# Seminar on Algorithms and Geometry – Handout 6

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## Today's topics

We describe an alternative proof of the Johnson-Lindenstrauss lemma, this time from the perspective of measure concentration. Along the way, we will see the following two theorems, which were first proved by P. Levy (1951). Throughout, $\mu$ is the uniform (Haar) probability measure on the unit sphere $S^{n-1}$. We make no attempt to optimize the constants.

**Theorem 1.** *Let $A \subset S^{n-1}$ be a measurable set, and let $B \subset S^{n-1}$ be a cap such that $\mu(A) = \mu(B)$. Then for all $\varepsilon \geq 0$, $\mu(A_\varepsilon) \geq \mu(B_\varepsilon)$. Here, $A_\varepsilon = \{x \in S^{n-1} : \ d(x, A) \leq \varepsilon\}$ and similarly for $B_\varepsilon$.*

**Theorem 2.** *Let $A \subset S^{n-1}$ be a measurable set with $\mu(A) \geq 1/2$. Then $\mu(A_\varepsilon) \geq 1 - 2e^{-\varepsilon^2 n/2}$.*

**Reading material.** For more details, see Matousek's book, and the lecture notes by Goemans and by Barvinok. The course webpage will contain exact details and links for these references.

## Homework

1. Let $x \in S^{n-1}$ be drawn randomly according to $\mu$. Give a high-probability bound for its largest coordinate in absolute value, namely find $t = t(n)$ such that

   $$\Pr_{x \in \mu}[\|x\|_\infty \geq t] \leq 1/4.$$

   How many standard deviation is it away from the expectation of a single coordinate $x_1$?

2. Show how to extend the JL lemma to triangles, i.e. a dimension reduction theorem for a finite set $X \in \ell_2$, such that for every 3 points in $X$, the area of the triangle they form is distorted by at most $1 + \varepsilon$.

   Hint: First find an example for a triangle and an embedding of it with distortion $1 + \varepsilon$ (for all its pairwise distances), yet this embedding distorts its area by a much larger factor. Second, add to $X$ additional points, and show that preserving distances in this bigger set within $1 + \varepsilon'$ implies that all triangles in $X$ are preserved within $1 + \varepsilon$.

3. Show an analogue of Theorem 2 for expander graphs, as follows. For fixed $r, \alpha > 0$, consider all graphs $G = (V, E)$ with maximum degree at most $r$ and edge-expansion $\alpha > 0$. Show for $1 - \mu(A_t)$ an upper bound that drops exponentially in $t$ (with dependence on $r$ and $\alpha$, but not on $G$ or $|V|$).

   Here, edge-expansion means that $e(S, \bar{S}) \geq \alpha |S|$ for all $S \subseteq V$ with $0 < |S| \leq |V|/2$. $\mu$ is the uniform distribution over the vertices, i.e. $\mu(v) = 1/|V|$ for all $v \in V$, and $A_t$ defined as above with respect to shortest-path distance in $G$.