Proximity Oblivious Testing

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Proximity Oblivious Testing?

- A tester with additional constraints
  - Must work in repeated basic rounds
  - Sees a constant number of locations each round
  - Each round is executed with random locations
  - Rejects if one of the rounds rejects
  - Accepts if all rounds accept
  - **Forgets everything between rounds**

Formal Definition

- Let $\Pi = \bigcup_{n\in\mathbb{N}} \Pi_n$ (Π_n - instances if size n)
- Let ρ: $(0,1] \to (0,1]$ (success given ϵ)
- T is a Proximity-Oblivious tester with detection probability ρ when:
  - $f \in \Pi_n \Rightarrow \Pr\left[T^f(n) = 1\right] = 1$
  - $f \notin \Pi_n \Rightarrow \Pr\left[T^f(n) = 0\right] \geq \rho\left(\underbrace{\delta_{\Pi_n}(f)}_{\epsilon}\right)$
- We will assume T has constant query complexity

Properties we'd look at

- Query Complexity
  - Note that it will always be a constant
- The detection probability ρ
- Comparison to regular testers

Testing graph properties under the adjacency matrix model

- The tested function $f: [N] \times [N] \to \{0,1\}$ is the adjacency matrix
- Being ϵ-far means having to change ϵ fraction of the table entries
- Best used when the graph is dense

Example – Clique Collection

- Detecting if the graph is a single clique
  - Requires a single query
  - $\rho(\epsilon) = \epsilon$
- Detecting if the graph is complete bipartite
  - Requires three queries
  - $\rho(\epsilon) = \epsilon$
- Generalizing $CC^{\leq c}$ for $c \geq 3$
  - Using $\binom{c+1}{2}$ queries
  - $\rho(\epsilon) = \epsilon^{c+1+o(1)}$
  - Can't do better than $\rho(\epsilon) = \omega\left(\epsilon^{c/2}\right)$
  - Regular testers are superior!

Is everything obliviously testable?

- No!
- Testing bi-partiteness is not obliviously testable!
- But bi-partiteness is very easy to test in the old fashioned way!
- Proof on board…

What is obliviously testable?

- Theorem: Π has a proximity oblivious tester **if and only if** $\exists$ constant c and an infinite sequence $\bar{\mathcal{F}} = \{\mathcal{F}_N\}_{N\in\mathbb{N}}$ such that
  - Each $\mathcal{F}_N$ contains graphs of size at most c
  - Π_N equals the set of N-vertex $\mathcal{F}_N$-free graphs
- Characterizes proximity-oblivious testers for the adjacency matrix model

Proof idea

- The proof uses results from 2 other papers
- Is in fact very intuitive
- A proximity oblivious tester actually decides everything based on possible constant views
- Identical to looking for forbidden subgraphs
- Looking for forbidden sub-graphs of constant size is clearly proximity-oblivious
- Specifies a minimal detection probability!

A special case – graph freeness

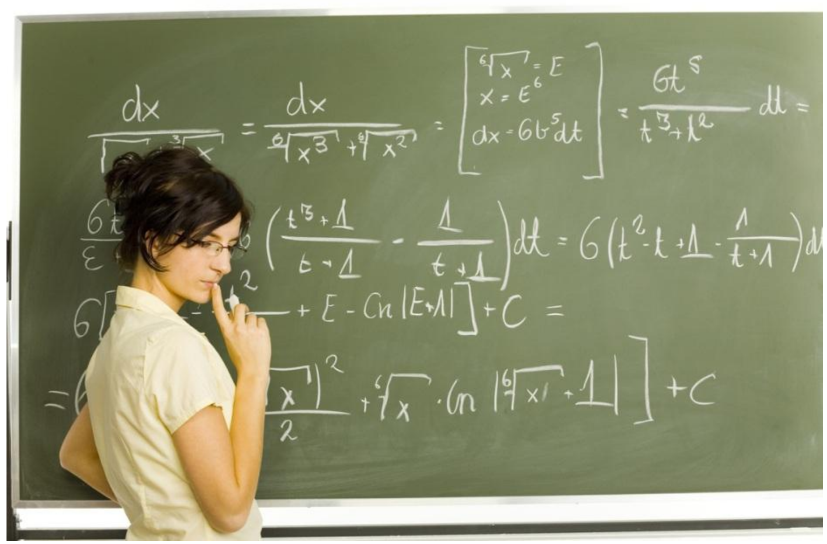
- Consider the special case in which $\mathcal{F}_n = \mathcal{F}_{n+1}$
- Can achieve better complexity
  - Using $\binom{c}{2}$ queries
  - Detection Probability $\rho_{\mathcal{F}}$
- Any other tester has detection probability $\Omega(\rho_{\mathcal{F}})$

Testing graph properties under the bounded degree model

- All degrees are bounded by the constant d
- The tested function $f: [N] \times [d] \rightarrow \{0, \dots, N\}$ is the adjacency list
- Being ϵ-far means having to change ϵ of the entries (similar to the previous case)
- Good for sparse graphs

Is there a difference?

- Surprisingly, there is!
- The bounded degree model achieves a very interesting characterisation
- Provides more power than the dense model



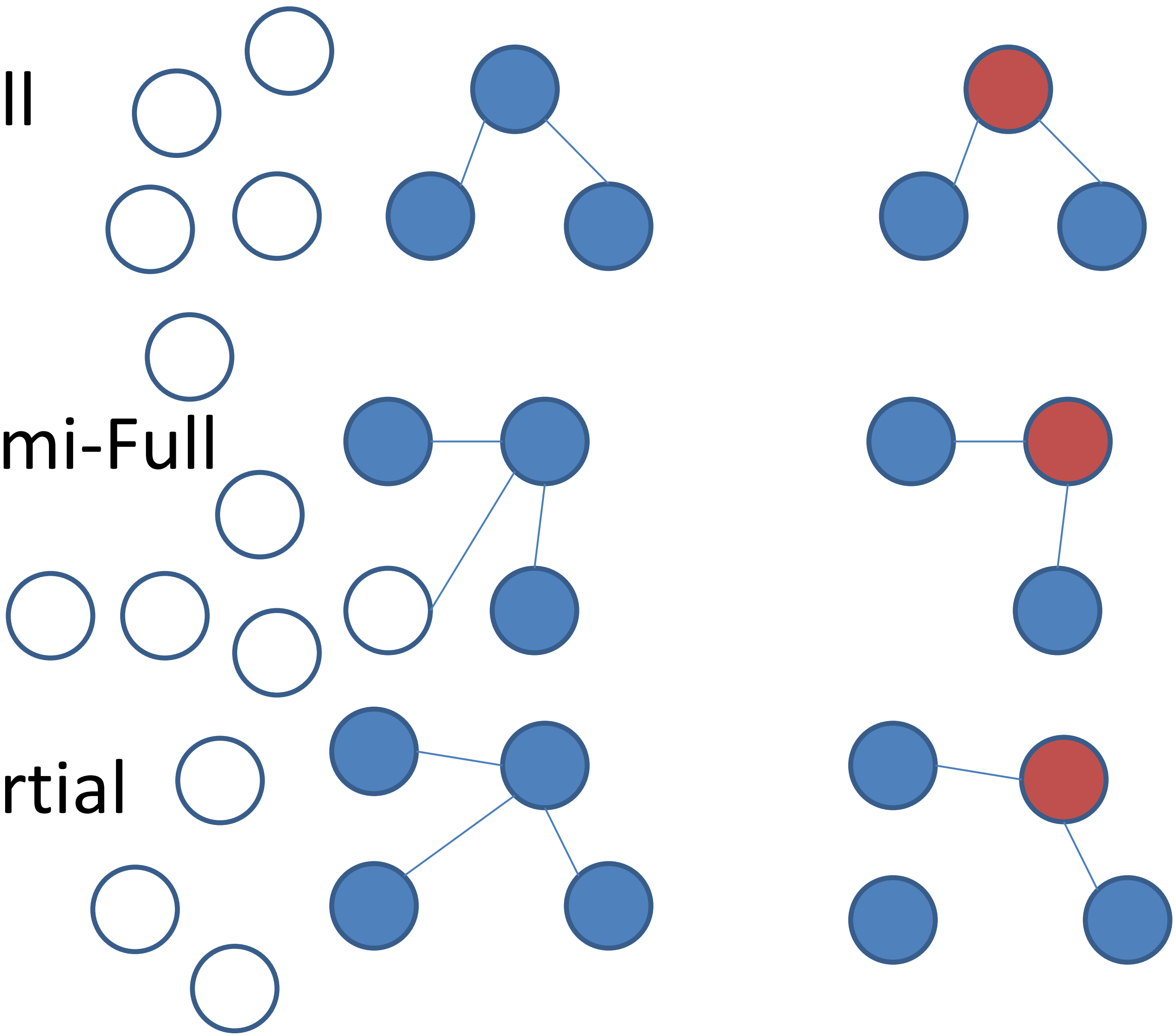
Characterisation of the bounded degree model

- A marked graph is a graph in which every vertex is marked with either full, semi-full or partial
- Instead of looking for forbidden sub-graphs – look for a forbidden embedding

- A sub-graph (induced or not) is a special case of embedding

What is an embedding?

- Full
- Semi-Full
- Partial



Characterisation of the bounded degree model (cont.)

- The property Π is called **local** if $\exists s$ and an infinite sequence $\bar{\mathcal{F}} = \{\mathcal{F}_N\}_{N\in\mathbb{N}}$ such that $\forall N$
  - $\mathcal{F}_N$ is a set of marked graphs of size at most s
  - Π_N cannot have an $\mathrm{F} \in \mathcal{F}_N$ embedded into it
- In this case, we say Π is $\bar{\mathcal{F}}$-local
- Being $\mathcal{F}_N$-free is also local!
- So, is being local the charactrisation of the bounded degree model?

Characterisation of the bounded degree model (cont.)

- Proof requires an additional property to exist
- The property is **non-propagation** (shall be defined soon)
- Still remains an open problem whether locality implies "non-propagatation"

Non-propagating condition

- For a graph $G = ([N], E)$, we say $B \subset [N]$ covers $\mathcal{F}_N$ in G if $\forall F \in \mathcal{F}_N$ for every embedding of F in G, at least one vertex of F is mapped to a vertex in B
- Can think of B as the "inescapable set"

Non-propagating condition (cont.)

- We say that $\bar{\mathcal{F}}$ is **non-propagating** if there exists a non-decreasing function $\tau: (0,1] \to (0,1]$ such that:
  - $\forall \epsilon > 0 \; \exists \beta$ such that $\tau(\beta) < \epsilon$
  - For every graph $G = ([N], E)$ and every $B \subset [N]$ that covers G, either G is $\tau\left(\frac{|B|}{N}\right)$-close to being $\mathcal{F}_N$ free, or there are no N-vertex graphs that are $\mathcal{F}_N$-free.

Interesting observations

- For every bounded degree $d \geq 3$ we can find an $\bar{\mathcal{F}}$ that is **not** non-propagating
  – Proof on the board… (if time permits)
- **Induced** subgraph freeness **is** non-propagating
  – Proof on the board… (if time permits)
- Can find non-hereditary properties that are non-propagating
  – Proof on board… (if time permits)

Main Theorem

- Theorem: A graph property Π has a constant query proximity oblivious tester **if and only if** Π is local and non-propagating

Proof idea

- $\Rightarrow$ Being $\tau(\beta)$-far implies poly(β) fraction of the possible choices makes the tester reject.
- $\Leftarrow$ A proximity-oblivious tester can be converted to test constant surroundings that are equivalent to a non propagating sequence.

Sum things up…

- In the adjacency matrix model we saw:
  - A proximity-oblivious tester might be inferior
  - A proximity-oblivious tester may not even exist (while still being testable the ordinary way)
  - Characterisation of being proximity-oblivious testable
- In the bounded degree model we saw:
  - Charactarisation of being proximity-testable is very special
  - The bounded degree model contains the adjacency matrix model
  - Can test non-hereditary properties
  - Poses an interesting open question

Thank you!