proof. We prove the claim for e_λ^+. Since e_λ^+ is central in FA_n and $e_{T_\lambda} + e'_{T_\lambda}$ is an element of FA_n we have $e_\lambda^+(e_{T_\lambda} + e'_{T_\lambda}) = (e_{T_\lambda} + e'_{T_\lambda})e_\lambda^+$. Assume for a contradiction that $e_\lambda^+(e_{T_\lambda} + e'_{T_\lambda}) = 0$. Since $n \geq 2$ and λ is self-conjugate, therefore $\lambda \neq (1^n)$. Thus, without loss of generality we assume the tableau T_λ has a first row starting with entries one and two. Multiply the equation $(e_{T_\lambda} + e'_{T_\lambda})e_\lambda^+ = 0$ from the left by $a := (1 + (1,2))$; since $aR_{T_\lambda}^+ = 2R_{T_\lambda}^+$ while $aR_{T_\lambda}^- = 0$ (see Lemma 3.3), together with $(e_{T_\lambda} + e'_{T_\lambda})e_\lambda^+ = 0$ it implies that $e_{T_\lambda}e_\lambda^+ = 0$, which in turn implies $(FS_n e_{T_\lambda})e_\lambda^+ = 0$. This provides the contradiction: By the character theory of A_n the simple FS_n-module $FS_n e_{T_\lambda}$ splits over FA_n into two (non-zero) irreducible modules, one of which is $(FS_n e_{T_\lambda})e_\lambda^+$ (see [7], Section 2.5). ∎

Lemma 3.5 *Suppose λ is self-conjugate and let T_λ be a tableau of shape λ. Then $FA_n g_{T_\lambda}$ is the direct sum of two non-zero left ideals (we show later that these are minimal left ideals):*

$$FA_n g_{T_\lambda} = FA_n e_\lambda^+ g_{T_\lambda} \oplus FA_n e_\lambda^- g_{T_\lambda}. \tag{4}$$

Proof. In Lemma 3.4 we have seen that $e_\lambda^\pm g_{T_\lambda} \neq 0$. Hence the two ideals on the right-hand-side of Equation (4) are non-zero. Their sum is direct since the elements $e_\lambda^\pm$ are central in FA_n with $e_\lambda^+ \cdot e_\lambda^- = 0$. The right-hand-side of Equation (4) is included in the left-hand-side since $FA_n e_\lambda^\epsilon g_{T_\lambda} \subseteq FA_n g_{T_\lambda}$. Also, since $e_\lambda^+ + e_\lambda^- = e_\lambda^{S_n}$ and hence $g_{T_\lambda} = e_\lambda^+ g_{T_\lambda} + e_\lambda^- g_{T_\lambda}$, we obtain that the left-hand-side of Equation (4) is contained in the right-hand-side. ∎

Lemma 3.6 1. *Let λ be a self-conjugate partition. Then the following sum of left ideals of FA_n is direct*

$$\sum_{i=1}^{h} FA_n g_{T_i} = \bigoplus_{i=1}^{h} FA_n g_{T_i}.$$

2. *Let λ be a non-self-conjugate partition. Then the following sum of left ideals of FA_n is direct*

$$\sum_{T_\lambda \text{ standard}} FA_n g_{T_\lambda} = \bigoplus_{T_\lambda \text{ standard}} FA_n g_{T_\lambda}.$$

Proof. Let $f : FS_n \to FS_n$ be as in Section 2.5.

1. Recall Lemma 3.3 and Definition 3.1. Assume that $\sum_{i=1}^{h} a_i g_{T_i} = 0$ for some elements $a_i \in FA_n$. We multiply this equation by $(1 - (1,2))$ from the right. Since $C_{T_i}^-(1 - (1,2)) = 2C_{T_i}^-$ and $C_{T_i}^+(1 - (1,2)) = 0$, it implies

$$0 = \sum_{i=1}^{h} a_i g_{T_i}(1 - (1,2)) = 2\sum_{i=1}^{h} a_i e_{T_i}.$$

But since the sum $\sum FS_n e_{T_\lambda}$ running over all standard tableaux T_λ is a direct sum, it implies that all $a_i e_{T_i} = 0$. Next, deduce that $\sum_{i=1}^{h} a_i e'_{T_i} = 0$ and apply the automorphism f: $0 = f(\sum_{i=1}^{h} a_i e'_{T_i}) = \sum_{i=1}^{h} f(a_i) e_{T_i}$. By the same argument as above all $f(a_i)e_{T_i} = f(a_i e'_{T_i}) = 0$ hence all $a_i e'_{T_i} = 0$. Since $g_{T_i} = e_{T_i} + e'_{T_i}$ it follows that all $a_i g_{T_i} = 0$.

2. Note that for λ non-self-conjugate $e_\lambda^{S_n} g_{T_\lambda} = e_{T_\lambda}$ since $e'_{T_\lambda} \in I_{\lambda'}$ and $e_\lambda^{S_n} I_{\lambda'} = 0$. Assume that $\sum a_{T_\lambda} g_{T_\lambda} = 0$ where the sum runs over all standard tableaux T_λ and $a_{T_\lambda} \in FA_n$. We multiply this equation with the central element $e_\lambda^{S_n}$ and obtain $0 = \sum e_\lambda^{S_n} a_{T_\lambda} g_{T_\lambda} = \sum a_{T_\lambda} e_{T_\lambda}$. But since the sum $\sum FS_n e_{T_\lambda}$ running over all standard tableaux T_λ is a direct sum, it implies that all $a_{T_\lambda} e_{T_\lambda} = 0$. Applying f and arguing as in the first part, deduce that all $a_{T_\lambda} g_{T_\lambda} = 0$. Hence

$$\sum_{T_\lambda \text{ standard}} FA_n g_{T_\lambda} = \bigoplus_{T_\lambda \text{ standard}} FA_n g_{T_\lambda}.$$

■

Proof of Theorem 3.2.

1. Note that for λ non-self-conjugate $\bar{I}_\lambda$ is isomorphic to the matrix algebra $M_{d_\lambda}(F)$, hence any direct sum of d_λ non-zero left ideals in $\bar{I}_\lambda$ must equal $\bar{I}_\lambda$, and these ideals necessarily are irreducible. The proof now follows from part 2 of the last lemma.

2. Since $e_\lambda^{S_n} = e_\lambda^+ + e_\lambda^-$ we obtain using Lemma 3.5 and Lemma 3.6

$$\begin{aligned}
\sum_{i=1}^{h}(FA_n e_\lambda^+ g_{T_i} + FA_n e_\lambda^- g_{T_i}) &\supseteq \sum_{i=1}^{h} FA_n g_{T_i} = \bigoplus_{i=1}^{h} FA_n g_{T_i} \\
&= \bigoplus_{i=1}^{h}(FA_n e_\lambda^+ g_{T_i} \oplus FA_n e_\lambda^- g_{T_i}) \\
&= \bigoplus_{i=1}^{h}(FA_n e_\lambda^+ g_{T_i}) \oplus \bigoplus_{i=1}^{h}(FA_n e_\lambda^- g_{T_i})
\end{aligned}$$

and by dimensionality we obtain equality in this last equation. The equation also implies (multiplying by e_λ^+) that

$$\sum_{i=1}^{h} FA_n e_\lambda^+ g_{T_i} = \bigoplus_{i=1}^{h} FA_n e_\lambda^+ g_{T_i}.$$

Since $\bar{I}_\lambda^+$ is isomorphic to the matrix algebra $M_h(F)$, any direct sum of h non-zero left ideals in $\bar{I}_\lambda^+$ must equal $\bar{I}_\lambda^+$, and these ideals necessarily are irreducible. Similarly for $\bar{I}_\lambda^-$. ■

Corollary 3.7 *Let T_λ be any tableau of shape λ then* $\dim FA_n g_{T_\lambda} = d_\lambda$*, the cardinality of the standard tableaux of shape λ.*

Proof.

1. Let λ be non-self-conjugate. Since $\bar{I}_\lambda \cong M_{d_\lambda}(F)$, all minimal left ideals in the decomposition in Theorem 3.2 (1) are d_λ dimensional.

2. Let λ be self-conjugate and $\epsilon = \pm$. If $T_\lambda = T_i$ this follows since

$$\dim FA_n e_\lambda^\epsilon g_{T_i} = h = (1/2)d_\lambda$$

and $FA_n g_{T_i} = FA_n e_\lambda^+ g_{T_i} \oplus FA_n e_\lambda^- g_{T_i}$. Let now T_λ be any tableau of shape λ, then there exist $\sigma \in S_n$ such that $T_\lambda = \sigma T_1$. It follows that $g_{T_\lambda} = \sigma g_{T_1} \sigma^{-1}$, and since $\sigma A_n \sigma^{-1} = A_n$, $FA_n g_{T_\lambda} = \sigma FA_n \sigma^{-1} \sigma g_{T_1} \sigma^{-1} = \sigma FA_n g_{T_1} \sigma^{-1} \cong FA_n g_{T_1}$, a vector space isomorphism $x \to \sigma x \sigma^{-1}$. ∎

We can now prove

Proposition 3.8 *Let λ be any partition of n with corresponding minimal two-sided ideal I_λ. Let T_λ be a tableau of shape λ with corresponding e_{T_λ} and g_{T_λ}, and let $J_\lambda \subseteq I_\lambda$ be a minimal left ideal in FS_n. Then*

1. *As FA_n-left modules $J_\lambda \cong FA_n g_{T_\lambda}$.*

2. *$FS_n e_{T_\lambda} = FA_n e_{T_\lambda}$.*

3. *$FS_n g_{T_\lambda} = FS_n e_{T_\lambda} \oplus FS_n e'_{T_\lambda}$ and in particular* $\dim FS_n g_{T_\lambda} = 2\dim FS_n e_{T_\lambda} = 2d_\lambda$*.*

Proof. Let $\eta = id + f$ where f was defined in Section 2.5, then $\eta : FS_n \to FA_n$ is an FA_n-module homomorphism, and $\eta(e_{T_\lambda}) = g_{T_\lambda}$. As FS_n and therefore also as left FA_n-modules, $J_\lambda \cong FS_n e_{T_\lambda}$. Apply η to show that $FS_n e_{T_\lambda} \cong FA_n g_{T_\lambda}$: $\eta(FA_n e_{T_\lambda}) = FA_n g_{T_\lambda}$ therefore η is surjective, hence 3.7 implies that $\eta : FS_n e_{T_\lambda} \to FA_n g_{T_\lambda}$ is an isomorphism, which proves the first part. This also shows that $\eta(FS_n e_{T_\lambda}) = \eta(FA_n e_{T_\lambda})$ and since η is an isomorphism, we can deduce the second part. To prove the third part, notice that $FA_n g_{T_\lambda}$ is a left ideal in FA_n, and for any non-zero left ideal $L \subseteq FA_n$, $\dim FS_n L = 2\dim L$ (since $FS_n L = L \oplus (1,2)L$). Since $FS_n g_{T_\lambda} = FS_n(FA_n g_{T_\lambda})$, deduce that $\dim FS_n g_{T_\lambda} = 2\dim FA_n g_{T_\lambda} = 2d_\lambda$. Now $FS_n g_{T_\lambda} \subseteq FS_n e_{T_\lambda} + FS_n e'_{T_\lambda}$ and since both sides have the same dimension, they are equal and the sum is direct. ∎

3.3. Let $T_1, \ldots, T_{d_\lambda}$ be the standard tableaux of shape λ, ordered as in Section 2 in [4] (i.e. lexicographically, by the rows), and with e_{T_i} the corresponding semi-idempotents: $e_{T_i}^2 = \alpha e_{T_i}$ where α equals the product of the hook numbers in λ (with $1 \le i \le d_\lambda$), and $e_{T_j} e_{T_i} = 0$ if $i < j$. If λ is non-self-conjugate then also $g_{T_i}^2 = \alpha g_{T_i}$. In this case denote

$$E_{T_i} = \frac{1}{\alpha} e_{T_i} \text{ and } G_{T_i} = \frac{1}{\alpha} g_{T_i},$$

then E_{T_i} and G_{T_i} are idempotents. Write $e_\lambda^{S_n} = \sum_{i=1}^{d_\lambda} a_i E_{T_i}$, then $\{a_1 E_{T_1}, \ldots, a_{d_\lambda} E_{T_{d_\lambda}}\}$ is a set of mutually orthogonal idempotents. The following explicit expression for the $a_i E_{T_i}$ is given in [4], Lemma 3:

$$a_i E_{T_i} = (1 - \sum_{j<i} E_{T_j} + \sum_{j<k<i} E_{T_j} E_{T_k} - + \ldots + (-1)^{i-1} E_{T_1} \cdots E_{T_{i-1}}) E_{T_i}.$$

In the non-self-conjugate case this extends to the elements G_{T_i} as follows.

Proposition 3.9 *Let λ be non-self-conjugate and $T_1, \ldots, T_{d_\lambda}$ be the standard tableaux of shape λ, ordered as above, and let $e_\lambda^{A_n} = \sum_{i=1}^{d_\lambda} a_i G_{T_i}$. Then $\{a_1 G_{T_1}, \ldots, a_{d_\lambda} G_{T_{d_\lambda}}\}$ forms a set of mutually orthogonal idempotents with*

$$a_i G_{T_i} = (1 - \sum_{j<i} G_{T_j} + \sum_{j<k<i} G_{T_j} G_{T_k} - + \ldots + (-1)^{i-1} G_{T_1} \cdots G_{T_{i-1}}) G_{T_i}.$$

Proof. Recall that $g_{T_\lambda} = e_{T_\lambda} + f(e_{T_\lambda})$, hence $G_{T_\lambda} = E_{T_\lambda} + f(E_{T_\lambda})$, and $e_\lambda^{A_n} = e_\lambda^{S_n} + f(e_\lambda^{S_n})$, where $f(\sigma) = \mathrm{sgn}(\sigma)\sigma$ for $\sigma \in S_n$. Adding the above expressions for $a_i E_{T_i}$ and $f(a_i E_{T_i})$, we only need to verify that, for example, $E_{T_j} E_{T_k} + f(E_{T_j} E_{T_k}) = (E_{T_j} + f(E_{T_j}))(E_{T_k} + f(E_{T_k}))$ (and similarly for longer products). This follows from $E_{T_j} f(E_{T_k}) = f(E_{T_j}) E_{T_k} = 0$, since $E_{T_j} \in I_\lambda$, $f(E_{T_k}) \in I'_\lambda$ and λ non-self-conjugate. ■

4 Semi-idempotents g_{T_λ}

4.1. *Semi-idempotents g_{T_λ}.* We assume the notation as established in Section 3. As $e_{T_\lambda}^2 = \alpha e_{T_\lambda}$, hence also $e'^2_{T_\lambda} = \alpha e'_{T_\lambda}$ (see Definition 3.1). So if $e_{T_\lambda} e'_{T_\lambda} = 0$ then also $e'_{T_\lambda} e_{T_\lambda} = 0$ and it follows that $g_{T_\lambda}^2 = \alpha g_{T_\lambda}$, that is, g_{T_λ} is semi-idempotent. However, if $e_{T_\lambda} e'_{T_\lambda} + e'_{T_\lambda} e_{T_\lambda} \neq 0$ then g_{T_λ} is not semi-idempotent. These properties are independent of the particular tableau T_λ but depend only on the shape λ itself since, if $\bar{T}_\lambda$ is another tableau of shape λ, then $\bar{T}_\lambda = \eta T_\lambda$ for some $\eta \in S_n$, and $g_{\eta T_\lambda} = \eta g_{T_\lambda} \eta^{-1}$. We henceforth look for conditions on λ such that $e_{T_\lambda} e'_{T_\lambda} = 0$ and hence g_{T_λ} is a semi-idempotent.

If λ is non-self-conjugate then $e_{T_\lambda} e'_{T_\lambda} = 0$ since $e_{T_\lambda} \in I_\lambda$ and $e'_{T_\lambda} \in I_{\lambda'}$, hence g_{T_λ} is semi-idempotent. We therefore consider for the rest of this subsection self-conjugate partitions λ. In this case not every element g_{T_λ} is a semi-idempotent. For example, let $\lambda = (2,1)$ and let the tableau T_λ be given by

$$T_{2,1} = \begin{array}{ll} 1 & 2 \\ 3 & \end{array}$$

then $g_{T_\lambda} = 2 - 2 \cdot (1,3,2)$, and it is easy to see that this is not a semi-idempotent.

Theorem 4.1 and Proposition 4.2 give different (and independent) conditions on λ, which guarantee that g_{T_λ} is a semi-idempotent. Since most λ's satisfy these conditions, most g_{T_λ}'s are semi-idempotents. Nevertheless Proposition 4.3 yields an infinite sequence of λ's with g_{T_λ} not semi-idempotent.

Theorem 4.1 *Let λ be a partition of n lying in a (k,k)-hook and assume $\lambda_1 \geq 2k+1$ (this in particular holds if $n > 3k^2$). Let T_λ be a tableau of shape λ with corresponding $e = e_{T_\lambda}$ and $e' = e'_{T_\lambda}$. Then $ee' = e'e = 0$, hence g_{T_λ} is semi-idempotent.*

We shall give two different proofs of Theorem 4.1; because of the length and complexity of these proofs we postpone it to the Appendix. The following proposition can be applied in many cases to show that $e_{T_\lambda} e'_{T_\lambda} = 0$.

Proposition 4.2 *Let λ be self-conjugate and assume there exist at least two rows (hence also columns) in λ with equal and odd length (this holds in particular if $\lambda_1 - \lambda_2 \geq 2$), and let T_λ be a tableau of shape λ. Then $e_{T_\lambda} e'_{T_\lambda} = 0$, hence g_{T_λ} is semi-idempotent.*

Proof. Recall that $e_{T_\lambda} e'_{T_\lambda} = R^+_{T_\lambda} C^-_{T_\lambda} R^-_{T_\lambda} C^+_{T_\lambda}$. We show that $C^-_{T_\lambda} R^-_{T_\lambda} C^+_{T_\lambda} = 0$. The two rows of equal and odd lengh (say length r) split T_λ into three parts T_1, T_2 and T_3 as indicated in Figure 4. The parts T_1 and T_3 might be empty. Part T_2 consisits of the two rows of equal and odd length r. Suppose these two rows have entries a_i and b_i respectively. Let $\sigma = (a_1, b_1) \cdots (a_r, b_r)$ be the product of the r transpositions

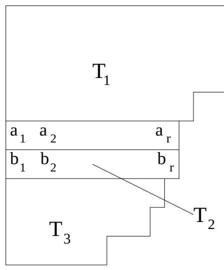

Figure 1: The diagram T_λ.

(a_i, b_i) defined by T_2. Then σ lies in C_{T_λ} with $\sigma^2 = id$ and $\text{sgn}(\sigma) = -1$. The row stabilizer of T_λ factors as $R_{T_\lambda} = R_{T_1} \times R_{T_2} \times R_{T_3}$. Since σ is disjoint from the entries of T_1 and T_3, therefore $\sigma\tau = \tau\sigma$ for any $\tau \in R_{T_1} \cup R_{T_3}$. Also,

$$\begin{aligned} R_{T_2} &= S_r(a_1, \ldots, a_r) \times S_r(b_1, \ldots, b_r), \\ \eta R_{T_2} \eta^{-1} &= S_r(\eta(a_1), \ldots, \eta(a_r)) \times S_r(\eta(b_1), \ldots, \eta(b_r)) \end{aligned}$$

where $S_r(x_1, \ldots, x_r)$ denotes the symmetric group on the r symbols x_i and $\eta \in S_n$ arbitrary. It follows that $\sigma R_{T_2} \sigma^{-1} = R_{T_2}$. In particular, this implies that $\sigma R^-_{T_2} = R^-_{T_2} \sigma$. Since $R^-_{T_\lambda} = R^-_{T_1} \cdot R^-_{T_2} \cdot R^-_{T_3}$ we conclude that

$$(1-\sigma) R^-_{T_\lambda} = R^-_{T_\lambda} (1-\sigma).$$

We apply Lemma 3.3 and its proof to $\sigma \in C_{T_\lambda}$; hence we can write $C^+_{T_\lambda} = (1+\sigma) q^+_1$ and $C^-_{T_\lambda} = q^-_2 (1-\sigma)$ for some $q^+_1, q^-_2 \in FS_n$. Since $(1-\sigma)(1+\sigma) = 0$ this implies

$$\begin{aligned} C^-_{T_\lambda} R^-_{T_\lambda} C^+_{T_\lambda} &= q^-_2 (1-\sigma) R^-_{T_\lambda} (1+\sigma) q^+_1 \\ &= q^-_2 R^-_{T_\lambda} (1-\sigma)(1+\sigma) q^+_1 = 0. \ \blacksquare \end{aligned}$$

Application. Note that for a large n, a random λ would have a *tail* of (many) 1's, hence g_{T_λ} is semi-idempotent.

4.2. *Non-semi-idempotents* g_{T_λ}. The next proposition shows that for infinitely many partitions λ we obtain $e_{T_\lambda}e'_{T_\lambda} + e'_{T_\lambda}e_{T_\lambda} \neq 0$ and hence g_{T_λ} is not a semi-idempotent. Let $t_1, \ldots, t_k \in V_0$ and $u_1, \ldots, u_k \in V_1$ be bases of V_0 and V_1 respectively with $V = V_0 \oplus V_1$. Call $w = z_1 \otimes \cdots \otimes z_n \in W$ *monomial* if $z_1, \ldots, z_n \in \{t_1, \ldots, t_k, u_1, \ldots, u_k\}$. Following [12] (see also Section 3 in [1]), T_λ identifies a monomial in $V^{\otimes n}$ with a (k,k)-tableau of shape λ on $t_1, \ldots, t_k, u_1, \ldots, u_k$, and a row or a column permutation of T_λ acts on such a tableau (i.e. monomial) accordingly.

Proposition 4.3 *Call λ a stair-case partition if $\lambda = (m, m-1, \ldots, 1)$. Let λ be such a partition and let T_λ be a tableau of shape λ. Then $e_{T_\lambda}e'_{T_\lambda} + e'_{T_\lambda}e_{T_\lambda} \neq 0$.*

Proof. Let $n = 1 + \ldots + m$ with λ a stair-case partition of n, and let T_λ be a tableau on $1, \ldots, n$. In this proof we only use the t_i's, hence we identify $t_i \equiv i$, for $1 \leq i \leq k$. Denote by $w_0 = w_0(t)$ and $w_1 = w_1(t)$ the monomials identified by T_λ with the following tableaux:

$$w_0 = \begin{array}{cccc} m & \ldots & 2 & 1 \\ \vdots & \cdot & \cdot & \\ 2 & \cdot & & \\ 1 & & & \end{array} \quad \text{and} \quad w_1 = \begin{array}{cccc} 1 & 2 & \ldots & m \\ \vdots & \vdots & \cdot & \\ 1 & 2 & & \\ 1 & & & \end{array}$$

Denote $R = R_{T_\lambda}$, $C = C_{T_\lambda}$, $e = e_{T_\lambda}$ and $e' = e'_{T_\lambda}$. We will prove that $e'ew_1 = 0$ while $ee'w_1 \neq 0$, thus proving that $ee' + e'e \neq 0$. Note that $C^-w_1 = 0$ since there are repetitions in the columns of w_1. Hence we obtain $e'ew_1 = R^-C^+R^+C^-w_1 = 0$. We show next that $ee'w_1 \neq 0$. For $r := m!(m-1)!\cdots 1!$ we have $C^+w_1 = r \cdot w_1$, and since $ee' = R^+C^-R^-C^+$ it suffices to show that $R^+C^-R^-w_1 \neq 0$. We need the following two properties of w_0 and w_1.

Property 1. Let $p \in R$, let $M = pw_1$ and assume M has no repetitions in its columns. Then $M = w_0$. To see this, check the rows of M from the bottom to the top: The bottom row in w_1 is 1, which is also the bottom row in $M = pw_1$. For the same reason, the next row in M contains one and two, and since M has no repetitions in its columns, this row has to be $2, 1$. Continue this process to obtain $M = w_0$.

Property 2. Let $p \in R$, $q \in C$ such that $pqw_0 = w_0$; we claim $p = q = 1$. To prove this we show first that q must fix the first column of w_0; otherwise, in the first column of qw_0 there is some j *below* the $m-j+1$th row, and therefore in pqw_0 there is some j below the $m-j+1$th row, contradicting $pqw_0 = w_0$. By the same argument q must also fix the second column of w_0. Continue to conclude firstly that $q = 1$, then secondly that $p = 1$.

We can now show that $R^+C^-R^-w_1 \neq 0$: There is a unique element $p \in R$ such that $pw_1 = w_0$. Therefore, $R^-w_1 = \pm w_0 +$ sum of monomials M with column repetitions (by Property 1). For each such monomial M, $C^-M = 0$, which implies $C^-R^-w_1 = \pm C^-w_0$. Thus $R^+C^-R^-w_1 = \pm R^+C^-w_0$. By the second property, $R^+C^-w_0 = w_0 + \text{other monomials}$, hence $R^+C^-R^-w_1 = \pm R^+C^-w_0 \neq 0$. ■

4.3. *Von Neumann's Lemma.* An important tool in analyzing the structure of FS_n is von Neumann's Lemma. It implies that the e_{T_λ}'s are semi-idempotents. In this section we discuss a von-Neumann-type lemma for FA_n. However, it does not imply that the g_{T_λ}'s are semi-idempotents. For convinience we restate the result for S_n before proving an analogue for A_n.

Theorem 4.4 (Von Neumann's Lemma for S_n) *Let λ be a partition of n, T be a tableau of shape λ and let $a \in FS_n$. Then $paq = \operatorname{sgn}(q)a$ for all $p \in R_T$ and $q \in C_T$, if and only if $a = \alpha e_T$ for some $\alpha \in F$.*

Proposition 4.5 (Von Neumann's Lemma for A_n) *Let λ be a partition of n, $T = T_\lambda$ be a tableau of shape λ and let $a \in FA_n$. Then $paq = \operatorname{sgn}(q)a$ for all $p \in R_T$ and $q \in C_T$ such that $pq \in A_n$, if and only if $a = \alpha g_T$ for some $\alpha \in F$.*

Proof. The proof is essentially identical to the proof in the S_n-case. Denote $R = R_T$ and $C = C_T$. First, since $\operatorname{sgn}(p) = \operatorname{sgn}(q)$, we have

$$pg_Tq = p(R^+C^- + R^-C^+)q = \operatorname{sgn}(q)R^+C^- + \operatorname{sgn}(p)R^-C^+ = \operatorname{sgn}(q)g_T.$$

Next, let $a = \sum_{\sigma \in A_n} \alpha_\sigma \sigma$ and assume a satisfies the above condition. Deduce that $\alpha_{p\sigma q} = \operatorname{sgn}(q)\alpha_\sigma$ for all $\sigma \in A_n$. If $\sigma = 1$ then $\alpha_{pq} = \operatorname{sgn}(q)\alpha$ where $\alpha = \alpha_1$. If σ is not of the form $\sigma = pq$ for any $p \in R$ and $q \in C$ then there exist two transpositions $p \in R$ and $q \in C$ – hence $pq \in A_n$ – such that $\sigma = p\sigma q$ (see [2], Theorem 2.8). This implies that $\alpha_\sigma = 0$. ■

5 The Weyl modules for $S^*_A(V, n)$

5.1. The (super) Weyl modules for the symmetric group of degree n (denoted in [1] as $W * e_{T_\lambda}$) are given by $\varphi^*(e_{T_\lambda})W = e_{T_\lambda}W$. Recall from Corollary 3.18 of [1] that these are irreducible over $S^*(V, n)$ and that by [1], Section 3

$$W = \bigoplus_{\lambda \in H(k,k,n)} W_\lambda \quad \text{and} \quad W_\lambda = \bigoplus_{T_\lambda \text{ standard}} e_{T_\lambda}W. \tag{5}$$

We give an explicit construction of $S^*_A(V, n)$-submodules $L_{T_\lambda} \subseteq V^{\otimes n}$ which are natural generalisations of these Weyl modules. We also call these modules *Weyl modules*. When λ is non-self-conjugate then L_{T_λ} is irreducible over $S^*_A(V, n)$, whereas when λ is self-conjugate, L_{T_λ} is a sum of two irreducible and non-isomorphic submodules $S^*_A(V, n)$-modules. We show that $V^{\otimes n}$ is a direct sum of such L_{T_λ}.

We introduce the map $\mathcal{T} : W \longrightarrow W$, which is the map T in Lemma 1.8 in [8]. We recall its definition and basic properties. Let $t_1, \ldots, t_k \in V_0$ and $u_1, \ldots, u_k \in V_1$ be bases of V_0 and V_1 respectively with $V = V_0 \oplus V_1$. Let $\tau \in \mathrm{End}_{\mathrm{F}}(\mathrm{V})$ be given by $\tau t_i = u_i$ and $\tau u_i = t_i$ for $i = 1, \ldots, k$. Note that the degree of τ in $\mathrm{End}_{\mathrm{F}}(\mathrm{V})$ is one. Now define $\mathcal{T} = \tau^{\otimes n}$. For example, if $n = 2$ and $y_1, y_2 \in V$ are homogeneous,

$$\mathcal{T}(y_1 \otimes y_2) = (\tau \otimes \tau)(y_1 \otimes y_2) = (-1)^{\deg y_1} \cdot \tau y_1 \otimes \tau y_2.$$

It follows that $\varphi^*_\sigma \circ T = \mathrm{sgn}(\sigma) T \circ \varphi^*_\sigma$ for $\sigma \in S_n$, which clearly implies that $\mathcal{T} \in S^*_A(V, n)$. It also implies that for $w \in W$, $\mathcal{T} e_{T_\lambda} w = e'_{T_\lambda} \mathcal{T} w$ hence $\mathcal{T} e_{T_\lambda} W = e'_{T_\lambda} W$, and that $\mathcal{T} : W_\lambda \longrightarrow W_{\lambda'}$ which is a vector space isomorphism.

5.2. We begin by describing the isotypic decomposition of $V^{\otimes n}$ as an $S^*_A(V, n)$-module. Using Section 2.1 (and notations of Sections 2.2 and 2.4) we deduce:

Proposition 5.1 *Let λ be a partition lying in the (k, k)-hook. Then the following holds:*

1. *Suppose λ is non-self-conjugate. As FA_n-modules (i.e. as $R = \varphi^*(FA_n)$-modules), W_λ and $W_{\lambda'}$ are isomorphic, and $W_\lambda \oplus W_{\lambda'}$ is an isotypic component of W both as as $\varphi^*(FA_n)$-module and as $S^*_A(V, n)$-module.*
2. *Suppose λ is self-conjugate. As FA_n-module and as $S^*_A(V, n)$-module, W_λ is the direct sum of two isotypic components $W_\lambda = W^+_\lambda \oplus W^-_\lambda$ with $W^+_\lambda = e^+_\lambda W$ and $W^-_\lambda = e^-_\lambda W$.*

So, the isotypic decomposition of W as $\varphi^*(FA_n)$-module and as $S^*_A(V, n)$-module is

$$W = [\bigoplus_{\lambda \in H_1(k,k;n)} (W_\lambda \oplus W_{\lambda'})] \oplus [\bigoplus_{\lambda \in H_2(k,k;n)} (W^+_\lambda \oplus W^-_\lambda)].$$

5.3. The super-Weyl-modules $e_{T_\lambda} W$ are irreducible over $S^*(V, n)$, see [1]. Replacing e_{T_λ} by g_{T_λ} we obtain the analogous $S^*_A(V, n)$-modules. These may be reducible, and are given in the following definition.

Definition 5.2 *Let λ be a partition of n and T_λ a tableau of shape λ. Then define $L_{T_\lambda} = g_{T_\lambda} W = (e_{T_\lambda} + e'_{T_\lambda}) W$. Since g_{T_λ} is an element of FA_n hence L_{T_λ} is an $S^*_A(V, n)$-module; we call it the Weyl module for the Schur super algebra $S^*_A(V, n)$ of the alternating groups, corresponding to the tableau T_λ.*

We prove that L_{T_λ} satisfies the following properties:

Theorem 5.3 *Let λ be a partition of $n \geq 2$ and let T_λ be a tableau of shape λ with L_{T_λ} the corresponding Weyl module for $S^*_A(V, n)$. Then the following holds:*

1. *$L_{T_\lambda} = e_{T_\lambda} W \oplus e'_{T_\lambda} W$.*
2. *If λ is non-self-conjugate then L_{T_λ} is an irreducible $S^*_A(V, n)$-module, $L_{T_\lambda} \subseteq W_\lambda \oplus W_{\lambda'}$, and any irreducible $S^*_A(V, n)$-submodule of $W_\lambda \oplus W_{\lambda'}$ is isomorphic to L_{T_λ}.*

3. *If λ is self-conjugate then L_{T_λ} splits: $L_{T_\lambda} = L^+_{T_\lambda} \oplus L^-_{T_\lambda}$, a direct sum of two irreducible and non-isomorphic $S^*_A(V,n)$-modules. Here $L^+_{T_\lambda} = e^+_\lambda g_{T_\lambda} W$ and $L^-_{T_\lambda} = e^-_\lambda g_{T_\lambda} W$ with* $\dim L^+_{T_\lambda} = \dim L^-_{T_\lambda}$. *Moreover, any irreducible $S^*_A(V,n)$-submodule of W_λ is isomorphic to either $L^+_{T_\lambda}$ or $L^-_{T_\lambda}$.*

Note that by Proposition 5.1, up to isomorphism, the irreducible $S^*_A(V,n)$-modules from 2. and 3. above are the only irreducible $S^*_A(V,n)$-submodules of W.

Proof. To prove part 1, we show first that the sum $e_{T_\lambda}W + e'_{T_\lambda}W$ is direct. If λ is non-self-conjugate this follows since $e_{T_\lambda}W \subseteq W_\lambda$ and $e'_{T_\lambda}W \subseteq W'_\lambda$, and moreover $W_\lambda \cap W'_\lambda = 0$. If λ is self-conjugate then as $n \geq 2$, it follows that $\lambda \neq (1^n)$ and so R_{T_λ} is non-trivial. Let $(a,b) \in R_{T_\lambda}$ then, by Lemma 3.3, $(1+(a,b))e_{T_\lambda} = 2e_{T_\lambda}$ while $(1+(a,b))e'_{T_\lambda} = 0$. This implies that the sum is direct.

Decompose the identity $1 = \sum_{\lambda \in H(k,k;n)} 1_\lambda$, where $FS_n = \oplus_{\lambda \vdash n} I_\lambda$, $\varphi^*(FS_n) = \oplus_{\lambda \in H(k,l;n)} \varphi^*(I_\lambda)$, and each $1_\lambda \in \varphi^*(I_\lambda)$. Now, $S^*(V,n) = \oplus_{\lambda \in H(k,k;n)} S(\lambda)$ where each $S(\lambda)$ is isomorphic to a matrix algebra. Since $S^*(V,n)$ is the centralizer of $\varphi^*(FS_n)$ it also follows that each 1_λ is the identity of $S(\lambda)$. Since $S^*(V,n) \subseteq S^*_A(V,n)$ and L_{T_λ} is an $S^*_A(V,n)$-module, $1_\lambda L_{T_\lambda} \subseteq L_{T_\lambda}$.

Obviously $L_{T_\lambda} \subseteq e_{T_\lambda}W + e'_{T_\lambda}W$. We show that equality between these modules holds, and this will complete the proof of part 1.

Consider first a non-self-conjugate partition λ. Then $1_\lambda L_{T_\lambda} \subseteq L_{T_\lambda}$ while $1_\lambda g_{T_\lambda} = e_{T_\lambda}$ therefore $e_{T_\lambda}W \subseteq L_{T_\lambda}$, and similarly, $e'_{T_\lambda}W \subseteq L_{T_\lambda}$. Hence $L_{T_\lambda} = e_{T_\lambda}W + e'_{T_\lambda}W$ for λ non-self-conjugate.

We consider next a self-conjugate partition λ. Let $\epsilon = \pm$. Since $e^\epsilon_\lambda \in S^*_A(V,n)$, $e^\epsilon_\lambda L_{T_\lambda} \subseteq L_{T_\lambda}$. By Lemma 3.4, $e^\epsilon_\lambda g_{T_\lambda} \neq 0$ and since $e^+_\lambda e^-_\lambda = 0$, we deduce that $e^+_\lambda g_{T_\lambda} W \oplus e^-_\lambda g_{T_\lambda} W \subseteq L_{T_\lambda} \subseteq e_{T_\lambda}W + e'_{T_\lambda}W$. Since $e^\epsilon_\lambda g_{T_\lambda} W$ is an $S^*_A(V,n)$-submodule, it is also an $S^*(V,n)$-module. Since $e_{T_\lambda}W$ and $e'_{T_\lambda}W$ are irreducible over $S^*(V,n)$, any $S^*(V,n)$-submodule of $e_{T_\lambda}W + e'_{T_\lambda}W$ which is a direct sum of two non-zero modules must equal $e_{T_\lambda}W + e'_{T_\lambda}W$. It follows that

$$e^+_\lambda g_{T_\lambda} W \oplus e^-_\lambda g_{T_\lambda} W = L_{T_\lambda} = e_{T_\lambda}W + e'_{T_\lambda}W.$$

Also, the above equality and the irreducibility of $e_{T_\lambda}W$ and $e'_{T_\lambda}W$ as $S^*(V,n)$-modules imply that the $S^*_A(V,n)$-modules $e^+_\lambda g_{T_\lambda} W$ and $e^-_\lambda g_{T_\lambda} W$ are irreducible. In particular, $\dim e^\epsilon_\lambda g_{T_\lambda} W = \dim e_{T_\lambda} W$. This completes the proof of part 1, as well as of part 3.

To complete the proof of the theorem we only need to show that the Weyl module L_{T_λ} is irreducible over $S^*_A(V,n)$ for λ non-self-conjugate. Suppose $z \neq 0$ is an arbitrary element of L_{T_λ}, say $z = x + \mathcal{T}y$ with $x, y \in e_{T_\lambda}W$ (see Section 5.1). We show that $S^*_A(V,n)z = L_{T_\lambda}$. Without loss of generality we assume that $x \neq 0$. Then for any $\lambda \neq \mu$ we have $1_\lambda W_\mu = 0$. In particular $1_\lambda W_{\lambda'} = 0$. It follows that $1_\lambda z = x$.

Since $1_\lambda \in S^*(V,n) \subseteq S^*_A(V,n)$, we have $0 \neq x \in S^*_A(V,n)z$. Since $x \in e_{T_\lambda}W$ and $e_{T_\lambda}W$ is an irreducible $S^*(V,n)$-module we have $e_{T_\lambda}W = S^*(V,n)x \subseteq S^*_A(V,n)z$. Applying $\mathcal{T}$ we also have $\mathcal{T}e_{T_\lambda}W = e'_{T_\lambda}W \subseteq S^*_A(V,n)z$ and hence $L_{T_\lambda} \subseteq S^*_A(V,n)z$. The opposite inclusion is obvious. ■

5.4. In Section 5.2 we defined Weyl modules for $S^*_A(V,n)$. We now give an explicit decomposition of each isotypic component of W as a direct sum of Weyl modules.

Proposition 5.4 *Let λ be non-self-conjugate and let $T_1, \cdots, T_{d_\lambda}$ be the standard tableaux of shape λ. Then*

$$W_\lambda \oplus W_{\lambda'} = \bigoplus_{j=1}^{d_\lambda} L_{T_j}.$$

Proof. Recall the expression for W_λ given in Equation (5). Applying the vector space isomorphism $\mathcal{T} : W_\lambda \to W_{\lambda'}$ (see Section 5.1) to this expression, we obtain a similar equation for $W_{\lambda'}$. Using Theorem 5.3 this implies

$$W_\lambda \oplus W_{\lambda'} = \bigoplus_{j=1}^{d_\lambda} (e_{T_j}W \oplus \mathcal{T}e_{T_j}W) = \bigoplus_{j=1}^{d_\lambda} L_{T_j},$$

which is a direct sum of simple modules. ■

We consider henceforth a self-conjugate partition λ. Let $T_1, \ldots, T_h$ be the standard tableaux of shape λ whose first row starts with the entries one and two. As remarked in Section 3.1, the number of standard tableaux of shape λ is $d_\lambda = 2h$.

Proposition 5.5 *Let λ be self-conjugate and let $T_1, \ldots, T_h$ be the standard tableaux of shape λ with one and two in the first row. Then $\sum_{i=1}^h L_{T_i} = \oplus_{i=1}^h L_{T_i}$ and hence*

$$W_\lambda = \bigoplus_{i=1}^{h} L_{T_i}.$$

Proof. By a standard argument, the sum $\sum_{i=1}^h e_{T_i}W$ is direct. Since $\mathcal{T}$ is a vector space isomorphism (see Section 5.1), the sum $\sum_{i=1}^h \mathcal{T}e_{T_i}W$ is also direct. The proof of this proposition follows once we show that

$$\bigoplus_{i=1}^{h} e_{T_i}W \cap \bigoplus_{i=1}^{h} \mathcal{T}e_{T_i}W = 0.$$

Note that for each $1 \leq i \leq j$ we have $(1+(1,2))R^+_{T_i} = 2R^+_{T_i}$ and $(1+(1,2))R^-_{T_i} = 0$ (see Lemma 3.3). Suppose x is an element in the intersection

$$(e_{T_1}W \oplus \ldots \oplus e_{T_h}W) \cap \mathcal{T}(e_{T_1}W \oplus \ldots \oplus e_{T_h}W).$$

We calculate the element $(1+(1,2))x$ in two different ways: The element x has the form $x = \sum_{i=1}^h R^+_{T_i}C^-_{T_i}w_i$ for some elements $w_i \in W$ and, by the above, we obtain $(1+(1,2))x = 2x$. On the other hand, since x also has the form $x = \sum_{i=1}^h R^-_{T_i}C^+_{T_i}w'_i$ for some $w'_i \in W$ we obtain $(1+(1,2))x = 0$. Thus $2x = 0$ implies that $x = 0$. ■

6 Splitting the Weyl modules via eigenvalues

6.1. In this section we discuss the splitting of the reducible Weyl modules via eigenvalues. Let $Z(FA_n)$ denote the center of FA_n. Recall from Section 2.5 that when acting on W, FS_n means $\varphi^*(FS_n)$. Also, recall the notation and facts given in Section 2.3 on the conjugacy classes C_μ of A_n and its characters χ^λ and $\chi^{\lambda\pm}$. In case we need to distinguish the minimal central idempotents e of A_n and S_n we write instead of e either e^{A_n} or e^{S_n}. We begin with the following simple lemma on the left multiplication:

Lemma 6.1 *Let λ be a partition of n lying in the (k,k)-hook and let $f_a : W \to W$ be the left multiplication $f_a(x) = ax$.*

1. *If $a \in Z(FS_n)$ then the restriction of f_a to the $S^*(V,n)$-Weyl module $e_{T_\lambda}W$ is an endomorphism.*

2. *If $a \in Z(FA_n)$ then the restriction to the $S^*_A(V,n)$-Weyl module L_{T_λ} satisfies $aL_{T_\lambda} \subseteq L_{T_\lambda}$. Let α be an eigenvalue of the restriction f_a to L_{T_λ} and let L_α be the corresponding eigenspace. Then L_α is an $S^*_A(V,n)$-submodule of L_{T_λ}.*

Proof. We prove only the second part. It follows from the definition of $S^*_A(V,n)$ that $Z(FA_n) \subseteq S^*_A(V,n)$, hence, since L_{T_λ} is a left $S^*_A(V,n)$-module, the restriction of f_a to L_{T_λ} is $f_a : L_{T_\lambda} \to L_{T_\lambda}$. Also $sa = as$ for any $s \in S^*_A(V,n)$, which implies for an element $x \in L_{T_\lambda}$ that $f_a(sx) = asx = sax = \alpha(sx)$. ∎

Since F denotes the field of complex numbers, such f_a has precisely one eigenvalue on the $S^*(V,n)$-module $e_{T_\lambda}W$ and at least one eigenvalue on the $S^*_A(V,n)$-module L_{T_λ}. The irreducibility of L_{T_λ} when $\lambda \neq \lambda'$, and its being the direct sum of two irreducibles when $\lambda = \lambda'$, imply that the linear map f_a has precisely one eigenvalue for λ non-self-conjugate and at most two distinct eigenvalues in the other case.

6.2. It is well-known that the class sums of a finite group form a basis for the center of the group algebra. We proceed with describing the map f_a operating on the Weyl modules of $S^*(V,n)$ and $S^*_A(V,n)$ where a is a class sum for either S_n or A_n. We fix the following notation: For an eigenvalue α of f_a we denote the eigenspace of f_a with respect to α by L_α, and moreover, we define $\alpha(\lambda,\mu)$ by

$$\alpha(\lambda,\mu) = \frac{\mid C_\mu \mid \chi^\lambda(\mu)}{\chi^\lambda(1)}. \tag{6}$$

In this equation the involved characters are S_n-characters and the conjugacy class is an S_n-conjugacy class.

Theorem 6.2 (The S_n-case) *The map $f_{c_\mu} : e_{T_\lambda}W \to e_{T_\lambda}W$ is given by $f_{c_\mu}(x) = \alpha x$ and $e_{T_\lambda}W = L_\alpha$ for $\alpha = \alpha(\lambda,\mu)$.*

Proof. Denote by e_ν the minimal central idempotent of S_n which is the identity of I_ν. By the orthogonality relations for the irreducible characters of S_n, we can express (see [3], page 237) the class sum c_μ of the conjugacy class C_μ by

$$c_\mu = \sum_\nu \frac{\mid C_\mu \mid \chi^\nu(\mu)}{\chi^\nu(1)} e_\nu. \tag{7}$$

As $e_{T_\lambda} \in I_\lambda$ we obtain $e_\nu e_{T_\lambda} = 0$ for $\nu \neq \lambda$ and $e_\nu e_{T_\lambda} = e_{T_\lambda}$ for $\nu = \lambda$. This implies $c_\mu e_{T_\lambda} = \alpha e_{T_\lambda}$ with $\alpha = \alpha(\lambda, \mu)$ defined as in Equation (6). Hence, using Lemma 6.1, we obtain $f_{c_\mu}(x) = \alpha(\lambda, \mu)x$ for any element $x \in e_{T_\lambda} W$. ∎

Theorem 6.3 (The A_n-case) *Consider the map $f_{c_\mu} : L_{T_\lambda} \to L_{T_\lambda}$ for an even partition μ of n.*

1. *Assume that the S_n-conjugacy class C_μ does not split in A_n. Then $f_{c_\mu}(x) = \alpha x$ for $x \in L_{T_\lambda}$ and $L_{T_\lambda} = L_\alpha$ for $\alpha = \alpha(\lambda, \mu)$.*
2. *Assume that the conjugacy class C_μ splits in A_n. Let either λ be non-self-conjugate or let λ be self-conjugate such that $\mu \neq h(\lambda)$ (see Section 2.3). Then for $x \in L_{T_\lambda}$ we have $f_{c_{\mu^\pm}}(x) = \alpha x$ and $L_{T_\lambda} = L_\alpha$ for $\alpha = \frac{1}{2}\alpha(\lambda, \mu)$.*

Proof.

1. By Theorem 6.2, $f_{c_\mu}(x) = \alpha(\lambda, \mu)x$ for any $x \in e_{T_\lambda} W$ and where $\alpha(\lambda, \mu)$ is given as in Equation (6). Next, let $x \in \mathcal{T} e_{T_\lambda} W = e'_{T_\lambda} W$ (see Section 5.1) where $e'_{T_\lambda} \in I_{\lambda'}$. Similar as before, by applying Equation (7), we obtain that $f_{c_\mu}(x) = \alpha(\lambda', \mu)x$. Since μ is even, $\alpha(\lambda, \mu) = \alpha(\lambda', \mu)$.
2. We give the proof only for the class sum c_{μ^+} (a similar proof holds for c_{μ^-}).

   (a) Suppose first that λ is non-self-conjugate. Rewriting Equation (7) for A_n we obtain

   $$c_{\mu^+} = \frac{\mid C_{\mu^+} \mid \chi^\lambda(\mu^+)}{\chi^\lambda(1)} e_\lambda^{A_n} + \text{ other summands}$$

   where the other summands involve $e_\nu^{A_n}$ with $\nu \neq \lambda, \lambda'$. Since λ is non-self-conjugate and by Section 2.4 we obtain $e_\lambda^{A_n} e_{T_\lambda} = e_\lambda^{S_n} e_{T_\lambda} = e_{T_\lambda}$. Thus for $x = e_{T_\lambda} w \in e_{T_\lambda} W$ we have

   $$c_{\mu^+} x \quad = \quad c_{\mu^+} e_{T_\lambda} w = \frac{\mid C_{\mu^+} \mid \chi^\lambda(\mu^+)}{\chi^\lambda(1)} e_{T_\lambda} w = \frac{1}{2}\alpha(\lambda, \mu) e_{T_\lambda} w$$

   where the last equation follows since $\chi^\lambda(\mu^+) = \frac{1}{2}\chi^\lambda(\mu)$ (see Section 2.3). If $x \in \mathcal{T} e_{T_\lambda} W = e'_{T_\lambda} W$ the argument is similar and the above statement for a non-self-conjugate partitions λ follows.

(b) Suppose next that λ is self-conjugate with $\mu \neq h(\lambda)$. We have $e_\lambda^{S_n} = e_\lambda^+ + e_\lambda^-$ where $e^\pm$ are minimal central idempotents in A_n and the irreducible S_n-character splits into two irreducible A_n-chracters $\chi^\lambda = \chi^{\lambda+} + \chi^{\lambda-}$. Then

$$c_{\mu^+} = \frac{\mid C_{\mu^+} \mid \chi^{\lambda^+}(\mu^+)}{\chi^{\lambda^+}(1)} e_\lambda^+ + \frac{\mid C_{\mu^+} \mid \chi^{\lambda^-}(\mu^+)}{\chi^{\lambda^-}(1)} e_\lambda^- + \text{ other summands.}$$

Since $\mu \neq h(\lambda)$ and $\chi^{\lambda^\pm}(\mu^\pm) = \frac{1}{2}\chi^\lambda(\mu)$ it follows that

$$c_{\mu^+} = \frac{1}{2}\alpha(\lambda,\mu)e_\lambda^{S_n} + \text{other summands}$$

and the proof follows by continuing as above. ∎

6.3. So far we described the conditions on λ and μ such that the map $f_{c_\mu} : L_{T_\lambda} \to L_{T_\lambda}$ has one eigenvalue and is given by a scalar multiple of the identity matrix. In case λ is self-conjugate with $\mu = h(\lambda)$ we obtain two eigenvalues for $f_{c_\mu^\pm}$ and this case is considered in this subsection. Define for $\epsilon = \pm$

$$\alpha^\epsilon = \alpha^\epsilon(\lambda, h(\lambda)) = \frac{\mid C_{h(\lambda)} \mid}{\chi^\lambda(1)} \cdot \chi^{\lambda^\epsilon}(h(\lambda)^+). \tag{8}$$

Theorem 6.4 (The A_n-case) *The left multiplications $f_{c_{h(\lambda)^\pm}} : L_{T_\lambda} \to L_{T_\lambda}$ both have the two eigenvalues α^ϵ defined in Equation (8). They are given by*

$$f_{c_{h(\lambda)^+}}(x) = \begin{cases} \alpha^+ x & \text{for } x \in L_{T_\lambda}^+ \\ \alpha^- x & \text{for } x \in L_{T_\lambda}^- \end{cases} \quad \text{and} \quad f_{c_{h(\lambda)^-}}(x) = \begin{cases} \alpha^+ x & \text{for } x \in L_{T_\lambda}^- \\ \alpha^- x & \text{for } x \in L_{T_\lambda}^+ \end{cases}$$

where in the first case $L_{T_\lambda}^\epsilon = L_{\alpha^\epsilon}$ and in the second case $L_{T_\lambda}^\epsilon = L_{\alpha^{-\epsilon}}$.

Proof. Note that e_λ^+ annihilates elements in $L_{T_\lambda}^-$ and acts as identity on elements in $L_{T_\lambda}^+$. Similarly, e_λ^- annihilates $L_{T_\lambda}^+$ and acts as identity on $L_{T_\lambda}^-$. Moreover,

$$c_{h(\lambda)^+} = \frac{\mid C_{h(\lambda)^+} \mid \chi^{\lambda^+}(h(\lambda)^+)}{\chi^{\lambda^+}(1)} e_\lambda^+ + \frac{\mid C_{h(\lambda)^+} \mid \chi^{\lambda^-}(h(\lambda)^+)}{\chi^{\lambda^-}(1)} e_\lambda^- + \text{ other summands.}$$

With this it follows that $c_{h(\lambda)^+}x = \alpha^\epsilon x$ for an element $x \in L_{T_\lambda}^\epsilon$. A similar argument holds for $c_{h(\lambda)^-}$ using that $\chi^{\lambda+}(h(\lambda)^+) = \chi^{\lambda-}(h(\lambda)^-)$ (see Section 2.3). ∎

7 Bases for the Weyl modules

In this section we discuss bases of Weyl modules for the Schur algebra of the alternating groups. Theorem 7.1 describes a basis for every L_{T_λ}, Theorem 7.3 describes another such basis when n is large enough. When λ is self-conjugate, Theorem 7.2 describes bases for $L_{T_\lambda}^+$ and $L_{T_\lambda}^-$. These bases allow us to attach to each irreducible module a certain super symmetric function. Unlike in the S_n-case, in the case of a self-conjugate partition these functions do not characterize the corresponding modules.

7.1. Based on Corollary 3.18 of [1] we now describe an explicit basis for the Weyl module L_{T_λ}. Let $t_1, \ldots, t_k \in V_0$ and $u_1, \ldots, u_k \in V_1$ be a fixed basis of $V_0 \oplus V_1$. Recall the definition of *monomial* given in Section 4.2. Then a monomial $w \in W$ is called *semi-standard* if its corresponding (k,k)-tableau is semi-standard. Call a monomial $w \in W$ *dual-semi-standard* if T_λ identifies w with a tableau T which becomes semi-standard upon exchanging the t's and the u's (that is after applying $\mathcal{T}$, see Section 5.1). Denote the dual of a semi-standard tableau w by $\bar{w}$.

Theorem 7.1 *Let λ be a partition of n lying in a (k,k)-hook and let T_λ be a tableau of shape λ. Then L_{T_λ} has basis*

$$\{e_{T_\lambda} w, e'_{T_\lambda} \bar{w} \mid w \in W \text{ is } (k,k)\text{-semi-standard of shape } \lambda\}.$$

Proof. By Corollary 3.18 of [1], $\{e_{T_\lambda} w \mid w \text{ is } (k,k)\text{-semi-standard}\}$ is a basis of the vecor space $e_{T_\lambda} W$. Now, up to a $\pm$ sign, $\mathcal{T}$ interchanges between the t's and the u's in a monomial, hence the vector space isomorphism $e_{T_\lambda} W \cong \mathcal{T} e_{T_\lambda} W = e'_{T_\lambda} \mathcal{T} W = e'_{T_\lambda} W$ implies that $\{e'_{T_\lambda} w \mid w \text{ is } (k,k)\text{-dual-semi-standard }\}$ is a basis of the vector space $\mathcal{T} e_{T_\lambda} W = e'_{T_\lambda} W$. The proof follows from the first part of 5.3. ∎

For a self-conjugate partition λ the Weyl module splits by Theorem 5.3 into two non-isomorphic irreducible components. Explicit bases for these irreducible components are given in the following theorem.

Theorem 7.2 *Let λ be a self-conjugate partition, T_λ a tableau of shape λ and $e^\epsilon_\lambda \in FA_n$ with $\epsilon = \pm$ be the corresponding minimal central idempotents. Then a basis of $L^\epsilon_{T_\lambda}$ is given by*

$$\{e^\epsilon_\lambda e_{T_\lambda} w \mid w \in W \text{ is } (k,k)\text{-semi-standard of shape } \lambda\}.$$

Proof. We show the theorem for $\epsilon = +$. Suppose the tableau has first row starting with the entries a and b. Denote by τ the transposition $\tau = (a,b)$. Then $\tau \in R_{T_\lambda}$ and hence $\tau R^+_{T_\lambda} = R^+_{T_\lambda}$ and $\tau e_{T_\lambda} = e_{T_\lambda}$. It is well-known that for any odd permutation η in S_n we have $e^-_\lambda = \eta e^+_\lambda \eta^{-1}$. See for example [7], 2.5.7. In particular, we therefore have $e^-_\lambda = \tau e^+_\lambda \tau$ which also can be seen directly from the expressions for e^ϵ, see Section 2.4. Let X denote the set of the (k,k)-semi-standard tableaux of shape λ. Assume that

$$\sum_{w \in X} \alpha_w e^+_\lambda e_{T_\lambda} w = 0. \tag{9}$$

We calculate from this assumption the value for the same equation with e^+_λ replaced by e^-_λ:

$$\sum_{w \in X} \alpha_w e^-_\lambda e_{T_\lambda} w = \sum_{w \in X} \alpha_w \tau e^+_\lambda (\tau e_{T_\lambda}) w = \tau \sum_{w \in X} \alpha_w e^+_\lambda e_{T_\lambda} w = 0.$$

Hence Equation (9) implies that also

$$\sum_{w \in X} \alpha_w e_\lambda^- e_{T_\lambda} w = 0. \tag{10}$$

Recall that $e_\lambda^+ + e_\lambda^- = e_\lambda$ and $e_\lambda e_{T_\lambda} = e_{T_\lambda}$. Thus adding Equations (9) and (10) we obtain $\sum_{w \in X} \alpha_w e_{T_\lambda} w = 0$. By Theorem 7.1 this implies that all $\alpha_w = 0$. ■

7.2. The elements g_{T_λ}, introduced and discussed in Sections 3 and 4 give the decomposition of the two-sided ideals of FA_n into minimal left ideals (see Theorem 3.2) and therefore play a similar role in the A_n-theory as the elements e_{T_λ} in the S_n-theory. As a basis for the super Weyl modules in the S_n-case is described by the elements e_{T_λ} it suggests to look for a basis of the super Weyl modules L_{T_λ} in the A_n-case described by the g_{T_λ}.

Theorem 7.3 *Let $n > 7k^2$ and let λ be a partition of n lying in a (k,k)-hook and T_λ a tableau of shape λ. Then a basis of the super Weyl module L_{T_λ} of $S_A^*(V,n)$ is given by*

$$\{g_{T_\lambda} w, g_{T_\lambda} \bar{w} \mid w \in W \text{ is } (k,k)\text{-semi-standard of shape } \lambda\}.$$

Note that in the case $\lambda = (3,1)$ and $k = 1$ the corresponding set is not a basis for L_{T_λ}.

Proof. Let X denote the set of the (k,k)-semi-standard tableaux of shape λ. By Corollary 3.15 of [1] the elements $\{e_{T_\lambda} w \mid w \in X\}$ are linearly independent. We show first that the elements $\{g_{T_\lambda} w \mid w \in X\}$ are linearly independent.

The case $\lambda = (1^n)$ is easy: Let $w \in X$. Since $n > 2k$ and each of $t_1 \ldots, t_k$ appears at most once in w, there exist a u_j which appears at least twice in w. Thus $C_{T_\lambda}^+ w = 0$ and hence $g_{T_\lambda} w = e_{T_\lambda} w$, which clearly implies the above linear independence.

Assume next that $\lambda \neq (1^n)$, that is $\lambda_1 \geq 2$, and let $(a,b) \in R_{T_\lambda}$. Then $(1 + (a,b)) g_{T_\lambda} = 2 e_{T_\lambda}$ (see Lemma 3.3) and so $(1 + (a,b)) g_{T_\lambda} w = 2 e_{T_\lambda} w$ and we deduce that the elements $\{g_{T_\lambda} w \mid w \in X\}$ are also linearly independent. Deduce now that this holds also for the elements in the set $\{g_{T_\lambda} \bar{w} \mid w \in X\}$: Indeed, here (see Section 5.1)

$$e_{\lambda'}^{S_n} g_{T_\lambda} \bar{w} = e'_{T_\lambda} \bar{w} = \pm e'_{T_\lambda} \mathcal{T} w = \pm \mathcal{T} e_{T_\lambda} w,$$

and independance follows since $\mathcal{T} : e_{T_\lambda} W \to \mathcal{T} e_{T_\lambda} W$ is a vector space isomorphism.

Recall that each g_{T_λ} is a signed sum of the permutations pq where $p \in R_{T_\lambda}$, $q \in C_{T_\lambda}$ (and $pq \in A_n$). Thus a linear combination of elements of the set $\{g_{T_\lambda} w \mid w \in X\}$ is a linear combination of such monomials pqw. Similarly for $\{g_{T_\lambda} \bar{w} \mid w \in X\}$. Let S denote the linear span of $\{g_{T_\lambda} w \mid w \in X\}$ and $\bar{S}$ that of $\{g_{T_\lambda} \bar{w} \mid w \in X\}$. The theorem will be proved once we show that $S \cap \bar{S} = 0$, and this will follow once we show that no monomial of the form $p_1 q_1 w_1$ can equal a monomial

$p_2q_2\bar{w}_2$ where $p_1, p_2 \in R_{T_\lambda}$ and $q_1, q_2 \in C_{T_\lambda}$ and $w_1, w_2 \in X$. For a contradiction we assume the opposite, that is assume that $p_1q_1w_1 = \pm p_2q_2\bar{w}_2$.

As usual, the fixed tableau T_λ identifies a monomial w with its (k,k)-tableau. Suppose μ denotes the first k rows of a tableau of shape λ. Then we call this the *μ-part* of this tableau. Consider the μ-part of $p_1q_1w_1$. First, the number of u_i's in that μ-part of w_1 is less than or equal to k^2 since w_1 is semi-standard. Now, the column permutation q_1 might bring at most k^2 new u_i's into that μ-part, while the row permutation p_1 does not change the amount of u_i's in that μ-part. It follows that the number of u_i's in the μ-part of $p_1q_1w_1$ is less than or equal to $2k^2$: $\deg_u[(p_1q_1w_1)_\mu] \le 2k^2$. The total t_i's degree of the μ-part therefore satisfies $\deg_t[(p_1q_1w_1)_\mu] \ge |\mu| - 2k^2$.

Now, in a dual semi-standard tableau, the role of the t_i's and the u_i's is reversed. Hence, by the same argument, $\deg_t[(p_2q_2\bar{w}_2)_\mu] \le 2k^2$. It follows that if $|\mu| > 4k^2$ then $\deg_t[(p_1q_1w_1)_\mu] > \deg_t[(p_2q_2\bar{w}_2)_\mu]$ and hence $p_1q_1w_1 \ne p_2q_2\bar{w}_2$.

Next, denote the first k columns of λ by ν. By the same argument, if $|\nu| > 4k^2$ then $p_1q_1w_1$ cannot be of the form $p_2q_2\bar{w}_2$. Finally, if $|\lambda| > 7k^2$ then either $|\mu| > 4k^2$ or $|\nu| > 4k^2$, and the proof of the theorem follows. ∎

7.3. *The super symmetric functions.* We follow [1], Section 6, whose basic idea is due to I. Schur. Let $P = diag(x_1, \dots, x_k, y_1, \dots, y_k)$ be an endomorphism in End(V). Given an irreducible $S_A^*(V,n)$-submodule M of W, one calculates the trace of $P^{\otimes n}$ on M to obtain a super-symmetric function $S_M(x;y) = S_M(x_1, \dots, x_k; y_1, \dots, y_k)$.

1. Let λ be non-self-conjugate and $M = L_{T_\lambda}$. By Theorem 7.1 and by Lemma 6.9 in [1] it follows that $S_{L_{T_\lambda}}(x;y) = HS_\lambda(x;y) + HS_\lambda(y;x)$. Since $HS_\lambda(y;x) = HS_{\lambda'}(x;y)$ we get
$$S_{L_{T_\lambda}}(x;y) = HS_\lambda(x;y) + HS_{\lambda'}(x;y).$$
2. Let λ be self-conjugate. For either $M = L^+_{T_\lambda}$ or $M = L^-_{T_\lambda}$ Theorem 7.3 and again Lemma 6.9 in [1] imply that
$$S_{L^+_{T_\lambda}}(x;y) = S_{L^-_{T_\lambda}}(x;y) = HS_\lambda(x;y).$$

Remark. In the S_n-case the functions $HS_\lambda(x;y)$ characterize the irreducible $S^*(V,n)$-submodules of W (see Section 6 in [1]). In the A_n-case this however is not true: it is still true for the non-self-conjugate partitions, but for a self-conjugate partition λ, to the two irreducible and *non*-isomorphic submoduls $L^+_{T_\lambda}$ and $L^-_{T_\lambda}$ corresponds the same super symmetric function $HS_\lambda(x;y)$.

A Appendix written together with T. Seeman.

In this Appendix we give two proofs of Theorem 4.1. We recall, this theorem says:

Theorem A.1 *Let λ be a self-conjugate partition lying in a (k,k)-hook and assume $\lambda_1 \geq 2k+1$ (this in particular holds if $n > 3k^2$). Let T_λ be a tableau of shape λ with corresponding elements $e = e_{T_\lambda}$ and $e' = e'_{T_\lambda}$. Then $ee' = e'e = 0$ and hence g_{T_λ} is a semi-idempotent.*

First proof. We show that $ee' = 0$. Here we employ the classical action of S_n on $V^{\otimes n}$, i.e $V = V_0$ and $V_1 = 0$. Assume dim $V \geq n$ and let $M \in V^{\otimes n}$ denote the monomial $M = x_1 \otimes \cdots \otimes x_n \equiv x_1 \cdots x_n$, where $x_1, \ldots, x_n \in V$ are linearly independent. For simplicity we denote $\varphi_\sigma(M) = \sigma M$. As vector spaces, $FS_n \cong \mathrm{span}\{\sigma M \mid \sigma \in S_n\}$; hence for $a \in FS_n$, $a = 0$ if and only if $aM = 0$. As was remarked in Section 4.1 to prove that $ee' = 0$, we can choose any T_λ, then prove that $e_{T_\lambda} e'_{T_\lambda} = 0$. Here we choose

$$T_\lambda = \begin{array}{llll} 1 & h_1+1 & \cdot & \cdot \quad \cdot \\ 2 & h_1+2 & & \cdot \\ \vdots & \quad\vdots & \cdot & \\ \cdot & h_1+h_2 & & \\ h_1 & & & \end{array}$$

where $\lambda'_j = h_j$, $j = 1, 2, \ldots$ Now, T_λ identifies monomials with tableaux. Consider the monomials $r = x_1^{\lambda'_1} x_2^{\lambda'_2} \cdots$ and the above M. Rename $x_1, \ldots, x_n$ by $x_{i,j}$'s so that M and r are identified with the following tableaux:

$$M = \begin{array}{llll} x_{1,1} & x_{2,1} & \cdot & \cdot \quad \cdot \\ x_{1,2} & x_{2,2} & & \cdot \\ \cdot & \cdot & \cdot & \\ \cdot & x_{2,h_2} & & \\ x_{1,h_1} & & & \end{array} \quad \text{and} \quad r = \begin{array}{llll} x_1 & x_2 & \cdot & \cdot \quad \cdot \\ x_1 & x_2 & & \cdot \\ \cdot & \cdot & \cdot & \\ \cdot & x_2 & & \\ x_1 & & & \end{array}$$

Linearization. Let $m \in V^{\otimes n}$ be a monomial in the variables $x_1, \ldots, x_k$ of degree $d_i = \partial_{x_i}(m)$ in x_i, where $i = 1, \ldots, k$. Write $m = m_1 x_i m_2 x_i m_3 \cdots m_{d_i} x_i m_{d_i+1}$ where $m_1, \ldots, m_{d+1}$ do not involve x_i. Then the operator L_{x_i} (linearization of x_i) is defined by linearity and by:

$$L_{x_i}(m) = \sum_{\sigma \in S_{d_i}} m_1 x_{i,\sigma(1)} m_2 x_{i,\sigma(2)} \cdots x_{i,\sigma(d_i)} m_{d_i+1}.$$

It is easy to verify that for $i \neq j$ we have $L_{x_i} L_{x_j} = L_{x_j} L_{x_i}$. Now define the linearization operator L by linearity and by

$$L(m) = L_{x_1} \cdots L_{x_k}(m).$$

For example, using the identification of monomials $m \in V^{\otimes n}$ and tableaux, induced by the above T_λ, the above monomials r and M are related by $L(r) = C^+ M$ where $C = C_{T_\lambda}$. Note the following basic property: Let $a \in FS_n$ then, as operators on $V^{\otimes n}$, $aL = La$. (Indeed we may assume that $a = \sigma \in S_n$ then verify, for a given i, that $\sigma L_{x_i} = L_{x_i} \sigma$, which is straightforward, since σ permutes the places in a monomial.) In Lemma A.2 we deduce the basic tool for proving Theorem 4.1. The proof of Theorem 4.1 then follows from Lemma A.3. ∎

Lemma A.2 *Recall that* $ee' = R^+C^-R^-C^+$ *and let the tableau* r *of shape* λ *be defined as above. If* $R^+C^-R^-r = 0$ *then* $ee' = 0$ *.*

Proof. Firstly, $ee' = 0$ if and only if $ee'M = 0$. Since by above $C^+M = L(r)$ and $(R^+C^-R^-)L = L(R^+C^-R^-)$ we deduce that $ee'M = R^+C^-R^-C^+M = (R^+C^-R^-)L(r) = L(R^+C^-R^-r)$, and the proof follows. ∎

Lemma A.3 *Assume that* $\lambda_1 \geq 2k+1$ *and let* r *be as above. Then* $C^-R^-r = 0$.

Proof. Identify r with the tableau

$$r = \begin{matrix} A_1 & A_2 \\ A_3 & \end{matrix} \qquad \text{where} \qquad A_3 = \begin{matrix} 1 & 2 & \cdot & \cdot & k \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 2 & \cdot & \cdot & k \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & & \\ 1 & & & & \end{matrix}$$

and the number of ones in the first column of A_3 is at least $k+1$. Then $R^-r = \sum \pm m_i$ where

$$m_i = \begin{matrix} A_{i,1} & A_{i,2} \\ A_{i,3} & \end{matrix}$$

Since each row in $A_{i,3}$ is a permutation of $1, \ldots, k$, each row contains a one; hence the number of ones in $A_{i,3}$ is greater than or equal to $k+1$. Since the number of columns in $A_{i,3}$ is $\leq k$, it follows that there is at least one column in $A_{i,3}$ – and therefore also in m_i – with at least *two* ones. This implies that $C^-m_i = 0$.∎

Second proof. Let $R = R_{T_\lambda}$, $C = C_{T_\lambda}$, and recall that $ee' = R^+C^-R^-C^+$. We show that $ee' = 0$ by proving (see the beginning of Section 4.1) that

$$C^-R^-C^+T_\lambda = \sum_{\substack{q,\overline{q}\in C,\\ p\in R}} \operatorname{sgn}(qp)qp\overline{q}\ T_\lambda = \sum_{p\in R} \operatorname{sgn}(p) \sum_{q,\overline{q}\in C} \operatorname{sgn}(q)qp\overline{q}\ T_\lambda = 0.$$

To do so it suffices to prove the following: Given $p \in R$, $q, \overline{q} \in C$, let $S = \{(q_1, \overline{q}_1), \ldots, (q_m, \overline{q}_m)\}$ be the set of all pairs $q_i, \overline{q}_i \in C$ satisfying $q_ip\overline{q}_iT_\lambda = qp\overline{q}T_\lambda$. Then $\sum_{i=1}^m \operatorname{sgn}(q_i) = 0$.

Let T_1 be the first k rows of T_λ, and T_2 the remaining rows. It follows from the assumptions on λ that T_2 consists of at most k columns, with the leftmost column of T_2 being of length $k+1$ or greater.

A pair of elements is said to be *co-column* if they appear in the same column in a given tableau. We claim that for every $i \in \{1, \ldots, m\}$, there exists a pair of elements which are co-column in both $\overline{q}_iT_\lambda$ and $p\overline{q}_iT_\lambda$. Define $T_1(\overline{q}_i)$ to be the first k rows of $\overline{q}_iT_\lambda$, and $T_2(\overline{q}_i)$ the remaining rows, and similarly define $T_1(p\overline{q}_i)$ and $T_2(p\overline{q}_i)$. Consider the first $k+1$ elements in the leftmost column of $T_2(\overline{q}_i)$. Since $p \in R$, each of these elements appears in $T_2(p\overline{q}_i)$. But $T_2(p\overline{q}_i)$ consists of at most k columns, hence it follows by the pigeonhole principle that at least two of these $k+1$ elements must also be co-column in $T_2(p\overline{q}_i)$.

We can therefore let a and b be a pair of elements which are co-column in both $\overline{q}_1 T_\lambda$ and $p\overline{q}_1 T_\lambda$, with a being higher than b in $\overline{q}_1 T_\lambda$. Let x be the number of pairs $(q_i, \overline{q}_i)$ such that a and b are co-column in both $\overline{q}_i T_\lambda$ and $p\overline{q}_i T_\lambda$, with a higher than b in $\overline{q}_i T_\lambda$. Similarly, let y be the number of pairs $(q_i, \overline{q}_i)$ such that a and b are co-column in both $\overline{q}_i T_\lambda$ and $p\overline{q}_i T_\lambda$, but with b higher than a in $\overline{q}_i T_\lambda$. Without loss of generality, the above set S is ordered so that a and b are co-column in both $\overline{q}_i T_\lambda$ and $p\overline{q}_i T_\lambda$ for all $i \in \{1, \ldots, x+y\}$. Furthermore, if $i \in \{1, \ldots, x\}$, then a is higher than b in $\overline{q}_i T_\lambda$; if $i \in \{x+1, \ldots, x+y\}$, however, then b is higher than a. We claim that $y = x$, and $\sum_{i=1}^{x+y} \operatorname{sgn}(q_i) = 0$.

Given $(q_i, \overline{q}_i)$, $i \leq x$, let $\tau_i \in C$ be the permutation that, when applied to $\overline{q}_i T_\lambda$, swaps a and b and leaves all other entries unchanged. Then $\overline{q}_i T_\lambda$ and $\tau_i \overline{q}_i T_\lambda$ are identical except with a and b in opposite locations; hence $p\overline{q}_i T_\lambda$ and $p\tau_i \overline{q}_i T_\lambda$ are also identical except with a and b in opposite locations. It follows that a and b are co-column in both $\tau_i \overline{q}_i T_\lambda$ and $p\tau_i \overline{q}_i T_\lambda$, with b higher than a in $\tau_i \overline{q}_i T_\lambda$.

Now, given such a pair $(q_i, \overline{q}_i)$ with τ_i as above, let σ_i be the column permutation that swaps a and b in $p\overline{q}_i T_\lambda$ and leaves all other entries unchanged. Thus $p\overline{q}_i T_\lambda = \sigma_i p \tau_i \overline{q}_i T_\lambda$, hence $(q_i \sigma_i) p (\tau_i \overline{q}_i) T_\lambda = q_i p \overline{q}_i T_\lambda = qp\overline{q} T_\lambda$. Note that $\sigma_i, \tau_i \in C$ implies $q_i \sigma_i, \tau_i \overline{q}_i \in C$, hence $(q_i \sigma_i, \tau_i \overline{q}_i) \in S$. Moreover, since a and b are co-column in both $\tau_i \overline{q}_i T_\lambda$ and $p\tau_i \overline{q}_i T_\lambda$, with b higher than a in $\tau_i \overline{q}_i T_\lambda$, $(q_i \sigma_i, \tau_i \overline{q}_i) \neq (q_j, \overline{q}_j)$ for all $1 \leq j \leq x$, hence we can reorder S such that $(q_{x+i}, \overline{q}_{x+i}) = (q_i \sigma_i, \tau_i \overline{q}_i)$.

For all $i \leq x$, both τ_i and σ_i are column permutations which swap two elements in a tableau, leaving all other elements fixed. Thus $\tau_i^2 = \sigma_i^2 = 1$, so $(q_i, \overline{q}_i) = (q_{x+i} \sigma_i, \tau_i \overline{q}_{x+i})$. This defines a bijection between the sets $\{(q_1, \overline{q}_1), \ldots, (q_x, \overline{q}_x)\}$ and $\{(q_{x+1}, \overline{q}_{x+1}), \ldots, (q_{x+y}, \overline{q}_{x+y})\}$, which implies that $y = x$. Moreover, it also implies that $\operatorname{sgn}(q_{x+i} \sigma_i) = \operatorname{sgn}(q_i)$. Thus $\sum_{i=1}^{x+y} \operatorname{sgn}(q_i) = \sum_{i=1}^{2x} \operatorname{sgn}(q_i) = 0$.

Recall that $S = \{(q_1, \overline{q}_1), \ldots, (q_m, \overline{q}_m)\}$. Thus if $x + y = m$, then we are done. If $x + y < m$, however, then repeat this procedure on $\{(q_{x+y+1}, \overline{q}_{x+y+1}), \ldots, (q_m, \overline{q}_m)\}$. ■

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