A-Codimensions and A-Cocharacters

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Abstract

The codimensions and the cocharacters of a p.i. algebra arise from the group algebra FS_n of the symmetric group S_n, where F is an algebraically closed field of characteristic zero. The subalgebra FA_n of the alternating subgroup A_n gives rise to the corresponding A-codimensions and A-cocharacters. Some general properties of these invariants are studied. In particular, the A-codimensions and the A-cocharacters of the infinite dimensional Grassmann (exterior) algebra E are calculated.

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1 Introduction

Let $A_n \subseteq S_n$ denote the corresponding alternating and symmetric groups, and let F be a field of characteristic zero. We will be interested in A_n-representations, hence it is crucial to assume here that F is algebraically closed. The multi-linear polynomials of degree n

$$V_n = \{ \sum_{\sigma \in S_n} a_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)} \mid a_\sigma \in F\}$$

are identified with the group algebra FS_n via

$$\sigma \equiv x_{\sigma(1)} \cdots x_{\sigma(n)}.$$

This identification $V_n \equiv FS_n$ gives rise to the cocharacters and codimensions of algebras satisfying polynomial identities *(p.i. algebras)*, and we review these invariants in the next section.

The identification $V_n \equiv FS_n$ and the inclusion $FA_n \subseteq FS_n$ lead to the subspace $FA_n \equiv V_n^A \subseteq V_n$ of the multi-linear polynomials supported by the even

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permutations $\sigma \in A_n$. We call such polynomials *even* or *A-polynomials*. This leads naturally to the introduction of the notions of the A-identities (even identities), cocharacters and codimensions of a p.i. algebra; precise definitions are given in the next section.

Note that A-identities are abundant! For example, let R be a p.i. algebra and assume R satisfies $p(x_1, x_2, \ldots, x_d) \in V_d$, then R satisfies the A-identity $p(x_1x_2, x_3x_4, \ldots, x_{2d-3}x_{2d-2}, x_{2d-1}) \in V_{2d-1}$.

Example 1.1 *Let E denote the infinite dimensional Grassmann (exterior) algebra. It is well known that E satisfies the polynomial $[[x_1, x_2], x_3]$ where $[a, b] = ab - ba$, and this identity is of minimal degree. Thus, E satisfies, for example, the A-identity $[[x_1x_2, x_3x_4], x_5]$, namely $[[b_1b_2, b_3b_4], b_5] = 0$ for any $b_1, \ldots, b_5 \in E$. In this case there are no A-identities of degree 3, but, as we show here,*

$$p(x) = p(x_1, \ldots, x_4) = [x_1, x_2x_3]x_4 - x_4[x_1, x_3x_2]$$

is an A-identity of E, hence of minimal degree. Trivially, $p(x)$ is an A-polynomial. To show that $p(x)$ is an identity for E, substitute $x_i \to a_i \in E$ where each a_i is of degree either zero or one. If either a_1 or a_2a_3 is of degree zero then it is central in E hence $p(a) = 0$. So we may assume next that both a_1 and a_2a_3 have degree 1, and without loss of generality, that a_2 has degree one and a_3 has degree zero. Then $[a_1, a_2a_3] = [a_1, a_3a_2]$ and is central, hence again $p(a) = 0$.

Conjecture 1.2 *Let $n \geq 4$. The polynomial $[x_1, x_2x_3]x_4 - x_4[x_1, x_3x_2]$ generates the multilinear A-identities of E in the following (strong) sense: Given $\sigma \in A_n$ and $0 \leq r \leq n-4$, let $f_{r,\sigma}$ denote the following (A-)polynomial*

$$f_{r,\sigma} = p_{r,\sigma}([x_{\sigma(r+1)}, x_{\sigma(r+2)}x_{\sigma(r+3)}]x_{\sigma(r+4)} - x_{\sigma(r+4)}[x_{\sigma(r+1)}, x_{\sigma(r+3)}x_{\sigma(r+2)}])q_{r,\sigma},$$

where $p_{r,\sigma} = x_{\sigma(1)} \cdots x_{\sigma(r)}$ and $q_{r,\sigma} = x_{\sigma(r+5)} \cdots x_{\sigma(n)}$. Then any multilinear A-identity of E is a linear combination of the A-polynomials

$$\{f_{r,\sigma} \mid 0 \leq r \leq n-4, \quad \sigma \in A_n\}.$$

Computer calculations have verified Conjecture 1.2 for $n = 4, 5, 6$ and 7. Note that a similar phenomen does *not* hold for the ordinary identities of E. For example, the 48 (multilinear) polynomials

$$\{\ [[x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}]x_{\sigma(4)},\ x_{\sigma(1)}[[x_{\sigma(2)}], x_{\sigma(3)}]x_{\sigma(4)}]\}$$

span a vector space of dimension 14, while the multilinear identities of E of degree 4 form a space of dimension 16=24-8.

Example 1.3 *By the Amitsur-Levitzki theorem the $k \times k$ matrices in $M_k(F)$ satisfy the standard identity $s_{2k}[x_1, \ldots, x_{2k}]$ which implies the obvious A-identity of degree $4k-1$. It is of some interest to find the minimal degree of such identities of $M_k(F)$. It is not difficult to show that $M_k(F)$ does not satisfy any multilinear identity of degree $2k+1$. Applying computer calculations, D. Zeilberger showed that $M_2(F)$ satisfies many A-identities of degree 6. This leads us to the following conjecture.*

Conjecture 1.4 *$M_k(F)$ satisfies A-identities of degree $2k+2$.*

Let R denote a general p.i. algebra. Corresponding to the A-identities of R there are the A-cocharacters $\chi_n^A(R)$ and A-codimensions $c_n^A(R)$. In section 2 we prove some general results about $\chi_n^A(R)$ and $c_n^A(R)$, in particular in comparison with the ordinary cocharacters $\chi_n(R)$ and codimensions $c_n(R)$ of R.

In the rest of the paper we calculate the A-cocharacters $\chi_n^A(E)$ and codimensions $c_n^A(E)$ of the infinite dimensional Grassmann algebra E. In particular we prove that $c_n^A(E) = 2^{n-1} - 1$ whereas by [4], $c_n(E) = 2^{n-1}$.

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2 Generalities

2.1. *Generalities.* All modules in this paper are left modules. An explicit decomposition of FA_n into a direct sum of two-sided and of one-sided ideals is given in [2], and we apply it here. The partitions are totally ordered by the left lexicographic order. For example, $(2,2,1) \lneqq (3,2)$ and also $(2,2,1) \lneqq (3)$. As usual, given a partition λ of n (denoted $\lambda \vdash n$), let λ' denote its conjugate partition. To $\lambda \vdash n$ corresponds the S_n-irreducible character χ^λ, whose degree is denoted by $\deg \chi^\lambda = d_\lambda = f^\lambda$. If χ is any S_n-character, we denote by $\bar{\chi}$ its restriction to A_n. Thus, for any partition λ of n, $\bar{\chi}^\lambda = \bar{\chi}^{\lambda'}$. If $\lambda' \lneqq \lambda$ in the left lexicographic order, we usually (and rather arbitrarily) chose $\bar{\chi}^\lambda$ to represent that A_n-character.

The following basic facts about the characters of A_n are well known, see for example [3].

1. The case $\lambda \neq \lambda'$: Here $\bar{\chi}^\lambda$ is A_n-irreducible and $\deg \bar{\chi}^\lambda = \deg \chi^\lambda = d_\lambda$.
2. The case $\lambda = \lambda'$: Here $\bar{\chi}^\lambda = \bar{\chi}^{\lambda+} + \bar{\chi}^{\lambda-}$ where $\bar{\chi}^{\lambda\pm}$ are A_n-irreducible (and conjugate) and $\deg \bar{\chi}^{\lambda\pm} = \frac{1}{2} d_\lambda$.
3. These are all the irreducible A_n-characters. In other words, the set of the irreducible A_n-characters is

$$\{\bar{\chi}^\lambda \mid \lambda \vdash n \ , \ \lambda' \lneqq \lambda\} \cup \{\bar{\chi}^{\lambda\pm} \mid \lambda \vdash n \ , \ \lambda' = \lambda\}.$$

The following decomposition is well known:

$$FS_n = \bigoplus_{\lambda \vdash n} I_\lambda \tag{1}$$

where I_λ are the minimal two-sided ideals of FS_n. Similarly (see [2])

$$FA_n = \left[\bigoplus_{\lambda\vdash n\ ,\ \lambda'\gneqq\lambda} \bar{I}_\lambda \right] \oplus \left[\bigoplus_{\lambda\vdash n\ ,\ \lambda'=\lambda} (\bar{I}_\lambda^+ \oplus \bar{I}_\lambda^-) \right] \tag{2}$$

where the above $\bar{I}_\lambda$'s and $\bar{I}_\lambda^\pm$'s are the minimal two-sided ideals of FA_n – with the corresponding irreducible A_n-characters.

1. The case $\lambda \neq \lambda'$. Here $\bar{I}_\lambda \subset I_\lambda \oplus I_{\lambda'}$ and $\bar{I}_\lambda \cong I_\lambda \cong I_{\lambda'}$ as left FA_n-modules. Also, if $J_\lambda \subseteq I_\lambda$ is a minimal left ideal (in FS_n) then it is an FA_n-irreducible module, and $\dim J_\lambda = \deg \chi^\lambda = d_\lambda$.
2. The case $\lambda = \lambda'$. Here $\bar{I}_\lambda^+ \oplus \bar{I}_\lambda^- \subset I_\lambda$ and $\dim \bar{I}_\lambda^+ = \dim \bar{I}_\lambda^- = \frac{1}{4}\dim I_\lambda$. If $J_\lambda \subseteq I_\lambda$ is a minimal left ideal in FS_n then, as FA_n-modules, $J_\lambda \cong \bar{J}_\lambda^+ \oplus \bar{J}_\lambda^-$, with both $\bar{J}_\lambda^+$ and $\bar{J}_\lambda^-$ being FA_n-irreducible modules. Also $\dim \bar{J}_\lambda^\pm = \deg \bar{\chi}^{\lambda\pm} = \frac{1}{2}d_\lambda$.

2.2. *The A-cocharacters and A-codimensions.* Let R be a p.i. algebra with $Q = \mathrm{Id}(R) \subseteq F\langle x\rangle$ its ideal of identities. Recall the definition of its S_n-cocharacters $\chi_n(R)$ and codimensions $c_n(R)$: $FS_n \cap Q$ is a left ideal, hence a left submodule of FS_n, and $\chi_n(R)$ is the S_n-character of the quotient module:

$$\chi_n(R) \quad = \quad \chi_{S_n}\Big(\frac{FS_n}{FS_n \cap Q}\Big).$$

The *codimensions* are the corresponding degrees:

$$c_n(R) \quad = \quad \deg \chi_n(R) = \dim \Big(\frac{FS_n}{FS_n \cap Q}\Big).$$

Thus

$$\chi_n(R) = \sum_{\lambda\vdash n} m_\lambda(R)\chi^\lambda \qquad \text{and} \qquad c_n(R) = \sum_{\lambda\vdash n} m_\lambda(R) d_\lambda \tag{3}$$

where $m_\lambda(R)$ denotes the multiplicity of the irreducible character χ^λ occuring as constituent in $\chi_n(R)$. Replacing S_n by A_n leads to the following definition.

Definition 2.1 *Let R be a p.i. algebra with $Q = \mathrm{Id}(R) \subseteq F\langle x\rangle$ its ideal of identities. We define its n-th A-cocharacter $\chi_n^A(R)$ to be the following A_n-character*

$$\chi_n^A(R) = \chi_{A_n}\Big(\frac{FA_n}{FA_n \cap Q}\Big),$$

and its n-th A-codimension $c_n^A(R)$ to be

$$c_n^A(R) = \deg \chi_n^A(R) = \dim \Big(\frac{FA_n}{FA_n \cap Q}\Big).$$

The A_n-analogue of Equation (3) is hence given by

$$\chi_n^A(R) = \sum_{\lambda\vdash n\ ,\ \lambda'\gneqq\lambda} m_\lambda(R)\bar{\chi}^\lambda + \sum_{\lambda\vdash n\ ,\ \lambda'=\lambda} (m_\lambda^+(R)\bar{\chi}^{\lambda+} + m_\lambda^-(R)\bar{\chi}^{\lambda-}), \tag{4}$$

with a corresponding expression for $c_n^A(R)$.

Recall that $H(k,l;n) = \{(\lambda_1,\lambda_2,\ldots)\vdash n \mid \lambda_{k+1}\leq l\}$ denotes the partitions whose diagrams are contained in the (k,l)-hook. Thus, in particular, $H(1,1;n) = \{(n-j,1^j) \mid 0\leq j\leq n-1\}$. The main result in this paper is the following theorem.

Theorem 2.2 (The A-codimensions and A-cocharacters) *Let E denote the infinite dimensional Grassmann (exterior) algebra.*

1. *Then $c_n^A(E) = 2^{n-1}-1$.*
2. *Let $H^*(1,1;n) = H(1,1;n)\setminus\{(n),(1^n)\}$, then*

$$\chi_n^A(E) = \bar{\chi}^{(n)} + \sum_{\lambda\in H^*(1,1;n)} \bar{\chi}^\lambda.$$

In terms of the irreducible A_n-characters this says the following:

(a) *If $n=2k$ then*

$$\chi_n^A(E) = \bar{\chi}^{(n)} + \sum_{j=1}^{k-1} 2\cdot\bar{\chi}^{(n-j,1^j)}.$$

(b) *If $n=2k+1$ then*

$$\chi_n^A(E) = \bar{\chi}^{(n)} + \sum_{j=1}^{k-1} 2\cdot\bar{\chi}^{(n-j,1^j)} + \bar{\chi}^{(k+1,1^k)+} + \bar{\chi}^{(k+1,1^k)-}.$$

Recall from [4] that $c_n(E) = 2^{n-1}$. To prove part 1 of Theorem 2.2 we first show, in Lemma 3.1, that either $c_n^A(E) = 2^{n-1}-1$ or $c_n^A(E)\leq 2^{n-1}-4$. The proof is then completed by Lemma 4.1 which asserts that $c_n^A(E)\geq 2^{n-1}-3$. Part 2 then follows from part 1, as shown in Corollary 2.9. The proof also requires some general machinery, of interest on its own, which we develop in this section.

2.3. *Local cocharacters and codimensions.* To study the above invariants we need to introduce local cocharacters and codimensions. Recall that $FS_n = \oplus_{\lambda\vdash n} I_\lambda$ where the I_λ's are the minimal two-sided ideals in FS_n.

Definition 2.3 *(Local cocharacters and codimensions, the S_n-case). Denote*

$$\chi_\lambda(R) = \chi_{S_n}\Big(\frac{I_\lambda}{I_\lambda\cap Q}\Big) \qquad \textit{and} \qquad c_\lambda(R) = \deg\chi_\lambda(R) = \dim\Big(\frac{I_\lambda}{I_\lambda\cap Q}\Big).$$

Remark. Recall that $Q = Id(R)$, hence $Q \cap FS_n$ is a left ideal in FS_n and by applying the central idempotents $e_\lambda \in I_\lambda$ it is easy to show that $(\oplus_{\lambda\vdash n} I_\lambda) \cap Q = \oplus_{\lambda\vdash n}(I_\lambda \cap Q)$. Therefore $\chi_n(R) = \sum_{\lambda\vdash n} \chi_\lambda(R)$ and $c_n(R) = \sum_{\lambda\vdash n} c_\lambda(R)$, and in the notation of Equation (3), $\chi_\lambda(R) = m_\lambda(R)\chi^\lambda$, and $c_\lambda(R) = m_\lambda(R)d_\lambda$.

Definition 2.4 *(Local cocharacters and codimensions, the A_n-case). As in Equation (2), write FA_n as the direct sum of its minimal two-sided ideals and let λ be a partition.*

1. *When $\lambda \neq \lambda'$, then define*
$$\chi_\lambda^A(R) = \chi_{A_n}\Big(\frac{\bar{I}_\lambda}{\bar{I}_\lambda \cap Q}\Big) \quad \textit{and} \quad c_\lambda^A(R) = \deg \chi_\lambda^A(R) = \dim\Big(\frac{\bar{I}_\lambda}{\bar{I}_\lambda \cap Q}\Big).$$
2. *When $\lambda = \lambda'$, then define*
$$\chi_\lambda^{A\pm}(R) = \chi_{A_n}\Big(\frac{\bar{I}_\lambda^\pm}{\bar{I}_\lambda^\pm \cap Q}\Big) \quad \textit{and} \quad c_\lambda^{A\pm}(R) = \deg \chi_\lambda^{A\pm}(R) = \dim\Big(\frac{\bar{I}_\lambda^\pm}{\bar{I}_\lambda^\pm \cap Q}\Big).$$
Also denote
$$c_\lambda^A(R) = c_\lambda^{A+}(R) + c_\lambda^{A-}(R).$$

Remark. By the same reasoning as before, $\chi_n^A(R)$ is the sum of the local cocharacters and $c_n^A(R)$ is the sum of the local codimensions, given by Equation (4).
Thus
$$\chi_n^A(R) = \sum_{\lambda\vdash n\ ,\ \lambda' \gneqq \lambda} \chi_\lambda^A(R) + \sum_{\lambda\vdash n\ ,\ \lambda'=\lambda} (\chi_\lambda^{A+}(R) + \chi_\lambda^{A-}(R)),$$
$$c_n^A(R) = \sum_{\lambda\vdash n\ ,\ \lambda' \gneqq \lambda} c_\lambda^A(R) + \sum_{\lambda\vdash n\ ,\ \lambda'=\lambda} (c_\lambda^{A+}(R) + c_\lambda^{A-}(R)),$$
and also
$$c_n^A(R) = \sum_{\lambda\vdash n\ ,\ \lambda' \gneqq \lambda} c_\lambda^A(R) + \sum_{\lambda\vdash n\ ,\ \lambda'=\lambda} c_\lambda^A(R).$$

Recall (see Section 2.1), if λ is non-self-conjugate, then the dimension of any irreducible FA_n-submodule or quotient of I_λ or $I_{\lambda'}$ is equal to d_λ. If λ is self-conjugate, then the dimension of any irreducible FA_n-submodule or quotient of I_λ is equal to $\frac{1}{2}d_\lambda$.

Proposition 2.5 *Let R be a p.i. algebra. Recall that $\bar{\chi}$ denotes the restriction of the S_n character χ to A_n.*

1. *Let λ be non-self-conjugate, then either $c_\lambda^A(R) = c_\lambda(R) + c_{\lambda'}(R)$, in which case we have $\chi_\lambda^A(R) = \bar{\chi}_\lambda(R) + \bar{\chi}_{\lambda'}(R)$, or we have $c_\lambda^A(R) \leq c_\lambda(R) + c_{\lambda'}(R) - d_\lambda$. Thus always,*
$$c_\lambda^A(R) \leq c_\lambda(R) + c_{\lambda'}(R).$$
Also, both $c_\lambda(R)\,,\, c_{\lambda'}(R) \leq c_\lambda^A(R)$.

2. *Let λ be self-conjugate, then either $c_\lambda^A(R) = c_\lambda(R)$, in which case we have $\chi_\lambda^A(R) = \bar{\chi}_\lambda(R)$, or $c_\lambda^A(R) \le c_\lambda(R) - \frac{1}{2}d_\lambda$.*

Proof. Again let $Q = \mathrm{Id}(R)$.

1. Let λ be non-self-conjugate. The inclusion $\bar{I}_\lambda \subset I_\lambda \oplus I'_\lambda$ implies the embedding

$$\frac{\bar{I}_\lambda}{\bar{I}_\lambda \cap Q} \longrightarrow \frac{I_\lambda \oplus I_{\lambda'}}{(I_\lambda \oplus I_{\lambda'}) \cap Q} \cong \frac{I_\lambda}{I_\lambda \cap Q} \oplus \frac{I_{\lambda'}}{I_{\lambda'} \cap Q}.$$

As FA_n-modules, both sides are isotypic, corresponding to the same irreducible FA_n-module whose dimension equals d_λ. The case of equality is the case when the embedding is surjective. Otherwise this implies the inequality $c_\lambda^A(R) \le c_\lambda(R) + c_{\lambda'}(R) - d_\lambda$.

Next we show, for example, that $c_\lambda(R) \le c_\lambda^A(R)$. Recall that in this case $\bar{I}_\lambda \cong I_\lambda$ as FA_n-modules. Define $\psi : \bar{I}_\lambda \to I_\lambda$ as follows. Let $x \in \bar{I}_\lambda \subset I_\lambda \oplus I_{\lambda'}$, so $x = x_\lambda + x_{\lambda'}$ with $x_\lambda \in I_\lambda$ and $x_{\lambda'} \in I_{\lambda'}$, then define $\psi(x) = x_\lambda$. It easily follows that $\psi : \bar{I}_\lambda \to I_\lambda$ is indeed an isomorphism, and we show that ψ induces a surjective homomorphism of FA_n-modules

$$\psi^* : \frac{\bar{I}_\lambda}{\bar{I}_\lambda \cap Q} \longrightarrow \frac{I_\lambda}{I_\lambda \cap Q},$$

which would clearly imply our claim. If $x \in Q$ then both $x_\lambda, x_{\lambda'} \in Q$ (left-multiply $x = x_\lambda + x_{\lambda'}$ by the central idempotent $e_\lambda \in I_\lambda$: $x_\lambda = e_\lambda x$). This easily implies that ψ^*, given by

$$x \in \bar{I}_\lambda, \qquad \psi^*(x + (\bar{I}_\lambda \cap Q)) = \psi(x) + (I_\lambda \cap Q),$$

is a well defined homomorphism. Since $\psi : \bar{I}_\lambda \to I_\lambda$ is surjective (in fact, an isomorphism), also ψ^* is surjective.

2. Let $\lambda = \lambda'$ and let $\bar{I}_\lambda = \bar{I}_\lambda^+ \oplus \bar{I}_\lambda^- \subset I_\lambda$. The inclusion $\bar{I}_\lambda \subset I_\lambda$ implies the embedding

$$\frac{\bar{I}_\lambda}{\bar{I}_\lambda \cap Q} \longrightarrow \frac{I_\lambda}{I_\lambda \cap Q}.$$

Again, the case of equality is the case when the embedding is surjective, and otherwise this implies the inequality $c_\lambda^A(R) \le c_\lambda(R) - \frac{1}{2}d_\lambda$ by an argument similar to that in the previous case. Also, in the case of equality we get that $c_\lambda^{A\pm}(R) = \frac{1}{2}c_\lambda(R)$.

Recall that finite dimensional algebras satisfy Capelli identities: if $\dim R = k$ then R satisfies the $k+1$-th Capelli polynomial

$$c_{k+1}[x; y] = \sum_{\sigma \in S_{k+1}} \mathrm{sgn}(\sigma) x_{\sigma(1)} y_1 x_{\sigma(2)} y_2 \cdots y_k x_{\sigma(k+1)}.$$

For example, the $r\times r$ matrices in $M_r(F)$ satisfy the identity $c_{r^2+1}[x;y]$. As a corollary we deduce the following theorem.

Theorem 2.6 *Let R be a p.i. algebra satisfying the $k+1$-th Capelli polynomial $c_{k+1}[x;y]$, and let $n \geq k^2+1$. Then $c_n^A(R) = c_n(R)$.*

Proof. Since R satisfies $c_{k+1}[x;y]$, we obtain from [6], Theorem 2 that

$$c_n(R) = \sum_{\lambda \in H(k,0;n)} c_\lambda(R).$$

Let $\lambda \in H(k,0;n)$ and let $n \geq k^2+1$, then $\lambda' \notin H(k,0;n)$ and therefore $c_{\lambda'}(R) = 0$ and $\lambda \neq \lambda'$. Applying part 1 of Proposition 2.5 deduce, for $\lambda \in H(k,0;n)$, that

$$c_\lambda(R) \leq c_\lambda^A(R) \leq c_\lambda(R) + c_{\lambda'}(R) = c_\lambda(R),$$

hence $c_\lambda(R) = c_\lambda^A(R)$. This implies the proof. ∎

Definition 2.7 *Let $\Gamma(n,R)$ denote the partitions of n for which the corresponding local codimensions are not equal: $\Gamma(n,R) = \Gamma_{nsc}(n,R) \cup \Gamma_{sc}(n,R)$ where*

$$\Gamma_{nsc}(n,R) = \{\lambda \vdash n \;\mid\; \lambda' \lneqq \lambda \;\; \textit{and} \;\; c_\lambda^A \neq c_\lambda(R) + c_{\lambda'}(R)\}$$

and

$$\Gamma_{sc}(n,R) = \{\lambda \vdash n \;\mid\; \lambda' = \lambda \;\; \textit{and} \;\; c_\lambda^A(R) \neq c_\lambda(R)\}.$$

Here nsc stands for non-self-conjugate and sc stands for self-conjugate.

Then Proposition 2.5 clearly implies the following upper bound for $c_n^A(R)$.

Corollary 2.8 *Let R be a p.i. algebra, then*

$$c_n^A(R) \leq c_n(R) - \sum_{\lambda \in \Gamma_{nsc}(n,R)} d_\lambda - \frac{1}{2} \sum_{\lambda \in \Gamma_{sc}(n,R)} d_\lambda.$$

Corollary 2.9 *In Theorem 2.2, part 1 implies part 2.*

Proof. By the Corollary to Theorem 3.1 in [4] and by Theorem 2.7 in [5] we know that

$$c_n(E) = 2^{n-1} \qquad \text{and} \qquad \chi_n(E) = \sum_{\lambda \in H(1,1;n)} \chi^\lambda. \tag{5}$$

By the remarks at the begining of Section 3 we may assume that $n \geq 4$. Then for any $\lambda \in H(1,1;n)$ which is not equal to (n) or to (1^n) , $d_\lambda \geq 3$. By part 1 of Theorem 2.2 and by Corollary 2.8, for any $\lambda' \leq \lambda$ partition of n not equal to (n) or to (1^n), the corresponding local codimensions are equal:

1. If λ is non-self-conjugate, then $c_\lambda^A(E) = c_\lambda(E) + c_{\lambda'}(E)$.
2. If λ is self-conjugate, then $c_\lambda^A(E) = c_\lambda(E)$.

By Proposition 2.5 this implies the equality of the corresponding local cocharacters. It also follows that for $\lambda = (n)$ the multiplicity $m_\lambda^A(E)$ equals 1 (see also the proof of Lemma 3.1). The proof of part 2 of Theorem 2.2 now follows from the Equations in (5). ∎

3 An upper bound for $c_n^A(E)$

Recall that E is the infinite dimensional Grassmann (exterior) algebra. By the Corollary to Theorem 4.1 in [4] the ideal $Q = Id(E)$ of the identities of E is generated by the double commutator $[[x_1, x_2], x_3]$. It clearly follows that $FA_3 \cap Q = 0$, hence $c_3^A(E) = \dim FA_3 = 3$.

Lemma 3.1 *Let $n \geq 4$, then either $c_n^A(E) = 2^{n-1} - 1$ or $c_n^A(E) \leq 2^{n-1} - 4$.*

Proof. By Equation (5), $c_\lambda(E) = d_\lambda$ if λ lies in $H(1,1;n)$ and $c_\mu(E) = 0$ if μ lies not $H(1,1;n)$. In particular, $c_{(1^n)}(E) = c_{(n)}(E) = 1$. Recall that $\bar{I}_{(n)} \subset I_{(n)} \oplus I_{(1^n)}$ and $\bar{I}_{(n)} \cong I_{(n)}$, so $\dim \bar{I}_{(n)} = 1$. By Equation (5), $(I_{(n)} \oplus I_{(1^n)}) \cap Id(E) = 0$, and it follows that $c_{(n)}^A(E) = 1$. But $c_{(n)}(E) + c_{(1^n)}(E) = 2$, hence (see Definition 2.7) $(n) \in \Gamma_{\rm nsc}(n, E)$, with

$$(c_{(n)}(E) + c_{(1^n)}(E)) - c_{(n)}^A(E) = 1. \tag{6}$$

Note that if $\lambda \in H(1,1;n)$ then $\lambda = (n-j, 1^j)$ for some j, and $d_{(n-j,1^j)} = \binom{n-1}{j}$. If $1 \leq j \leq n-2$ then $d_{(n-j,1^j)} \geq n-1 \geq 3$ since, by assumption $n \geq 4$. Also, if n is odd, $n = 2m+1$ then $m \geq 2$ and for the unique self-conjugate hook $(m+1, 1^m)$, $\frac{1}{2} d_{(m+1,1^m)} \geq 3$.

Case 1. Assume $\Gamma(n, E) = \{(n)\}$. We show that in this case $c_n^A(E) = 2^{n-1} - 1$. Indeed, $c_n(E) = c_{(n)}(E) + c_{(1^n)}(E) + c_n^*(E) = 2 + c_n^*(E)$, where

$$c_n^*(E) = \sum_{\lambda \in H(1,1,n) \setminus \{(n),(1^n)\}} c_\lambda(E).$$

So $c_n^A(E) = 1 + c_n^*(E)$ and therefore by Equation (5), $c_n^A(E) = c_n(E) - 1 = 2^{n-1} - 1$.

Case 2. Assume there exist $\lambda \in \Gamma(n, E)$, $\lambda \neq (n)$. Show that in this case $c_n^A(E) \leq 2^{n-1} - 4$. By Definition 2.7 and by (5), $\Gamma(n, E) \subseteq H(1,1;n)$, therefore $\lambda = (n-j, 1^j)$ for some $1 \leq j \leq n-2$. Now either $\lambda \neq \lambda'$, or n is odd, $n = 2m+1$ and $\lambda = (m+1, 1^m)$. In the first sub-case $c_\lambda^A(E) \leq c_\lambda(E) + c_{\lambda'}(E) - d_\lambda$ by Proposition 2.5, hence $c_n^A(E) \leq c_n(E) - 1 - d_\lambda \leq 2^{n-1} - 4$ since $d_\lambda \geq 3$. In the second sub-case, similarly, $c_n^A(E) \leq c_n(E) - 1 - \frac{1}{2} d_{(m+1,m)} \leq 2^{n-1} - 4$. ■

4 Strategy of the proof of the lower bound.

We shall prove that $c_n^A(E) = 2^{n-1} - 1$ by proving the following lemma:

Lemma 4.1 *The following two equivalent statements hold (see Lemma 3.1):*

1. *Let $n \geq 3$, then $c_n^A(E) \geq 2^{n-1} - 1$.*
2. *Let $n \geq 3$, then $c_n^A(E) \geq 2^{n-1} - 3$.*

The proof of the above lemma is similar in structure – and in some of its details – to the proof in [4] of the lower bound $c_n(E) \geq 2^{n-1}$. In this section we review this proof, then explain the strategy for proving Lemma 4.1.

4.1. *Review and reformulation.* The main step in the proof of the lower bound for $c_n^A(E)$ is similar to the proof in [4] of the lower bound for $c_n(E)$. We therefore review now this particular step in [4] and reformulate it for the A_n-situation.

We define the functions $f_I : S_n \to \{\pm 1\}$. Here $I \subseteq \{1, \ldots, n\}$, $\sigma \in S_n$ and $f_I(\sigma)$ is the sign of the permutation that σ induces on the elements of I – when it permutes $1, \ldots, n$.

Example 4.2 *Let $n \geq 2$ and let τ be the 3-cycle $\tau = (n-1, n, n+1) \in S_{n+1}$, then $\tau : [n-1, n, n+1] \to [n, n+1, n-1]$, where $[a, b, c]$ denotes the ordered 3-tuple of a, b and c. Thus $f_{\{n-1,n\}}(\tau) = -1$ since on the subset $\{n-1, n\}$, τ is a transposition. Similarly $f_{\{n-1,n+1\}}(\tau) = -1$, $f_{\{n,n+1\}}(\tau) = +1$ and $f_{\{n-1,n,n+1\}}(\tau) = +1$. It also is clear that $f_{\{i\}}(\tau) = +1$ for any i. More generally, let $W \subseteq \{1, 2, \ldots, n-2\}$ and $A \subseteq \{n-1, n, n+1\}$, then $f_{W \cup A}(\tau) = f_A(\tau)$.*

Lemma 4.3 (Lemma 1.2 of [1]) *Let $I \subseteq \{1, \ldots, n\}$ and $\sigma, \eta \in S_n$. Then*

$$f_I(\sigma\eta) = f_I(\sigma) f_{\sigma^{-1}(I)}(\eta).$$

Lemma 4.4 (Lemma 2.1 of [4]) *Let $H^{(n)}$ be the $2^n \times n!$-matrix $(f_I(\sigma))$ which has its rows indexed by the subsets $I \subseteq \{1, \ldots, n\}$ and its columns indexed by the permutations $\sigma \in S_n$; its entries are $f_I(\sigma)$. Then $c_n(E) =$ rank $H^{(n)}$.*

Restricting to A_n, we obtain the following Lemma:

Lemma 4.5 *Let $K^{(n)} = (f_I(\sigma))$ has its rows indexed by the subsets $I \subseteq \{1, \ldots, n\}$ and its columns indexed by the permutations $\sigma \in A_n$; its entries are $f_I(\sigma)$. Then $c_n^A(E) =$ rank $K^{(n)}$.*

The proof of Lemma 4.5 is identical to the proof of Lemma 2.1 in [4]. Using this last lemma we obtain that Lemma 4.1 is hence equivalent to

Lemma 4.6

1. *Let $n \geq 3$. There exist a $2^n \times (2^{n-1} - 1)$ sub-matrix $P^{(n)}$, made from $2^{n-1} - 1$ columns of $K^{(n)}$, such that rank $P^{(n)} = 2^{n-1} - 1$.*
   The equivalence of parts 1 and 2 in Lemma 4.1 implies that this is equivalent to

2. *Let $n \geq 3$. There exist a $2^n \times (2^{n-1} - 2)$ sub-matrix $L^{(n)}$, made from $2^{n-1} - 2$ columns of $K^{(n)}$, such that rank $L^{(n)} = 2^{n-1} - 2$.*

It is this last part that we shall prove in the next section. We first describe the strategy for such a proof.

4.2. *The strategy of the proof of Lemma 4.6.* In this section we prove part 2 of Lemma 4.6 (and with that prove Lemma 4.1). In this subsection we will describe the strategy of this proof.

The matrices $M^{(n)}$ of [4] are constructed, inductively, by choosing from the matrix $H^{(n)}$ the 2^{n-1} rows corresponding to the subsets $I \subseteq \{1, \ldots, n\}$ with $1 \in I$, and certain 2^{n-1} columns. The lower bound $c_n(E) \geq 2^{n-1}$ is obtained by showing that rank $H^{(n)} = 2^{n-1}$.

Here we construct, again inductively, the matrices $L^{(n)}$ by choosing *all* the rows of $K^{(n)}$ and certain $2^{n-1} - 2$ of its columns. We then show that rank $L^{(n)} = 2^{n-1} - 2$. The choice of these columns is done by induction as follows:

First, $A_n \subset A_{n+1}$ is the natural embedding: $\sigma(n+1) = n+1$ for $\sigma \in A_n$. Assume now that Lemma 4.6 holds for a certain n, $n \geq 3$. Part 1 of Lemma 4.6 then gives (by induction) the matrix $P^{(n)}$ with the corresponding $2^{n-1} - 1$ even permutations $\{\sigma^j \in A_n \mid 1 \leq j \leq 2^{n-1} - 1\}$ that index the columns of $P^{(n)}$. (Note that ^{j} is used to index the elements.) Moreover, rank$P^{(n)} = 2^{n-1} - 1$. Let τ be the following 3-cycle $\tau = (n-1, n, n+1)$ and note that $\tau \in A_{n+1}$, hence $\tau\sigma^j \in A_{n+1}$ for $1 \leq j \leq 2^{n-1} - 1$. The $2^n - 2$ even permutations that index the columns of $L^{(n+1)}$ are (given in their order of occurence)

$$\{\sigma^j \mid 1 \leq j \leq 2^{n-1} - 1\} \cup \{\tau\sigma^j \mid 1 \leq j \leq 2^{n-1} - 1\} \subseteq A_{n+1}. \tag{7}$$

We then prove that rank $L^{(n+1)} = 2^n - 2$. This of course completes the proof, since by the equivalence of the two parts of Lemma 4.6, this implies the existence also of $P^{(n+1)}$.

Note that in [4] the choice of permutations (i.e. columns) for constructing $M^{(n)}$ is recursively explicit. Here, on the other hand, the choice of permutations for constructing $P^{(n)}$ is *not* recursively explicit: the one additional permutation in the transition from $L^{(n+1)}$ to $P^{(n+1)}$ exists, but we do not calculate it.

5 The proof of Lemma 4.1.

Here we shall prove part 2 of Lemma 4.6. As was explained in the previous section, this is equivalent to Lemma 4.1. We begin by describing the matrices $L^{(n)}$ and $P^{(n)}$, starting with $P^{(3)}$. The entries of these matrices are certain $f_I(\sigma)$, as described in Section 4. We represent $\sigma \in A_n$ by the n-tuple of its values $[\sigma(1), \ldots, \sigma(n)]$. Thus, for example, $f_{\{1,3\}}([2,3,1]) = -1$. We obtain the matrix:

$P^{(3)} =$

	$[1,2,3]$	$[2,3,1]$	$[3,1,2]$
ϕ	$+1$	$+1$	$+1$
$\{1\}$	$+1$	$+1$	$+1$
$\{2\}$	$+1$	$+1$	$+1$
$\{1,2\}$	$+1$	-1	$+1$
$\{3\}$	$+1$	$+1$	$+1$
$\{1,3\}$	$+1$	-1	-1
$\{2,3\}$	$+1$	$+1$	-1
$\{1,2,3\}$	$+1$	$+1$	$+1$

The four rows indexed by the subsets $W \subseteq \{1,2,3\}$ with $1 \in W$ yield the matrix

$$\begin{pmatrix} +1,+1,+1 \\ +1,-1,+1 \\ +1,-1,-1 \\ +1,+1,+1 \end{pmatrix}$$

of rank $3 = 2^{3-1} - 1$, which implies that the rank of $P^{(3)}$ equals 3. Construct now $L^{(4)}$ from $P^{(3)}$ as explained in Section 4.2. The 16 rows of $L^{(4)}$ are indexed by the 16 subsets $W \subseteq \{1,2,3,4\}$: first list the 8 subsets $W \subseteq \{1,2,3,\}$ then the other 8 subsets W with $4 \in W$. The columns of $L^{(4)}$ are indexed by the following 6 permutations, listed in the following order: Here $\tau = (2,3,4)$. So using Equation (7), the first three indices are the permutations that indexed the columns of $P^{(3)}$, now viewed in A_4: these are $[1,2,3,4]$, $[2,3,1,4]$, $[3,1,2,4]$. The next three indices are $[1,3,4,2]$, $[3,4,1,2]$, $[4,1,3,2]$ since

$$\begin{aligned} (2,3,4)[1,2,3,4] &= [1,3,4,2], \\ (2,3,4)[2,3,1,4] &= [3,4,1,2], \\ (2,3,4)[3,1,2,4] &= [4,1,3,2]. \end{aligned}$$

The reader can verify that rank $L^{(4)} = 6 = 2^{4-1} - 2$; to simplify, choose the 8 rows corresponding to the subsets $W \subseteq \{1,2,3,4\}$ with $1 \in W$. Turn now to the general case.

Definition 5.1 *Let $W \subseteq \{1,\dots,m\}$ and let $a,b > m$. Denote $W \cup \{a\} = \{W,a\}$ and $W \cup \{a,b\} = \{W,a,b\}$. Let $n \geq 3$ and let $\{W_i \mid 1 \leq i \leq 2^{n-2}\}$ be some ordering on the subsets $W \subseteq \{1,\dots,n-2\}$. Then order the subsets $W \subseteq \{1,\dots,n\}$ as follows:*

$$\{W_i \mid 1 \leq i \leq 2^{n-2}\} \quad \textit{then} \quad \{\{W_i, n-1\} \mid 1 \leq i \leq 2^{n-2}\} \quad \textit{then}$$

$$\{\{W_i, n\} \mid 1 \leq i \leq 2^{n-2}\} \quad \textit{then} \quad \{\{W_i, n-1, n\} \mid 1 \leq i \leq 2^{n-2}\}.$$

With this order of its rows, $P^{(n)}$ breaks into the following four blocks, each of size $2^{n-2} \times (2^{n-1}-1)$, as shown in the following table.

$$P^{(n)} = (A,B,C,D)^t = \begin{array}{|c|c|}\hline & \sigma^j \\ \hline W_i & A \\ \hline \{W_i, n-1\} & B \\ \hline \{W_i, n\} & C \\ \hline \{W_i, n-1, n\} & D \\ \hline \end{array} \tag{8}$$

Next, construct $L^{(n+1)}$ from $P^{(n)}$ as follows: the columns of $L^{(n+1)}$ are indexed by the permutations as in (7), and in that corresponding order; the first 2^n rows of $L^{(n+1)}$ are indexed by the same subsets as in $P^{(n)}$, while the next 2^n rows are indexed by the corresponding subsets with $n+1$ added. The blocks of $L^{(n+1)}$ are then denoted by $L_{i,j}$. Thus

$$L^{(n+1)} = \begin{array}{|c|c|c|}\hline & \sigma^j & \tau\sigma^j \\ \hline W_i & L_{1,1} & L_{1,2} \\ \hline \{W_i, n-1\} & L_{2,1} & L_{2,2} \\ \hline \{W_i, n\} & L_{3,1} & L_{3,2} \\ \hline \{W_i, n-1, n\} & L_{4,1} & L_{4,2} \\ \hline \{W_i, n+1\} & L_{5,1} & L_{5,2} \\ \hline \{W_i, n-1, n+1\} & L_{6,1} & L_{6,2} \\ \hline \{W_i, n, n+1\} & L_{7,1} & L_{7,2} \\ \hline \{W_i, n-1, n, n+1\} & L_{8,1} & L_{8,2} \\ \hline \end{array}$$

As shown below, the proof that $\text{rank} L^{(n+1)} = 2^n - 2$ follows from the following lemma.

Lemma 5.2 *Let $P^{(n)} = (A,B,C,D)^t$ be defined as in Equation (8), and construct $L^{(n+1)}$ from $P^{(n)}$ as above. Then the above sub-matrices $L_{i,j}$ of $L^{(n+1)}$ are given by the following table:*

$$L^{(n+1)} = \begin{array}{|c|c|c|}\hline & \sigma^j & \tau\sigma^j \\ \hline W_i & A & A \\ \hline \{W_i, n-1\} & B & A \\ \hline \{W_i, n\} & C & B \\ \hline \{W_i, n-1, n\} & D & -B \\ \hline \{W_i, n+1\} & A & C \\ \hline \{W_i, n-1, n+1\} & B & -C \\ \hline \{W_i, n, n+1\} & C & D \\ \hline \{W_i, n-1, n, n+1\} & D & D \\ \hline \end{array}$$

The proof of Lemma 5.2 is given in the next section. This lemma now implies the following proposition:

Proposition 5.3 *The rank of $L^{(n+1)}$ is $2^n - 2$.*

Proof. We apply some row operations on $L^{(n+1)}$. First note that $L^{(n+1)} = \begin{pmatrix} U & V \\ W & X \end{pmatrix}$ where $U = W = P^{(n)}$. Subtracting the upper from the lower half we obtain

$\begin{pmatrix} U & V \\ 0 & X-V \end{pmatrix}$ and $X - V = \begin{pmatrix} C-A \\ -C-A \\ D-B \\ D+B \end{pmatrix}$. Simple row-block operations now transform $X-V$ into $P^{(n)}$, then transform $\begin{pmatrix} V \\ P^{(n)} \end{pmatrix}$ into $\begin{pmatrix} 0 \\ P^{(n)} \end{pmatrix}$. Thus by row operations $L^{(n+1)}$ is transformed into $\begin{pmatrix} P^{(n)} & 0 \\ 0 & P^{(n)} \end{pmatrix}$ which has rank $2^n - 2$ since, by induction, $\mathrm{rank} P^{(n)} = 2^{n-1} - 1$. ■

6 The proof of Lemma 5.2.

The four blocks $L_{1,1}, \ldots, L_{4,1}$ are given by the table defining $P^{(n)}$. Calculate the next four blocks $L_{5,1}, \ldots, L_{8,1}$. Let $I \subseteq \{1, \ldots, n\}$ and let $\sigma \in S_n$ (hence also $\sigma \in S_{n+1}$). It then follows from the definition of the sign function f that $f_{\{I,n+1\}}(\sigma) = f_I(\sigma)$. This implies that $L_{i+4,1} = L_{i,1}$, $1 \le i \le 4$, which verifies the statement of Lemma 5.2 for the first block-column of $L^{(n+1)}$.

Turn now to the second block-column. Recall that each $W_i \subseteq \{1, \ldots, n-2\}$ (see Definition 5.1), each $\sigma^j \in A_n$ (recall Equation (7)), and τ is the 3-cycle $\tau = (n-1, n, n+1)$ (see Example 4.2).

1. Show that $L_{1,2} = A$. This is equivalent to showing that for all i ($1 \le i \le 2^{n-2}$) $f_{W_i}(\tau\sigma^j) = f_{W_i}(\sigma^j)$. Indeed, since $W_i \subseteq \{1, \ldots, n-2\}$, $\tau^{-1}(W_i) = W_i$ and also $f_{W_i}(\tau) = +1$. Therefore by Lemma 4.3 $f_{W_i}(\tau\sigma^j) = f_{W_i}(\tau) f_{\tau^{-1}(W_i)}(\sigma^j) = f_{W_i}(\sigma^j)$.

2. Show that $L_{2,2} = A$, which is equivalent to showing that for all i, $f_{\{W_i,n-1\}}(\tau\sigma^j) = f_{W_i}(\sigma^j)$. Here, again, $f_{\{W_i,n-1\}}(\tau) = +1$ while $\tau^{-1}(\{W_i, n-1\}) = \{W_i, n+1\}$. Finally, $f_{\{W_i,n+1\}}(\sigma^j) = f_{W_i}(\sigma^j)$ since $\sigma^j(n+1) = n+1$. Therefore $f_{\{W_i,n-1\}}(\tau\sigma^j) = f_{\{W_i,n-1\}}(\tau) f_{\tau^{-1}(\{W_i,n+1\})}(\sigma^j) = f_{W_i}(\sigma^j)$.

3. Show that $L_{3,2} = B$, which is equivalent to showing that for all i, $f_{\{W_i,n\}}(\tau\sigma^j) = f_{\{W_i,n-1\}}(\sigma^j)$. The argument here similar: $f_{\{W_i,n\}}(\tau) = +1$ and $\tau^{-1}(\{W_i, n\}) = \{W_i, n-1\}$, and the proof follows.

4. Show that $L_{4,2} = -B$, which is equivalent to showing that for all i, $f_{\{W_i,n-1,n\}}(\tau\sigma^j) = -f_{\{W_i,n-1\}}(\sigma^j)$. Since $\tau = [1, \ldots, n-2, n, n+1, n-1]$, $f_{\{W_i,n-1,n\}}(\tau) = -1$. Also, $\tau^{-1}(\{W_i, n-1, n\}) = \{W_i, n-1, n+1\}$ and $f_{\{W_i,n-1,n+1\}}(\sigma^j) = f_{\{W_i,n-1\}}(\sigma^j)$.

5. Show that $L_{5,2} = C$, namely that $f_{\{W_i,n+1\}}(\tau\sigma^j) = f_{\{W_i,n\}}(\sigma^j)$. This follows since $f_{\{W_i,n+1\}}(\tau) = +1$ and $\tau^{-1}(\{W_i, n+1\}) = \{W_i, n\}$.

6. Show that $L_{6,2} = -C$, namely that $f_{\{W_i,n-1,n+1\}}(\tau\sigma^j) = -f_{\{W_i,n\}}(\sigma^j)$. Here $f_{\{W_i,n-1,n+1\}}(\tau) = -1$. The proof follows since $\tau^{-1}(\{W_i, n-1, n+1\}) = \{W_i, n, n+1\}$ and $f_{\{W_i,n,n+1\}}(\sigma^j) = f_{\{W_i,n\}}(\sigma^j)$.

7. Show that $L_{7,2} = D$, namely that $f_{\{W_i,n,n+1\}}(\tau\sigma^j) = f_{\{W_i,n-1,n\}}(\sigma^j)$. Here $f_{\{W_i,n,n+1\}}(\tau) = +1$ and $\tau^{-1}(\{W_i, n, n+1\}) = \{W_i, n-1, n\}$, and the proof follows.

8. Show that $L_{8,2} = D$, namely that $f_{\{W_i,n-1,n,n+1\}}(\tau\sigma^j) = f_{\{W_i,n-1,n\}}(\sigma^j)$. Here $f_{\{W_i,n-1,n,n+1\}}(\tau) = \mathrm{sgn}(\tau) = +1$ and $\tau^{-1}(\{W_i, n-1, n, n+1\})$ is equal to $\{W_i, n-1, n, n+1\}$ and $f_{\{W_i,n-1,n,n+1\}}(\sigma^j) = f_{\{W_i,n-1,n\}}(\sigma^j)$.

The proof of Lemma 5.2 is complete. ■

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